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Some results on majorization and their applications

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HIGHLIGHTS

- Some useful results on majorization are developed.
- This enriches the theory of majorization.
- As applications, some distributions have been studied.

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ABSTRACT

Majorization is a key concept in studying the Schur-convex property of a function, which is very useful in the study of stochastic orders. In this paper, some results on Schur-convexity have been developed. We have studied the conditions under which a function φ defined by $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ will be Schur-convex. This fills some gap in the theory of majorization. The results so developed have been used in the case of generalized exponential and gamma distributions. During this, we have also developed some stochastic properties of order statistics.

1. Introduction

The notion of a stochastic order based on majorization, called majorization order, is quite useful in establishing various useful inequalities. The method of majorization which is used in finding some nice and applicable inequalities is also useful in understanding the insight of the theory. This concept deals with the diversity of the components of vector in $\mathbb{R}^n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be any two real vectors. We arrange the components of $\mathbf{x}$ and $\mathbf{y}$ in ascending order as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ respectively. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}$, for $j = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. In this case we also say that $\mathbf{x}$ is more than $\mathbf{y}$ in majorization order. This is a partial order on $\mathbb{R}^n$, and $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$ tells that the components of $\mathbf{x}$ are more dispersed compared to those of $\mathbf{y}$ (although average is the same for both the vectors). Majorization order and its variants are used for the last couple of decades at an accelerated rate, in many diverse areas of mathematics, statistics, economics, physics and so on. It has also been used in reliability viz. optimal component allocation in parallel–series as well as in series–parallel systems, allocation of standby in series and parallel systems, and so on, see, for instance, [1]. It has also been used in the

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Table 1
Schur-convexity/concavity of $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$.

g	$\mathbf{u}, \mathbf{x}$			
	$\mathbf{u} \in \mathcal{D}_+, \mathbf{x} \in \mathcal{D}$	$\mathbf{u} \in \mathcal{D}_+, \mathbf{x} \in \mathcal{E}$	$\mathbf{u} \in \mathcal{E}_+, \mathbf{x} \in \mathcal{D}$	$\mathbf{u} \in \mathcal{E}_+, \mathbf{x} \in \mathcal{E}$
Increasing convex	Schur-convex	Inconclusive	Inconclusive	Schur-convex
Decreasing convex	Inconclusive	Schur-convex	Schur-convex	Inconclusive
Increasing concave	Inconclusive	Schur-concave	Schur-concave	Inconclusive
Decreasing concave	Schur-concave	Inconclusive	Inconclusive	Schur-concave

context of minimal repair of two-component parallel system with exponentially distributed lifetimes by Boland and El-Neweihi [2]. Majorization is also used as a measure of income inequality, species diversity and bio-diversity. One may also notice the usefulness of majorization ordering in pair comparisons, phase-type distributions, disease transmission, statistical mechanics and so on. The details of these applications may be obtained in [3]. For an overview of majorization and some more applications, one may refer to [4].

One question raised and also partially solved by Marshall et al. [4] on majorization is that what condition(s) a function $g : \mathbb{R}^n \to \mathbb{R}$ should satisfy such that, given a monotonic sequence of real numbers $\{u_1, u_2, \dots, u_n\}$, the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, defined by $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ is Schur-convex/Schur-concave (for definition, see Section 2). They have shown that if $u_1 \geq u_2 \geq \cdots \geq u_n \geq 0$, and $g(\cdot)$ is increasing and convex, then $\varphi(\mathbf{x})$ is Schur-convex. They have also mentioned that if $\mathbf{u}$ is decreasing, and $g(\cdot)$ is decreasing and convex, then $\varphi(\mathbf{x})$ is Schur-convex. However, in the last statement there is a typographical error and the term 'decreasing and convex' should be 'decreasing and concave'. The other questions related to φ could be: what can we say about φ when g is increasing and concave or decreasing and convex, and also $\mathbf{u}$ is in increasing order, i.e., $u_1 \leq u_2 \leq \cdots \leq u_n$? This paper deals with this type of questions. We have shown that in some cases the property of φ can be explicitly mentioned whereas in some cases no conclusion on Schur-convexity or Schur-concavity of φ can be made.

The application of these results has been studied for order statistics generated from heterogeneous generalized exponential distributions. The usefulness of this distribution has been seen in different studies available in the literature, see, for instance, Gupta and Kundu [5]. Further, it is well known that there is a one-to-one correspondence between an order statistic and the lifetime of a k-out-of-n system, which has a lot of applications in statistics, reliability theory, applied probability, and so on and so forth. A k-out-of-n system is a special kind of coherent system, which survives as long as at least k of the n components of the system survive. For details of this kind of systems one may refer to [6]. It is to be mentioned here that we denote by $X_{k:n}$, the kth order statistic formed from $X_1, X_2, \dots, X_n$. Although different properties of order statistics from homogeneous populations have been studied in detail in the literature, very less amount of work has been done so far for order statistic from non-homogeneous populations, due to its complicated nature of expressions. For properties of order statistics for independent and non-identically distributed random variables, one may refer to [7].

The applications of other majorization orders viz. different weak majorization orders (defined in Section 2) have been discussed in connection with heterogeneous gamma populations for series systems.

The paper is organized as follows. Section 2 deals with different notations and definitions of different majorization orders used in this paper. The interrelations among different orders are also given here. Section 3 deals with the main findings of the paper. The details of the findings are given in Table 1. Section 4 deals with applications of different majorization orders. This is discussed in connection with generalized exponential and gamma populations, whereas Section 5 concludes.

Throughout the paper, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\mathbb{R}$ denotes the set of real numbers $\{x : -\infty < x < \infty\}$. We also write $a \overset{sign}{=} b$ to mean that a and b have the same sign. Further, by $a \overset{\text{def}}{=} b$ we mean that b is defined as a. For any differentiable function $k(\cdot)$, we write $k'(t)$ to denote the first derivative of $k(t)$ with respect to t. The random variables considered in this paper are all nonnegative.

2. Notations and preliminaries

For an absolutely continuous random variable X, we denote the probability density function by $f_X(\cdot)$, the cumulative distribution function by $F_X(\cdot)$, the hazard rate function by $r_X(\cdot)$, and the reversed hazard rate function by $\tilde{r}_X(\cdot)$. The survival or reliability function of the random variable X is written as $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$.

The following definitions may be obtained in [4].

Definition 2.1. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ be any two vectors.

(i) The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \quad j = 1,\ 2,\ \dots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

(ii) The vector $\mathbf{x}$ is said to weakly supermajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

(iii) The vector $\mathbf{x}$ is said to weakly submajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \succeq_{\mathrm{w}} \mathbf{y}$) if

$$\sum_{i=j}^{n} x_{(i)} \ge \sum_{i=j}^{n} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

(iv) The vector $\mathbf{x}$ is said to be p-larger than the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

(v) The vector $\mathbf{x}$ is said to reciprocally majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \ge \sum_{i=1}^{j} \frac{1}{y_{(i)}} \quad \text{for } j = 1, 2, \ldots, n. \quad \square$$

It is not so difficult to show that $\mathbf{x} \overset{\mathrm{m}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{\mathrm{rm}}{\succeq} \mathbf{y}$.

Definition 2.2. Let $I \subseteq \mathbb{R}$. A function $\psi : I^n \to \mathbb{R}$ is said to be Schur-convex (resp. Schur-concave) on I^n if

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \ge (\text{resp. } \le)\ \psi(\mathbf{y}) \quad \text{for all } \mathbf{x},\ \mathbf{y} \in I^n.$$

Notation 2.1. *Let us include the following notations. The first and the third are borrowed from* [4].

(i) $\mathcal{D} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \ge x_2 \ge \cdots \ge x_n\}$.
(ii) $\mathcal{E} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \le x_2 \le \cdots \le x_n\}$.
(iii) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}$.
(iv) $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le x_2 \le \cdots \le x_n\}$.

In order to compare different order statistics, stochastic orders are used for fair and reasonable comparison. In the literature many different kinds of stochastic orders have been developed and studied. Here we consider different stochastic orders. For the definitions, motivations and usefulness of these stochastic orders, readers may see [8–10]. However, for the sake of completeness and readers' convenience we give the definitions below.

Definition 2.3. Let X and Y be two absolutely continuous random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then, X is said to be smaller than Y in

(a) likelihood ratio (lr) order, denoted as $X \le_{lr} Y$, if

$$\frac{f_Y(t)}{f_X(t)} \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y);$$

(b) hazard rate (hr) order, denoted as $X \le_{hr} Y$, if

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ is increasing in } t \in (-\infty, \max(u_X, u_Y));$$

(c) up shifted hazard rate ($hr \uparrow$) order, denoted as $X \le_{hr\uparrow} Y$, if $X - x \le_{hr} Y$, for all $x \ge 0$;
(d) down shifted hazard rate ($hr \downarrow$) order, denoted as $X \le_{hr\downarrow} Y$, if $X \le_{hr} [Y - x | Y > x]$, for all $x \ge 0$;
(e) reversed hazard rate (rhr) order, denoted as $X \le_{rhr} Y$, if

$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (\min(l_X, l_Y), \infty);$$

(f) up shifted reversed hazard rate ($rhr \uparrow$) order, denoted as $X \le_{rhr\uparrow} Y$, if $X - x \le_{rhr} Y$, for all $x \ge 0$;
(g) dispersive ($disp$) order, denoted as $X \le_{disp} Y$, if

$$F_X^{-1}(b) - F_X^{-1}(a) \le F_Y^{-1}(b) - F_Y^{-1}(a) \quad \text{for all } 0 < a \le b < 1,$$

where, for any distribution function F, F^{-1} is the right continuous inverse of F;
(h) usual stochastic (st) order, denoted as $X \le_{st} Y$, if $\bar{F}_X(t) \le \bar{F}_Y(t)$ for all $t \in \mathbb{R}$. $\square$

In the following diagram we present a chain of implications among the above mentioned stochastic orders.

$$\begin{array}{ccccc} X \le_{hr\downarrow} Y & \to & X \le_{hr} Y & & \\ & \nearrow & \uparrow & \searrow & \\ X \le_{hr\uparrow} Y & & X \le_{lr} Y & \to & X \le_{st} Y. \\ & & \downarrow & \nearrow & \\ X \le_{rhr\uparrow} Y & \to & X \le_{rhr} Y & & \end{array}$$

Like stochastic orders, stochastic ageing is also another important concept which has many applications in reliability theory. Different stochastic ageing properties describe how a system improves or deteriorates with age. Here, we particularly consider increasing likelihood ratio (ILR), decreasing likelihood ratio (DLR), increasing failure rate in average (IFRA) and decreasing failure rate in average (DFRA) classes. The definitions and usefulness of these ageing classes could be found in [6] and [11].

