The operator sum-difference representation for quantum maps: application to the two-qubit amplitude damping channel

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On account of the Abel-Galois no-go theorem for the algebraic solution to quintic and higher order polynomials, the eigenvalue problem and the associated characteristic equation for a general noise dynamics in dimension d via the Choi-Jamiolkowski approach cannot be solved in general via radicals. We provide a way around this impasse by decomposing the Choi matrix into simpler, not necessarily positive, Hermitian operators that are diagonalizable via radicals, which yield a set of 'positive' and 'negative' Kraus operators. The price to pay is that the sufficient number of Kraus operators is d^4 instead of d^2, sufficient in the Kraus representation. We consider various applications of the formalism: the Kraus repesentation of the 2-qubit amplitude damping channel, the noise resulting from a 2-qubit system interacting dissipatively with a vacuum bath; defining the maximally dephasing and purely dephasing components of the channel in the new representation, and studying their entanglement breaking and broadcast properties.

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I. INTRODUCTION

Any practical use of a quantum operation involves taking into account the effect of the ambient environment, and the systematic study of such an effect is called open quantum systems. This now prevades a vast arena of studies, see for e.g., the Refs. [1, 2] for two distinct flavors of the subject. The effect of the environment, interchangeably called the reservoir or the bath, effects the system dynamics, in general, in two ways depending upon the commutability of the system and interaction Hamiltonian. If the two commute, then the process is a quantum nondemolition one, that is, there is dephasing without any energy exchange [3]; while if they do not commute, then there is dephasing along with dissipation [4]. These effects have been brought within the ambit of practical implementation by a number of very impressive experiments, for e.g., [5] involving ion traps and [6] using high-Q cavity quantum electrodynamics.

As a result, the use of ideas from open quantum systems have become widespread in quantum information processing [7]. A very useful tool in this regard is the Kraus representation [8] which encodes the effect of the environment on the system of interest. Since a quantum operation that can be represented in the Kraus representation is guaranteed to be completely positive (CP), it is of importance to find the Kraus representation pertaining to different open system models. The main aim of this paper is to develop Kraus representations for general N-qubit open system models, and use it specifically on a two-qubit model much discussed in the literature [9]. The model considered is that of two qubits interacting with a bath, for e.g., an electromagnetic field in a squeezed thermal state, via the dipole interaction, which has been considered in detail for both pure dephasing [10] as well as dissipative [11] system-reservoir interactions. The system-reservoir coupling constant is dependent upon the position of the qubit, leading to interesting dynamical consequences. Basically this allows a classification of the dynamics into two regimes: the independent (or localized) decoherence regime, where the inter-qubit distances are such that each qubit sees an individual bath or the collective decoherence regime, where the qubits are close enough to justify a collective interaction with the bath.

When we attempt to obtain the Kraus operators, for both the independent as well as collective regimes for a version of the dissipative model [11], described below, we encounter a problem which has its origin in the famous Abel-Galois

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irreducibility theorem. We are able to circumvent this mathematical obstruction, in the case considered by resorting to the inherent symmetries in the model. This is accomplished by introducing the concept of extended Kraus operators. We can thus coin the word *Abel-Galois integrable* for such systems. It would be of interest, here, to note that the label *Abel-Galois integrable* could be ascribed to infinite dimensional systems also, such as the dissipative harmonic oscillator [12], which due to their inherent symmetries can be solved by quadratures.

The plan of the paper is as follows. In the next section, we briefly discuss, the Abel-Galois irreducibility theorem, due to its relevance to our work. We then, in anticipation of its need for the Kraus operators of the general two-qubit dissipative model, introduce the concept of extended Kraus operators. This is followed by some illustrative applications to a typical single-qubit channel. The use of this formalism is, of course, not needed for the single qubit case, where it is possible to obtain the usual Kraus operators, but is intended to serve as an illustrative example to the main application of extended Kraus operators to two-qubit dissipative interaction with a vacuum bath, the 2AD (amplitude damping) channel. We then indicate why this would not work for the case of a bath at finite temperature and bath squeezing, the two-qubit squeezed generalized amplitude damping channel (2SGAD). Next, we discuss some features of the 2AD channel. Finally, we make our conclusions.

II. ABEL-GALOIS IRREDUCIBILITY THEOREM

The first famous no-go result in algebra, the irreducibility theorem, discovered independently by mathematicians Niels Henrik Abel and Jean Evariste Gallois (and anticipated earlier by Ruffini) states that polynomial equations of degree 5 or higher do not *in general* have solutions that can be expressed algebraically, i.e., in terms of addition, subtraction, multiplication, division and taking roots to a given finite order over the equation's coefficients and rational numbers (and any finite number of irrationals). Obviously there are infinitely many examples where solutions thus expressible *do* exist, a trivial example being $f_5(x) \equiv (x-c)^5 = 0$, where c is a real or complex number, and where the solution $x = c$ is manifest. Similarly solutions to a product of a quartic and a linear polynomial, $f_4(x)f_1(x) = 0$, can be obtained by solving those polynomials separately. For the general case of $n \geq 5$, one would have to resort to numerical methods like the Laguerre method or the Newton-Raphson method.

Briefly, the argument, which is part of Galois theory, runs thus [13]: a polynomial equation over rational numbers (or more generally, over the base field of given constants) admits a solution by radicals precisely if its *Galois group* is a *solvable group*. For polynomials upto quartic degree, the associated Galois group is solvable, but for quintic and above, unsolvable cases exist.

