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Stabilization of a Fractional Order Chain of Integrators: A Contraction Based Approach

S. Kamal[1], *B. Bandyopadhyay* [1], *S. Spurgeon* [2],
[1] *IDP in Systems and Control Engg., Indian Institute of Technology Bombay, India*
[2] *School of Engineering and Digital Arts, University of Kent, UK*

In this paper, stabilization of a chain of fractional order integrators is attempted. The stability is proved using contraction analysis. A numerical example is presented to illustrate the proposed method.

Keywords: Fractional order integrators, Contraction theory, Stabilization

1. Introduction

The idea of fractional calculus has been established since the development of the regular integer order calculus, with the first reference being associated with Leibniz and L'Hospital in 1695 where half order derivatives were mentioned. In the last two decades, fractional differential equations have been used to model various physical phenomena. We can refer to Machado *et al.* (2010) and Sabatier *et al.* (2013) for the recent history of fractional calculus and state space representation respectively. Since it plays an incessantly important role in modeling significant phenomenon in science and engineering, the study of stability of fractional differential equations has attracted much attention by Li *et al.* (2011). Furthermore, in recent years, increasing attention has been given to fractional order controllers, and a great deal of progress has been achieved in Kamal *et al.* (2012), Ortigueira *et al.* (2008) and Dingy *et al.* (2002) and the references cited therein.

There are several types of stability concepts proposed in control theory depending on how and when the system is stabilized. One of the most classic concepts regarding stability is obtained by introducing a weighted norm, or more generally by defining a Lyapunov function, which is decreasing at each instant and is strongly related to the norm of the system states. The revisiting concept of contraction leads to the introduction of suitable Riemann metrics or more generally Finsler metrics. Recently, inspired from fluid mechanics and differential geometry, Lohmiller and Slotine proposed a new method of stability analysis known as *contraction theory* (Lohmiller *et al.* (1998), Aylward *et al.* (2008)) and (Jouffroy *et al.* (2010)). This theory is based on the concept that the stability can be analyzed differentially by analyzing if the nearby trajectories converge to one another, rather than through finding some implicit motion integral as in Lyapunov theory, or through some global state transformation as in feedback linearization (see Angeli (2002)).

Making use of the concepts of contraction theory we have attempted to design a globally exponentially stable controller for fractional order systems. To the best of the author's knowledge this is the first attempt in this direction.

The paper has been organized in the following way: In Section II some preliminaries of fractional calculus have been presented. Section III introduces the concept of contraction theory and the motivation behind present work is given in Section IV. The main result is presented in Section V. In Section VI, a numerical example is presented to illustrate the proposed method. Finally, some concluding remarks are included in Section VII.

2. Preliminaries of Fractional order calculus

Fractional-order integration and differentiation are the generalization of their integer-order counterparts. Efforts to extend the specific definition of the traditional integer-order to the more general arbitrary order produced different definitions for fractional derivatives. One of the most common definitions used is the Reimann-Liouville definition (see Podlubny *et al.* (1999)).

DEFINITION 2.1 The α^{th}-order fractional integration of the function $f(t)$ with respect to t and the terminal value t_0 is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}} d\tau, \tag{2.1} $$

and, the R-L definition of the α^{th}-order fractional derivative is given by:

$$ D^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(\alpha-m+1)}} d\tau, \tag{2.2} $$

and, the Caputo definition of the α^{th}-order fractional derivative is given by:

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{(\alpha-m+1)}} d\tau, \tag{2.3} $$

where m is the first integer larger than α and $\Gamma(\cdot)$ is Euler's Gamma function.

Properties A few important properties of fractional derivatives and integrals follows (see Chen *et al.* (2009)):-

- For $\alpha = n$, where n is an integer, the operation $D^{\alpha} f(t)$ gives the same result as classical differentiation of integer order n.
- For $\alpha = 0$ the operation $D^{\alpha} f(t)$ is the identity operator:

$$ D^{\alpha} f(t) = f(t). \tag{2.4} $$

- Fractional differentiation and integration are linear operations:

$$ D^{\alpha}(af(t) + bg(t)) = aD^{\alpha} f(t) + bD^{\alpha} g(t). \tag{2.5} $$

- The additive index law (semigroup property)

$$ D^{\alpha} D^{\beta} f(t) = D^{\beta} D^{\alpha} f(t) = D^{\alpha+\beta} f(t), \tag{2.6} $$

  holds under some reasonable constraints on the function $f(t)$.