Definition 2.4. A random variable X is said to be

1. increasing likelihood ratio (ILR) (resp. decreasing likelihood ratio (DLR)) if $f_X(t)$ is log-concave (resp. log-convex) in t;
2. increasing failure rate in average (IFRA) (resp. decreasing failure rate in average (DFRA)) if $\frac{1}{t}\int_0^t r_X(x)\,dx$ is increasing (resp. decreasing) in t. □

3. Some results on majorization

The following lemma may be obtained in [4, p. 83], where the parenthetical statement is not given.

Lemma 3.1. *Let $\varphi : \mathcal{D} \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{D}$. Then, for $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \ge (\text{resp. } \le)\ \varphi(\mathbf{y})$$

if, and only if,

$$\varphi_{(k)}(\mathbf{z}) \text{ is decreasing (resp. increasing)} \quad \text{in } k = 1, 2, \ldots, n,$$

where $\varphi_{(k)}(\mathbf{z}) = \partial\varphi(\mathbf{z})/\partial z_k$ denotes the partial derivative of φ with respect to its kth argument. □

On using the above lemma, we have the following.

Lemma 3.2. *Let $\varphi(\mathbf{x}) = \sum_{i=1}^n g_i(x_i)$ with $\mathbf{x} \in \mathcal{D}$, where $g_i : \mathbb{R} \to \mathbb{R}$ is differentiable, for all $i = 1, 2, \ldots, n$. Then $\varphi(\mathbf{x})$ is Schur-concave on $\mathcal{D}$ if, and only if,*

$$g_i'(a) \le g_{i+1}'(b) \text{ whenever } a \ge b, \quad \text{for all } i = 1, 2, \ldots, n-1,$$

where $g'(a) = \left.\frac{dg(x)}{dx}\right|_{x=a}$. □

Now, we are in a position to prove the following theorem.

Theorem 3.1. *Let $\varphi(\mathbf{x}) = \sum_{i=1}^n u_i g(x_i)$ with $\mathbf{x} \in \mathcal{D}$, and let $I \subseteq \mathbb{R}$ be an interval. Consider a function $g : I \to \mathbb{R}$.*

(a) *If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{D}_+$ and*
  (i) *$g(\cdot)$ is increasing and convex then $\varphi(\mathbf{x})$ is Schur-convex on $\mathcal{D}$;*
  (ii) *$g(\cdot)$ is decreasing and concave then $\varphi(\mathbf{x})$ is Schur-concave on $\mathcal{D}$.*
(b) *If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{E}_+$ and*
  (i) *$g(\cdot)$ is increasing and concave then $\varphi(\mathbf{x})$ is Schur-concave on $\mathcal{D}$;*
  (ii) *$g(\cdot)$ is decreasing and convex then $\varphi(\mathbf{x})$ is Schur-convex on $\mathcal{D}$.*

Proof. We give a proof for (i) only. The proof of (ii) follows by taking $-g$ in place of g in (i).

(a) Let $g_i(x_i) = u_i g(x_i)$. If $g(\cdot)$ is increasing and convex then, for all $a \ge b$, $g'(a) \ge g'(b) \ge 0$. Again, if $\mathbf{u} \in \mathcal{D}_+$ then $u_i g'(a) \ge u_{i+1} g'(b)$. Hence, by Proposition H.2 of Marshall et al. [4], (i) is proved.
(b) Note that $g(\cdot)$ is increasing and concave implies that, for all $a \ge b$, $g'(a) \ge 0$, $g'(b) \ge 0$ and $g'(a) \le g'(b)$. So, if $\mathbf{u} \in \mathcal{E}_+$ then $u_i g'(a) \le u_{i+1} g'(b)$. Hence, by Lemma 3.2, (i) is proved. □

The following counterexample shows that if $g(\cdot)$ is increasing and convex, and $\mathbf{u} \in \mathcal{E}_+$, then $\varphi(\mathbf{x})$ may not be Schur-convex or Schur-concave on $\mathcal{D}$.

Counterexample 3.1. *Let $g(x) = e^x$, $\mathbf{x} = (30, 8, 2) \in \mathcal{D}_+$ and $\mathbf{y} = (18, 12, 10) \in \mathcal{D}_+$. So, clearly $g(x)$ is increasing and convex, and $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$. Now, if $\mathbf{u} = (1, 2, 3) \in \mathcal{E}_+$ is taken, then it can be easily checked that*

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = 1.068640854 \times 10^{13} > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, if $\mathbf{x} = (4, 3, 1) \in \mathcal{D}_+$ and $\mathbf{y} = (3, 3, 2) \in \mathcal{D}_+$ are taken then, for $\mathbf{u} = (1, 2, 30) \in \mathcal{E}_+$ and for the same function $g(\cdot)$, it can be easily checked that, although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = -105.610615 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{D}_+$. □

That nothing can be said about the Schur-convexity of $\varphi(\mathbf{x})$ on $\mathcal{D}$ when $g(\cdot)$ is increasing and concave, and $\mathbf{u} \in \mathcal{D}_+$, is shown in the next counterexample.

Counterexample 3.2. *For $\mathbf{x} = (30, 8, 2) \in \mathcal{D}_+$, $\mathbf{y} = (18, 12, 10) \in \mathcal{D}_+$ and $\mathbf{u} = (3, 2, 1) \in \mathcal{D}_+$, if $g(x) = \ln x$ is taken, which is increasing and concave, then*

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = -0.887891257 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. Again, for $\mathbf{x} = (4, 3, 1) \in \mathcal{D}_+$, $\mathbf{y} = (3, 3, 2) \in \mathcal{D}_+$ and $\mathbf{u} = (30, 2, 1) \in \mathcal{D}_+$ and, for the same function $g(\cdot)$, it can be seen that

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = 7.937314993 > 0,$$

satisfying the claim. □

The counterexample given below shows that $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{D}$ if the function $g(\cdot)$ is decreasing and convex, and $\mathbf{u} \in \mathcal{D}_+$.

Counterexample 3.3. *Let $g(x) = e^{-x}$, which is decreasing and convex, and $\mathbf{x} = (4, 3, 1) \in \mathcal{D}_+$ and $\mathbf{y} = (3, 3, 2) \in \mathcal{D}_+$. Now, if $\mathbf{u} = (3, 2, 1) \in \mathcal{D}_+$ is taken, then it can be easily verified that*

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = 0.138129869 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, if we take $\mathbf{x} = (3, 2, 1) \in \mathcal{D}_+$ and $\mathbf{y} = (2, 2, 2) \in \mathcal{D}_+$ then, for $\mathbf{u} = (26, 2, 1) \in \mathcal{D}_+$ and for the same function $g(\cdot)$, it can be easily checked that, although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = -1.991709429 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{D}_+$. □

The following counterexample shows that if $g(\cdot)$ is decreasing and concave and $\mathbf{u} \in \mathcal{E}_+$, then $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{D}$.

Counterexample 3.4. *Let $g(x) = 1 - e^{-9x^{-0.4}}$, which is decreasing and concave for all $x \in [0, 10]$. Now, if we take $\mathbf{x} = (8, 4, 3) \in \mathcal{D}_+$, $\mathbf{y} = (7, 4, 4) \in \mathcal{D}_+$ and $\mathbf{u} = (10, 10.2, 10.4) \in \mathcal{E}_+$, then it can also be checked that although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,*

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = -0.0108009 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. Again, for the same function $g(\cdot)$ and for same $\mathbf{x}$, $\mathbf{y}$, if $\mathbf{u} = (1, 20, 30) \in \mathcal{E}_+$ is taken then

$$\sum_{i=1}^{3} u_i g\left(x_i\right) - \sum_{i=1}^{3} u_i g\left(y_i\right) = 0.0759684 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{D}_+$. □

From Counterexample 3.3, it is clear that there is a typographical error in the parenthetical statement of Proposition H.2.b in [4]. This can also be verified by observing that the parenthetical statement of Proposition H.2.b with $u_i \equiv 1$ contradicts Proposition C.1 of Marshall et al. [4, p. 92].

In all the above discussion we take $\mathbf{x} \in \mathcal{D}$. An immediate question that arises is—what will be the behaviour of φ, under different conditions on g, when $\mathbf{x} \in \mathcal{E}$? In order to answer this question we take help of the following lemmas. The proof of Lemma 3.3 is analogous to that of Theorem A.3 of Marshall et al. [4, p. 83], whereas that of Lemma 3.4 is based on Lemma 3.3.

Lemma 3.3. *Let $\varphi : \mathcal{E} \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{E}$. Then, for $\mathbf{x}, \mathbf{y} \in \mathcal{E}$,*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \geq (\text{resp. } \leq)\ \varphi(\mathbf{y})$$

if, and only if,

$$\varphi_{(k)}(\mathbf{z}) \text{ is increasing (resp. decreasing)} \quad \text{in } k = 1, 2, \ldots, n,$$

where $\varphi_{(k)}(\mathbf{z}) = \partial\varphi(\mathbf{z})/\partial z_k$ denotes the partial derivative of φ with respect to its kth argument.