Let the base field be the set $\mathbb{Q}$ of all rationals (augmented by at most a finite number of irrational constants, which we ignore for simplicity). Suppose we are given a polynomial

$$f(x) = \sum_{j=0}^{5} \alpha_j x^j \in \mathbb{Q}[x]. \tag{1}$$

If x_k are the solutions to $f(x) = 0$, then:

$$f(x) = \Pi_{k=1}^{5}(x - x_k) \in \mathbb{E}, \tag{2}$$

where the solutions x_k exist in general in the *splitting field* $\mathbb{E}$, the extension field of $\mathbb{Q}$, which is the smallest subfield of $\mathbb{C}$ containing the roots of $f(x)$.

Multiplying out Eq. (2) and comparing it with Eq. (1) shows the α_j's to be elementary symmetric functions of the roots x_j:

$$\begin{aligned} \beta_4 &= \sum_j x_j; \beta_3 = \sum_{j\neq k} x_j x_k; \beta_2 = \sum_{j\neq k, j\neq l, k\neq l} x_j x_k x_l, \\ \beta_1 &= \sum_j \frac{x_1 x_2 x_3 x_4 x_5}{x_j}; \beta_0 = \Pi_j x_j, \end{aligned} \tag{3}$$

where $\alpha_j = (-1)^{j+1}\beta_j$. We observe that every permutation σ of the roots, which is clearly an automorphism of $\mathbb{E}$, leaves $\beta_j \in \mathbb{Q}$, and hence $\mathbb{Q}$, fixed. It can be shown that this is the only automorphism of $\mathbb{E}$ that leaves $\mathbb{Q}$ fixed. The group of automorphisms σ of $\mathbb{E}$ such that $\sigma(q) = q$, with $q \in \mathbb{Q}$ is called the Galois group associated with the polynomial, and denoted $\text{Gal}(\mathbb{E}/\mathbb{Q}) \subseteq S_n$, the group of permutations on the roots of $f(x)$.

A subgroup N of group G is called normal if and only if $gN = Ng$ for all $g \in G$, or equivalently, $g^{-1}Ng = N$, that is the normal subgroup is invariant under conjugation of elements of G. The relationship is denoted $N \triangleleft G$. The normal subgroup defines the quotient or factor group, denoted G/N. A subnormal series (or tower) of a group G is a sequence of subgroups:

$$\{e\} \triangleleft A_0 \triangleleft A_1 \triangleleft A_2 \cdots \triangleleft A_n = G, \tag{4}$$

terminating in the trivial subgroup. A *composition series* is a subnormal tower such that $A_j \neq A_{j+1}$ and A_j is the maximal normal subgroup of A_{j+1}. The group G is solvable only if the factor groups A_{j+1}/A_j in the composition series are Abelian. The Abel-Galois theorem can be stated as follows: a polynomial equation is solvable by radicals iff its Galois group is solvable.

For all n, the maximal normal subgroup of S_n is A_n, the alternating group, which is the subgroup of even permutations on S. For $n \leq 4$, all subgroups of S_n are solvable. However for $n \geq 5$, A_n is non-Abelian, as is the factor group $A_n/\{e\}$, so that S_n is not solvable, and we find that polynomials that are quintic or of higher degrees are in general not solvable.

The significance of the Abel-Galois theorem for us is that our work on deriving the Kraus operators, for general two-qubit systems, requires diagonalizing a (density) matrix in a Hilbert space of dimension $d^2 = 16$, which involves solving the characteristic equation of a self-adjoint matrix of degree 16. In general, the relevant Galois group is S_{16}, which is unsolvable. We circumvent this problem, in any dimension, by making use of symmetry properties of the matrix to circumvent the problem of diagonalization. At times, it may happen that even if the Galois group is solvable, diagonalization can be so tedious that our alternative is preferable.

III. EXTENDED KRAUS OPERATORS

A transformation of a quantum state is a CP map if and only if it evolves a density operator according to the prescription [14]:

$$\rho \longrightarrow \rho' = \sum_j A_j \rho A_j^\dagger, \tag{5}$$

where A_j are at most d^2 operators that satisfy the completeness condition $\sum_j A_j^\dagger A_j = \mathbb{I}$. Here we will study the conditions under which a CP map allows a description of the form:

$$\rho \longrightarrow \mathcal{E}(\rho) = \rho' = \sum_{j=1}^{\mu} A_j^+ \rho \left(A^+\right)_j^\dagger - \sum_{k=1}^{\nu} A_k^- \rho \left(A^-\right)_k^\dagger, \tag{6}$$

where $\mu + \nu \geq d^2$ and the extended Kraus operators $A_j^\pm$ must satisfy the new completeness condition

$$\sum_{j=1}^{\mu} \left(A^+\right)_j^\dagger A_j^+ - \sum_{k=1}^{\nu} \left(A^-\right)_k^\dagger A_k^- = \mathbb{I}. \tag{7}$$

Let $|\mathcal{A}_j^\pm\rangle$ represent a vector 'unfolding' of $A_j^\pm$. Then the necessary and sufficient condition for the extended operator sum representation to be CP is that:

$$\sum_{j=1}^{\mu} |\mathcal{A}_j^+\rangle\langle\mathcal{A}_j^+| - \sum_{k=1}^{\nu} |\mathcal{A}_k^-\rangle\langle\mathcal{A}_k^-| \equiv \mathcal{B}^+ - \mathcal{B}^- = \mathcal{B}, \tag{8}$$

where $\mathcal{B}$ is the Choi matrix for the map.

