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On the perturbed Q-curvature problem on $\mathbb{S}^4$

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ABSTRACT

Let g_0 denote the standard metric on $\mathbb{S}^4$ and $P_{g_0} = \Delta_{g_0}^2 - 2\Delta_{g_0}$ denote the corresponding Paneitz operator. In this work, we study the following fourth order elliptic problem with exponential nonlinearity

$$P_{g_0} u + 6 = 2Q(x)e^{4u} \quad \text{on } \mathbb{S}^4.$$

Here Q is a prescribed smooth function on $\mathbb{S}^4$ which is assumed to be a perturbation of a constant. We prove existence results to the above problem under assumptions only on the “shape” of Q near its critical points. These are more general than the non-degeneracy conditions assumed so far. We also show local uniqueness and exact multiplicity results for this problem. The main tool used is the Lyapunov–Schmidt reduction.

1. Introduction

Fourth order operators arise in the applications in the areas of conformal geometry, thermionic emission, gas combustion and gauge theory. Prompted by questions in quantum field theory, Paneitz discovered a fourth order conformally covariant operator in dimension $N \geqslant 4$. Let (M, g) be a Riemannian manifold with $dim(M) \geqslant 4$. Let Δ_g be the Laplace Beltrami operator, div_g the divergence

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operator, d the differential and S_g, Ric_g denote the scalar curvature and Ricci tensor of the metric g respectively. When $N = 4$, the Paneitz operator P_g can be written in the form

$$P_g\psi = \Delta_g^2\psi + div_g\left(\frac{2}{3}S_g - 2Ric_g\right)d\psi,$$

where $\psi \in C^\infty(M)$ (see Paneitz [17], Chang and Yang [6]).

If $dim(M) = 4$, the analogue of the Gauss curvature for a surface is the so-called Q-curvature function given as

$$Q_g = -\frac{1}{12}\left(\Delta_g S_g - S_g^2 + 3|Ric_g|^2\right).$$

In fact, Paneitz operator was generalized by T. Branson for $N \geqslant 3$ (see [3]).

Let us now consider the question:

Given a smooth function Q on $\mathbb{S}^4$, does there exist a metric g conformal to the standard metric g_0 such that $Q = Q_g$?

If we assume a conformal transformation of the form $g = e^{4w}g_0$, the answer to the above question is "yes" iff we can solve for w in the equation

$$P_{g_0}w + 2Q_{g_0} = 2Qe^{4w} \quad \text{on } \mathbb{S}^4.$$

It can be checked that $Q_{g_0} \equiv 3$ and that the Paneitz operator on $(\mathbb{S}^4, g_0)$ is given by $P_{g_0} = \Delta_{g_0}^2 - 2\Delta_{g_0}$. Hence, we look to solve for w in the problem

$$\left(\Delta_{g_0}^2 - 2\Delta_{g_0}\right)w + 6 = 2Qe^{4w} \quad \text{on } \mathbb{S}^4. \tag{1.1}$$

Integrating (1.1) over $\mathbb{S}^4$, one obtains that the total Q-curvature of $(\mathbb{S}^4, g_0)$ denoted by k_{g_0}, which is a conformal invariant, satisfies

$$k_{g_0} = \int_{\mathbb{S}^4} Qe^{4w} = \int_{\mathbb{S}^4} Q_{g_0} = 3\,vol\left(\mathbb{S}^4\right).$$

Furthermore, if g is conformal to g_0, the Weyl tensor of $(\mathbb{S}^4, g)$ vanishes identically and the following Gauss–Bonnet type formula holds

$$\int_{\mathbb{S}^4} Q_g = 4\pi^2\chi\left(\mathbb{S}^4\right) = 8\pi^2 \tag{1.2}$$

where χ is the Euler characteristic. This immediately gives the first obstruction: If $Q \leqslant 0$, then (1.1) has no solution. More subtle obstructions similar to the Kazdan–Warner identities [14] can be shown in the case of (1.1) as well (see Section 5 for details). The problem (1.1) is variational and the solutions can be characterized as critical points of the following functional on $H^2(\mathbb{S}^4)$

$$J(u) = \frac{1}{vol(\mathbb{S}^4)}\int_{\mathbb{S}^4}\left(uP_{g_0}u + 4u\right)d\mu_{g_0} - 3\log\left(\frac{1}{vol(\mathbb{S}^4)}\int_{\mathbb{S}^4} Qe^{4u}\,d\mu_{g_0}\right).$$

However, the functional fails to satisfy Palais Smale condition. Hence, for these reasons, solvability of (1.1) is not straight forward.

Using ideas similar to the ones used in [4,5,7] to solve Nirenberg's problem on $\mathbb{S}^N$, Wei and Xu [20] proved existence of solutions of (1.1) when $Q > 0$ satisfies the non-degeneracy condition

$$\big(\Delta Q(x)\big)^2 + \big|\nabla Q(x)\big|^2 \neq 0, \tag{1.3}$$

and the vector field $G : \mathbb{S}^N \to \mathbb{R}^{N+1}$ defined by

$$G(x) = \big(-\Delta Q(x), \nabla Q(x)\big) \tag{1.4}$$

has $deg(\frac{G}{|G|}, \mathbb{S}^N) \neq 0$. Later, in the work [20], they extended their results to very general pseudo-differential operators on $\mathbb{S}^N$ which look like $(-\Delta)^{\frac{N}{2}}$ when N is odd. To our knowledge it seems that the non-degeneracy condition (1.3) is crucially required in [7,19,20] to obtain a-priori estimates for the solution of (1.1).

The other approach is via the heat-flow as done in [18,2,15]. In particular, Malchiodi and Struwe [15], proved existence of a solution to (1.1) assuming that Q is a Morse function (i.e., has only non-degenerate critical points p) with Morse Index $ind(Q, p)$ such that $\Delta Q(p) \neq 0$ and satisfies the index count

$$\sum_{\nabla Q(p)=0,\ \Delta Q(p)<0} (-1)^{ind(Q,p)} \neq 1.$$

Consider the inverse of the stereographic projection

$$\Pi : \mathbb{R}^4 \to \mathbb{S}^4$$

given by

$$x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1}\right).$$

The round metric g_0 is given in terms of the stereographic co-ordinate system as

$$g_0 = \frac{4\,dx^2}{(1+|x|^2)^2}.$$

By a direct computation,

$$P_{g_0}\Phi(u) = \frac{(1+|x|^2)^4}{16}\Delta^2 u \quad \text{for all } u \in C^\infty\big(\mathbb{R}^4\big)$$

where

$$\Phi(u)(y) = u(x) + \log\big(1+|x|^2\big) - \log 2, \quad y = \Pi(x).$$

Then (1.1) reduces to

$$\Delta^2 u = 2\tilde{Q}(x)e^{4u} \quad \text{in } \mathbb{R}^4 \text{ where } \tilde{Q} = Q \circ \Pi. \tag{1.5}$$

We would like to study the problem (1.1) by taking Q to be a perturbation of a constant function. More precisely, we let $Q = 3(1 + \varepsilon h)$ where h is a smooth function on $\mathbb{S}^4$ and $\varepsilon > 0$ is a small parameter. Using the stereographic projection from $\mathbb{S}^4$ to $\mathbb{R}^4$, we transform (1.1) (with f denoting the transformed function h) to the following problem

$$\Delta^2 u = 6\big(1 + \varepsilon f(x)\big)e^{4u} \quad \text{in } \mathbb{R}^4. \tag{1.6}$$

Note that the problem (1.6) is a perturbation of the following problem

$$\begin{cases} \Delta^2 U = 6e^{4U} & \text{in } \mathbb{R}^4, \\ \displaystyle\int_{\mathbb{R}^4} e^{4U} < +\infty \end{cases} \tag{1.7}$$

whose solutions in the space E (see below for definition of E) are classified by Lin [12] as

$$U_{\delta,y}(x) = \log \frac{2\delta}{\delta^2 + |x - y|^2}, \quad \text{with } (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^4. \tag{1.8}$$

We remark that, if $U = U_{1,0}$ solves (1.7), then so does the function $w(x) = U_{1,0}(\frac{x}{|x|^2}) - 2\log|x|$.

In this work, taking advantage of the fact that we are in a perturbative situation, we show existence of a solution to (1.6) without assuming that Q (and hence f) satisfies the non-degeneracy conditions as in (1.3). In particular, we do not assume Q to be a Morse function. What we assume is something about the "shape" of Q near the critical points (see the definition of the quantity $C_{\beta,\xi}$ in Section 8). As in the previous works, the main idea is to define a suitable vector field $\mathcal{V}_0$ on $\mathbb{R}^+ \times \mathbb{R}^N$ (see (1.14)). A stable zero (see Definition 1.5) $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N$ of $\mathcal{V}_0$ will make the corresponding $U_{\delta,y}$ a "bifurcation point" for a continuum of solutions to (1.6) as $\varepsilon \to 0$. For a precise statement of this fact see Theorem 1.1 below. If we assume that this zero is "stable" in the more standard sense, we can show that this "bifurcation" branch from $U_{\delta,y}$ is locally unique; this also leads to an exact multiplicity result for (1.6) for all small $\varepsilon > 0$. For a precise statement of such uniqueness and multiplicity see Theorems 1.3 and 1.4 below.

It is not possible to study (1.6) directly in a variational framework as $\Delta U \notin L^2(\mathbb{R}^4)$. Due to this fact we will work in a non-variational framework using weighted Sobolev spaces as in [16,10,20] to perform the Lyapunov–Schmidt reduction.

Let $\omega(x) = (1 + |x|^2)$. We introduce the following weighted Sobolev spaces:

Definition 1.1. Let $E = \{u \in W^{4,2}_{loc}(\mathbb{R}^4) \mid \omega^2\Delta^2 u, \omega^{-2}u \in L^2(\mathbb{R}^4)\}$ equipped with the inner product $\langle u, v\rangle_E = \int_{\mathbb{R}^4} \omega^4 \Delta^2 u \Delta^2 v + \int_{\mathbb{R}^4} \omega^{-4} uv$.

Definition 1.2. Let

$$H = \big\{u \in W^{4,2}_{loc}\big(\mathbb{R}^4\big) \bigm| \omega^2\Delta^2 u, \omega\big|\nabla(\Delta u)\big|, \Delta u, \omega^{-1}|\nabla u|, \omega^{-2}u \in L^2\big(\mathbb{R}^4\big)\big\}$$

with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}^4} \omega^4 \Delta^2 u \Delta^2 v + \int_{\mathbb{R}^4} \omega^2 \nabla(\Delta u) \cdot \nabla(\Delta v) + \int_{\mathbb{R}^4} \Delta u \Delta v + \int_{\mathbb{R}^4} \omega^{-2}\nabla u \cdot \nabla v + \int_{\mathbb{R}^4} \omega^{-4} uv.$$

Definition 1.3.

$$\tilde{H} = \{u \in L^2_{loc}(\mathbb{R}^4) \mid \omega^2 u \in L^2(\mathbb{R}^4)\}$$

with the inner product

$$\langle u, v\rangle_{\tilde{H}} = \int_{\mathbb{R}^4} \omega^4 uv\, dx.$$

Finally,

Definition 1.4. Let $\omega_{\delta,y}(x) = (\delta^2 + |x-y|^2)$. We define $E_{\delta,y}$, $H_{\delta,y}$ and $\tilde{H}_{\delta,y}$ by replacing the weight ω by $\omega_{\delta,y}$ in the definitions of E, H and $\tilde{H}$ respectively.