REMARK 2.1 *In the fractional calculus, Caputo derivative and Riemann-Liouville are mostly used Li et al. (2011). It seems that the former is more acceptable, since the initial value of fractional differential equation with Caputo derivative is the same as that of the integer differential equation. For example, the initial value $x(t)$ of fractional differential equation ${}_0^C D_t^{\alpha} x(t) = f(t,x)$ with $\alpha \in (0,1), t > 0$ is assumed as $x(0) \equiv x_0$.*

But for the fractional differential equation ${}_0^{RL}D_t^{\alpha}x(t) = f(t,x)$ with $\alpha \in (0,1), t > 0$, the initial value of x(t) involves fractional integral, (and/ or derivative), its initial condition is given as $[{}_0^{RL}D_t^{\alpha-1}x(t)]_{t=0} = x_0'$.

However, Caputo definition is not able to capture exact physical behavior of the system which is illustrated by Sabatier et al. (2010), that system trajectories generated by Caputo definition, when initial condition is nonzero differs from the actual. Physical and geometrical interpretations for fractional derivatives and initial value condition can be found in Heymans et al. (2008) and more coherent way in Sabatier et al. (2010).

Another way (see Zhang et al. (2009)), fractional-order initial value conditions for RL-type differential equation can be given as follows:- For example, the initial value condition for the ${}_0^{RL}D_t^{\alpha}x(t) = f(t,x)$ with $\alpha \in (0,1),\ t > 0$ is $[{}_0^{RL}D_t^{\alpha-1}x(t)]_{t=0} = x_0'$ which can be replaced by $[t^{1-\alpha}x(t)]_{t=0} = \frac{x_0'}{\Gamma(\alpha)}$. However, the calculation of $[{}_0^{RL}D_t^{\alpha-1}x(t)]_{t=0} = x_0'$ is not feasible always. Above mathematical foundation is explained using the following Theorem (Kilbas et al. (2006))

THEOREM 2.1 (Kilbas *et al.* (2006)) Let $0 < \alpha < 1$ and let $x(t) \in \mathbb{C}_{1-\alpha}([0,b])$

a If

$$\lim_{t\to 0^+}\left[t^{1-\alpha}x(t)\right] = c,\ \ c \in \mathbb{R} \tag{2.7}$$

Then

$$I^{1-\alpha}x(0^+) = \lim_{t\to 0^+} I^{1-\alpha}x(t) = c\Gamma(\alpha) \tag{2.8}$$

b If

$$\lim_{t\to 0^+} I^{1-\alpha}x(t) = b,\ \ b \in \mathbb{R} \tag{2.9}$$

and if there exists the limit, $\lim_{t\to 0^+}\left[t^{1-\alpha}x(t)\right]$, then

$$\lim_{t\to 0^+}\left[t^{1-\alpha}x(t)\right] = \frac{b}{\Gamma(\alpha)}. \tag{2.10}$$

RL definition is considered henceforth unless otherwise specified. In the next section we present a brief review of contraction analysis of the dynamical system, which is essential for proving the the main result of this paper.

3. Contraction analysis of dynamical systems (Aylward *et al.* (2008))

Consider an autonomous dynamical system

$$\dot{x} = f(x(t)), \tag{3.1}$$

where f is a nonlinear vector field and $x(t)$ is an n-dimensional state vector. It is assumed that all the quantities are real and smooth and thus all the required derivatives exist and are continuous. In contraction analysis, the state x is slightly modified to see the change in velocity vector $\dot{x}$, that is termed

as the virtual displacement (infinitesimal displacement at fixed time) δx and virtual velocity $\delta \dot{x}$ respectively, introduced by Lagrange. Due to this infinitesimal displacement of the state, virtual dynamics are introduced into the system, which can be represented by,

$$\delta \dot{x} = \delta f(x(t)) = \frac{\partial f(x(t))}{\partial x} \delta x. \tag{3.2}$$