To see this, we note that the Choi matrix corresponding to noise $\mathcal{E}$ can be written as:

$$\mathcal{B} = \sum_{jk} |j\rangle\langle k| \otimes \mathcal{E}\left(|j\rangle\langle k|\right). \tag{9}$$

Let $\mathcal{B} = \sum_j \mathcal{B}_j$ be any Hermitian partition of $\mathcal{B}$. The partition elements $\mathcal{B}_j$ can be spectrally decomposed with positive or negative eigenvalues. Let their corresponding eigenvectors, normalized to the absolute value of the eigenvalues be, $|\mathcal{A}_i^+\rangle$ and $|\mathcal{A}_i^-\rangle$, respectively. It follows from the properties of matrices that [15]:

$$|\mathcal{A}_i^\pm\rangle\langle\mathcal{A}_i^\pm| = \sum_{j,k} \left(|j\rangle\langle k| \otimes \left(A_i^\pm |j\rangle\langle k| A_i^{\pm\dagger}\right)\right). \tag{10}$$

Inserting these values in Eq. (8) and comparing the result with the expression in Eq. (9), we have:

$$\begin{aligned}
\mathcal{B} &= \sum_{j=1}^{\mu} |\mathcal{A}_j^+\rangle\langle\mathcal{A}_j^+| - \sum_{k=1}^{\nu} |\mathcal{A}_k^-\rangle\langle\mathcal{A}_k^-| \\
&= \sum_{j,k} \left(|j\rangle\langle k| \otimes \left[\sum_{i=1}^{\mu} A_i^+ \left(|j\rangle\langle k| A_i^{+\dagger}\right)\right] - \left[\sum_{i=1}^{\nu} A_i^- \left(|j\rangle\langle k| A_i^{-\dagger}\right)\right]\right) \\
&= \sum_{jk} |j\rangle\langle k| \otimes \mathcal{E}\left(|j\rangle\langle k|\right), \qquad (11)
\end{aligned}$$

from which Eq. (6) follows, because the sum-difference operation on each element $|j\rangle\langle k|$ reproduces the effect of $\mathcal{E}$ on that element, and thus also on any linear combination, such as ρ, of those elements.

We consider an example of the above idea, applied to the generalized amplitude damping channel (GAD) [4]. The Choi matrix corresponding to GAD channel with elements $\left[\sqrt{p}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\lambda}\end{pmatrix}, \sqrt{p}\begin{pmatrix}0 & 0\\ \sqrt{\lambda} & 0\end{pmatrix}, \sqrt{1-p}\begin{pmatrix}\sqrt{1-\lambda} & 0\\ 0 & 1\end{pmatrix}, \sqrt{1-p}\begin{pmatrix}0 & \sqrt{\lambda}\\ 0 & 0\end{pmatrix}\right]$ is

$$\mathcal{B} = \begin{pmatrix} 1-\lambda+p\lambda & 0 & 0 & \sqrt{1-\lambda} \\ 0 & p\lambda & 0 & 0 \\ 0 & 0 & (1-p)\lambda & 0 \\ \sqrt{1-\lambda} & 0 & 0 & 1-p\lambda \end{pmatrix}. \qquad (12)$$

A decomposition of the above matrix is $\mathcal{B} = \mathcal{B}^+ - \mathcal{B}^-$, with

$$\mathcal{B}^+ = \begin{pmatrix} 1-\lambda+p\lambda+\frac{\sqrt{1-\lambda}}{2} & 0 & 0 & \frac{3\sqrt{1-\lambda}}{4} \\ 0 & p\lambda & 0 & 0 \\ 0 & 0 & (1-p)\lambda & 0 \\ \frac{3\sqrt{1-\lambda}}{4} & 0 & 0 & 1-p\lambda+\frac{\sqrt{1-\lambda}}{2} \end{pmatrix}; \mathcal{B}^- = \begin{pmatrix} \frac{\sqrt{1-\lambda}}{2} & 0 & 0 & -\frac{\sqrt{1-\lambda}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sqrt{1-\lambda}}{4} & 0 & 0 & \frac{\sqrt{1-\lambda}}{2} \end{pmatrix}. \qquad (13)$$

Both the matrices are seen to be Hermitian. The positive Kraus operators are

$$\begin{aligned}
K_1^+ &= \frac{\sqrt{4+2\sqrt{1-\lambda}-2\lambda-\sqrt{a}}}{2}\begin{pmatrix} -\frac{2\lambda(1-2p)+\sqrt{a}}{3\sqrt{1-\lambda}} & 0 \\ 0 & 1\end{pmatrix}, \qquad (14) \\
K_2^+ &= \frac{\sqrt{4+2\sqrt{1-\lambda}-2\lambda+\sqrt{a}}}{2}\begin{pmatrix} -\frac{2\lambda(1-2p)-\sqrt{a}}{3\sqrt{1-\lambda}} & 0 \\ 0 & 1\end{pmatrix}, \\
K_3^+ &= \begin{pmatrix} 0 & \sqrt{(1-p)\lambda} \\ 0 & 0 \end{pmatrix}, \qquad K_4^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{p\lambda} & 0 \end{pmatrix},
\end{aligned}$$

where $a = 9(1-\lambda) + 4\lambda^2(1-2p)^2$. The negative Kraus operators are

$$K_1^- = \frac{(1-\lambda)^{1/4}}{2} I, \qquad K_2^- = \frac{\sqrt{3}(1-\lambda)^{1/4}}{2}\sigma_z. \qquad (15)$$

Of course, the extended Kraus formalism used here is not necessary, since the usual Kraus operators for these channels can be obtained analytically. The purpose here was to serve as a simple example that sets the scene, in what follows, for the non-trivial application of the formalism in the two-qubit case.