Remark 1.1. It is easy to see that $U_{\delta,y} \in E_{\delta,y}$ for all (δ, y).

Remark 1.2. We can easily check that the spaces $H_{\delta,y}$, $E_{\delta,y}$ and $\tilde{H}_{\delta,y}$ are uniformly equivalent as Hilbert spaces to H, E and $\tilde{H}$ respectively as (δ, y) varies over a compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^4$.

Remark 1.3. It is easy to see that $H_{\delta,y}$ is continuously embedded in $E_{\delta,y}$.

We denote the derivatives of $U_{\delta,y}$ as follows $(i = 1, 2, 3, 4)$

$$\begin{cases} \psi^{(0)}_{\delta,y}(x) = \dfrac{\partial U_{\delta,y}}{\partial \delta} = \dfrac{(|x-y|^2 - \delta^2)}{\delta(\delta^2 + |x-y|^2)}, \\ \psi^{(i)}_{\delta,y}(x) = \dfrac{\partial U_{\delta,y}}{\partial x_i} = -\dfrac{2(x_i - y_i)}{(\delta^2 + |x-y|^2)}. \end{cases} \tag{1.9}$$

As noted before, the solutions of (1.7) form a five dimensional manifold which we denote by

$$\mathcal{M} = \{U_{\delta,y}\colon (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^4\}.$$

For any compact $K \subset \mathbb{R}^+ \times \mathbb{R}^4$ define

$$d(u, \mathcal{M}_K) = \inf_{(\delta,y)\in K} \|u - U_{\delta,y}\|_{H_{1,0}}.$$

Let the vector field $\mathcal{V}_0 : \mathbb{R}^+ \times \mathbb{R}^4 \to \mathbb{R}$ be defined as

$$\mathcal{V}_0(\delta, y) = \left(\int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \psi^{(0)}_{\delta,y}(x)\, dx, \ldots, \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \psi^{(4)}_{\delta,y}(x)\, dx \right). \tag{1.10}$$

We note that $\mathcal{V}_0$ is a gradient vector field as

$$\mathcal{V}_0(\delta, y) = \nabla J(\delta, y) \quad \text{where } J(\delta, y) = \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}}\, dx. \tag{1.11}$$

We make the following definition of a stable vector field:

Definition 1.5. Let $\Omega \subset \mathbb{R}^N$ be an open set. We call a point $P \in \Omega$ as a stable zero for a vector field $\mathcal{V}_0 \in C(\Omega; \mathbb{R}^N)$ if $\mathcal{V}_0(P) = 0$ and for any sequence of vector fields $\mathcal{V}_\varepsilon \in C(\Omega; \mathbb{R}^N)$ converging uniformly to $\mathcal{V}$ in a neighborhood of P, there exist a zero P_ε of $\mathcal{V}_\varepsilon$ with $P_\varepsilon \to P$ as $\varepsilon \to 0$.

We now state the theorems we will prove.

Theorem 1.1 *("Bifurcation" from a stable zero). Let $K \subset \mathbb{R}^+ \times \mathbb{R}^4$ be a compact set with a nonempty interior. Let $(\delta, y) \in K$ be a stable zero of the vector field $\mathcal{V}_0$. Then there exists an $\varepsilon_0 > 0$ depending on K such that (1.6) admits a solution u_ε for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, $u_\varepsilon = U_{\delta_\varepsilon, y_\varepsilon} + \phi_\varepsilon$ with $\|\phi_\varepsilon\|_{H_{\delta,y}} = O(\varepsilon)$ and $(\delta_\varepsilon, y_\varepsilon) \to (\delta, y)$.*

Theorem 1.2 *(Necessary condition). Let u_ε be a sequence of solution of (1.6) such that $\|u_\varepsilon - U_{\delta,y}\|_{H_{\delta,y}} \to 0$. Then $\mathcal{V}_0(\delta, y) = 0$.*

Theorem 1.3 *(Local uniqueness). Let $K \subset \mathbb{R}^+ \times \mathbb{R}^4$ with a nonempty interior. Let $(\delta, y) \in K$ be a zero of the vector field $\mathcal{V}_0(\delta, y)$ such that $D^2 J(\delta, y)$ is invertible. Furthermore, suppose f satisfies*

$$\left|\nabla f(x)\right| \leqslant C. \tag{1.12}$$

If $\{u_{\varepsilon,i}\}$, $i = 1, 2$ are two sequences of solutions of (1.6) such that

$$\|u_\varepsilon - U_{\delta,y}\|_{H_{\delta,y}} \to 0 \quad \text{as } \varepsilon \to 0,$$

then there exists $\varepsilon_0(K) > 0$ depending on K such that for all $\varepsilon \in (0, \varepsilon_0)$ we obtain $u_{\varepsilon,1} \equiv u_{\varepsilon,2}$.

Theorem 1.4 *(Exact multiplicity). Let $\mathcal{V}_0$ have only finitely many zeroes all of which are stable and contained in a compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^4$. Suppose that at any stable zero of $\mathcal{V}_0$ the Hessian $D^2 J$ is invertible. Then there exists a $\rho_0 = \rho_0(K) > 0$ and $\varepsilon_0 = \varepsilon_0(\rho_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem (1.6) has exactly the same number of solutions u with $d(u, \mathcal{M}_K) < \rho_0$ as the number of stable zeroes of $\mathcal{V}_0$.*

Remark 1.4. The proof of the above theorems are done using Lyapunov–Schmidt reduction carried out for the nonlinear solution operator (see (2.6)) between the spaces $H_{\delta,y}$ and $\tilde{H}_{\delta,y}$. The calculations for this reduction are given in Sections 2 and 3.

Remark 1.5. Consider the problem

$$\Delta^2 u = 6e^{4u} + \varepsilon \Psi(x, u) \quad \text{in } \mathbb{R}^4 \tag{1.13}$$

where $\Psi : \mathbb{R}^4 \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and twice differentiable in the second variable and satisfies

$$\sup_{x \in \mathbb{R}^4} \left[\left|\Psi(x, u)\right| + \left|\Psi_u(x, u)\right| + \left|\Psi_{uu}(x, u)\right|\right] \leqslant Ce^{4u};$$

$$\left|\nabla_x \Psi(x, u)\right| \leqslant Ce^{4u}.$$

An inspection of the proofs of Theorems 1.1–1.4 shows that they hold for the problem (1.13) as well if we replace the vector field $\mathcal{V}_0$ by the following

$$\tilde{\mathcal{V}}_0(\delta, y) = \left(\int_{\mathbb{R}^4} \Psi(x, U_{\delta,y}) \psi_{\delta,y}^{(0)}(x)\, dx, \ldots, \int_{\mathbb{R}^4} \Psi(x, U_{\delta,y}) \psi_{\delta,y}^{(4)}(x)\, dx \right). \tag{1.14}$$

Remark 1.6. A similar kind of result was obtained by Grossi [9] for single peak solutions of the subcritical singularly perturbed nonlinear Schrödinger equation

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + u^p = 0 & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.15}$$

By exploiting the "shape" of the potential $V \in C^1(\mathbb{R}^N)$ near its critical points, the author obtained exact multiplicity results for (1.15) whenever $\varepsilon > 0$ is sufficiently small. In addition, if P is a non-degenerate critical point of V, the author showed that there is a unique solution concentrating at P.

Remark 1.7. Moreover, Theorems 1.1–1.4 hold for the equation

$$(-\Delta)^m u = (2m-1)!\big(1 + \varepsilon f(x)\big)e^{2mu} \quad \text{in } \mathbb{R}^{2m} \tag{1.16}$$

where $m \in \mathbb{N}$. The construction of solution follows from Wei and Xu [21].

Remark 1.8. The following problem was studied by Felli [8]

$$\begin{cases} \Delta^2 u = \big(1 + \varepsilon f(x)\big)u^{\frac{N+4}{N-4}} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \end{cases} \tag{1.17}$$

for $N \geqslant 5$. Existence to the above problem is shown in [8] assuming a suitable "shape" for f near a critical point. In particular, an expansion of the form

$$f(x) = f(\eta) + \sum a_j |y - \eta|^\beta + o\big(|y - \eta|^\beta\big) \quad \text{as } y \to \eta,\ \beta \in (1, N)$$

is assumed at a critical point η. We remark that the problem (1.17) is variational and can be handled in the Sobolev space $\mathcal{D}^{2,2}(\mathbb{R}^N)$.

2. Preliminaries

Let $\log^+ |x| = \max\{0, \log |x|\}$.

Lemma 2.1. *There exists a positive constant C such that*

$$\sup_{\mathbb{R}^4} \big|v(x)\big| \leqslant C\|v\|_E\big(|x| + \log^+ |x| + 1\big), \quad \forall v \in E, \tag{2.1}$$

$$\sup_{\mathbb{R}^4} \big|v(x)\big| \leqslant C\|v\|_H\big(\log^+ |x| + 1\big), \quad \forall v \in H. \tag{2.2}$$

Proof. Note that the fundamental solution of the biharmonic operator in $\mathbb{R}^4$ is given by

$$F(x, y) = \frac{1}{8\pi^2} \log \frac{1}{|x - y|}.$$

For v in E with $\|v\|_E = 1$ we set $\Delta^2 v = g$. By definition of the space E, the function $g \in \tilde{H}$. Then we can write $v = v_0 + v_1$ where $\Delta^2 v_0 = 0$ and $v_1(x) = \int_{\mathbb{R}^4} F(x, y)g(y)\,dy$. We now estimate

$$\begin{aligned}
\left|v_1(x)\right| &= \left|\int\limits_{\mathbb{R}^4} F(x,y)g(y)\,dy\right| \\
&\leqslant \frac{1}{8\pi^2}\int\limits_{\mathbb{R}^4} \left|\log|x-y|\right|\left|g(y)\right|dy \\
&\leqslant \frac{1}{8\pi^2}\left(\int\limits_{\mathbb{R}^4} \left(1+|y|^2\right)^4\left|g(y)\right|^2\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^4} \frac{|\log|x-y||^2}{(1+|y|^2)^4}\,dy\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{8\pi^2}\|v\|_E\left(\int\limits_{\mathbb{R}^4} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\begin{aligned}
I :&= \int\limits_{\mathbb{R}^4} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy \\
&= \int\limits_{\{|y|\leqslant 1\}} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy + \int\limits_{\{|y|\geqslant 1\}} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy \\
&= I_1 + I_2.
\end{aligned}$$

Now we estimate

$$I_1 = \int\limits_{\{|y|\leqslant 1\}} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy \leqslant C \int\limits_{\{|y|\leqslant 1\}} \left|\log|y|\right|^2 dy < +\infty.$$