State dependent local and virtual change of coordinates using nonsingular transformation matrix Θ

$$\delta z = \Theta \delta x(t), \tag{3.3}$$

produce virtual dynamics (3.2) converted into δz-coordinates as

$$\frac{d}{dt}\delta z = \dot{\Theta}\delta x(t) + \Theta \delta \dot{x}(t) = F \delta z, \tag{3.4}$$

where F is termed the generalized Jacobian, given by

$$F = \left(\dot{\Theta} + \Theta \frac{\partial f}{\partial x} \right) \Theta^{-1}. \tag{3.5}$$

The rate of change of squared length is given by

$$\frac{d}{dt}(\delta z^T \delta z) = 2 \delta z^T F \delta z, \tag{3.6}$$

A more general definition of infinitesimal length can be given by

$$\delta x^T(t) M(x(t)) \delta x(t) \tag{3.7}$$

where $M(x(t))$ is a symmetric, uniformly positive definite and continuously differentiable metric (formally, this defines a Riemannian manifold). Using the more general definition of infinitesimal length, one can calculate its rate of change as

$$\begin{aligned} &\frac{d}{dt}(\delta x^T(t) M(x(t)) \delta x(t)) \\ &= \delta x^T(t) \left(\frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M} \right) \delta x(t), \end{aligned} \tag{3.8}$$

Based on the above observation, Lohmiller *et al.* (1998) give the following definition of the contraction region as:

DEFINITION 3.1 : Given the n-dimensional system equations $\dot{x} = f(x(t))$, a region of the state space is called a contraction region with respect to a uniformly positive definite metric $M(x(t)) = \Theta^T \Theta$, if equivalently F in equation (3.5) or $\frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M}$ are uniformly negative definite in that region.

Using the definition above, Lohmiller *et al.* (1998) generalized the convergence result as:

THEOREM 3.1 : *Given the system equations $\dot{x} = f(x(t))$, any trajectory, which starts in a ball of constant radius with respect to the metric $M(x(t))$, centered at a given trajectory and contained at all times in a contraction with respect to $M(x(t))$, remains in that ball and converges exponentially to this trajectory.*

Proof. - See Lohmiller and Slotine (1998). □
A similar theorem (see Aylward *et al.* (2008)) shows that contraction metrics can be used to prove convergence to a single trajectory, and thus existence and/or uniqueness of equilibria.

THEOREM 3.2 : *Consider the autonomous system $\dot{x} = f(x(t))$. If a contraction metric exists for the system over the entire state-space and a finite equilibrium exists, then this equilibrium is unique and all trajectories converge to this equilibrium. If the system is exponentially contracting, there exists a unique finite equilibrium, and all trajectories converge to this equilibrium.*

In terms of the Jacobian matrix, Lohmiller *et al.* (1998) have given the following definition for finding a contraction region:

DEFINITION 3.2 : Given the system equations $\dot{x} = f(x,t)$, a region of the state space is called a contraction or semi-contraction region, if the Jacobian matrix $\partial f/\partial x$ is uniformly negative definite or negative semi-definite in that region.

The main result of this paper is inspired from the finite time stabilization of an integrator chain (Bhat *et al.* (2005)), which is presented in next section.

4. Motivation: Finite time stabilization of an integrator chain (Bhat *et al.* (2005))

Consider the nominal system (4.1), which is represented by SISO independent integrator chains, defined as follows

$$\begin{cases} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{n-1} & = z_n \\ \dot{z}_n & = u. \end{cases} \tag{4.1}$$