IV. INTRODUCTION TO TWO-QUBIT DYNAMICS

Consider the Hamiltonian, describing the dissipative interaction of two qubits with the bath via the dipole interaction as [9]

$$\begin{aligned}
H &= H_S + H_R + H_{SR} \\
&= \sum_{n=1}^{2} \hbar\omega_n S_n^z + \sum_{\vec{k}s} \hbar\omega_k (b_{\vec{k}s}^\dagger b_{\vec{k}s} + 1/2) - i\hbar \sum_{\vec{k}s}\sum_{n=1}^{2} [\vec{\mu}_n . \vec{g}_{\vec{k}s}(\vec{r}_n)(S_n^+ + S_n^-) b_{\vec{k}s} - h.c.]. \qquad (16)
\end{aligned}$$

Here H_S is the system, H_R the bath, and H_{SR} the interaction Hamiltonians, respectively, and $\vec{\mu}_n$ are the transition dipole moments, dependent on the different atomic positions $\vec{r}_n$. Also, $S_n^{\pm}$ are the dipole raising and lowering operators, respectively while S_n^z is the energy operator of the nth atom, and $b^{\dagger}_{\vec{k}s}$, $b_{\vec{k}s}$ are the creation and annihilation operators of the (bath) field mode $\vec{k}s$ with the wave vector $\vec{k}$, frequency ω_k and polarization index $s = 1, 2$ with the system-reservoir (S-R) coupling constant

$$\vec{g}_{\vec{k}s}(\vec{r}_n) = (\frac{\omega_k}{2\varepsilon\hbar V})^{1/2}\vec{e}_{\vec{k}s}e^{i\vec{k}.r_n}. \tag{17}$$

In Eq. (17) V is the normalization volume and $\vec{e}_{\vec{k}s}$ is the unit polarization vector of the field. It can be seen from Eq. (17) that the S-R coupling constant is dependent on the atomic position r_n leading to the possibility of considering the dynamics in the independent or collective regimes, depending on whether the qubits are far apart or close with respect to the length scales set by the environment. Assuming separable initial conditions, and taking a trace over the bath the reduced density matrix of the two-qubit system can be obtained [11]. We will now attempt to obtain the Kraus operators for this model, in a unified way for both the independent as well as collective regimes, first for the case of a vacuum bath, that is, the 2AD channel and then, point out the difficulties when we encounter finite temperature and bath squeezing.

The two-qubit density matrix in dressed state basis is

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{es} & \rho_{ea} & \rho_{eg} \\ \rho_{se} & \rho_{ss} & \rho_{sa} & \rho_{sg} \\ \rho_{ae} & \rho_{as} & \rho_{aa} & \rho_{ag} \\ \rho_{ge} & \rho_{gs} & \rho_{ga} & \rho_{gg} \end{pmatrix}. \tag{18}$$

The time-evolution of the two-qubit density matrix ρ to $\rho' = \mathcal{E}(\rho)$ is given by:

$$\rho' = \begin{pmatrix} A\rho_{ee} & J\rho_{es} & M\rho_{ea} & L\rho_{eg} \\ J^*\rho_{se} & B\rho_{ss} + C\rho_{ee} & P\rho_{sa} & T\rho_{sg} + (U+iV)\rho_{es} \\ M^*\rho_{ae} & P^*\rho_{as} & D\rho_{aa} + E\rho_{ee} & Q\rho_{ag} + (iS-R)\rho_{ea} \\ L^*\rho_{ge} & T^*\rho_{gs} + (U^*-iV^*)\rho_{se} & Q^*\rho_{ga} + (-iS^*-R^*)\rho_{ae} & \rho_{gg} + F\rho_{ss} + G\rho_{aa} + H\rho_{ee} \end{pmatrix}, \tag{19}$$

where $A, B, C, \ldots, U, V, S, R$ are given Appendix A.

The Choi matrix for the above interaction, which is the density operator $(\mathcal{E} \otimes \mathbb{I})(|\Phi\rangle\langle\Phi|)$, where $|\Phi\rangle \equiv |00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle$, is given by:

$$\mathcal{B} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & L \\ 0 & C & 0 & 0 & 0 & 0 & 0 & U+iV & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & iS-R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ J^* & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 & T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U^*-iV^* & 0 & 0 & 0 & 0 & F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M^* & 0 & 0 & 0 & 0 & P^* & 0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 0 & Q \\ 0 & 0 & -iS^*-R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L^* & 0 & 0 & 0 & 0 & T^* & 0 & 0 & 0 & 0 & Q^* & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

It turns out (as can be found using Mathematica software) that the relevant characteristic equation is cubic, so that the Galois group is a subset of S_3. This is so because the matrix in Eq. (20) is sparse, and is found not to be the case otherwise. However the solutions are so tediously long that they would be hardly of practical interest, and it is advantageous to use our method, which is applicable quite generally (even for non-sparse Hermitian matrices).