Lemma 3.4. *Let $\varphi(\mathbf{x}) = \sum_{i=1}^{n} g_i(x_i)$ with $\mathbf{x} \in \mathcal{E}$, where $g_i : \mathbb{R} \to \mathbb{R}$ is differentiable, for all $i = 1, 2, \ldots, n$. Then $\varphi(\mathbf{x})$ is Schur-convex (resp. Schur-concave) on $\mathcal{E}$ if, and only if,*

$$g'_{i+1}(a) \geq (\text{resp. } \leq)\ g'_i(b) \text{ whenever } a \geq b, \quad \text{for all } i = 1, 2, \ldots, n-1,$$

where $g'(a) = \frac{dg(x)}{dx}\big|_{x=a}$. □

Below we give a theorem whose proof, with the help of Lemma 3.4, follows in the same line as in Theorem 3.1.

Theorem 3.2. *Let $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ with $\mathbf{x} \in \mathcal{E}$, and let $I \subseteq \mathbb{R}$ be an interval. Consider a function $g : I \to \mathbb{R}$.*

(a) *If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{D}_+$ and*
  (i) *$g(\cdot)$ is increasing and concave then $\varphi(\mathbf{x})$ is Schur-concave on $\mathcal{E}$;*
  (ii) *$g(\cdot)$ is decreasing and convex then $\varphi(\mathbf{x})$ is Schur-convex on $\mathcal{E}$.*

(b) *If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{E}_+$ and*
  (i) *$g(\cdot)$ is increasing and convex then $\varphi(\mathbf{x})$ is Schur-convex on $\mathcal{E}$;*
  (ii) *$g(\cdot)$ is decreasing and concave then $\varphi(\mathbf{x})$ is Schur-concave on $\mathcal{E}$.* □

The following counterexample shows that if $g(\cdot)$ is increasing and convex, and $\mathbf{u} \in \mathcal{D}_+$, then $\varphi(\mathbf{x})$ may not be Schur-convex or Schur-concave on $\mathcal{E}$.

Counterexample 3.5. *Let $g(x) = e^x$, $\mathbf{x} = (3, 7, 10) \in \mathcal{E}_+$ and $\mathbf{y} = (4, 7, 9) \in \mathcal{E}_+$. So, clearly $g(x)$ is increasing and convex and $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$. Now, if $\mathbf{u} = (3, 2, 1) \in \mathcal{D}_+$ is taken, then it can be easily checked that*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 13819.8 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, if $\mathbf{x} = (1, 2, 3) \in \mathcal{E}_+$ and $\mathbf{y} = (1.5, 1.5, 3) \in \mathcal{E}_+$ are taken then, for $\mathbf{u} = (30, 3, 2) \in \mathcal{D}_+$ and for the same function $g(\cdot)$, it can be easily checked that, although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -44.1801 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{E}_+$. □

That nothing can be said about the Schur-convexity of $\varphi(\mathbf{x})$ on $\mathcal{E}$ when $g(\cdot)$ is increasing and concave, and $\mathbf{u} \in \mathcal{E}_+$, is shown in the next counterexample.

Counterexample 3.6. *For $\mathbf{x} = (4, 5, 6) \in \mathcal{E}_+$, $\mathbf{y} = (5, 5, 5) \in \mathcal{E}_+$ and $\mathbf{u} = (2, 3, 5) \in \mathcal{E}_+$, if $g(x) = \ln x$ is taken, which is increasing and concave, then*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 0.465321 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, for $\mathbf{x} = (7, 8, 10) \in \mathcal{E}_+$, $\mathbf{y} = (7, 9, 9) \in \mathcal{E}_+$ and $\mathbf{u} = (2, 11, 11.1) \in \mathcal{E}_+$ and for the same function $g(\cdot)$, it can be seen that

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -0.126112 < 0,$$

satisfying the claim. □

The counterexample given below shows that $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{E}$ if the function $g(\cdot)$ is decreasing and convex, and $\mathbf{u} \in \mathcal{E}_+$.

Counterexample 3.7. *Let $g(x) = e^{-x}$, which is decreasing and convex, and $\mathbf{x} = (4, 6, 8) \in \mathcal{E}_+$ and $\mathbf{y} = (4, 7, 7) \in \mathcal{E}_+$. Now, if $\mathbf{u} = (1, 12, 12.1) \in \mathcal{E}_+$ is taken, then it can be easily verified that*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 0.0118278 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, if we take $\mathbf{x} = (7, 9, 11) \in \mathcal{E}_+$ and $\mathbf{y} = (8, 9, 10) \in \mathcal{E}_+$ then, for $\mathbf{u} = (1, 3, 150) \in \mathcal{E}_+$ and for the same function $g(\cdot)$, it can be easily checked that, although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -0.00372823 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{E}_+$. □

The following counterexample shows that if $g(\cdot)$ is decreasing and concave and $\mathbf{u} \in \mathcal{D}_+$, then $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{E}$.

Counterexample 3.8. *Let $g(x) = 1 - e^{-9x^{-0.4}}$, which is decreasing and concave for all $x \in [0, 10]$. Now, if we take $\mathbf{x} = (3, 6, 8) \in \mathcal{E}_+$, $\mathbf{y} = (5, 5, 7) \in \mathcal{E}_+$ and $\mathbf{u} = (15, 3, 2) \in \mathcal{D}_+$, then it can also be checked that although $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 0.0690984 > 0,$$

giving that $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$. Again, for the same function $g(\cdot)$ and for $\mathbf{x} = (3, 4, 5) \in \mathcal{E}_+$, $\mathbf{y} = (3, 4.5, 4.5) \in \mathcal{E}_+$, if $\mathbf{u} = (16, 15, 14.9) \in \mathcal{D}_+$ is taken then

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -0.00135098 < 0,$$

giving that $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$. So, $\varphi(\mathbf{x})$ is neither Schur-convex nor Schur-concave on $\mathcal{E}_+$. □

The observations, from Theorems 3.1 and 3.2, and the counterexamples, are reported in Table 1. The statements in the body of the table are regarding the function given by $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$.

4. Applications of majorization

In this section, we give two applications of majorization in the context of stochastic comparison of parallel/series systems of components. The first one deals with parallel systems of heterogeneous generalized exponential (GE) components and the other does the same with series systems of heterogeneous gamma components. Different orderings between parallel systems of heterogeneous exponential components through majorization was first studied, to the best of our knowledge, by Dykstra et al. [12]. Later, [13] extended the results for the multiple outlier heterogeneous exponential model. Similar problem for parallel systems of heterogeneous gamma components was addressed by Zhao [14].

4.1. Application with generalized exponential model

A random variable X is said to have GE distribution with parameters (λ, θ), written as GE(λ, θ), if the distribution function of X is given by

$$F_X(x) = \left(1 - e^{-\lambda x}\right)^{\theta}, \quad x > 0,\ \lambda > 0,\ \theta > 0,$$

where θ is the shape parameter and λ is the scale parameter. Clearly, this distribution is a generalization of exponential distribution in the sense that one can obtain exponential distribution from this distribution by taking $\theta = 1$. Unlike exponential distribution, this distribution has increasing (decreasing) failure rate for $\theta > (<)1$ for any fixed λ. Therefore, if it is known that the data are from a regular maintenance environment, it may make more sense to fit GE distribution than exponential distribution.

Now, suppose X_i (resp. Y_i), $i = 1, 2, \ldots, n$, be n independent random variables following GE distribution with parameters (λ_i, θ_i) (resp. (δ_i, θ_i)). Write $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. Then the distribution functions of $X_{n:n}$ and $Y_{n:n}$ can be written respectively as

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} F_{X_i}(x) = \prod_{i=1}^{n} \left(1 - e^{-\lambda_i x}\right)^{\theta_i},$$

and

$$F_{Y_{n:n}}(x) = \prod_{i=1}^{n} F_{Y_i}(x) = \prod_{i=1}^{n} \left(1 - e^{-\delta_i x}\right)^{\theta_i}.$$

It is to be mentioned here that Theorem 4.1 given below gives comparison of two parallel systems under p-larger order (which is weaker than majorization order). But before that we give a lemma, which is similar to the one in [15].

Lemma 4.1. *Let $\psi : \mathcal{E}_+$ (resp. $\mathcal{D}_+$) $\to \mathbb{R}$ be a function. Then, for $\mathbf{x}, \mathbf{y} \in \mathcal{E}_+$ (resp. $\mathbf{x}, \mathbf{y} \in \mathcal{D}_+$),*

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Rightarrow \psi(\mathbf{x}) \geq \psi(\mathbf{y})$$

if, and only if,

(i) *$\psi(e^{a_1}, \ldots, e^{a_n})$ is Schur-convex in $(a_1, \ldots, a_n) \in \mathcal{E}$ (resp. $(a_1, \ldots, a_n) \in \mathcal{D}$),*
(ii) *$\psi(e^{a_1}, \ldots, e^{a_n})$ is decreasing in a_i, for $i = 1, \ldots, n$,*

where $a_i = \ln x_i$, for $i = 1, \ldots, n$. □

In the following theorem we show that, if $\boldsymbol{\lambda}$ is p-larger than $\boldsymbol{\delta}$ then $X_{n:n}$ is superior to $Y_{n:n}$ in the usual stochastic order.