THEOREM 4.1 (Bhat *et al.* (2005)) *Let $k_1, ..., k_n > 0$ be such that the polynomial $\lambda^n + k_n\lambda^{n-1} + ... + k_2\lambda + k_1$ is Hurwitz. Consider system (4.1). There exists $\varepsilon \in (0,1)$ such that, for every $\alpha \in (1-\varepsilon, 1)$, the origin is a globally finite time stable equilibrium for the system under the feedback*

$$u(z) = -k_1 sgn(z_1)|z_1|^{\alpha_1} - ... - k_n sgn(z_n)|z_n|^{\alpha_n}, \tag{4.2}$$

where $\alpha_1, ..., \alpha_n$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_1}, \quad i = 2, ..., n, \quad with \quad \alpha_{n+1} = 1. \tag{4.3}$$

REMARK 4.1 Above result is only limited for the integer order systems. But in the last two decades, fractional chain of integrators have been used to model various stable physical phenomena. Therefore, in next section we generalized same kind of result for the fractional order.

5. Main Result

Consider the following fractional order system:

$$\begin{cases} D^{\alpha}x_1 & = x_2 \\ D^{\alpha}x_2 & = x_3 \\ & \vdots \\ D^{\alpha}x_{n-1} & = x_n \\ D^{\alpha}x_n & = u. \end{cases} \tag{5.1}$$

This can be also represented as:

$$D^{\alpha}x = Ax + Bu,$$

where $D^{\alpha}x = [D^{\alpha}x_1\ D^{\alpha}x_2\ ...\ D^{\alpha}x_n]^T = D^{\alpha}\,[x_1\ x_2\ ...\ x_n]^T$,

$$A = \begin{pmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 0 & 0 & ... & 1 \end{pmatrix} \tag{5.2}$$

and $x(t) \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}$ and $\alpha \in (0,1)$. A and B are system matrices of appropriate dimensions. Note that (A,B) are controllable.

REMARK 5.1 (Djamah *et al.* (2009)) In the commensurate case the above state space model remains valid with $D^{\alpha}x = D^{\alpha}\,[x_1\ x_2\ ...\ x_n]^T$ same as integer order.

Following Threorem presents the main result of this paper stabilization of fractional chain of integrators.

THEOREM 5.1 *Let $k_1,...,k_n > 0$ be such that the polynomial $\lambda^n + k_n\lambda^{n-1} + ... + k_2\lambda + k_1$ is Hurwitz (in the sense of fractional order systems (see the Appendix)). Consider the system (5.1). There exists $\varepsilon \in (0,1)$ such that, for every $\beta_i \in (1-\varepsilon, 1)$, where i=1,2...n, the origin is a globally exponential stable equilibrium for the system under the feedback:*

$$u = \left[\sum_{i=1}^{n} -k_i |x_i|^{\beta_i} sgn(x_i) - \sum_{j=1}^{n} \frac{(x_j)t^{\alpha-1}}{\Gamma(\alpha)} \right] \Omega, \tag{5.3}$$

where $\Omega = (t-\delta)^{(1-\alpha)}$. Here, δ is interpreted as the step size for numerical control implementation. The β_i's satisfy

$$\beta_{i-1} = \frac{\beta_i \beta_{i+1}}{2\beta_{i+1} - \beta_1}, \quad i = 2,...,n, \quad \text{with} \quad \beta_{n+1} = 1. \tag{5.4}$$

Proof. Consider,

$$D^{\alpha}x = Ax + Bu \tag{5.5}$$

Applying $D^{1-\alpha}$ to both side of (5.5), one can write

$$D^{1-\alpha}D^{\alpha}x = D^{1-\alpha}Ax + D^{1-\alpha}Bu,$$
$$\dot{x} = D^{1-\alpha}Ax + D^{1-\alpha}Bu.$$

Consider now two neighboring trajectories in the flow field of the above equation and the virtual displacement δx between them. This yields the following

$$\delta \dot{x} = \left[\frac{\partial D^{1-\alpha} Ax}{\partial x} + \frac{\partial D^{1-\alpha} Bu}{\partial x} \right] \delta x.$$

The squared distance between these two trajectories can be defined as $\delta x^T \delta x$, the rate of change of which is given by

$$\begin{aligned} \frac{d\delta x^T \delta x}{dt} &= 2\delta x^T \delta \dot{x} \\ &= 2\delta x^T \left[\frac{\partial D^{1-\alpha} Ax}{\partial x} + \frac{\partial D^{1-\alpha} Bu}{\partial x} \right] \delta x. \end{aligned}$$