As a result, the formalism of Section (III) will be used to derive extended Kraus operators. We find the following decomposition convenient:

$$\mathcal{B} \equiv \mathcal{B}_{\rm diag} + \mathcal{B}_J + \mathcal{B}_M + \mathcal{B}_L + \mathcal{B}_P + \mathcal{B}_Q + \mathcal{B}_T + \mathcal{B}_U + \mathcal{B}_V + \mathcal{B}_R + \mathcal{B}_S, \tag{21}$$

where $\mathcal{B}_{\rm diag}$ is the submatrix of $\mathcal{B}$ consisting of precisely the diagonal entries in $\mathcal{B}$, and 0's elsewhere; $\mathcal{B}_J$ is the submatrix consisting only of the conjugate terms J, J^*, and 0's elsewhere, and so on.

A. Diagonal terms

It is straightforward to see that by diagonalizing $\mathcal{B}_{diag} = (A|0000\rangle\langle 0000| + C|0001\rangle\langle 0001| + E|0010\rangle\langle 0010| + H|0011\rangle\langle 0011| + B|0101\rangle\langle 0101| + F|0111\rangle\langle 0111| + D|1010\rangle\langle 1010| + G|1011\rangle\langle 1011| + |1111\rangle\langle 1111|)$, we have

$$\mathcal{B}_{diag} \equiv \sum_{j=1}^{9} |\mathcal{K}_j^+\rangle\langle\mathcal{K}_j^+|. \tag{22}$$

Then the Kraus operators obtained by 'folding' each eigenvector $\mathcal{K}_j^+$ is:

$$\begin{aligned} K_H^+ &= \sqrt{H}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\ K_G^+ = \sqrt{G}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},\ K_F^+ = \sqrt{F}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}, \\ K_E^+ &= \sqrt{E}\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\ K_D^+ = \sqrt{D}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\ K_C^+ = \sqrt{C}\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \\ K_A^+ &= \sqrt{A}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ K_1^+ = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\ K_B^+ = \sqrt{B}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \end{aligned} \tag{23}$$

These operators lead to the evolution $\mathcal{E}_{\text{diag}}$, which transforms only the diagonal elements, killing off the rest:

$$\rho'_{\text{diag}} \equiv \mathcal{E}_{\text{diag}}(\rho) = \sum_{j\in\mathbf{T}} K_j^+ \rho \left(K_j^+\right)^\dagger = \begin{pmatrix} A\rho_{ee} & 0 & 0 & 0 \\ 0 & B\rho_{ss} + C\rho_{ee} & 0 & 0 \\ 0 & 0 & D\rho_{aa} + E\rho_{ee} & 0 \\ 0 & 0 & 0 & \rho_{gg} + F\rho_{ss} + G\rho_{aa} + H\rho_{ee} \end{pmatrix}, \tag{24}$$

$j \in \mathbf{T} \equiv \{H, G, F, E, D, C, A, B, 1\}$.

B. Off-diagonal terms

By diagonalizing $\mathcal{B}_J = J|0000\rangle\langle 0100| + J^*|0100\rangle\langle 0000|$, we have:

$$\mathcal{B}_J \equiv |\mathcal{K}_J^+\rangle\langle\mathcal{K}_J^+| - |\mathcal{K}_J^-\rangle\langle\mathcal{K}_J^-|. \tag{25}$$

By 'folding' the vectors, we obtain the Kraus operators

$$K_J^- = \sqrt{\frac{|J|}{2}}\begin{pmatrix}1&0&0&0\\0&-e^{i\phi_J}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ K_J^+ = \sqrt{\frac{|J|}{2}}\begin{pmatrix}1&0&0&0\\0&e^{i\phi_J}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \tag{26}$$

where $\phi_J = -(\omega_0 - \Omega_{12})t$. The evolution produced by these operators transforms two conjugate elements of ρ into the two corresponding elements of ρ', while annihilating all other elements in the density operator. Thus:

$$\rho'_J = K_J^+ \rho (K_J^+)^\dagger - K_J^- \rho \left(K_{10}^-\right)^\dagger = \begin{pmatrix} 0 & e^{i\phi_J}|J|\rho_{es} & 0 & 0 \\ e^{-i\phi_J}|J|\rho_{se} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

Proceeding thus with the other terms in Eq. (21), we obtain pairs of positive and negative Kraus operators:

$$
\begin{aligned}
K^{\pm}_M &= \sqrt{\frac{|M|}{2}}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&\pm e^{i\phi_M}&0\\0&0&0&0\end{pmatrix},\; K^{\pm}_L = \sqrt{\frac{|L|}{2}}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&\pm e^{i\phi_L}\end{pmatrix},\; K^{\pm}_P = \sqrt{\frac{|P|}{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&\pm e^{i\phi_P}&0\\0&0&0&0\end{pmatrix},\\
K^{\pm}_T &= \sqrt{\frac{|T|}{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&\pm e^{i\phi_T}\end{pmatrix},\; K^{\pm}_U = \sqrt{\frac{|U|}{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&\pm e^{i\phi_U}&0&0\end{pmatrix},\; K^{\pm}_V = \sqrt{\frac{|V|}{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&\pm e^{i\phi_V}&0&0\end{pmatrix},\\
K^{\pm}_Q &= \sqrt{\frac{|Q|}{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&\pm e^{i\phi_Q}\end{pmatrix},\; K^{\pm}_S = \sqrt{\frac{|S|}{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&\pm e^{i\phi_S}&0\end{pmatrix},\; K^{\pm}_R = \sqrt{\frac{|R|}{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&\pm e^{i\phi_R}&0\end{pmatrix},
\end{aligned} \tag{28}
$$