Also for $|y| \geqslant 2|x|$, we have

$$|y-x| \geqslant |y| - |x| \geqslant \frac{1}{2}|y|$$

and as a result we must have

$$\begin{aligned}
I_2 &= \int\limits_{\{|y|\geqslant 1\}\cap\{|y|\geqslant 2|x|\}} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy + \int\limits_{\{|y|\geqslant 1\}\cap\{|y|<2|x|\}} \frac{|\log|y||^2}{(1+|x-y|^2)^4}\,dy \\
&\leqslant C\left(1+\left(\log^+|x|\right)^2\right).
\end{aligned}$$

Since $\omega^{-2}v$, $\omega^{-2}v_1$ are in $L^2(\mathbb{R}^4)$ so is $\omega^{-2}v_0$ and hence v_0 is a tempered distribution in $\mathbb{R}^4$. Using Fourier transform and the fact that $\omega^{-2}v_0 \in L^2(\mathbb{R}^4)$ we obtain $\sup_{\mathbb{R}^4}|v_0(x)| \leqslant C(1+|x|)$ for some $C > 0$. Putting together the estimates for I_1, I_2 and v_0 we get (2.1). If $v \in H$ with $\|v\|_H = 1$, we note that the corresponding biharmonic function $v_0 \in H$ and hence is uniformly bounded in $\mathbb{R}^4$. The estimate for v_1 can be obtained as above to get (2.2). $\square$

Lemma 2.2 *(Non-degeneracy). The kernel of the linearized operator*

$$\Delta^2 - 24e^{4U_{\delta,y}}$$

in $E_{\delta,y}$ is five dimensional and is generated by

$$\left\{\frac{\partial U_{\delta,y}}{\partial \delta}, \frac{\partial U_{\delta,y}}{\partial x_1}, \frac{\partial U_{\delta,y}}{\partial x_2}, \frac{\partial U_{\delta,y}}{\partial x_3}, \frac{\partial U_{\delta,y}}{\partial x_4}\right\}.$$

Proof. Without loss of generality, let $\delta = 1$ and $y = 0$. Consider the problem

$$\Delta^2\psi - 24e^{4U}\psi = 0 \tag{2.3}$$

where $\psi \in E_{1,0}$. Then $\psi \in W^{4,2}_{loc}(\mathbb{R}^4)$ and by a boot-strap argument $\psi \in C^\infty_{loc}(\mathbb{R}^4)$. Now we claim that every ψ satisfying (2.3) with at most linear growth has to be bounded. Let $|\psi| \leqslant C|x|$ for $|x| \gg 1$. Then define the Kelvin transform of ψ be

$$\hat{\psi}(x) = \psi\left(\frac{x}{|x|^2}\right) \quad \text{in } \mathbb{R}^4 \setminus \{0\}. \tag{2.4}$$

Then $\hat{\psi}(x) \leqslant C|x|^{-1}$ near the origin and satisfies

$$\Delta^2\hat{\psi} - \frac{1}{(1+|x|^2)^4}\hat{\psi} = 0 \quad \text{in } \mathbb{R}^4 \setminus \{0\}. \tag{2.5}$$

But $\hat{\psi} \in L^2_{loc}(\mathbb{R}^4)$ and hence by regularity $\hat{\psi} \in C^\infty_{loc}(\mathbb{R}^4)$. Hence $\hat{\psi}$ is bounded near the origin and hence ψ is bounded at infinity. As a result, we must have $|\psi| \leqslant C$ for $|x| \gg 1$. Hence $\sup_{\mathbb{R}^4}|\psi(x)| \leqslant C\|\psi\|_E(\log^+|x| + 1)$ and we can apply the method of Lin and Wei [13] in Lemma 2.6 to conclude the non-degeneracy. □

We want to find solutions to (1.6) of the form $u_\varepsilon = U_{\delta,y} + \varphi_\varepsilon$ such that $\varphi_\varepsilon \to 0$ as $\varepsilon \to 0$ in $H_{\delta,y}$. If we plug this ansatz in (1.6) then we have

$$\Delta^2\varphi_\varepsilon = 6e^{4U_{\delta,y}}\left(e^{4\varphi_\varepsilon} - 1\right) + 6\varepsilon f(x)e^{4(U_{\delta,y}+\varphi_\varepsilon)}.$$

This motivates us to introduce the following nonlinear operator $\mathcal{B}^{\delta,y}_\varepsilon$ from a small ball B around the origin in $H_{\delta,y}$ into $\tilde{H}_{\delta,y}$

$$\mathcal{B}^{\delta,y}_\varepsilon : B \subset H_{\delta,y} \mapsto \tilde{H}_{\delta,y}$$

given by

$$\mathcal{B}^{\delta,y}_\varepsilon(v) = \Delta^2 v - 6e^{4U_{\delta,y}}\left(e^{4v} - 1\right) - 6\varepsilon f(x)e^{4(U_{\delta,y}+v)}. \tag{2.6}$$

Therefore finding solutions u_ε of (1.6), bifurcating from $U_{\delta,y}$ for some $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^4$ is equivalent to proving the following lemma.

Lemma 2.3. *There exists a suitable value $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^4$ for which one can find $\varphi_\varepsilon \in H_{\delta,y}$ with $\|\varphi_\varepsilon\|_{H_{\delta,y}} \to 0$ as $\varepsilon \to 0$ and $\mathcal{B}^{\delta,y}_\varepsilon(\varphi_\varepsilon) = 0$.*

We now show some basic properties of $\mathcal{B}_\varepsilon^{\delta,y}$.

Lemma 2.4. *Let $B_\rho(0) \subset H_{\delta,y}$. Then for $\rho > 0$ small enough we have*

$$\mathcal{B}_\varepsilon^{\delta,y}\big(B_\rho(0)\big) \subset \tilde{H}_{\delta,y}.$$

Proof. Let $\|v\|_{H_{\delta,y}} < \rho$. Then using (2.1), we have

$$\int_{\mathbb{R}^4} \big(\delta^2 + |x-y|^2\big)^4 e^{8(U_{\delta,y}+v)} \leqslant C_1 \int_{\mathbb{R}^4} \frac{e^{8v}}{(\delta^2 + |x-y|^2)^4} \leqslant C_1 \int_{\mathbb{R}^4} \frac{e^{c_2\|v\|_{H_{\delta,y}}(1+\log^+|x|)}}{(\delta^2 + |x-y|^2)^4} < +\infty$$

provided ρ is sufficiently small. Hence, $e^{4(U_{\delta,y}+v)} \in \tilde{H}_{\delta,y}$. It follows that $\mathcal{B}_\varepsilon^{\delta,y}$ maps $B_\rho(0)$ into $\tilde{H}_{\delta,y}$. □

Theorem 2.1. *Let $B_\rho(0) \subset H_{\delta,y}$, with $\rho > 0$ small. Then for any $\varepsilon > 0$,*

$$\mathcal{B}_\varepsilon^{\delta,y} \in C^1\big(B_\rho(0), \tilde{H}_{\delta,y}\big).$$

Proof. First we prove that

$$\mathcal{B}_\varepsilon^{\delta,y} \in C^0\big(B_\rho(0), \tilde{H}_{\delta,y}\big).$$

Let $v_n \to v$ in $H_{\delta,y}$ where $v_n, v \in B_\rho(0)$. This implies that $\Delta^2 v_n \to \Delta^2 v$ in $\tilde{H}_{\delta,y}$ and $v_n \to v$ in $C_{loc}(\mathbb{R}^4)$. Hence, again by the estimate (2.1) and dominated convergence theorem we obtain

$$6\big(1 + \varepsilon f(x)\big)e^{4(U_{\delta,y}+v_n)} \to 6\big(1 + \varepsilon f(x)\big)e^{4(U_{\delta,y}+v)} \quad \text{in } \tilde{H}_{\delta,y}.$$

Now we prove that $\mathcal{B}_\varepsilon^{\delta,y}$ is continuously differentiable in $B_\rho(0)$. We claim that its derivative is given by

$$\begin{cases} \big\langle \big(\mathcal{B}_\varepsilon^{\delta,y}\big)'(v), h\big\rangle = \Delta^2 h - 24\big(1 + \varepsilon f(x)\big)e^{4(U_{\delta,y}+v)}h & \text{in } \mathbb{R}^4, \\ h \in H_{\delta,y}, \quad v \in B_\rho(0). \end{cases} \tag{2.7}$$

Let $A_v^\varepsilon : H_{\delta,y} \to \tilde{H}_{\delta,y}$ be defined by $A_v^\varepsilon(h) = \Delta^2 h - 24(1 + \varepsilon f(x))e^{4(U_{\delta,y}+v)}h$. Then A_v^ε is a continuous linear map for all $v \in B_\rho(0)$. To see this, let $h_n \to h$ in $H_{\delta,y}$. Then $\Delta^2 h_n \to \Delta^2 h$ in $\tilde{H}_{\delta,y}$ as well as $h_n \to h$ in $C_{loc}(\mathbb{R}^4)$. As a result we must have

$$\big(\delta^2 + |x-y|^2\big)^4\big(1 + \varepsilon f(x)\big)^2 e^{8(U_{\delta,y}+v)}h_n^2 \leqslant C\frac{e^{8v}h_n^2}{(\delta^2 + |x-y|^2)^4} \leqslant \frac{C\|h_n\|_{H_{\delta,y}}^2(1 + \log^+|x|)^2}{(\delta^2 + |x-y|^2)^4} e^{c_1\|v\|_{H_{\delta,y}}(1+\log^+|x|)}.$$

Hence by the dominated convergence theorem, for $\rho > 0$ small enough,

$$e^{4(U_{\delta,y}+v)}h_n \to e^{4(U_{\delta,y}+v)}h \quad \text{in } \tilde{H}_{\delta,y}.$$

This shows the continuity of A_v^{ε}. Now we claim that

$$\left(\mathcal{B}_{\varepsilon}^{\delta,y}\right)'(v) = A_v^{\varepsilon}.$$

We have

$$\begin{aligned}
\left|\mathcal{B}_{\varepsilon}^{\delta,y}(v+h) - \mathcal{B}_{\varepsilon}^{\delta,y}(v) - A_v^{\varepsilon}h\right| &= 6e^{4(U_{\delta,y}+v)}\left(1+\varepsilon f(x)\right)\left(e^{4h} - 1 - 4h\right) \\
&\leqslant Ce^{4(U_{\delta,y}+v)}e^{4|h|}h^2 \\
&\leqslant Ce^{c_1\|h\|_{H_{\delta,y}}(1+\log^+|x|)} \frac{\|h\|^2_{H_{\delta,y}}(1+\log^+|x|)^2}{(\delta^2+|x-y|^2)^{4-c_2\|v\|_{H_{\delta,y}}}}.
\end{aligned}$$

This implies for $\|v\|_{H_{\delta,y}}$ and $\|h\|_{H_{\delta,y}}$ small

$$\left\|\mathcal{B}_{\varepsilon}^{\delta,y}(v+h) - \mathcal{B}_{\varepsilon}^{\delta,y}(v) - A_v^{\varepsilon}h\right\|_{\tilde{H}_{\delta,y}} \leqslant C\|h\|^2_{H_{\delta,y}}$$

and hence we obtain the required result. □

Let $\mathcal{K} = Ker(\mathcal{B}_0^{\delta,y})'(0)$ and $\mathcal{R} = Im(\mathcal{B}_0^{\delta,y})'(0)$. Then by Lemma 2.2

$$\mathcal{K} = \left\{\frac{\partial U_{\delta,y}}{\partial \delta}, \frac{\partial U_{\delta,y}}{\partial x_1}, \frac{\partial U_{\delta,y}}{\partial x_2}, \frac{\partial U_{\delta,y}}{\partial x_3}, \frac{\partial U_{\delta,y}}{\partial x_4}\right\}.$$

Define

$$\mathcal{R}^{\perp} = \left\{\psi \in \tilde{H}_{\delta,y} \colon \langle\psi,\zeta\rangle_{\tilde{H}_{\delta,y}} = 0;\ \zeta \in \mathcal{R}\right\}.$$

We define for $i = 0, 1, 2, 3, 4$

$$\Phi_{\delta,y}^{(i)} = \omega_{\delta,y}^{-4}\psi_{\delta,y}^{(i)}.$$

Lemma 2.5. $\mathcal{R}^{\perp} = span\{\Phi_{\delta,y}^{(0)}, \Phi_{\delta,y}^{(1)}, \ldots, \Phi_{\delta,y}^{(4)}\}$.