Consider,

$$D^{1-\alpha} Ax = \frac{1}{\Gamma(1-(1-\alpha))} \frac{d}{dt} \int_0^t \frac{Ax(\tau)}{(t-\tau)^{((1-\alpha)-1+1)}} d\tau$$

$$\begin{aligned} \frac{\partial D^{1-\alpha} Ax}{\partial x} &= \frac{1}{\Gamma(\alpha))} \frac{d}{dt} \int_0^t \frac{A}{(t-\tau)^{(1-\alpha)}} d\tau \\ &= \frac{At^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

For the proposed controller,

$$u = \left[\sum_{i=1}^{n} -k_i |x|^{\beta_i} sgn(x) - \sum_{j=1}^{n-1} \frac{(x_{j+1}) t^{\alpha-1}}{\Gamma(\alpha)} \right] \Omega,$$

where $\Omega = (t-\delta)^{(1-\alpha)}$.

$$D^{1-\alpha} Bu = \frac{1}{\Gamma(1-(1-\alpha))} \frac{d}{dt} \int_0^t \frac{Bu(\tau)}{(t-\tau)^{((1-\alpha)-1+1)}} d\tau.$$

Substituting,

$$D^{1-\alpha} Bu = \frac{1}{\Gamma(1-(1-\alpha))} \frac{d}{dt} \int_0^t Bp(x) d\tau,$$

where

$$p(x) = \left[\sum_{i=1}^{n} -k_i |x_i|^{\beta_i} sgn(x_i) - \sum_{j=1}^{n} \frac{(x_j) t^{\alpha-1}}{\Gamma(\alpha)} \right].$$

$$\begin{aligned} D^{1-\alpha} Bu &= \frac{1}{\Gamma(\alpha))} \frac{d}{dt} \int_0^t p(x) d\tau, \\ \frac{\partial D^{1-\alpha} Bu}{\partial x} &= \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{\partial Bp(x)}{\partial x} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \frac{\partial Bp(x)}{\partial x} \\ &= \frac{BP(x)}{\Gamma(\alpha)}. \end{aligned}$$

where,

$$P(x) = \begin{pmatrix} 0 & 0 & ... & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \vdots & 0 \\ -k_1|x_1|^{(\beta_1-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} & ... & 0 & -k_n|x_n|^{(\beta_n-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \tag{5.6}$$

The Jacobian is represented by:

$$\begin{aligned} J &= \left[\frac{\partial D^{1-\alpha}Ax}{\partial x} + \frac{\partial D^{1-\alpha}Bu}{\partial x}\right] \\ &= \frac{At^{\alpha-1}}{\Gamma(\alpha)} + \frac{BP(x)}{\Gamma(\alpha)} \end{aligned} \tag{5.7}$$

One can easily observe that as $t \to \infty$, the Jacobian J becomes

$$J = \begin{pmatrix} 0 & \to 0 & ... & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \vdots & 0 \\ (-)\text{ve} & ... & (-)\text{ve} & (-)\text{ve} \end{pmatrix} . \tag{5.8}$$

which is negative semi-definite, only when $k_i > 0, i = 1, 2..., n$. Hence, the rate of change of the distance between the two considered trajectories is negative. As a result, the distance progressively decreases. By path integration, this implies that the length of any finite path converges exponentially to zero.