Theorem 4.1. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i) respectively, for $i = 1, 2, \ldots, n$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$ holds. Then*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\delta} \text{ implies } X_{n:n} \geq_{st} Y_{n:n}.$$

Proof. Let the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ hold. Further, let $F_E(\cdot)$ and $\tilde{r}_E(\cdot)$ be the distribution function and the reversed hazard rate function of the standard exponential distribution, respectively. Then the survival function of $X_{n:n}$ is given by

$$\begin{aligned} \bar{F}_{X_{n:n}}(x) &= 1 - \prod_{i=1}^{n} [F_E(e^{a_i})]^{\theta_i} \\ &= \Psi(e^{a_1}, e^{a_2}, \ldots, e^{a_n}), \text{ say,} \end{aligned}$$

where $a_i = \ln(\lambda_i x)$, for $i = 1, 2, \ldots, n$. Note that $\Psi(e^{a_1}, e^{a_2}, \ldots, e^{a_n})$ is decreasing in each a_i, for $i = 1, 2, \ldots, n$. Further, for $1 \leq p \leq q \leq n$,

$$\begin{aligned} \frac{\partial \Psi}{\partial a_p} - \frac{\partial \Psi}{\partial a_q} &= \prod_{i=1}^{n} [F_E(e^{a_i})]^{\theta_i} \left[\theta_q e^{a_q} \tilde{r}_E(e^{a_q}) - \theta_p e^{a_p} \tilde{r}_E(e^{a_p})\right] \\ &\leq 0, \end{aligned}$$

where the inequality follows because $x\tilde{r}_E(x)$ is decreasing in $x > 0$. Thus, by Lemma 3.3 we have that $\Psi(e^{a_1}, e^{a_2}, \ldots, e^{a_n})$ is Schur-convex in $(a_1, a_2, \ldots, a_n) \in \mathcal{E}$ whenever $(\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{D}_+$. Hence the result follows from Lemma 4.1. The result follows in a similar way under the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$. □

The following counterexample shows that Theorem 4.1 does not hold under the sets of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$, even if the condition of p-larger order is replaced by the majorization order.

Counterexample 4.1. *Let X_1, X_2, Y_1 and Y_2 follow GE distribution with respective parameters (λ_1, θ_1), (λ_2, θ_2), (δ_1, θ_1) and (δ_2, θ_2), where $(\lambda_1, \lambda_2) = (1, 5) \in \mathcal{E}_+$, $(\delta_1, \delta_2) = (2, 4) \in \mathcal{E}_+$, and $(\theta_1, \theta_2) = (2, 100) \in \mathcal{E}_+$. Clearly, $(\lambda_1, \lambda_2) \overset{m}{\succeq} (\delta_1, \delta_2)$. Now,*

$$\begin{aligned} k_1(x) &\overset{\text{def}}{=} \bar{F}_{X_{2:2}}(x) - \bar{F}_{Y_{2:2}}(x) \\ &= (1 - e^{-2x})^2 (1 - e^{-4x})^{100} - (1 - e^{-x})^2 (1 - e^{-5x})^{100}. \end{aligned}$$

It can be shown that $k_1(x)$ changes sign. Thus, $X_{2:2} \not\geq_{st} Y_{2:2}$. Further, let $(\lambda_1, \lambda_2) = (6, 2) \in \mathcal{D}_+$, $(\delta_1, \delta_2) = (5, 3) \in \mathcal{D}_+$, and $(\theta_1, \theta_2) = (60, 2) \in \mathcal{D}_+$. Clearly, $(\lambda_1, \lambda_2) \overset{m}{\succeq} (\delta_1, \delta_2)$. Now,

$$\begin{aligned} k_2(x) &\overset{\text{def}}{=} \bar{F}_{X_{2:2}}(x) - \bar{F}_{Y_{2:2}}(x) \\ &= (1 - e^{-5x})^{60} (1 - e^{-3x})^2 - (1 - e^{-6x})^{60} (1 - e^{-2x})^2. \end{aligned}$$

It is easy to verify that $k_2(x)$ changes sign. Thus, $X_{2:2} \not\geq_{st} Y_{2:2}$. □

The following counterexample shows that the condition of p-larger order given in Theorem 4.1 cannot be replaced by reciprocal majorization order.

Counterexample 4.2. *Let X_1, X_2, Y_1 and Y_2 follow GE distribution with respective parameters (λ_1, θ_1), (λ_2, θ_2), (δ_1, θ_1) and (δ_2, θ_2), where $(\lambda_1, \lambda_2) = (2, 4) \in \mathcal{E}_+$, $(\delta_1, \delta_2) = (2.4, 3) \in \mathcal{E}_+$, and $(\theta_1, \theta_2) = (1, 0.99) \in \mathcal{D}_+$. Clearly, $(\lambda_1, \lambda_2) \overset{rm}{\succeq} (\delta_1, \delta_2)$ and $(\lambda_1, \lambda_2) \overset{p}{\not\succeq} (\delta_1, \delta_2)$. Writing $k_3(x) = \bar{F}_{X_{2:2}}(x) - \bar{F}_{Y_{2:2}}(x)$ we have*

$$k_3(x) = (1 - e^{-2.4x})(1 - e^{-3x})^{0.99} - (1 - e^{-2x})(1 - e^{-4x})^{0.99}.$$

It can be shown that $k_3(x)$ changes sign. Thus, $X_{2:2} \not\geq_{st} Y_{2:2}$. □

Below we give a lemma without proof, which will be used in proving the upcoming theorems.

Lemma 4.2. *For $x > 0$, $g(y) = y\,(e^{yx} - 1)^{-1}$ is decreasing and convex in $y > 0$.* □

The following lemma to be used in proving the next theorem, may be obtained in [4, p. 87], where the parenthetical statements are not given.

Lemma 4.3. *Let $S \subseteq \mathbb{R}^n$. Further, let $\varphi : S \to \mathbb{R}$. Then*

$$(a_1, a_2, \ldots, a_n) \succeq_w (b_1, b_2, \ldots, b_n) \text{ implies } \varphi(a_1, a_2, \ldots, a_n) \geq (\text{resp. } \leq)\ \varphi(b_1, b_2, \ldots, b_n)$$

if, and only if, φ is increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on S. Similarly,

$$(a_1, a_2, \ldots, a_n) \overset{w}{\succeq} (b_1, b_2, \ldots, b_n) \text{ implies } \varphi(a_1, a_2, \ldots, a_n) \geq (\text{resp. } \leq)\ \varphi(b_1, b_2, \ldots, b_n)$$

if, and only if, φ is decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on S. □

The following theorem shows that, if $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ weakly supermajorizes $(\delta_1, \delta_2, \ldots, \delta_n)$, then $X_{n:n}$ dominates $Y_{n:n}$ in reversed hazard rate ordering.

Theorem 4.2. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i) respectively, for $i = 1, 2, \ldots, n$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ holds. Then*

$$\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\delta} \text{ implies } X_{n:n} \geq_{rhr} Y_{n:n}.$$

Proof. The reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$ are given respectively by

$$\tilde{r}_{X_{n:n}}(x) = \sum_{i=1}^{n} \frac{\theta_i \lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} = \sum_{i=1}^{n} \theta_i g(\lambda_i), \tag{4.1}$$

and

$$\tilde{r}_{Y_{n:n}}(x) = \sum_{i=1}^{n} \frac{\theta_i \delta_i e^{-\delta_i x}}{1 - e^{-\delta_i x}} = \sum_{i=1}^{n} \theta_i g(\delta_i), \tag{4.2}$$

where $g(\cdot)$ is as defined in Lemma 4.2. By Lemma 4.2 and Theorem 3.1 b(ii) we have that $\tilde{r}_{X_{n:n}}(x)$ is Schur-convex in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Further, by Lemma 4.2 and Theorem 3.2 a(ii) we also get that $\tilde{r}_{X_{n:n}}(x)$ is Schur-convex in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Again, note that $\tilde{r}_{X_{n:n}}(x)$ is decreasing in each λ_i. Thus, the result follows from Lemma 4.3.

Remark 4.1. Theorem 4.2 improves Theorem 3.2 of Dykstra et al. [12] in the sense that the latter can be obtained from the former by taking $\theta_1 = \theta_2 = \cdots = \theta_n = 1$.

Remark 4.2. Counterexample 4.1 shows that Theorem 4.2 does not hold under the sets of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$. □

It is well known that p-larger order is weaker than the weak supermajorization order. Then a natural question arises—whether the result discussed in Theorem 4.2 holds under p-larger order. The following counterexample answers this question in negative.

Counterexample 4.3. *Let X_1, X_2, Y_1 and Y_2 follow GE distribution with respective parameters (λ_1, θ_1), (λ_2, θ_2), (δ_1, θ_1) and (δ_2, θ_2), where $(\lambda_1, \lambda_2) = (1, 7) \in \mathcal{E}_+$, $(\delta_1, \delta_2) = (2, 5) \in \mathcal{E}_+$, and $(\theta_1, \theta_2) = (2.1, 2) \in \mathcal{D}_+$. Clearly, $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\delta_1, \delta_2)$ and $(\lambda_1, \lambda_2) \overset{w}{\not\succeq} (\delta_1, \delta_2)$. Writing $k_4(x) = \tilde{r}_{X_{2:2}}(x) - \tilde{r}_{Y_{2:2}}(x)$ we have*

$$k_4(x) = \frac{2.1}{e^x - 1} + \frac{14}{e^{7x} - 1} - \frac{4.2}{e^{2x} - 1} - \frac{10}{e^{5x} - 1}.$$

It can be shown that $k_4(x)$ changes sign. Thus, $X_{2:2} \not\geq_{rhr} Y_{2:2}$. □

In the following theorem we generalize the above result for the up shifted reversed hazard rate order.

Theorem 4.3. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i) respectively, for $i = 1, 2, \ldots, n$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\delta} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\delta} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ holds. Then*

$$\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\delta} \text{ implies } X_{n:n} \geq_{rhr\uparrow} Y_{n:n}.$$

Proof. Note that $\tilde{r}_{n:n}(x)$ is decreasing in $x > 0$, and hence $X_{n:n}$ has log-concave distribution function. Then, on using Theorem 4.2, the result follows from Theorem 2.2 of Di Crescenzo and Longobardi [10].