where $\phi_L = -2\omega_0 t, \phi_M = -(\omega_0+\Omega_{12})t, \phi_P = -2\Omega_{12}t, \phi_T = -(\omega_0+\Omega_{12})t = \phi_U, \phi_V = -(\omega_0+\Omega_{12})t + \pi/2, \phi_Q = -(\omega_0-\Omega_{12})t, \phi_S = -(\omega_0-\Omega_{12})t+\pi/2, \phi_R = -(\omega_0-\Omega_{12})t+\pi$.
These operators lead, analogously to Eq. (27) to the partial evolutions $\rho^\prime_M, \rho^\prime_L, \rho^\prime_P, \rho^\prime_T, \rho^\prime_X, \rho^\prime_Q$ and $\rho^\prime_Y$, which satisfy:

$$
\rho^\prime = \rho^\prime_{\rm diag} + \rho^\prime_J + \rho^\prime_M + \rho^\prime_L + \rho^\prime_P + \rho^\prime_T + \rho^\prime_U + \rho^\prime_V + \rho^\prime_Q + \rho^\prime_R + \rho^\prime_S, \tag{29}
$$

while

$$
\mathcal{B} \;=\; \sum_{j=1}^{9} |\mathcal{K}^+_j\rangle\langle\mathcal{K}^+_j| + \sum_{j\in\mathbf{S}}\left(|\mathcal{K}^+_j\rangle\langle\mathcal{K}^+_j| - |\mathcal{K}^-_j\rangle\langle\mathcal{K}^-_j|\right), \tag{30}
$$

where $\mathbf{S} = \{J, M, L, P, T, U, V, Q, R, S\}$.

In short, our strategy to circumvent the Abel-Galois theorem is to replace the problem of diagonalizing the Choi matrix, by that of diagonalizing simpler Hermitian matrices that sum up to the Choi matrix. The elements of this Hermitian partition are diagonal or rank-two matrices, and thus trivially or readily diagonalized, in the case we considered. The price to pay is that the number of Kraus operators required to implement the CP map on a density operator in an d-dimensional Hilbert space can be as many as $d^4 = d^2 + \frac{d^2(d^2-1)}{2}\times 2$ (cf. Section IV C), which is in general more than d^2, the sufficient number of Kraus operators in the conventional formalism [7].

C. Abel-Galois non-integrability

In our method, $\mathcal{B}_{\rm diag}$ is a diagonal matrix while $\mathcal{B}_i$ $(i = J, M, L, P, Q, T, U, V, R, S)$ are all rank-2 Hermitian matrices, which are readily diagonalizable. We will assume that the base field is $\mathbb{Q}(\mathcal{B})$, i.e., $\mathbb{Q}$ augmented by the entries in $\mathcal{B}$. In that case, the Galois group of $\mathcal{B}_{\rm diag}$ is trivial (consisting of the identity permutation) in that the splitting field of the characteristic equation

$$
\chi_{\rm diag} \equiv (x-A)(x-C)(x-E)(x-H)(x-B)(x-F)(x-D)(x-G)(x-1), \tag{31}
$$

is the base field itself. The ten remaining component matrices of $\mathcal{B}$ are rank-2 matrices. The characteristic equation, for example, for $\mathcal{B}_J$ in Eq. (25), is:

$$
\chi_J \equiv x^2 - |J|^2, \tag{32}
$$

with solutions $\pm|J| = \pm\sqrt{J_R^2 + J_I^2}$, which are in general irrational. Hence the symmetry group is S_2 consisting of the identity element and an interchange.

Thus our method of circumventing the Abel-Galois theorem can be considered as reducing a problem requiring the solution of (the unsolvable) S_{d^2} to one requiring that of $(S_2)^{\times d^2(d^2-1)/2}$. It turns out that for the particular form (20) of the Choi matrix for the 2AD channel, the characteristic equation is cubic and indeed solvable (as can be found using Mathematica). However, the resulting eigenvalues and eigenvectors are so ponderous, that our method is preferable.

For the two-qubit squeezed generalized amplitude damping (2SGAD) channel [11], the problem is seen to be *Abel-Galois non-integrable*, i.e., it does not admit analytic Kraus operators.

V. AN APPLICATION OF THE OPERATOR SUM-DIFFERENCE FORMALISM

We indicate some features of the operator sum-difference formalism in identifying special cases of a given noise that may have special interest.

A. The maximally dephasing component

It may be interesting to note that the Kraus operators K_j^+, $j \in \mathbf{T} \equiv \{H, G, F, E, D, C, A, B, 1\}$, which correspond to the diagonal terms in the Choi matrix, by themselves constitute a CP trace-preserving (CPT) dynamics. Physically, this channel represents a dynamics of the populations (diagonal terms of the density operator) that is completely dephasing. The evolution under this channel, which may be called the maximally dephasing component (MDC) channel, is given by Eq. (24). Because all Kraus operators of this channel are of rank 1, it has the property of being *entanglement breaking*. In any dimension, it has an analytical operator sum (as against, sum-difference) representation.

A quantum channel $\mathcal{E}$ is called entanglement breaking and trace-preserving (EBT) if given any input state Γ, the state $(\mathbb{I} \otimes \mathcal{E})\Gamma$ is separable. This is equivalent to the separability of the Choi matrix $(I \otimes \mathcal{E})|\psi^+\rangle\langle\psi^+|$ and also to the condition that it can be expressed in the *Holevo form*

$$\mathcal{E}_{EB}(\rho) = \sum_i R_i Tr(F_i \rho), \tag{33}$$

where $\{R_i\}$ is a set of density operators and $\{F_i\}$ is a set of positive operator valued measures (POVM) [16].