More discussion is required for selection of k_i, because of the different scaling property satisfied by fractional order integral when compared to the integer order case. Due to this, a different kind of vector field analysis is required for characterizing the contraction behavior in the whole space. Mathematically, the above discussion can be expressed like this

$${}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}} d\tau,$$

One can write in form

$$\begin{aligned} {}_0I_t^{\alpha} f(t) &= \int_0^t f(\tau) dg_t(\tau) \\ g_t(\tau) &= \frac{1}{\Gamma(\alpha)} \{t^{\alpha} - (t-\tau)^{\alpha}\} \end{aligned}$$

The scaling property of $g_t(\tau)$ is given as

$$g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^{\alpha} g_t(\tau)$$

when $t_1 = kt$ and $\tau_1 = k\tau$. Due to this scaling property, the closed loop homogeneous vector field $v_{\text{fractional}}$ of this fractional order system is modified as

$$v_{\text{fractional}} = \frac{1}{\beta_1} x_1 \frac{\partial^{\alpha}}{\partial x_1^{\alpha}} + ... + \frac{1}{\beta_n} x_n \frac{\partial^{\alpha}}{\partial x_n^{\alpha}}$$

where $\beta_n; n = 1, ..., n$ are defined as in (5.4). Moreover, the above vector field is linear with the Hurwitz(in fractional sense) characteristic polynomial $\lambda^n + k_n\lambda^{n-1} + ... + k_2\lambda + k_1$. Therefore, a global uniform contraction region is obtained after applying the proposed control to the fractional chain of integrators. □

6. NUMERICAL RESULT

As an illustration of the results presented in this paper, a commensurate fractional order linear model of the long Aluminum rod heated from one of its sides is considered in Sabatier *et al.* (2007). The input $u(t)$ to such a model is the thermal flux applied at one end of the rod and the output is the actual temperature at a prescribed section of the rod. Consider the continuous fractional order state-space model of the specified system, which is given by

$$D^{0.5}x(t) = Ax(t) + bu(t) \tag{6.1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.0601251 & -0.42833 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.2}$$

The proposed controller for the above system (6.1) is represnted as $u = u_1 - A_0x(t)$, where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.0601251 & -0.42833 \end{bmatrix} \tag{6.3}$$

After applying the control $u = u_1 - A_0x(t)$, in (6.1), the closed loop system can be represnted as the fractional chain of integrtors

$$\begin{cases} D^\alpha x_1 & = x_2 \\ D^\alpha x_2 & = x_3 \\ D^\alpha x_3 & = u_1. \end{cases} \tag{6.4}$$

Now u_1 is given by (5.3)

$$u_1 = \left[\sum_{i=1}^{3} -k_i|x_i|^{\beta_i} sgn(x_i) - \sum_{j=1}^{3} \frac{(x_j)t^{\alpha-1}}{\Gamma(\alpha)} \right] \Omega, \tag{6.5}$$

where k_1, k_2 and k_3 are given by following polynomial

$$\lambda^3 + 0.78\lambda^2 + 0.4954\lambda + 0.7455 = 0 \tag{6.6}$$

Since all the coefficients of the above polynomial are positive, it is easy to check that $D(P) < 0$ (see the Appendix). Hence, the above polynomial is Routh Hurwitz in the fractional sense. $\beta_1 = 1/2, \beta_2 = 3/5, \beta_3 = 3/5$ and the initial conditions of state vector were chosen as $x_1 = \Gamma(0.5)$, $x_2 = -\Gamma(0.5)$, $x_3 = 1.5\Gamma(0.5)$. The numerical simulation for the states is depicted in Figure (1).

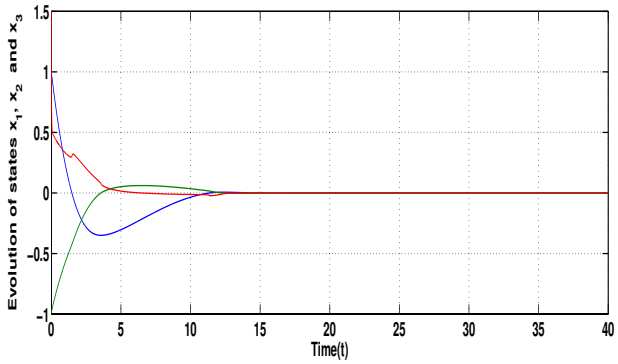

FIG. 1. Evolution of state(x_1 ,x_2 and x_3) w.r.t. time

7. Conclusions

The exponential stability of a fractional order system with the proposed controller is proved using contraction analysis. It can be seen that when $\alpha = 1$, the controller (5.3) takes the following form:

$$u|_{\alpha=1} = -(k_1 sgn(x_1)|x_1|^{\beta_1} + x_1) - (k_2 sgn(x_2)|x_2|^{\beta_2} + x_2) - \ldots - (k_n sgn(x_n|x_n|^{\beta_n} + x_n).$$

This controller is very similar to the one proposed by Bhat *et al.* (2005) which is discussed in section IV. The only difference is the addition of the proportional term. This proportional term will accelerate the convergence of the system states.