Proof. Let $\psi \in \mathcal{R}^{\perp}$. Then by definition we must have $\langle\psi, (\mathcal{B}_0^{\delta,y})'(0)\zeta\rangle_{\tilde{H}_{\delta,y}} = 0$, for all $\zeta \in C_0^{\infty}(\mathbb{R}^4)$. This implies that in the sense of distribution

$$\Delta^2\left(\omega_{\delta,y}^4\psi\right) - 24e^{4U_{\delta,y}}\omega_{\delta,y}^4\psi = 0.$$

By the elliptic regularity, $\psi \in W_{loc}^{4,2}(\mathbb{R}^4)$ and from the above equation $\omega_{\delta,y}^2\Delta^2(\omega_{\delta,y}^4\psi) \in L^2(\mathbb{R}^4)$. Hence $\omega_{\delta,y}^4\psi \in E_{\delta,y}$. Using Lemma 2.2, we obtain $\omega_{\delta,y}^4\psi \in \mathcal{K}$. We note that $\overline{C_0^{\infty}(\mathbb{R}^4)} = H_{\delta,y}$. Conversely, if $\phi \in \mathcal{K}$, we have $\langle\phi, \Delta^2\psi - e^{4U_{\delta,y}}\psi\rangle_{L^2} = 0$ for all $\psi \in C_0^{\infty}(\mathbb{R}^4)$. As a result, we must have $\omega_{\delta,y}^{-4}\phi \in \mathcal{R}^{\perp}$ for any $\phi \in \mathcal{K}$. Hence $\psi \in \mathcal{R}^{\perp}$ if and only if $\omega_{\delta,y}^4\psi \in \mathcal{K}$. □

Now we define the quotient spaces

$$M_{\delta,y} = H_{\delta,y}/\mathcal{K} \quad \text{and} \quad \tilde{M}_{\delta,y} = \tilde{H}_{\delta,y}/\mathcal{R}^{\perp}.$$

Then $(\mathcal{B}_0^{\delta,y})'(0) : M_{\delta,y} \to \tilde{M}_{\delta,y}$ is an isomorphism onto.

Now we are in situation to apply finite dimensional reduction.

3. Solving the reduced operator equation

Let $P_{\mathcal{K}^{\perp}}$ and $P_{\mathcal{R}}$ denote the projections

$$P_{\mathcal{K}^{\perp}} : H_{\delta,y} \to M_{\delta,y},$$
$$P_{\mathcal{R}} : \tilde{H}_{\delta,y} \to \tilde{M}_{\delta,y}.$$

For a ball $B_\rho(0) \subset M_{\delta,y}$ for $\rho > 0$ small enough, define the reduced solution operator

$$S_\varepsilon^{\delta,y} : B_\rho(0) \to \tilde{M}_{\delta,y} \quad \text{as } S_\varepsilon^{\delta,y}(v) = \big(P_{\mathcal{R}} \circ \mathcal{B}_\varepsilon^{\delta,y}\big)(v).$$

Then by Theorem 2.1, $S_\varepsilon^{\delta,y} \in C^1(B_\rho(0), \tilde{M}_{\delta,y})$ for small $\rho > 0$ and for any $\varepsilon > 0$.

For any $\phi \in B_\rho(0)$, we write

$$\mathcal{B}_\varepsilon^{\delta,y}(\phi) = \mathcal{B}_\varepsilon^{\delta,y}(0) + \big(\mathcal{B}_\varepsilon^{\delta,y}\big)'(0)\phi + Q_\varepsilon^{\delta,y}(\phi), \tag{3.1}$$

where

$$Q_\varepsilon^{\delta,y}(\phi) = -6\big(1 + \varepsilon f(x)\big)e^{4U_{\delta,y}}\big[e^{4\phi} - 1 - 4\phi\big]. \tag{3.2}$$

Applying the projection $P_{\mathcal{R}}$ on either side of (3.1) we obtain

$$\begin{aligned} S_\varepsilon^{\delta,y}(\phi) &= S_\varepsilon^{\delta,y}(0) + P_{\mathcal{R}}\big(\big(\mathcal{B}_\varepsilon^{\delta,y}\big)'(0)\phi\big) + P_{\mathcal{R}}\big(Q_\varepsilon^{\delta,y}(\phi)\big) \\ &= S_\varepsilon^{\delta,y}(0) + \big(S_\varepsilon^{\delta,y}\big)'(0)\phi + P_{\mathcal{R}}\big(Q_\varepsilon^{\delta,y}(\phi)\big). \end{aligned} \tag{3.3}$$

Therefore, solving

$$S_\varepsilon^{\delta,y}(\phi) = 0. \tag{3.4}$$

(3.3) reduces to solving

$$S_\varepsilon^{\delta,y}(0) + \big(S_\varepsilon^{\delta,y}\big)'(0)\phi + P_{\mathcal{R}}\big(Q_\varepsilon^{\delta,y}(\phi)\big) = 0.$$

We note that $(S_0^{\delta,y})'(0)$ is invertible and $(S_\varepsilon^{\delta,y})'(0) \to (S_0^{\delta,y})'(0)$ in the operator norm as $\varepsilon \to 0$. Therefore, we also obtain the invertibility of $(S_\varepsilon^{\delta,y})'(0)$ for all small $\varepsilon > 0$. Hence, solving (3.4) for small $\varepsilon > 0$ is equivalent to solving

$$\phi = -\big(\big(S_\varepsilon^{\delta,y}\big)'(0)\big)^{-1}\big[S_\varepsilon^{\delta,y}(0) + P_{\mathcal{R}}\big(Q_\varepsilon^{\delta,y}(\phi)\big)\big]. \tag{3.5}$$

Motivated by the above equation, define the map $\mathcal{G}_\varepsilon^{\delta,y} : B_\rho(0) \to M_{\delta,y}$ by

$$\mathcal{G}_\varepsilon^{\delta,y}(v) = -\big(\big(S_\varepsilon^{\delta,y}\big)'(0)\big)^{-1}\big[S_\varepsilon^{\delta,y}(0) + P_{\mathcal{R}}\big(Q_\varepsilon^{\delta,y}(v)\big)\big]. \tag{3.6}$$

Then solving (3.4) for small $\varepsilon > 0$ is equivalent to finding a fixed point of the map $\mathcal{G}_\varepsilon^{\delta,y}$. We do so in the lemma below, thereby solving the reduced operator equation:

Lemma 3.1. *Let K be a compact subset of $\mathbb{R}^+ \times \mathbb{R}^4$ and $\rho > 0$ be small. Then there exists $\varepsilon_0 = \varepsilon_0(K,\rho) > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and $(\delta,y) \in K$, there exists a fixed point $\phi_\varepsilon^{\delta,y} \in B_\rho(0)$ of the map $\mathcal{G}_\varepsilon^{\delta,y}$. That is, $S_\varepsilon^{\delta,y}(\phi_\varepsilon^{\delta,y}) = 0$ for all $\varepsilon \in (0,\varepsilon_0)$, $(\delta,y) \in K$.*

Proof. We use Banach fixed point theorem in order to prove the existence of ϕ_ε.

Claim 1. *Fix any $\varepsilon_0 > 0$. Then, for all $\varepsilon \in (0,\varepsilon_0)$ and $\phi \in B_\rho(0)$*

$$\big\|Q_\varepsilon^{\delta,y}(\phi)\big\|_{\tilde{H}_{\delta,y}} \leqslant C\|\phi\|^2_{H_{\delta,y}} \tag{3.7}$$

and for any $\phi_1, \phi_2 \in B_\rho(0)$

$$\big\|Q_\varepsilon^{\delta,y}(\phi_1) - Q_\varepsilon^{\delta,y}(\phi_2)\big\|_{\tilde{H}_{\delta,y}} \leqslant C\big(\|\phi_1\|_{H_{\delta,y}} + \|\phi_2\|_{H_{\delta,y}}\big)\|\phi_1 - \phi_2\|_{H_{\delta,y}}. \tag{3.8}$$

Proof. We have (see (3.2))

$$\begin{aligned}\big|Q_\varepsilon^{\delta,y}(\phi)\big|^2 &= 36\big|1 + \varepsilon f(x)\big|^2 e^{8U_{\delta,y}}\big|e^{4\phi} - 1 - 4\phi\big|^2 \\ &\leqslant C|\phi|^4 e^{8(U_{\delta,y}+|\phi|)}.\end{aligned}$$

Using Lemma 2.1 we have

$$\omega_{\delta,y}^4\big|Q_\varepsilon^{\delta,y}(\phi)\big|^2 \leqslant C\frac{\|\phi\|^4_{H_{\delta,y}}(1+\log^+|x|)^4 e^{c_1\|\phi\|_{H_{\delta,y}}(1+\log^+|x|)}}{(\delta^2 + |x-y|^2)^4}$$

which implies (3.7). Furthermore,

$$\big|Q_\varepsilon^{\delta,y}(\phi_1) - Q_\varepsilon^{\delta,y}(\phi_2)\big|^2 = \big|1 + \varepsilon f(x)\big|^2 e^{8U_{\delta,y}}\big|e^{4\phi_1} - e^{4\phi_2} - 4(\phi_1 - \phi_2)\big|^2 \tag{3.9}$$

and

$$e^{4\phi_1} - e^{4\phi_2} - 4(\phi_1 - \phi_2) = 16\int_0^1\left(\int_0^1 e^{4s(t\phi_1+(1-t)\phi_2)}\,ds\big(t\phi_1 + (1-t)\phi_2\big)\,dt\right)(\phi_1 - \phi_2). \tag{3.10}$$