The 2AD channel is asymptotically ($t \longrightarrow \infty$) EBT because only the Kraus operators elements of the MDC survive. In the asymptotic limit the channel is characterized by the fact that in (20), $H, F, G \longrightarrow 1$, whereas $A, B, C, D, E, J, K, L, M, P, Q, X, Y \longrightarrow 0$. Therefore, only the following 4 rank-1 Kraus operators, which are elements of the MDC channel, survive:

$$K_H^+(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad K_F^+(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad K_G^+(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad K_1^+(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

This channel produces the asymptotic state $|\Psi_\infty\rangle \equiv |11\rangle$, so that the corresponding Choi matrix has the separable form $|11\rangle\langle 11| \otimes I$ and the channel is entanglement breaking: for an arbitrary initial state of the 2-qubit system possibly entangled with any other system, asymptotically the 2-qubit system factors out to $|11\rangle$.

Furthermore, asymptotically it is of an extreme point of entanglement breaking, whereby the channel maps all input states to a single point in state space, the pure state $|11\rangle$: it is thus a *point channel* [16]. The Kraus operators of EB channel can be expressed as $K_i = \sqrt{R_i}|jk\rangle\langle lm|\sqrt{F_i}$, which are seen to reproduce the expression in Eq. (33). For the considered point channel $R_i = R = |11\rangle\langle 11|$, $F_i = \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$ and the resulting K_i are easily seen to conicide with the operators in Eq. (34).

A special case of the entanglement breaking channel is the quantum-to-classical (QC) measurement map, wherein the states R_i in Eq. (33) are orthogonal projectors $|e_j\rangle\langle e_j|$ of a fixed basis. Acting on any 2-system state, it produces a QC state, which has the form:

$$\rho^{QC} = \sum_j p_j \sigma_j \otimes |e_j\rangle\langle e_j|. \tag{35}$$

A fundamental result here, which refines the equivalent result for EBT channels, is that a channel Λ is of QC-type (meaning that $(\mathbb{I} \otimes \Lambda)(\rho_{AB})$ is a QC state for any bipartite state ρ_{AB}) if and only if the correponding Choi matrix is a QC state [17]. A particular feature of interest here is that, for any QC-type channel Λ, given any orthonormal basis $\{\phi_j\}$, there exists at least one state $\rho^*(\phi)$ diagonal in this basis, which is N-copy spectrum-broadcastable using the channel. If ϕ is identified with the channel basis $\{|e_j\rangle\}$, then $\rho^*(e)$ is fully broadcastable. Here the broadcast is state σ shared between Alice and N other parties, such that the reduced density matrix at each party has the same eigenvalue spectrum as $\rho^*(\phi)$ (spectrum broadcast) or is identical with $\rho^*(\phi)$ (full broadcast).

The question then arises of whether the MDC channel is also of QC-type. The answer, according the above quoted result, is in the negative, as seen from the matrix $\mathcal{B}_{\rm diag}$, formed by the diagonal terms of $\mathcal{B}$ in Eq. (20), which cannot be cast in the form (33) with R_i given by orthogonal projectors. To see this, we note that because only diagonal terms are present, both σ_j and $|e_j\rangle$ in Eq. (35) must be diagonal in the computational basis. Then it is clear why $\mathcal{B}_{\rm diag}$, though separable, is not QC. In particular, looking at the first 4 diagonal terms, we should have $C = H = 0$, which is not in general guaranteed.

B. The purely dephasing component

By contrast, a non-dissipative and purely dephasing operation is obtained from a channel comprising the extended Kraus operators $K_j^{\pm}$ $j \in \mathbf{S} \equiv \{J, M, L, P, T, U, V, Q, R, S\}$. This has the effect of dephasing the off-diagonal terms, but killing off the diagonal terms. Restoration of the diagonal components is obtained by use of positive Kraus operators given by all the projetors: $\Pi_{00} \equiv |00\rangle\langle 00|$, $\Pi_{01} \equiv |01\rangle\langle 01|$, $\Pi_{10} \equiv |10\rangle\langle 10|$ and $\Pi_{11} \equiv |11\rangle\langle 11|$, which collectively form the set $\mathbf{U}$. The elements of this set are just the Kraus operators K_D^+, K_A^+, K_B^+ and K_1^+ in Eq. (24), with $D, A, B = 1$.

Together the operators in the set $\mathbf{S}\cup\mathbf{U}$ constitute a CP map that is a purely dephasing component (PDC) of the 2AD channel, which leaves the populations (diagonal components) unchanged, but otherwise reproduces the (dephasing) effects of the 2AD channel. Its action is given by:

$$\begin{aligned}\rho'_{\rm pd} &= \sum_{j\in\mathbf{U}} K_j^+ \rho (K_j^+)^\dagger + \sum_{j\in\mathbf{S}} K_j^{\pm} \rho (K_j^{\pm})^\dagger \\ &= \begin{pmatrix} \rho_{ee} & J\rho_{es} & M\rho_{ea} & L\rho_{eg} \\ J^*\rho_{se} & \rho_{ss} & P\rho_{sa} & T\rho_{sg} + (U+iV)\rho_{es} \\ M^*\rho_{ae} & P^*\rho_{as} & \rho_{aa} & Q\rho_{ag} + (iS-R)\rho_{ea} \\ L^*\rho_{ge} & T^*\rho_{gs} + (U^*-iV^*)\rho_{se} & Q^*\rho_{ga} + (-iS^*-R^*)\rho_{ae} & \rho_{gg} \end{pmatrix}. \end{aligned} \tag{36}$$