Remark 4.3. Counterexample 4.3 shows that the weak supermajorization condition given in Theorem 4.3 cannot be replaced by p-larger order. □

Below we give another set of sufficient conditions under which the result given in Theorem 4.3 holds.

Theorem 4.4. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i), respectively, for $i = 1, 2, \ldots, n$. Further, let X_i's and Y_i's be independent. If $\delta_i \geq \lambda_i$ for all $i = 1, 2, \ldots, n$, then $X_{n:n} \geq_{rhr\uparrow} Y_{n:n}$.*

Proof. By Lemma 4.2 we have that $g(\cdot)$ is a decreasing function. Then $X_{n:n} \geq_{rhr} Y_{n:n}$ immediately follows from (4.1) and (4.2). Further, note that $\tilde{r}_{n:n}(x)$ is decreasing in $x > 0$. Hence, the result follows from Theorem 2.2 of Di Crescenzo and Longobardi [10]. □

The following corollaries are immediate.

Corollary 4.1. *If $\delta_i \geq \lambda_i$ for all $i = 1, 2, \ldots, n$, then $X_{n:n} \geq_{rhr} Y_{n:n}$.*

Corollary 4.2. *If $\min\{\delta_1, \delta_2, \ldots, \delta_n\} \geq \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $X_{n:n} \geq_{rhr} Y_{n:n}$.* □

Now the question arises whether Theorem 4.2 can be strengthened further by replacing reversed hazard rate order between $X_{n:n}$ and $Y_{n:n}$ by likelihood ratio order. For $n = 3$, the following counterexample gives a negative answer, even if the condition of weak supermajorization order is replaced by the majorization order.

Counterexample 4.4. *Let X_i (resp. Y_i) follow GE distributions with parameters (λ_i, θ_i) (resp. (δ_i, θ_i)), $i = 1, 2, 3$, where $(\lambda_1, \lambda_2, \lambda_3) = (6, 4, 2) \in \mathcal{D}_+$, $(\delta_1, \delta_2, \delta_3) = (5, 5, 2) \in \mathcal{D}_+$ and $(\theta_1, \theta_2, \theta_3) = (1, 2, 3) \in \mathcal{E}_+$. Clearly, $(\lambda_1, \lambda_2, \lambda_3) \overset{m}{\succeq} (\delta_1, \delta_2, \delta_3)$. Now, it can be shown that $f_{X_{3:3}}(x)/f_{Y_{3:3}}(x)$ is nonmonotone. Thus, there is no likelihood ratio order between $X_{3:3}$ and $Y_{3:3}$.* □

Below we see that there exists likelihood ratio order between $X_{n:n}$ and $Y_{n:n}$ if Y_i is a random variable following GE distribution with parameters $\left(\bar{\lambda}, \theta_i\right)$, $i = 1, 2, \ldots, n$, where $\bar{\lambda} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$.

Theorem 4.5. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and $\left(\bar{\lambda}, \theta_i\right)$, respectively, for $i = 1, 2, \ldots, n$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\boldsymbol{\lambda} \in \mathcal{D}_+, \boldsymbol{\theta} \in \mathcal{E}_+\}$ or $\{\boldsymbol{\lambda} \in \mathcal{E}_+, \boldsymbol{\theta} \in \mathcal{D}_+\}$ holds. Then $X_{n:n} \geq_{lr} Y_{n:n}$.*

Proof. To prove the result, we have to show that

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(x)} = c \sum_{i=1}^{n} \frac{\theta_i \lambda_i \left(e^{\bar{\lambda}x} - 1\right)}{\left(e^{\lambda_i x} - 1\right)} \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \text{ is increasing in } x > 0, \tag{4.3}$$

where c is a constant independent of x. Now, by Theorem 4.2, we have that

$$\frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \text{ is increasing in } x > 0.$$

So, from (4.3), it is only required to show that

$$\eta_3(x) \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{\theta_i\lambda_i\left(e^{\bar{\lambda}x}-1\right)}{\left(e^{\lambda_i x}-1\right)} \text{ is increasing in } x>0.$$

Differentiating $\eta_3(x)$ with respect to x, we have

$$\eta_3'(x) = \bar{\lambda}e^{\bar{\lambda}x}\sum_{i=1}^{n}\frac{\theta_i\lambda_i}{\left(e^{\lambda_i x}-1\right)} - \left(e^{\bar{\lambda}x}-1\right)\sum_{i=1}^{n}\frac{\theta_i\lambda_i^2 e^{\lambda_i x}}{\left(e^{\lambda_i x}-1\right)^2}. \tag{4.4}$$

It can be shown that each of $\lambda_i^2 e^{-\lambda_i x}/\left(1-e^{-\lambda_i x}\right)^2$ and $\left(1-e^{-\lambda_i x}\right)/\lambda_i$ is decreasing in λ_i. Thus, we have, on using Equation (1.5) of Mitrinović et al. [16]

$$\sum_{i=1}^{n}\frac{\theta_i\lambda_i}{e^{\lambda_i x}-1} \geq \frac{1}{n}\sum_{i=1}^{n}\frac{\theta_i\lambda_i^2 e^{-\lambda_i x}}{\left(1-e^{-\lambda_i x}\right)^2}\sum_{i=1}^{n}\frac{\left(1-e^{-\lambda_i x}\right)}{\lambda_i}.$$

Thus, $\eta_3(x)$ is increasing in x, if for all $x>0$,

$$\frac{\bar{\lambda}}{n}\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i} - \left(1-e^{-\bar{\lambda}x}\right) \geq 0,$$

which holds by judiciously using *AM-GM* inequality. Hence, the result follows.

Remark 4.4. The above theorem improves Theorem 2.1(b) of Dykstra et al. [12] in the sense that the latter can be obtained from the former by taking $\theta_1=\theta_2=\cdots=\theta_n=1$ and by noting the fact that likelihood ratio order is stronger than failure rate order. □

In case of multiple-outlier model, the following theorem shows that the restrictions on the parameters given in Theorem 4.2 can be relaxed. For more properties of this model, one may refer to Zhao and Balakrishnan [13].

Theorem 4.6. *Let* $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ *and* $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ *be two sets of independent random variables each following the multiple-outlier GE model such that* $X_i \sim GE(\lambda_1,\theta_i)$ *and* $Y_i \sim GE(\delta_1,\theta_i)$, *for* $i=1,2,\ldots,n_1$, $X_j \sim GE\left(\alpha,\theta_j\right)$ *and* $Y_j \sim GE\left(\alpha,\theta_j\right)$, *for* $j=n_1+1,n_1+2,\ldots,n_1+n_2\ (=n)$. *Further, let* X_i*'s and* Y_i*'s be independent. Then*

$$(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2}) \overset{w}{\succeq} (\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2}) \Rightarrow X_{n:n} \geq_{rhr} Y_{n:n}.$$

Proof. From (4.1) and (4.2) we have

$$\tilde{r}_{X_{n:n}}(x) = \theta g(\lambda_1) + \theta^* g(\alpha)$$

and

$$\tilde{r}_{Y_{n:n}}(x) = \theta g(\delta_1) + \theta^* g(\alpha),$$

where $\theta=\sum_{i=1}^{n_1}\theta_i$ and $\theta^*=\sum_{j=n_1+1}^{n}\theta_j$. Further,

$$(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2}) \overset{w}{\succeq} (\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2})$$

if, and only if, one of the following cases holds: (i) $\lambda_1\leq\delta_1\leq\alpha$, (ii) $\lambda_1\leq\alpha\leq\delta_1$, (iii) $\alpha\leq\lambda_1\leq\delta_1$. Note that, for all the above three cases, $\tilde{r}_{X_{n:n}}(x)\geq\tilde{r}_{Y_{n:n}}(x)$, and hence $X_{n:n}\geq_{rhr}Y_{n:n}$. □

We have shown that likelihood ratio ordering between $X_{n:n}$ and $Y_{n:n}$ with heterogeneous GE components does not exist for all n. Next theorem shows that a similar result still holds for multiple-outlier GE model.

Theorem 4.7. *Let* $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ *and* $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ *be two sets of independent random variables each following the multiple-outlier GE model such that* $X_i \sim GE(\lambda_1,\theta_1)$ *and* $Y_i \sim GE(\delta_1,\theta_1)$, *for* $i=1,2,\ldots,n_1$, $X_j \sim GE(\lambda_2,\theta_2)$ *and* $Y_j \sim GE(\delta_2,\theta_2)$, *for* $j=n_1+1,n_1+2,\ldots,n_1+n_2\ (=n)$. *Further, let* X_i*'s and* Y_i*'s be independent. Suppose that the set of conditions* $\{(\lambda_1,\lambda_2)\in\mathcal{D}_+,(\delta_1,\delta_2)\in\mathcal{D}_+,(\theta_1,\theta_2)\in\mathcal{E}_+\}$ *or* $\{(\lambda_1,\lambda_2)\in\mathcal{E}_+,(\delta_1,\delta_2)\in\mathcal{E}_+,(\theta_1,\theta_2)\in\mathcal{D}_+\}$ *holds. Then*

$$(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\lambda_2,\lambda_2,\ldots,\lambda_2}_{n_2}) \overset{m}{\succeq} (\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\delta_2,\delta_2,\ldots,\delta_2}_{n_2}) \Rightarrow X_{n:n} \geq_{lr} Y_{n:n}.$$

Proof. Write

$$\mathcal{F} = \{\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) : \ \mu_i = \lambda_1, \text{ for } 1 \le i \le n_1 \text{ and } \mu_i = \lambda_2, \text{ for } n_1+1 \le i \le n\},$$
$$\mathcal{G} = \{\boldsymbol{s} = (s_1, s_2, \ldots, s_n) : \ s_i = \delta_1, \text{ for } 1 \le i \le n_1 \text{ and } s_i = \delta_2, \text{ for } n_1+1 \le i \le n\}$$

and

$$\mathcal{H} = \{\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) : \ \theta_i = \theta_1, \text{ for } 1 \le i \le n_1 \text{ and } \theta_i = \theta_2, \text{ for } n_1+1 \le i \le n\}.$$