Appendix

On fractional Order Routh-Hurwitz Conditions (Ahmed *et al.* (2006))
The problem of interest is that all the roots of the polynomial equation

$$P(\lambda) = 0, \;\; P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n, \tag{7.1}$$

satisfy

$$|arg(\lambda)| > \alpha\pi/2, \tag{7.2}$$

where all the coefficients in (7.1) are real.
For integer order ($\alpha = 1$) the solution yields the Routh-Hurwitz conditions

$$0 < a_1, \quad 0 < \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \quad 0 < \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix} \ldots \tag{7.3}$$

For $\alpha \in [0,1)$ these conditions are sufficient but not necessary.

DEFINITION 7.1 The discriminant $D(f)$ of a polynomial

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n,$$

is defined by $D(f) = (-1)^{n(n-1)/2} R(f, f')$ where f' is the derivative of f and where $g(x) = x^n + b_1 x^{l-1} + b_2 x^{l-2} + ... + b_l$ and $R(f,g)$ is an $(n+l) \otimes (n+l)$ determinant.

Proposition

1. For $n = 1$ the condition for (7.2) is $a_1 > 0$.

2. For $n = 2$ the conditions for (7.2) are either the Routh-Hurwitz conditions or $a_1 < 0, 4a_2 > (a_1)^2$, $\left| tan^{-1}(\sqrt{4a_2 - (a_1)^2}/a_1 \right| > \alpha\pi/2$.

3. For $n = 3$ if the discriminant of $p(\lambda)D(P)$ is positive then the Routh-Hurwitz conditions are the necessary and sufficient conditions for (7.2), i.e.,

$$a_1 > 0, \quad a_3 > 0 \quad a_1 a_2 > a_3 \quad \text{if } D(P) > 0. \tag{7.4}$$

4. If $D(P) < 0, a_1 \geqslant 0, a_2 \geqslant 0, a_3 > 0$, $\alpha < (2/3)$ then (7.2) is satisfied.
Also if $D(P) < 0, a_1 < 0, a_2 < 0, \alpha > (2/3)$ then all roots of $P(\lambda)$ satisfy $|arg(\lambda)| < \alpha\pi/2$.

5. If $D(P) < 0, a_1 > 0, a_2 > 0, a_1 a_2 = a_3$ then (7.2) is satisfied for all $\alpha \in [0,1)$.

6. For general n, $a_n > 0$, is a necessary condition for (7.2).

7. If $\forall\lambda,\ P(\lambda) = P(-\lambda)$ then define $x = \lambda^2$ and the Routh-Hurwitz conditions for the resulting polynomial in x are necessary conditions for (7.2) for all $\alpha \in [0,1)$.

8. For $n > 1$, the necessary and sufficient condition for (7.2) is

$$\int_0^\infty dz/P(z)|_{C2} + \int_{-\infty}^0 dz/P(z)|_{C1} = 0, \tag{7.5}$$

where $C1$ is the curve $z = x(1 - i\tan\alpha\pi/2)$, and $C2$ is the curve $z = x(1 - i\tan\alpha\pi/2)$, $i = \sqrt{-1}$

Note:- Using the results of Mishina *et al.* (1965), if $D(f) > 0$ (< 0) then there is an even(odd) number of pairs of complex roots for the equation $f(x) = 0$. For $n = 3$ this implies that $D(f) > 0$ implies that all the roots are real and $D(f) < 0$ implies that there is only one real root and one complex root and its complex conjugate. For $n = 3$ we have

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2. \tag{7.6}$$

Proof. See Ahmed *et al.* (2006) □

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