Using (3.9) and (3.10) we have

$$\begin{aligned}\omega_{\delta,y}^4\big|Q_\varepsilon^{\delta,y}(\phi_1) - Q_\varepsilon^{\delta,y}(\phi_2)\big|^2 &\leqslant C\|\phi_1 - \phi_2\|^2_{H_{\delta,y}} e^{c_1(\|\phi_1\|_{H_{\delta,y}}+\|\phi_2\|_{H_{\delta,y}})(1+\log^+|x|)} \\ &\quad\times\frac{(1+\log^+|x|)^4}{(\delta^2+|x-y|^2)^4}\big(\|\phi_1\|^2_{H_{\delta,y}} + \|\phi_2\|^2_{H_{\delta,y}}\big)\end{aligned}$$

and we get (3.8). □

Claim 2. *For any compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^4$ and a ball $B_\rho(0) \subset M_{\delta,y}$ with $\rho > 0$ small we can choose $\varepsilon_0 = \varepsilon_0(K,\rho) > 0$ so that for any $\varepsilon \in (0,\varepsilon_0)$, $(\delta,y) \in K$, the operator $\mathcal{G}_\varepsilon^{\delta,y}$ defined by (3.6) has a unique fixed point $\phi_\varepsilon^{\delta,y} \in \overline{B_\rho(0)}$ for all $\varepsilon \in (0,\varepsilon_0)$. Moreover,*

$$\sup_{(\delta,y)\in K} \left\|\phi_\varepsilon^{\delta,y}\right\|_{H_{\delta,y}} = O(\varepsilon). \tag{3.11}$$

Proof. Let $(\delta,y) \in K$. For any $\phi \in B_\rho(0)$,

$$\left\|\mathcal{G}_\varepsilon^{\delta,y}(\phi)\right\|_{H_{\delta,y}} \leqslant \left\|\left(\left(S_\varepsilon^{\delta,y}\right)'(0)\right)^{-1}\right\|\left\{\left\|S_\varepsilon^{\delta,y}(0)\right\|_{\tilde{H}_{\delta,y}} + \left\|P_{\mathcal{R}}\left(Q_\varepsilon^{\delta,y}(\phi)\right)\right\|_{\tilde{H}_{\delta,y}}\right\}.$$

Now by Claim 1, there exists a constant $C > 0$ depending on K such that

$$\left\|\mathcal{G}_\varepsilon^{\delta,y}(\phi)\right\|_{H_{\delta,y}} \leqslant C\left[\varepsilon + \|\phi\|_{H_{\delta,y}}^2\right], \quad \forall(\delta,y) \in K. \tag{3.12}$$

If we choose $\|\phi\|_{H_{\delta,y}} \leqslant \rho$ where ρ is small enough and let $\varepsilon_0 = (\rho - C\rho^2)/C$, then for all $\varepsilon \in (0,\varepsilon_0)$

$$\left\|\mathcal{G}_\varepsilon^{\delta,y}(\phi)\right\|_{H_{\delta,y}} \leqslant \rho \quad \text{whenever } \|\phi\|_{H_{\delta,y}} \leqslant \rho,\ \forall(\delta,y) \in K.$$

Now we show that $\mathcal{G}_\varepsilon^{\delta,y}$ is a contraction

$$\begin{aligned}\left\|\mathcal{G}_\varepsilon^{\delta,y}(\phi_1) - \mathcal{G}_\varepsilon^{\delta,y}(\phi_2)\right\|_{H_{\delta,y}} &\leqslant \left\|\left(\left(S_\varepsilon^{\delta,y}\right)'(0)\right)^{-1}\right\|\left\{\left\|\left(Q_\varepsilon^{\delta,y}(\phi_1) - Q_\varepsilon^{\delta,y}(\phi_2)\right)\right\|_{\tilde{H}_{\delta,y}}\right\}\\ &\leqslant C\left(\|\phi_1\|_{H_{\delta,y}} + \|\phi_2\|_{H_{\delta,y}}\right)\|\phi_1 - \phi_2\|_{H_{\delta,y}}.\end{aligned}$$

Choosing $\phi_1, \phi_2 \in \overline{B_\rho(0)}$ with ρ small enough, we obtain $\mathcal{G}_\varepsilon^{\delta,y} : \overline{B_\rho(0)} \to \overline{B_\rho(0)}$ is a contraction map for all $(\delta,y) \in K$ and $\varepsilon \in (0,\varepsilon_0)$. Hence by Banach fixed point theorem we obtain a unique fixed point $\phi_\varepsilon^{\delta,y}$. Now, (3.11) follows from (3.12) by taking $\phi = \phi_\varepsilon^{\delta,y}$. This proves the claim. □

The proof of lemma follows from Claims 1 and 2. □

4. Existence of solution: Proof of Theorem 1.1

First, we have the following technical fact:

Proposition 4.1. *Let $\phi \in H_{\delta,y}$. Define*

$$\zeta(R) = \int\limits_{|x-y|=R\delta} \left(\omega_{\delta,y}^{-4}\phi^2 + \omega_{\delta,y}^{-2}|\nabla\phi|^2 + |\Delta\phi|^2 + \omega_{\delta,y}^2\left|\nabla(\Delta\phi)\right|^2\right) d\sigma.$$

Then there exist a sequence of real numbers $\{R_n\}$ with $R_n \to \infty$ such that

(i) $$\zeta(R_n) = O(1) \quad \text{as } n \to \infty,$$

(ii) $$\int\limits_{|x-y|=R_n\delta} |\phi|\, d\sigma = o\left(R_n^5\right) \quad \text{as } n \to \infty.$$

Proof. We note that $\int_0^\infty \zeta(r)\,dr \leqslant C\|\phi\|^2_{H_{\delta,y}} < \infty$. Given any $k > 0$, let $A_k = \{r \in (0,\infty)\colon\ \zeta(r) > k\}$. Clearly, $k|A_k| \leqslant C\|\phi\|^2_{H_{\delta,y}}$. Therefore, by choosing k large enough, we can ensure $|A_k| \leqslant 1$. Let $B_k = (0,\infty) \setminus A_k$. Then, it follows that $\zeta(r) \leqslant k$ for all $r \in B_k$. We claim a stronger version of (ii) holds, viz.,

$$\int\limits_{|x-y|=R_n\delta} |\phi|\,d\sigma = o\big(R_n^5\big) \quad \text{as } n \to \infty \text{ for any sequence } \{R_n\} \subset B_k,\ R_n \to \infty.$$

To prove this, we argue by contradiction i.e., suppose that there exist $c, R_0 > 0$ such that for all $R \in [R_0,\infty) \cap B_k$ we get

$$\int\limits_{|x-y|=R\delta} |\phi|\,d\sigma \geqslant cR^5 > 0. \tag{4.1}$$

By Hölder's inequality, we obtain

$$\int\limits_{|x-y|=R\delta} |\phi|\,d\sigma \leqslant \Bigg(\int\limits_{|x-y|=R\delta} \omega^4_{\delta,y}\,d\sigma\Bigg)^{\frac12}\Bigg(\int\limits_{|x-y|=R\delta} \omega^{-4}_{\delta,y}|\phi|^2\,d\sigma\Bigg)^{\frac12}. \tag{4.2}$$

But then, from (4.1) and (4.2),

$$\begin{aligned}\int\limits_{\mathbb{R}^4} \omega^{-4}_{\delta,y}|\phi|^2\,dx &= \delta^{-3}\int\limits_0^\infty\Bigg(\int\limits_{|x-y|=R\delta} \omega^{-4}_{\delta,y}|\phi|^2\,d\sigma\Bigg)dR\\ &\geqslant \delta^{-3}\int\limits_{[R_0,\infty)\cap B_k}\Bigg(\int\limits_{|x-y|=R\delta} \omega^{-4}_{\delta,y}|\phi|^2\,d\sigma\Bigg)dR\\ &\geqslant O(1)\int\limits_{[R_0,\infty)\cap B_k}\frac{1}{R}\,dR = +\infty,\end{aligned}$$

a contradiction. Hence (i), (ii) hold. □

The lemma below shows we can integrate by parts the functions in $H_{\delta,y}$ against $\psi^{(i)}_{\delta,y}$.

Lemma 4.1. *Let $\phi \in H_{\delta,y}$. Then, for $i = 0, 1, \ldots, 4$,*

$$\int\limits_{\mathbb{R}^4} \psi^{(i)}_{\delta,y}\Delta^2\phi = 24\int\limits_{\mathbb{R}^4} e^{4U_{\delta,y}}\psi^{(i)}_{\delta,y}\phi.$$

Proof. We prove the lemma for $i = 0$, the cases $i \geqslant 1$ are similar. As $\phi \in H_{\delta,y}$ we obtain

$$\int\limits_{\mathbb{R}^4} \omega^{-4}_{\delta,y}|\phi|^2\,dx < +\infty \quad \text{and} \quad \int\limits_{\mathbb{R}^4} |\Delta\phi|^2 < +\infty.$$

Let the sequence $\{R_n\}$ be as in the above proposition. Using (i), (ii) of this proposition, we deduce the following estimates

$$\int_{|x-y|=R_n\delta} |\phi|\, d\sigma = o\big(R_n^5\big), \tag{4.3}$$

$$\begin{aligned}\int_{|x-y|=R_n\delta} \left|\frac{\partial\phi}{\partial\nu}\right| d\sigma &\leqslant \Big(\int_{|x-y|=R_n\delta} \omega_{\delta,y}^{-2}|\nabla\phi|^2\, d\sigma\Big)^{\frac{1}{2}}\Big(\int_{|x-y|=R_n\delta} \omega_{\delta,y}^{2}\, d\sigma\Big)^{\frac{1}{2}} \\ &\leqslant O\big(R_n^{\frac{7}{2}}\big),\end{aligned} \tag{4.4}$$

$$\int_{|x-y|=R_n\delta} |\Delta\phi|\, d\sigma \leqslant O\big(R_n^{\frac{3}{2}}\big)\Big(\int_{|x-y|=R_n\delta} |\Delta\phi|^2\, d\sigma\Big)^{\frac{1}{2}} = O\big(R_n^{\frac{3}{2}}\big), \tag{4.5}$$

$$\begin{aligned}\int_{|x-y|=R_n\delta} \left|\frac{\partial\Delta\phi}{\partial\nu}\right| d\sigma &\leqslant \Big(\int_{|x-y|=R_n\delta} \big|\nabla(\Delta\phi)\big|^2\omega_{\delta,y}^{2}\, d\sigma\Big)^{\frac{1}{2}}\Big(\int_{|x-y|=R_n\delta} \omega_{\delta,y}^{-2}\, d\sigma\Big)^{\frac{1}{2}} \\ &\leqslant O\big(R_n^{-\frac{1}{2}}\big).\end{aligned} \tag{4.6}$$

Moreover, since $\phi \in H_{\delta,y}$, we obtain

$$\int_{\mathbb{R}^4} \psi_{\delta,y}^{(0)}\Delta^2\phi = \lim_{n\to\infty} \int_{|x-y|\leqslant R_n\delta} \psi_{\delta,y}^{(0)}\Delta^2\phi$$

and

$$\int_{\mathbb{R}^4} \psi_{\delta,y}^{(0)}e^{4U_{\delta,y}}\phi = \lim_{n\to\infty} \int_{|x-y|\leqslant R_n\delta} \psi_{\delta,y}^{(0)}e^{4U_{\delta,y}}\phi.$$