Unlike MDC, this is not entanglement breaking at finite time. To see this, suppose that the input state is $\psi_{\rm in} \equiv \frac{1}{\sqrt{2}}(|00,00\rangle + |11,11\rangle)$, i.e., a 1-bit entanglement between the given 2-qubit system, with another 2-qubit system. The effect of the MDC channel is seen to be

$$\rho_{\rm out} = \frac{1}{2}\left(|00,00\rangle\langle 00,00| + |11,11\rangle\langle 11,11| + J(e^{i\phi_L}|00,00\rangle\langle 11,11| + e^{-i\phi_L}|11,11\rangle\langle 00,00|)\right).$$

The above state lives in four dimensional Hilbert space, given by $\mathcal{H}_2 \otimes \mathcal{H}_2$, spanned by kets $\{|00\rangle, |11\rangle\}$ in each two-qubit system. The state is thus entirely equivalent to a correlated state of two qubits given by:

$$\rho'_{\rm out} = \frac{1}{2}\left(|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1| + J(e^{i\phi_L}|0,0\rangle\langle 1,1| + e^{-i\phi_L}|1,1\rangle\langle 0,0|)\right),$$

for which the concurrence [18], a measure of entanglement for a mixed 2-qubit system is J, implying that entanglement is not broken except asymptotically (when $J \to 0$).

VI. CONCLUSIONS

In the problem of deriving the Kraus representation of a given noise dynamics via the Choi-Jamiolkowski approach, we have developed a method that circumvents the impasse due to the Abel-Galois no-go theorem for the algebraic solution to quintic and higher order polynomials. Our idea is to obtain a Hermitian decomposition of the Choi matrix, which yield a set of 'positive' and 'negative' Kraus operators, satisfying the completeness condition.

The price to pay is that, in general, the sufficient number of Kraus operators for a d-dimensional system is d^4, rather than the usually sufficient number of d^2.

We have applied this formalism to three problems: the sum-difference representation of the 2-qubit amplitude damping channel; defining two mathematically interesting limits of any channel in this representation, namely the MDC and PDC channels, whose entanglement breaking property was studied. In particular, the MDC channel is EBT, whereas the PDC channel is not, except asymptotically.

Appendix A: The two-qubit amplitude damping channel

Information useful for the description of the 2AD channel, discussed in Sec. IV, are:

$A = e^{-2\Gamma t}$; $B = e^{-(\Gamma+\Gamma_{12})t}$; $C = \frac{\Gamma+\Gamma_{12}}{\Gamma-\Gamma_{12}}(1 - e^{-(\Gamma-\Gamma_{12})t})e^{-(\Gamma+\Gamma_{12})t}$; $D = e^{-(\Gamma-\Gamma_{12})t}$; $E = \frac{\Gamma-\Gamma_{12}}{\Gamma+\Gamma_{12}}(1 - e^{-(\Gamma+\Gamma_{12})t})e^{-(\Gamma-\Gamma_{12})t}$; $F = 1 - e^{-(\Gamma+\Gamma_{12})t}$; $G = 1 - e^{-(\Gamma-\Gamma_{12})t}$; $J = e^{-i(\omega_0-\Omega_{12})t}e^{(3\Gamma+\Gamma_{12})t/2}$; $L = e^{-i2\omega_0 t}e^{-\Gamma t}$;

$M = e^{-i(\omega_0+\Omega_{12})t}e^{-(3\Gamma-\Gamma_{12})t/2}$; $P = e^{-i2\Omega_{12}t}e^{-\Gamma t}$; $Q = e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}$; $T = e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}$;

$$\begin{aligned} H &= \frac{\Gamma+\Gamma_{12}}{2\Gamma}\left[1-\frac{2}{\Gamma-\Gamma_{12}}\left(\frac{\Gamma+\Gamma_{12}}{2}(1-e^{-(\Gamma-\Gamma_{12}t})+\frac{\Gamma-\Gamma_{12}}{2}\right)e^{-(\Gamma+\Gamma_{12})t}\right] \\ &+ \frac{\Gamma-\Gamma_{12}}{\Gamma+\Gamma_{12}}\left[(1-e^{-(\Gamma-\Gamma_{12})t})-\frac{\Gamma-\Gamma_{12}}{2\Gamma}(1-e^{-2\Gamma t})\right], \end{aligned} \tag{A1}$$

$$\begin{aligned} R &= \frac{\Gamma-\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}\left[2\Omega_{12}e^{-\Gamma t}\sin(2\Omega_{12}t)+\Gamma(1-e^{-\Gamma t}\cos(2\Omega_{12}t))\right], \\ S &= \frac{\Gamma-\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}\left[2\Omega_{12}(1-e^{-\Gamma t}\cos(2\Omega_{12}t))-\Gamma e^{-\Gamma t}\sin(2\Omega_{12}t))\right], \\ U &= \frac{\Gamma+\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}\left[2\Omega_{12}e^{-\Gamma t}\sin(2\Omega_{12}t)+\Gamma(1-e^{-\Gamma t}\cos(2\Omega_{12}t))\right], \\ V &= \frac{\Gamma+\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}\left[2\Omega_{12}(1-e^{-\Gamma t}\cos(2\Omega_{12}t))-\Gamma e^{-\Gamma t}\sin(2\Omega_{12}t))\right]. \end{aligned} \tag{A2}$$

The terms ω_0, Ω_{12}, Γ, Γ_{12} are as defined in [11].

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