Note that

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(x)} = \left.\left(\frac{\sum_{i=1}^{n} \frac{\theta_i \mu_i}{e^{\mu_i x}-1}}{\sum_{i=1}^{n} \frac{\theta_i s_i}{e^{s_i x}-1}}\right) \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)}\right|_{\boldsymbol{\mu}\in\mathcal{F},\ \boldsymbol{s}\in\mathcal{G},\ \boldsymbol{\theta}\in\mathcal{H}}.$$

By Theorem 4.2, we have that

$$\frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \text{ is increasing in } x > 0.$$

So, to prove the result, it suffices to show that, for $\boldsymbol{\mu} \in \mathcal{F}$, $\boldsymbol{s} \in \mathcal{G}$ and $\boldsymbol{\theta} \in \mathcal{H}$,

$$\Delta(x) \overset{\text{def}}{=} \frac{\sum_{i=1}^{n} \frac{\theta_i \mu_i}{e^{\mu_i x}-1}}{\sum_{i=1}^{n} \frac{\theta_i s_i}{e^{s_i x}-1}} = \frac{\sum_{i=1}^{n} \theta_i u(\mu_i x)}{\sum_{i=1}^{n} \theta_i u(s_i x)}$$

is increasing in x, where $u(x) = x/(e^x - 1)$. Now, $u'(x) = u(x)v(x)/x$, where $v(x) = (e^x - 1 - xe^x)/(e^x - 1)$. It can be easily shown that $u(x)$ and $v(x)$ are decreasing in $x > 0$. Now, differentiating $\Delta(x)$ with respect to x, we get

$$\Delta'(x) \overset{\text{sign}}{=} \sum_{i=1}^{n} \theta_i u(s_i x) \sum_{i=1}^{n} \theta_i u(\mu_i x) v(\mu_i x) - \sum_{i=1}^{n} \theta_i u(\mu_i x) \sum_{i=1}^{n} \theta_i u(s_i x) v(s_i x).$$

Clearly, $\Delta(x)$ is increasing in $x > 0$ if

$$\Omega(\mu_1, \mu_2, \ldots, \mu_n) \overset{\text{def}}{=} \frac{\sum_{i=1}^{n} \theta_i u(\mu_i x) v(\mu_i x)}{\sum_{i=1}^{n} \theta_i u(\mu_i x)}$$

is Schur-convex in $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathcal{F}$, for $\boldsymbol{\theta} \in \mathcal{H}$. Assume that $\omega(x) = u(x)v'(x)$, $\theta = \sum_{r=1}^{n_1} \theta_r$ and $\theta^* = \sum_{r=n_1+1}^{n} \theta_r$. Then, for $1 \le i \le n_1$,

$$\frac{\partial \Omega}{\partial \mu_i} = \frac{\theta_1 x[\theta^* u'(\lambda_1 x) u(\lambda_2 x)\{v(\lambda_1 x) - v(\lambda_2 x)\} + \omega(\lambda_1 x)\{\theta u(\lambda_1 x) + \theta^* u(\lambda_2 x)\}]}{(\theta u(\lambda_1 x) + \theta^* u(\lambda_2 x))^2}$$

and, for $n_1 + 1 \le j \le n$,

$$\frac{\partial \Omega}{\partial \mu_j} = \frac{\theta_2 x[\theta u'(\lambda_2 x) u(\lambda_1 x)\{v(\lambda_2 x) - v(\lambda_1 x)\} + \omega(\lambda_2 x)\{\theta^* u(\lambda_2 x) + \theta u(\lambda_1 x)\}]}{(\theta u(\lambda_1 x) + \theta^* u(\lambda_2 x))^2}.$$

Consider the following two cases:

Case I: Let $1 \le i, j \le n_1$ or $n_1 + 1 \le i, j \le n$. Then

$$\frac{\partial \Omega}{\partial \mu_i} - \frac{\partial \Omega}{\partial \mu_j} = 0.$$

Case II: Let $1 \le i \le n_1$ and $n_1 + 1 \le j \le n$. Then

$$\begin{aligned}\frac{\partial \Omega}{\partial \mu_i} - \frac{\partial \Omega}{\partial \mu_j} \overset{\text{sign}}{=} &\ \{v(\lambda_1 x) - v(\lambda_2 x)\}\{\theta_1 \theta^* u'(\lambda_1 x) u(\lambda_2 x) + \theta_2 \theta u'(\lambda_2 x) u(\lambda_1 x)\} \\ &+ \{\theta u(\lambda_1 x) + \theta^* u(\lambda_2 x)\}\{\theta_1 \omega(\lambda_1 x) - \theta_2 \omega(\lambda_2 x)\}.\end{aligned}$$

Again, as $v(x)$ is decreasing in x, on using Lemma 3.3 of Torrado and Kochar [17], we have that $w(x)$ is increasing and non-positive. Therefore, it follows that $\frac{\partial \Omega}{\partial \mu_i} - \frac{\partial \Omega}{\partial \mu_j} \ge 0$ whenever the set of conditions $\{(\lambda_1, \lambda_2) \in \mathcal{D}_+, (\delta_1, \delta_2) \in \mathcal{D}_+,$

$(\theta_1, \theta_2) \in \mathcal{E}_+\}$ holds. Further, $\frac{\partial \Omega}{\partial \mu_i} - \frac{\partial \Omega}{\partial \mu_j} \leq 0$ whenever the set of conditions $\{(\lambda_1, \lambda_2) \in \mathcal{E}_+, (\delta_1, \delta_2) \in \mathcal{E}_+, (\theta_1, \theta_2) \in \mathcal{D}_+\}$ holds. Hence the result follows from Lemmas 3.1 and 3.3. □

The following corollary immediately follows from the above theorem.

Corollary 4.3. *Let X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i), respectively, for $i = 1, 2$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\lambda_1 \geq \lambda_2, \delta_1 \geq \delta_2, \theta_1 \leq \theta_2\}$ or $\{\lambda_1 \leq \lambda_2, \delta_1 \leq \delta_2, \theta_1 \geq \theta_2\}$ holds. Then*

$$(\lambda_1, \lambda_2) \overset{m}{\succeq} (\delta_1, \delta_2) \text{ implies } X_{2:2} \geq_{lr} Y_{2:2}.$$

Remark 4.5. Counterexample 4.1 shows that Theorem 4.7 does not hold under the set of conditions $\{(\lambda_1, \lambda_2) \in \mathcal{D}_+, (\delta_1, \delta_2) \in \mathcal{D}_+, (\theta_1, \theta_2) \in \mathcal{D}_+\}$ or $\{(\lambda_1, \lambda_2) \in \mathcal{E}_+, (\delta_1, \delta_2) \in \mathcal{E}_+, (\theta_1, \theta_2) \in \mathcal{E}_+\}$.

Remark 4.6. The condition of majorization order given in Theorem 4.7 cannot be replaced by p-larger order as Counterexample 4.3 shows.

Remark 4.7. The above theorem improves Theorem 3.5 of Zhao and Balakrishnan [13] in the sense that the latter can be obtained from the former by taking $\theta_1 = \theta_2 = \cdots = \theta_n = 1$. □

Below we give another set of sufficient conditions under which the above theorem holds.

Theorem 4.8. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two sets of independent random variables each following the multiple-outlier GE model such that $X_i \sim GE\,(\lambda_1, \theta_i)$ and $Y_i \sim GE\,(\delta_1, \theta_i)$, for $i = 1, 2, \ldots, n_1$, $X_j \sim GE\left(\lambda_2, \theta_j\right)$ and $Y_j \sim GE\left(\delta_2, \theta_j\right)$, for $j = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$ $(=n)$. Further, let X_i's and Y_i's be independent. Then*

$$\min\{\delta_1, \delta_2\} \geq \max\{\lambda_1, \lambda_2\} \Rightarrow X_{n:n} \geq_{lr} Y_{n:n}.$$

Proof. Writing $\sum_{i=1}^{n_1} \theta_i = \theta$ and $\sum_{j=n_1+1}^{n} \theta_j = \theta^*$ we have

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(x)} = \left(\frac{\frac{\lambda_1 \theta}{e^{\lambda_1 x} - 1} + \frac{\lambda_2 \theta^*}{e^{\lambda_2 x} - 1}}{\frac{\delta_1 \theta}{e^{\delta_1 x} - 1} + \frac{\delta_2 \theta^*}{e^{\delta_2 x} - 1}} \right) \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)}.$$

As $\min\{\delta_1, \delta_2\} \geq \max\{\lambda_1, \lambda_2\}$, by Corollary 4.2, we have that

$$\frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \text{ is increasing in } x > 0.$$

Thus, to prove the result, it suffices to show that

$$\eta_4(x) \overset{\text{def}}{=} \frac{\frac{\lambda_1 \theta}{e^{\lambda_1 x} - 1} + \frac{\lambda_2 \theta^*}{e^{\lambda_2 x} - 1}}{\frac{\delta_1 \theta}{e^{\delta_1 x} - 1} + \frac{\delta_2 \theta^*}{e^{\delta_2 x} - 1}} \text{ is increasing in } x > 0.$$

Now, differentiating $\eta_4(x)$ with respect to x we have

$$\begin{aligned}
\eta_4'(x) &\overset{\text{sign}}{=} \theta^2 u(\lambda_1 x) u(\delta_1 x)[z(\delta_1 x) - z(\lambda_1 x)] + \theta\theta^* u(\lambda_1 x) u(\delta_2 x)[z(\delta_2 x) - z(\lambda_1 x)] \\
&\quad + \theta\theta^* u(\delta_1 x) u(\lambda_2 x)[z(\delta_1 x) - z(\lambda_2 x)] + \theta^{*2} u(\delta_2 x) u(\lambda_2 x)[z(\delta_2 x) - z(\lambda_2 x)] \\
&\geq 0,
\end{aligned}$$

where $u(x) = x/(e^x - 1) \geq 0$ and $z(x) = x/(1 - e^{-x}) \geq 0$. The inequality follows from the fact that $\min\{\delta_1, \delta_2\} \geq \max\{\lambda_1, \lambda_2\}$, and $z(x)$ is increasing in $x > 0$. Thus, the result is proved. □

In the following theorem we show that the above result holds under different set of sufficient conditions.