Using integration by parts, the last two equations and the above asymptotic estimates (4.3)–(4.6), we get

$$\begin{aligned}\int_{|x-y|\leqslant R_n\delta} \psi_{\delta,y}^{(0)}\Delta^2\phi &= 24\int_{|x-y|\leqslant R_n\delta} e^{4U_{\delta,y}}\psi_{\delta,y}^{(0)}\phi \\ &\quad + \int_{|x-y|=R_n\delta} \Big(\frac{\partial\Delta\phi}{\partial\nu}\psi_{\delta,y}^{(0)} - \frac{\partial\psi_{\delta,y}^{(0)}}{\partial\nu}\Delta\phi\Big)\, d\sigma \\ &\quad - \int_{|x-y|=R_n\delta} \Big(\frac{\partial\Delta\psi_{\delta,y}^{(0)}}{\partial\nu}\phi - \frac{\partial\phi}{\partial\nu}\Delta\psi_{\delta,y}^{(0)}\Big)\, d\sigma \\ &= 24\int_{|x-y|\leqslant R_n\delta} e^{4U_{\delta,y}}\psi_{\delta,y}^{(0)}\phi \\ &\quad + O(1)\int_{|x-y|=R_n\delta} \Big(\frac{|\Delta\phi|}{R_n^3} + \left|\frac{\partial\Delta\phi}{\partial\nu}\right|\Big)\, d\sigma\end{aligned}$$

$$+\,O\big(R_n^{-5}\big)\int\limits_{|x-y|=R_n\delta}|\phi|\,d\sigma+O\big(R_n^{-4}\big)\int\limits_{|x-y|=R_n\delta}\left|\frac{\partial\phi}{\partial\nu}\right|d\sigma$$

$$=24\int\limits_{|x-y|\leqslant R_n\delta}e^{U_{\delta,y}}\psi_{\delta,y}^{(0)}\phi+o_n(1).$$

This proves the lemma. □

By the previous section, for any compact set $K\subset\mathbb{R}^+\times\mathbb{R}^4$, $\rho>0$ small, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $(\delta,y)\in K$, there exists $\phi_\varepsilon^{\delta,y}\in B_\rho(0)\subset M_{\delta,y}$ such that $S_\varepsilon^{\delta,y}(\phi_\varepsilon^{\delta,y})=0$. For notational convenience, hereafter in this section we denote such a $\phi_\varepsilon^{\delta,y}$ simply as ϕ_ε.

Now we show that if (δ,y) is chosen carefully to be a stable zero of the vector field $\mathcal{V}_0$, then for a sequence $(\delta_\varepsilon,y_\varepsilon)\to(\delta,y)$, the function $\phi_\varepsilon^{\delta_\varepsilon,y_\varepsilon}$ is in fact a zero of the nonlinear operator $\mathcal{B}_\varepsilon^{\delta_\varepsilon,y_\varepsilon}$ and hence

$$u_\varepsilon=U_{\delta_\varepsilon,y_\varepsilon}+\phi_\varepsilon^{\delta_\varepsilon,y_\varepsilon}$$

will solve (1.6).

If $\phi_\varepsilon\in M_{\delta,y}$ solves $S_\varepsilon^{\delta,y}(\phi_\varepsilon)=0$, it follows that $\mathcal{B}_\varepsilon^{\delta,y}(\phi_\varepsilon)\in\mathcal{R}^\perp$. Hence by Lemma 2.5, there exist constants $c_{i,\varepsilon}$ such that for all $i=0,1,2,3,4$

$$\mathcal{B}_\varepsilon^{\delta,y}(\phi_\varepsilon)=\sum_{i=0}^{4}c_{i,\varepsilon}\Phi_{\delta,y}^{(i)}$$

and hence

$$\big\langle\mathcal{B}_\varepsilon^{\delta,y}(\phi_\varepsilon),\psi_{\delta,y}^{(i)}\big\rangle_{L^2(\mathbb{R}^4)}=c_{i,\varepsilon}\int\limits_{\mathbb{R}^4}\omega_{\delta,y}^{-4}\big|\psi_{\delta,y}^{(i)}\big|^2,\quad i=0,1,2,3,4,\tag{4.7}$$

holds.

Lemma 4.2. *Let $K\subset\mathbb{R}^+\times\mathbb{R}^4$ be a compact set. If ϕ_ε be obtained as in Lemma 3.1, then as $\varepsilon\to0$ we obtain for $i=0,1,\dots,4$*

$$\sup_{(\delta,y)\in K}\big|\big\langle\Delta^2\phi_\varepsilon-6e^{4U_{\delta,y}}\big(e^{4\phi_\varepsilon}-1\big),\psi_{\delta,y}^{(i)}\big\rangle_{L^2(\mathbb{R}^4)}\big|=O\big(\varepsilon^2\big)$$

and

$$\sup_{(\delta,y)\in K}\big|\big\langle f(x)\big(e^{4(U_{\delta,y}+\phi_\varepsilon)}-e^{4U_{\delta,y}}\big),\psi_{\delta,y}^{(i)}\big\rangle_{L^2(\mathbb{R}^4)}\big|=o_\varepsilon(1).$$

Proof. Let $K\subset\mathbb{R}^+\times\mathbb{R}^4$ be a compact set and $(\delta,y)\in K$. By (3.11), since $\phi_\varepsilon\to0$ in $H_{\delta,y}$, we obtain $\phi_\varepsilon\to0$ in $C^0_{loc}(\mathbb{R}^4)$. Using Lemma 4.1 and Theorem 2.1 we obtain

$$\int\limits_{\mathbb{R}^4}\big[\Delta^2\phi_\varepsilon-6e^{4U_{\delta,y}}\big(e^{4\phi_\varepsilon}-1\big)\big]\psi_{\delta,y}^{(i)}=-6\int\limits_{\mathbb{R}^4}e^{4U_{\delta,y}}\big[e^{4\phi_\varepsilon}-1-4\phi_\varepsilon\big]\psi_{\delta,y}^{(i)}$$

$$\leqslant C\|\phi_\varepsilon\|_{H_{\delta,y}}^2=O\big(\varepsilon^2\big).$$

Moreover, again by Theorem 2.1 and the dominated convergence theorem we get

$$\left\langle f(x)\left(e^{4(U_{\delta,y}+\phi_\varepsilon)}-e^{4U_{\delta,y}}\right),\psi^{(i)}_{\delta,y}\right\rangle_{L^2(\mathbb{R}^4)}\leqslant C\int_{\mathbb{R}^4} e^{4U_{\delta,y}}\left[e^{\phi_\varepsilon}-1\right]\psi^{(i)}_{\delta,y}=o_\varepsilon(1). \qquad \square$$

Define the matrix $\mathcal{A}_{\delta,y}=(A^{i,j}_{\delta,y})_{0\leqslant i,j\leqslant 4}$ by

$$A^{i,j}_{\delta,y}=\left\langle \Phi^{(i)}_{\delta,y},\psi^{(j)}_{\delta,y}\right\rangle_{L^2(\mathbb{R}^4)};\quad 0\leqslant i,j\leqslant 4$$

and the vector

$$c_\varepsilon=\begin{pmatrix} c_{0,\varepsilon}\\ c_{1,\varepsilon}\\ c_{2,\varepsilon}\\ c_{3,\varepsilon}\\ c_{4,\varepsilon}\end{pmatrix}.$$

We note that $\mathcal{A}_{\delta,y}$ is in fact an invertible diagonal matrix. Let $K\subset\mathbb{R}^+\times\mathbb{R}^4$ be a compact set with nonempty interior. Define the vector field

$$\mathcal{V}_\varepsilon(\delta,y)=\left(\frac{1}{\varepsilon}\int_{\mathbb{R}^4}\left(\Delta^2\phi_\varepsilon-6e^{4U_{\delta,y}}\left(e^{4\phi_\varepsilon}-1\right)\right)\psi^{(i)}_{\delta,y}-6\int_{\mathbb{R}^4} f(x)e^{4(U_{\delta,y}+\phi_\varepsilon)}\psi^{(i)}_{\delta,y}\right)_{i=0,1,\dots,4}.$$

Then from Lemma 4.2 we obtain $\mathcal{V}_\varepsilon(\delta,y)\to 6\mathcal{V}_0(\delta,y)$ in $C(K,\mathbb{R}^5)$. Now (4.7) can be written as

$$\mathcal{A}_{\delta,y}c_\varepsilon=\varepsilon\mathcal{V}_\varepsilon(\delta,y) \tag{4.8}$$

for $(\delta,y)\in K$.

Proof of Theorem 1.1. Let (δ,y) be a stable zero for the vector field $\mathcal{V}_0$. Since $\mathcal{V}_\varepsilon(\delta,y)\to 6\mathcal{V}_0(\delta,y)$ in $C(K,\mathbb{R}^5)$, we can find zeroes $(\delta_\varepsilon,y_\varepsilon)$ of $\mathcal{V}_\varepsilon$ such that $(\delta_\varepsilon,y_\varepsilon)\to(\delta,y)$. Take the solution $\phi^{\delta_\varepsilon,y_\varepsilon}_\varepsilon$ of $S^{\delta_\varepsilon,y_\varepsilon}_\varepsilon(\phi)=0$ given in Lemma 3.1 and write out the corresponding equations (4.7) and (4.8) for $\mathcal{A}_{\delta_\varepsilon,y_\varepsilon}$. Since $\mathcal{A}_{\delta_\varepsilon,y_\varepsilon}$ is invertible, we have $c_\varepsilon=0$ for all $\varepsilon>0$. Hence the corresponding $\phi^{\delta_\varepsilon,y_\varepsilon}_\varepsilon$ solves $\mathcal{B}^{\delta_\varepsilon,y_\varepsilon}_\varepsilon(\phi^{\delta_\varepsilon,y_\varepsilon})=0$ for all such ε. Defining $u_\varepsilon=U_{\delta_\varepsilon,y_\varepsilon}+\phi^{\delta_\varepsilon,y_\varepsilon}_\varepsilon$, we obtain that u_ε solves (1.6) for all $\varepsilon>0$ small. That $\|\phi^{\delta_\varepsilon,y_\varepsilon}_\varepsilon\|_{H_{\delta,y}}=O(\varepsilon)$ follows from Claim 2 in Lemma 3.1. $\square$

5. Necessary condition: Proof of Theorem 1.2

In this section we show that if there is a sequence of solutions u_ε of (1.6) "bifurcating" from some $U_{\delta,y}$, then necessarily $\mathcal{V}_0(\delta,y)=0$. The main tool to prove this result is a Pohozaev type identity for functions belonging to $H_{\delta,y}$. First, we prove the following sharp decay estimates:

Lemma 5.1. *Let u_ε be a sequence of solutions of (1.6) with $\|u_\varepsilon-U_{\delta,y}\|_{H_{\delta,y}}\to 0$ as $\varepsilon\to 0$ for some $(\delta,y)\in\mathbb{R}^+\times\mathbb{R}^4$. Then, uniformly as $\varepsilon\to 0$, we have the following decay estimates*

$$\lim_{|x|\to\infty} \frac{u_\varepsilon(x)}{\log|x|} = -2, \tag{5.1}$$

$$\lim_{|x|\to\infty} x\cdot\nabla u_\varepsilon = -2, \tag{5.2}$$

$$\lim_{|x|\to\infty} |x|^2\left|\Delta u_\varepsilon(x)\right| = 4, \tag{5.3}$$

$$\lim_{|x|\to\infty} x\cdot\nabla(x\cdot\nabla u_\varepsilon) = 0, \tag{5.4}$$

$$\lim_{|x|\to\infty} |x|^2 x\cdot\nabla(\Delta u_\varepsilon) = 8. \tag{5.5}$$

Proof. Let $\phi_\varepsilon = u_\varepsilon - U_{\delta,y}$. First note that $\|\phi_\varepsilon\|_{H_{\delta,y}} \to 0$ and hence

$$\frac{|u_\varepsilon - U_{\delta,y}|}{\log|x|} \leqslant C\|\phi_\varepsilon\|_{H_{\delta,y}}\left(1+\frac{1}{\log|x|}\right) \to 0 \tag{5.6}$$

as $|x| \to +\infty$. Using the fact that

$$\lim_{|x|\to\infty} \frac{U_{\delta,y}}{\log|x|} = -2,$$

we obtain (5.1). We use similar arguments in [12] to establish (5.2), (5.3), (5.4) and (5.5). Using (5.1) we obtain

$$\forall 0<\nu<2,\ \exists R(\nu)>0\colon\ u_\varepsilon(x) \leqslant (-2+\nu)\log^+|x|, \quad \forall |x|>R(\nu). \tag{5.7}$$

Then, since $\phi_\varepsilon \in H_{\delta,y}$ we can use (4.6) of Lemma 4.1 to conclude that for a suitable sequence $R_n \to \infty$,

$$\begin{aligned} 0 &= \lim_{R_n\to\infty} \int\limits_{\partial B_{R_n}(0)} \frac{\partial\Delta\phi_\varepsilon}{\partial\nu}\,d\sigma = \lim_{R_n\to\infty} \int\limits_{B_{R_n}(0)} \Delta^2(u_\varepsilon - U_{\delta,y}) \\ &= \lim_{R_n\to\infty} \int\limits_{B_{R_n}(0)} 6\big(1+\varepsilon f(x)\big)e^{4u_\varepsilon} - 6e^{4U_{\delta,y}} \\ &= \lim_{R_n\to\infty} \int\limits_{B_{R_n}(0)} 6\big(1+\varepsilon f(x)\big)e^{4u_\varepsilon} - 16\pi^2. \end{aligned} \tag{5.8}$$

Hence, we obtain

$$\forall \varepsilon>0, \quad \int\limits_{\mathbb{R}^4} \big(1+\varepsilon f(x)\big)e^{4u_\varepsilon} = \frac{8\pi^2}{3}. \tag{5.9}$$

We define v_ε by

$$v_\varepsilon(x) = \frac{1}{8\pi^2}\int\limits_{\mathbb{R}^4} \log\big(|x-y|\big)6\big(1+\varepsilon f(y)\big)e^{4u_\varepsilon(y)}\,dy.$$

It is easy to check that $\Delta^2 v_\varepsilon = -6(1+\varepsilon f(x))e^{4u_\varepsilon}$ in $\mathbb{R}^4$ and using (5.9) we obtain uniformly as $\varepsilon \to 0$,

$$\lim_{|x|\to\infty} \frac{v_\varepsilon(x)}{\log|x|} = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} (1+\varepsilon f(y)) e^{4u_\varepsilon(y)}\, dy = 2. \tag{5.10}$$

It can be shown, as in Lemma 2.1, that

$$\sup_{0<\varepsilon<1} \sup_{\mathbb{R}^4} \left| v_\varepsilon(x) \right| \leqslant C\left(\log^+|x| + 1\right).$$

Consider the function $w_\varepsilon = u_\varepsilon + v_\varepsilon$. Then $\Delta^2 w_\varepsilon = 0$ in $\mathbb{R}^4$. Hence Δw_ε is harmonic and by the mean value theorem, for any $r > 0$,

$$\Delta w_\varepsilon(x_0) = \frac{2}{\pi^2 r^4} \int_{B_r(x_0)} \Delta w_\varepsilon(x)\, dx = \frac{2}{\pi^2 r^4} \int_{\partial B_r(x_0)} \frac{\partial w_\varepsilon}{\partial r}(x)\, d\sigma.$$

Integrating along r we obtain

$$\frac{r^2}{8} \Delta w_\varepsilon(x_0) = \frac{1}{2\pi^2 r^3} \int_{\partial B_r(x_0)} w_\varepsilon\, d\sigma - w_\varepsilon(x_0).$$

From (5.7) and (5.10), it follows that w_ε and hence the absolute value of the RHS in the above equation grows at most like $\log r$ as $r \to \infty$. Hence, we obtain a contradiction if $\Delta w_\varepsilon(x_0) \neq 0$ at any x_0. Therefore, $\Delta w_\varepsilon = 0$ in $\mathbb{R}^4$. Further since w_ε has at most logarithmic growth at infinity, we conclude that $w_\varepsilon \equiv const.$ in $\mathbb{R}^4$. Successively differentiating v_ε and arguing in a similar way we obtain the relations (5.2)–(5.5). □

Corollary 5.1. *The following uniform estimates hold*

(i)
$$\limsup_{|x|\to\infty} |x| \left| \nabla u_\varepsilon(x) \right| < \infty,$$

(ii)
$$\limsup_{|x|\to\infty} |x|^2 \left| D^2 u_\varepsilon \right| < \infty.$$

Proof. We note that, from (5.1), we have the estimate $e^{4u_\varepsilon} \leqslant C(1+|x|)^{\nu-8}$ for any $\nu > 0$ and all $|x| \geqslant R = R(\nu)$. The conclusions (i) and (ii) follow by differentiating inside the integral sign in the definition of v_ε. □

We now develop two kinds of Pohozaev type identities.

Lemma 5.2. *Let $\{u_\varepsilon\}$ be a family of solutions to (1.6) such that $\|u_\varepsilon - U_{\delta,y}\|_{H_{\delta,y}} \to 0$ as $\varepsilon \to 0$ for some $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^4$. Then,*

$$\int_{\mathbb{R}^4} f(x) e^{4u_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} = 0, \quad i = 1, 2, 3, 4, \tag{5.11}$$

and

$$\int\limits_{\mathbb{R}^4} f(x)e^{4u_\varepsilon}\left[(x-y)\cdot\nabla u_\varepsilon+1\right]=0. \tag{5.12}$$

Proof. In order to prove (5.11) we multiply (1.6) by $\frac{\partial u_\varepsilon}{\partial x_i}$ and integrate by parts on the ball $B_R(0)$ to get

$$6\int\limits_{B_R(0)}\left(1+\varepsilon f(x)\right)e^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}=\int\limits_{\partial B_R(0)}\frac{\partial\Delta u_\varepsilon}{\partial\nu}\frac{\partial u_\varepsilon}{\partial x_i}\,d\sigma-\int\limits_{B_R(0)}\nabla(\Delta u_\varepsilon)\cdot\nabla\left(\frac{\partial u_\varepsilon}{\partial x_i}\right). \tag{5.13}$$

By (5.5) and Corollary 5.1(i), we obtain

$$\int\limits_{\partial B_R(0)}\left|\frac{\partial\Delta u_\varepsilon}{\partial\nu}\frac{\partial u_\varepsilon}{\partial x_i}\right|d\sigma=O\left(R^{-1}\right)\quad\text{as } R\to\infty. \tag{5.14}$$

Again, by suitable integration by parts and using (5.3) and Corollary 5.1(ii), we get as $R\to\infty$,

$$\int\limits_{B_R(0)}\nabla(\Delta u_\varepsilon)\cdot\nabla\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)=\int\limits_{\partial B_R(0)}\left\{\Delta u_\varepsilon\frac{\partial}{\partial\nu}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)-\frac{1}{2R}x_i|\Delta u_\varepsilon|^2\right\}d\sigma=O\left(R^{-1}\right). \tag{5.15}$$

Hence, from the last two relations,

$$\lim_{R\to\infty}\left\{\text{RHS of (5.13)}\right\}=0. \tag{5.16}$$

Again integrating by parts in another way,

$$\int\limits_{B_R(0)}(1+\varepsilon f)e^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}=\frac{1}{4R}\int\limits_{\partial B_R(0)}x_i e^{4u_\varepsilon}\,d\sigma+\varepsilon\int\limits_{B_R(0)}fe^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}. \tag{5.17}$$

Using the asymptotic relation (5.1) and Corollary 5.1(i), we may let $R\to\infty$ in the above equation to conclude

$$\lim_{R\to\infty}\int\limits_{B_R(0)}(1+\varepsilon f)e^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}=\varepsilon\int\limits_{\mathbb{R}^4}fe^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}. \tag{5.18}$$

Therefore we obtain, using (5.18) and (5.16),

$$6\varepsilon\int\limits_{\mathbb{R}^4}f(x)e^{4u_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}=\lim_{R\to\infty}\left\{\text{LHS of (5.13)}\right\}=0, \tag{5.19}$$

which proves (5.11). Now we are left to show (5.12). For this, we multiply (1.6) by $(x-y)\cdot\nabla u_\varepsilon+1$ on either side and integrate on the ball $B_R(y)$ as before to obtain,

$$6\int\limits_{B_R(y)}e^{4u_\varepsilon}\left(1+\varepsilon f(x)\right)\left((x-y)\cdot\nabla u_\varepsilon+1\right)=\int\limits_{B_R(y)}\Delta^2u_\varepsilon\left((x-y)\cdot\nabla u_\varepsilon+1\right). \tag{5.20}$$

Integrating by parts we obtain

$$\text{LHS of (5.20)} = \frac{3R}{2} \int_{\partial B_R(y)} e^{4u_\varepsilon}\, d\sigma + 6\varepsilon \int_{B_R(y)} f e^{4u_\varepsilon} \big((x-y)\cdot\nabla u_\varepsilon + 1\big). \tag{5.21}$$

We denote $r\frac{\partial}{\partial r} = (x-y)\cdot\nabla$. Again integrating by parts suitably,

$$\begin{aligned}\text{RHS of (5.20)} = \int_{\partial B_R(y)} \bigg\{ & R\left(\frac{1}{2}|\Delta u_\varepsilon|^2 + \left(\frac{\partial u_\varepsilon}{\partial r} + 1\right)\frac{\partial}{\partial r}(\Delta u_\varepsilon)\right) \\ & - \Delta u_\varepsilon \frac{\partial}{\partial r}\left(r\frac{\partial u_\varepsilon}{\partial r}\right)\bigg\}\, d\sigma.\end{aligned} \tag{5.22}$$