Theorem 4.9. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two sets of independent random variables each following the multiple-outlier GE model such that $X_i \sim GE\,(\lambda_1, \theta_i)$ and $Y_i \sim GE\,(\delta_1, \theta_i)$, for $i = 1, 2, \ldots, n_1$, $X_j \sim GE\left(\alpha, \theta_j\right)$ and $Y_j \sim GE\left(\alpha, \theta_j\right)$, for $j = n_1+1, n_1+2, \ldots, n_1+n_2$ $(=n)$. Further, let X_i's and Y_i's be independent. Suppose that $\lambda_1 \leq \min\{\delta_1, \alpha\}$. Then*

$$(\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{n_1}, \underbrace{\alpha, \alpha, \ldots, \alpha}_{n_2}) \overset{w}{\succeq} (\underbrace{\delta_1, \delta_1, \ldots, \delta_1}_{n_1}, \underbrace{\alpha, \alpha, \ldots, \alpha}_{n_2}) \Rightarrow X_{n:n} \geq_{lr} Y_{n:n}.$$

Proof. Write $\sum_{i=1}^{n_1}\theta_i=\theta$ and $\sum_{j=n_1+1}^{n}\theta_j=\theta^*$. Then

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(x)}=\left(\frac{\frac{\lambda_1\theta}{e^{\lambda_1 x}-1}+\frac{\alpha\theta^*}{e^{\alpha x}-1}}{\frac{\delta_1\theta}{e^{\delta_1 x}-1}+\frac{\alpha\theta^*}{e^{\alpha x}-1}}\right)\frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)}.$$

By Theorem 4.6, we have that

$$\frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \text{ is increasing in } x>0.$$

Thus, to prove the result, it suffices to show that

$$\zeta(x)\overset{\text{def}}{=}\frac{\frac{\lambda_1\theta}{e^{\lambda_1 x}-1}+\frac{\alpha\theta^*}{e^{\alpha x}-1}}{\frac{\delta_1\theta}{e^{\delta_1 x}-1}+\frac{\alpha\theta^*}{e^{\alpha x}-1}} \text{ is increasing in } x>0.$$

Now, differentiating $\zeta(x)$ with respect to x we have

$$\begin{aligned}\zeta'(x)&\overset{\text{sign}}{=}\theta^2u(\lambda_1x)u(\delta_1x)[z(\delta_1x)-z(\lambda_1x)]+\theta^*\theta u(\alpha x)u(\lambda_1x)[z(\alpha x)-z(\lambda_1x)]\\&\quad+\theta^*\theta u(\alpha x)u(\delta_1x)[z(\delta_1x)-z(\alpha x)]\\&=\gamma(x),\ \text{say},\end{aligned}\tag{4.5}$$

where $u(x)=x/(e^x-1)\ge 0$ and $z(x)=x/(1-e^{-x})\ge 0$. It is easy to show that $u(x)$ is decreasing in $x>0$, and $z(x)$ is increasing in $x>0$. Now from the hypothesis we have $\lambda_1\le\min\{\delta_1,\alpha\}$ and $(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2})\overset{w}{\succeq}(\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\alpha,\alpha,\ldots,\alpha}_{n_2})$, which is equivalent to the fact that $\lambda_1\le\alpha\le\delta_1$ or $\lambda_1\le\delta_1\le\alpha$.

Case I: $\lambda_1\le\alpha\le\delta_1$. Then we have $z(\lambda_1x)\le z(\alpha x)\le z(\delta_1x)$, which implies that $\gamma(x)\ge 0$.
Case II: $\lambda_1\le\delta_1\le\alpha$. Then we have $z(\lambda_1x)\le z(\delta_1x)\le z(\alpha x)$ and $u(\lambda_1x)\ge u(\delta_1x)\ge u(\alpha x)$. From (4.5) we have

$$\begin{aligned}\gamma(x)&\ge[z(\delta_1x)-z(\lambda_1x)][\theta^2u(\lambda_1x)u(\delta_1x)+\theta^*\theta u(\alpha x)u(\delta_1x)]\\&\ge 0,\end{aligned}$$

which gives that $\zeta'(x)\ge 0$. Thus, the result is proved.

Corollary 4.4. *Let X_1 and X_2 follow GE distributions with parameters (λ_1,θ_1) and (α,θ_2), and let Y_1 and Y_2 follow GE distributions with parameters (δ_1,θ_1) and (α,θ_2), respectively. Further, let X_i's and Y_i's be independent. Suppose that $\lambda_1\le\min\{\delta_1,\alpha\}$. Then*

$$(\lambda_1,\alpha)\overset{w}{\succeq}(\delta_1,\alpha) \text{ implies } X_{2:2}\ge_{lr}Y_{2:2}.$$

Below we cite a counterexample which shows that the condition $\lambda_1\le\min\{\delta_1,\alpha\}$ given in Theorem 4.9 cannot be relaxed.

Counterexample 4.5. *Let X_1 and X_2 follow GE distributions with respective parameters (λ_1,θ_1), (α,θ_2), and Y_1 and Y_2 follow GE distribution with respective parameters (δ_1,θ_1) and (α,θ_2), where $(\lambda_1,\alpha)=(2,1)\in\mathcal{D}_+$, $(\delta_1,\alpha)=(3,1)\in\mathcal{D}_+$ and $(\theta_1,\theta_2)=(6,6.1)\in\mathcal{E}_+$. Clearly, $\lambda_1\nleq\min\{\delta_1,\alpha\}$ and $(\lambda_1,\alpha)\overset{w}{\succeq}(\delta_1,\alpha)$. One can show that $f_{X_{2:2}}(x)/f_{Y_{2:2}}(x)$ is nonmonotone. Thus, $X_{2:2}\ngeq_{lr}Y_{2:2}$.* □

Before going into the next theorem we give the following lemma without proof.

Lemma 4.4. *If $\lambda_1\le\delta_1\le\delta_2\le\lambda_2$ or $\lambda_1\ge\delta_1\ge\delta_2\ge\lambda_2$, and $n_1\lambda_1+n_2\lambda_2=n_1\delta_1+n_2\delta_2$ then*

$$(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\lambda_2,\lambda_2,\ldots,\lambda_2}_{n_2})\overset{m}{\succeq}(\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\delta_2,\delta_2,\ldots,\delta_2}_{n_2}).\quad\square$$

The following theorem generalizes the result discussed in Theorem 4.9.

Theorem 4.10. *Let $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ and $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ be two sets of independent random variables each following the multiple-outlier GE model such that $X_i\sim GE(\lambda_1,\theta_1)$ and $Y_i\sim GE(\delta_1,\theta_1)$, for $i=1,2,\ldots,n_1$, $X_j\sim GE(\lambda_2,\theta_2)$ and $Y_j\sim GE(\delta_2,\theta_2)$, for $j=n_1+1,n_1+2,\ldots,n_1+n_2\ (=n)$. Further, let X_i's and Y_i's be independent. Suppose that the set of conditions $\{\lambda_1\le\delta_1\le\delta_2\le\lambda_2,\theta_1\ge\theta_2\}$ or $\{\lambda_1\ge\delta_1\ge\delta_2\ge\lambda_2,\theta_1\le\theta_2\}$ holds. Then*

$$(\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{n_1},\underbrace{\lambda_2,\lambda_2,\ldots,\lambda_2}_{n_2})\overset{w}{\succeq}(\underbrace{\delta_1,\delta_1,\ldots,\delta_1}_{n_1},\underbrace{\delta_2,\delta_2,\ldots,\delta_2}_{n_2})\Rightarrow X_{n:n}\ge_{lr}Y_{n:n}.$$

Proof. Suppose that the first set of conditions holds. The weak supermajorization order gives that $\lambda_1 \le \delta_1$ and $n_1\lambda_1 + r\lambda_2 \le n_1\delta_1 + r\delta_2$, for $r = 1, 2, \dots, n_2$. If $n_1\lambda_1 + n_2\lambda_2 = n_1\delta_1 + n_2\delta_2$ then the result follows from Theorem 4.7. Suppose that $n_1\lambda_1 + n_2\lambda_2 < n_1\delta_1 + n_2\delta_2$. Then there exists some λ such that $n_1\lambda + n_2\lambda_2 = n_1\delta_1 + n_2\delta_2$ and $\lambda_1 < \lambda \le \delta_1$. Let $X^*_{n:n}$ be the lifetime of a parallel system formed by n components $X^*_1, X^*_2, \dots, X^*_n$ where $X^*_i \sim GE(\lambda, \theta_1)$, for $i = 1, 2, \dots, n_1$ and $X^*_j \sim GE(\lambda_2, \theta_2)$, for $j = n_1+1, n_1+2, \dots, n_1+n_2$ $(=n)$. Then, on using Lemma 4.4, $X^*_{n:n} \ge_{lr} Y_{n:n}$ follows from Theorem 4.7. Further, note that $\lambda_1 \le \lambda \le \lambda_2$ and

$$(\underbrace{\lambda_1, \lambda_1, \dots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \lambda_2, \dots, \lambda_2}_{n_2}) \overset{w}{\succeq} (\underbrace{\lambda, \lambda, \dots, \lambda}_{n_1}, \underbrace{\lambda_2, \lambda_2, \dots, \lambda_2}_{n_2}).$$

Thus, $X_{n:n} \ge_{lr} X^*_{n:n}$ follows from Theorem 4.9. Hence $X_{n:n} \ge_{lr} Y_{n:n}$. The proof follows in a similar way under the second set of conditions.

Corollary 4.5. *If X_i and Y_i follow GE distributions with parameters (λ_i, θ_i) and (δ_i, θ_i) respectively, for $i = 1, 2$. Suppose that the set of conditions $\{\lambda_1 \le \delta_1 \le \delta_2 \le \lambda_2, \theta_1 \ge \theta_2\}$ or $\{\lambda_1 \ge \delta_1 \ge \delta_2 \ge \lambda_2, \theta_1 \le \theta_2\}$ holds. Then*

$$(\lambda_1, \lambda_2) \overset{w}{\succeq} (\delta_1, \delta_2) \text{ implies } X_{2:2} \ge_{lr} Y_{2:2}.$$

4.2. Application with gamma model

A random variable X is said to have gamma distribution with parameters (λ, α) if the density function of X is given by

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0,\ \lambda > 0,\ \alpha > 0,$$

where α is the shape parameter and λ is the scale parameter. As mentioned in the introduction, the comparison of parallel systems formed by gamma components with respect to different stochastic orders have been well studied in the literature (cf. [14,18], and the references therein). Here we consider series system in place of parallel system. We show that one series system dominates the other with respect to the hazard rate order whenever their scale parameters are ordered with respect to weak majorization order (which is weaker than majorization order). Below we give three lemmas. The first lemma may be obtained in [4, p. 92], whereas the second lemma is borrowed from [6, p. 116]. The proof of the third lemma is straightforward.

Lemma 4.5. *Let $I \subseteq \mathbb{R}$ be an interval, and let $g : I \to \mathbb{R}$ be convex (resp. concave). Then*

$$\varphi(\mathbf{x}) = \sum_{i=1}^{n} g(x_i)$$

is Schur-convex (resp. Schur-concave) on I^n.

Lemma 4.6. *Let Z be a random variable having IFRA (resp. DFRA) distribution. Then*

$$E(Z^2) \le (\text{resp. } \ge)\ 2[E(Z)]^2.$$

Lemma 4.7. *Let Z be a random variable having probability density function given by*

$$f_Z(t) = \frac{(1+t)^{\alpha-1} e^{-ty}}{\int_0^\infty (1+u)^{\alpha-1} e^{-uy} du}, \quad \text{for } t \in (0, \infty),$$

where α and y are positive constants. Then Z is DLR for $\alpha \in [0, 1]$, and is ILR for $\alpha \in [1, \infty)$. □

In the following theorem we compare $X_{1:n}$ and $Y_{1:n}$ with respect to the hazard rate order.

Theorem 4.11. *Let $X_1, X_2, \dots, X_n$ be independent gamma random variables with respective scale parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, and the same shape parameter α. Further, let $Y_1, Y_2, \dots, Y_n$ be another set of independent gamma random variables with respective scale parameters $\mu_1, \mu_2, \dots, \mu_n$, and the same shape parameter α. Suppose X_i's and Y_i's are independent. Then*

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \overset{\mathrm{w}}{\succeq} (\mu_1, \mu_2, \dots, \mu_n) \Rightarrow X_{1:n} \ge_{hr} Y_{1:n}, \quad \text{for } \alpha \in [0, 1],$$

and

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \succeq_{\mathrm{w}} (\mu_1, \mu_2, \dots, \mu_n) \Rightarrow X_{1:n} \le_{hr} Y_{1:n}, \quad \text{for } \alpha \in [1, \infty).$$

Proof. The hazard rate function of $X_{1:n}$ is given by $r_{X_{1:n}}(x) = \left(\sum_{i=1}^{n} \xi(\lambda_i x)\right)/x$, where $\xi(y) = \left[\int_0^\infty (1+u)^{\alpha-1} e^{-uy} du\right]^{-1} = 1/\vartheta(y)$, say. This gives that $\xi'(y) = -\vartheta'(y)/\vartheta^2(y)$, and $\vartheta(y)\xi''(y) = 2[E(Z)]^2 - E(Z^2)$, where Z, as defined in Lemma 4.7, is DFRA for $\alpha \in [0, 1]$, and IFRA for $\alpha \in [1, \infty)$ (cf. [11]). Thus, on using Lemma 4.6, we get $\xi''(y) \le 0$ for $\alpha \in [0, 1]$, and $\xi''(y) \ge 0$ for $\alpha \in [1, \infty)$. Hence, $\xi(y)$ is concave in y, for $\alpha \in [0, 1]$, and is convex in y, for $\alpha \in [1, \infty)$. So, by Lemma 4.5 we have that $r_{X_{1:n}}(x)$ is Schur-concave in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, for $\alpha \in [0, 1]$, and is Schur-convex in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, for $\alpha \in [1, \infty)$. Further, note that $\xi'(y) \ge 0$ for all $y > 0$, which implies that $r_{X_{1:n}}(x)$ is increasing in each λ_i, for all $\alpha \in (0, \infty)$. Thus, the result follows from Lemma 4.3. □

Because weak supermajorization order is superior to p-larger order, one may wonder whether weak supermajorization order given in Theorem 4.11 can be replaced by p-larger order. The following counterexample answers this question in negative.

Counterexample 4.6. *Let X_i (resp. Y_i) be independent gamma random variables with parameters (λ_i, α) (resp. (μ_i, α)), $i = 1, 2$, where $(\lambda_1, \lambda_2) = (1, 5) \in \mathcal{E}_+$, $(\mu_1, \mu_2) = (2, 3.9) \in \mathcal{E}_+$, and $\alpha = 0.1$. Clearly, $(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\mu_1, \mu_2)$ but $(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\not\succeq} (\mu_1, \mu_2)$. Now,*

$$k_7(x) \overset{\text{def}}{=} r_{X_{1:2}}(x) - r_{Y_{1:2}}(x)$$
$$= \frac{1}{x}\left[\xi(x) + \xi(5x) - \xi(2x) - \xi(3.9x)\right],$$

where $\xi(y)$ is as defined in Theorem 4.11. It can be shown that $k_7(x)$ changes sign. Thus, $X_{1:2} \not\ge_{hr} Y_{1:2}$. □

The following theorem shows that the above result also holds for the dispersive order.

Theorem 4.12. *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with respective scale parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the same shape parameter α. Further, let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with respective scale parameters $\mu_1, \mu_2, \ldots, \mu_n$ and the same shape parameter α. Suppose X_i's and Y_i's are independent. Then*

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\mathrm{w}}{\succeq} (\mu_1, \mu_2, \ldots, \mu_n) \Rightarrow X_{1:n} \ge_{disp} Y_{1:n}, \quad \text{for } \alpha \in [0, 1].$$

Proof. Note that $r_{X_{1:n}}(x)$ is decreasing in $x > 0$, for $\alpha \in [0, 1]$. Thus, on using Theorem 3.B.20(a) of Shaked and Shanthikumar [8], the result follows from Theorem 4.11. □

In the following theorem we show that Theorem 4.11 holds for the up shifted and the down shifted hazard rate orders.

Theorem 4.13. *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with respective scale parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the same shape parameter α. Further, let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with respective scale parameters $\mu_1, \mu_2, \ldots, \mu_n$, and the same shape parameter α. Suppose X_i's and Y_i's are independent. Then*

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\mathrm{w}}{\succeq} (\mu_1, \mu_2, \ldots, \mu_n) \Rightarrow X_{1:n} \ge_{hr\downarrow} Y_{1:n}, \quad \text{for } \alpha \in [0, 1],$$

and

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \succeq_{\mathrm{w}} (\mu_1, \mu_2, \ldots, \mu_n) \Rightarrow X_{1:n} \le_{hr\uparrow} Y_{1:n}, \quad \text{for } \alpha \in [1, \infty).$$

Proof. Note that $r_{X_{1:n}}(x)$ is decreasing in $x > 0$, for $\alpha \in [0, 1]$ and is increasing in $x > 0$, for $\alpha \in [1, \infty)$. Thus, on using Theorem 4.11, the result follows from Theorem 6.26 and Theorem 6.19 of Lillo et al. [9].

5. Conclusions

Although the concept of majorization started in order to compare the income inequalities, now-a-days one can find application of majorization in different branches of economics, reliability, engineering and many others. We have developed some useful results on majorization, viz. we have given answer to the question—what can we say about Schur-convexity/concavity of the function $\varphi(\cdot)$ defined by $\varphi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ if $g : I \to \mathbb{R}$ is increasing/decreasing and convex/concave, where I is an interval of $\mathbb{R}$? These results, which correct a typographical error in the book by Marshall et al. [4], also fill up some gap in the theory of majorization up to certain extent. We have also reported the cases when nothing can be said about Schur-convexity/concavity of φ. As an application of the results so developed, we have taken help of GE and gamma distributions. Here, we compare the lifetimes of two parallel systems formed by components having heterogeneous generalized exponentially distributed lifetimes. It is shown that if the vectors of parameters of the underlying distributions are ordered in the sense of p-larger order (resp. weak supermajorization order), then the life of one parallel system will be more than that of the other in usual stochastic order (resp. reversed hazard rate order). We also show, with the help of counterexample, that this result cannot be extended to likelihood ratio order. However, we have shown that, under certain restriction on the parameters, the result can be extended to the likelihood ratio order. Further, we prove that, in the multiple-outlier GE model, if one set of parameters majorizes another, a parallel system formed by the former will dominate that formed by the latter in the likelihood ratio order. While giving applications of the proven results in the gamma population, we compare two series systems formed by gamma components with respect to the hazard rate order, the up shifted hazard rate order, the down shifted hazard rate order and the dispersive order. Stochastic comparisons of series systems with heterogeneous GE components may be undertaken as a future research project.

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