We have used the relation (obtained from integrating by parts)

$$\int_{B_R(y)} \Delta u_\varepsilon (x-y)\cdot\nabla(\Delta u_\varepsilon) = \frac{R}{2}\int_{\partial B_R(y)} (\Delta u_\varepsilon)^2\, d\sigma - 2\int_{B_R(y)} (\Delta u_\varepsilon)^2\, dx$$

and the identity

$$\Delta\big((x-y)\cdot\nabla u_\varepsilon\big) = 2\Delta u_\varepsilon + (x-y)\cdot\nabla(\Delta u_\varepsilon)$$

to derive (5.22). Using the asymptotics (5.1)–(5.5), we obtain that

$$\lim_{R\to\infty}\{\text{LHS of (5.20)}\} = 6\varepsilon \int_{\mathbb{R}^4} f(x) e^{4u_\varepsilon}\big((x-y)\cdot\nabla u_\varepsilon + 1\big), \tag{5.23}$$

and

$$\lim_{R\to\infty}\{\text{RHS of (5.20)}\} = 0. \tag{5.24}$$

Hence (5.12) follows. □

Proof of Theorem 1.2. We note that $(x-y)\cdot\nabla_x U_{\delta,y} + 1 = -\delta\frac{\partial U_{\delta,y}}{\partial\delta}$. Since $u_\varepsilon \to U_{\delta,y}$ in $H_{\delta,y}$, the asymptotics in Lemma 5.1 allow us to pass to the limit as ε goes to 0 in (5.11) and (5.12). This means that $\mathcal{V}_0(\delta, y) = 0$. □

6. Local uniqueness: Proof of Theorem 1.3

In this section we show that a "strongly" stable zero of the vector field $\mathcal{V}_0(\delta, y)$ "bifurcates" at most one family of solutions to (1.6).

Proof of Theorem 1.3. We argue by contradiction. Let us suppose that for some sequence $\varepsilon_n \to 0$ there exist two distinct sequences of solutions $\{u_{1,\varepsilon_n}\}$ and $\{u_{2,\varepsilon_n}\}$ of (1.6) such that $\|u_{i,n} - U_{\delta,y}\|_{H_{\delta,y}} \to 0$ as $n\to\infty$ for $i = 1, 2$. For convenience, we denote $u_{i,n} = u_{i,\varepsilon_n}$. Set $\tilde{w}_n = u_{1,n} - u_{2,n}$. Then $\|\tilde{w}_n\|_{H_{\delta,y}} \to 0$ as $n\to\infty$. Then, we have the following two cases: either

Case (i): for any $\beta > 0$, for all large n, there exists $x_n \in \mathbb{R}^4$ such that $|\tilde{w}_n(x_n)| \geqslant \beta$, or

Case (ii): there exists $\beta > 0$ and a subsequence of $\{\tilde{w}_n\}$, which we still denote by $\{\tilde{w}_n\}$, such that $|\tilde{w}_n(x)| < \beta$ for all n and all $x \in \mathbb{R}^4$. In this case, let $x_n \in \mathbb{R}^4$ be such that $|\tilde{w}_n(x_n)| \geqslant \frac{1}{2}\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^4)}$.

If Case (i) holds, then we define $w_n = \frac{\tilde{w}_n}{\|\tilde{w}_n\|_{H_{\delta,y}}}$, and if Case (ii) holds then $w_n = \frac{\tilde{w}_n}{\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^4)}}$. Then w_n satisfies

$$\Delta^2 w_n = 24\big(1 + \varepsilon_n f(x)\big)c_n(x)w_n \quad \text{with } c_n(x) = \int_0^1 e^{4tu_{1,n}+(1-t)4u_{2,n}}\,dt. \tag{6.1}$$

We note that, from (5.1), we have the estimate

$$e^{4u_{i,n}} \leqslant C\big(1+|x|\big)^{\nu-8} \quad \text{for any } \nu > 0,\ \text{all } |x| \geqslant R = R(\nu),\ \text{and } \forall n. \tag{6.2}$$

Using Schauder estimates, we obtain $w_n \to w$ in $C^4_{loc}(\mathbb{R}^4)$ where w satisfies the problem

$$\Delta^2 w = 24e^{4U_{\delta,y}}w \quad \text{in } \mathbb{R}^4. \tag{6.3}$$

By non-degeneracy result in Lemma 2.2, $w = c_0\frac{\partial U_{\delta,y}}{\partial\delta} + \sum_{i=1}^4 c_i\frac{\partial U_{\delta,y}}{\partial x_i}$ for some $c_i \in \mathbb{R}$, $i = 0,1,\ldots,4$. We claim that $c_i = 0$ for all $i = 0,1,\ldots,4$. From the identity (5.11) we get

$$\int_{\mathbb{R}^4} f(x)e^{4u_{i,n}}\frac{\partial u_{i,n}}{\partial x_j} = 0, \quad i = 1,2;\ j = 1,2,3,4. \tag{6.4}$$

Using assumptions (1.12) and (6.2) we derive from (6.4)

$$\int_{\mathbb{R}^4} \frac{\partial f}{\partial x_j}e^{4u_{i,n}} = 0, \quad i = 1,2 \text{ and } j = 1,2,3,4. \tag{6.5}$$

Therefore,

$$\int_{\mathbb{R}^4} \left(\frac{\partial f}{\partial x_j}e^{4u_{1,n}} - \frac{\partial f}{\partial x_j}e^{4u_{2,n}}\right) = 0 \quad \text{for } j = 1,2,3,4, \tag{6.6}$$

which can be written as

$$\int_{\mathbb{R}^4} \frac{\partial f}{\partial x_j}c_n(x)w_n(x)\,dx = 0 \quad \text{for } j = 1,2,3,4. \tag{6.7}$$

Using (1.12) we can pass to the limit in (6.7) to obtain,

$$\int_{\mathbb{R}^4} \frac{\partial f}{\partial x_j}e^{4U_{\delta,y}}\left(c_0\frac{\partial U_{\delta,y}}{\partial\delta} + \sum_{i=1}^4 c_i\frac{\partial U_{\delta,y}}{\partial x_i}\right) = 0, \quad j = 1,2,3,4. \tag{6.8}$$

That is, integrating by parts again,

$$\int_{\mathbb{R}^4} f \frac{\partial}{\partial x_j}\left(e^{4U_{\delta,y}}\left\{c_0 \frac{\partial U_{\delta,y}}{\partial \delta} + \sum_{i=1}^{4} c_i \frac{\partial U_{\delta,y}}{\partial x_i}\right\}\right) = 0, \quad j = 1,2,3,4. \tag{6.9}$$

Similarly, using (1.12) and (6.2) we deduce from (5.12),

$$\int_{\mathbb{R}^4} \langle (x-y), \nabla f \rangle e^{4u_{i,n}} = 0 \quad \text{for } i = 1,2. \tag{6.10}$$

Then, arguing as above we get

$$\int_{\mathbb{R}^4} \langle (x-y), \nabla f \rangle e^{4U_{\delta,y}} w = 0.$$

Hence doing integration by parts we obtain that

$$-4\delta \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \frac{\partial U_{\delta,y}}{\partial \delta} w + \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \langle (x-y), \nabla w \rangle = 0. \tag{6.11}$$

Using the relations

$$\langle (x-y), \nabla w \rangle = -\left(\delta \frac{\partial w}{\partial \delta} + w\right),$$

and

$$\int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}(x)} w = 0 \quad \big(\text{from (6.8)}\big),$$

we rewrite (6.11) as

$$-4\delta \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \frac{\partial U_{\delta,y}}{\partial \delta} w - \delta \int_{\mathbb{R}^4} f(x) e^{4U_{\delta,y}} \frac{\partial w}{\partial \delta} = 0.$$

That is,

$$\int_{\mathbb{R}^4} f(x) \frac{\partial}{\partial \delta}\left(e^{4U_{\delta,y}}\left\{c_0 \frac{\partial U_{\delta,y}}{\partial \delta} + \sum_{i=1}^{4} c_i \frac{\partial U_{\delta,y}}{\partial x_i}\right\}\right) = 0. \tag{6.12}$$

Thus, from (6.9) and (6.12), we deduce $D^2 J(\delta, y)\mathbf{c} = 0$ where $\mathbf{c}$ is the column vector $(c_0, c_1, c_2, c_3, c_4)^T$. Since $D^2 J(\delta, y)$ is an invertible matrix, we deduce $c_0 = c_1 = c_2 = c_3 = c_4 = 0$. This implies $w \equiv 0$ in $\mathbb{R}^4$. Therefore, $w_n \to 0$ in $C^4_{loc}(\mathbb{R}^4)$ and hence we necessarily have $|x_n| \to \infty$. Let us use the Kelvin transform to define

$$\hat{u}_{i,n}(x) = u_{i,n}\left(\frac{x}{|x|^2}\right), \qquad \hat{w}_n(x) = w_n\left(\frac{x}{|x|^2}\right), \qquad \hat{c}_n(x) = c_n\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^4 \setminus \{0\}.$$

Clearly, we have $|\hat{w}_n(\frac{x_n}{|x_n|^2})| \geqslant \frac{1}{2}$ for all large n. It can be shown that $\hat{w}_n$ satisfies the following equation

$$\Delta^2 \hat{w}_n = \frac{24}{|x|^8}\hat{c}_n\left(1 + \varepsilon_n f\left(\frac{x}{|x|^2}\right)\right)\hat{w}_n \quad \text{in } \mathbb{R}^4 \setminus \{0\}. \tag{6.13}$$

In Case (i), using the growth estimate (2.1), we get that $|\hat{w}_n(x)| \leqslant C(1 - \log|x|)$ for all n and all $x \in B_1(0)$. Since $\hat{w}_n \to 0$ in $C^4_{loc}(\mathbb{R}^4 \setminus \{0\})$, by dominated convergence theorem we get that $\hat{w}_n \to 0$ in $L^p(B_1(0))$ for all $p \geqslant 1$. In Case (ii), we have again, $|\hat{w}_n| \leqslant 1$ and $\hat{w}_n \to 0$ in $C^4_{loc}(\mathbb{R}^4 \setminus \{0\})$. Hence $\hat{w}_n \to 0$ in $L^p(B_1(0))$ for any $p \geqslant 1$. Using the assumption $f \in L^\infty(\mathbb{R}^4)$ and the estimate (6.2) we get that

$$\left\{\frac{24}{|x|^8}\hat{c}_n\left(1 + \varepsilon_n f\left(\frac{x}{|x|^2}\right)\right)\right\}$$

is a bounded sequence in $L^p(B_1(0))$ for any $p > 1$. Therefore the RHS in Eq. (6.13) converges to 0 in $L^p(B_1(0))$ as $n \to \infty$ for any $p > 1$. We recall that $\hat{w}_n \to 0$ in $C^4_{loc}(\mathbb{R}^4 \setminus \{0\})$. Using the standard L^p regularity theory (see for example, Corollary 2.23 in [11]) and Sobolev embedding to Eq. (6.13) we obtain

$$\|\hat{w}_n\|_{L^\infty(B_1(0))} \to 0.$$

This gives a contradiction easily in Case (i) and as well in Case (ii) since

$$\|\hat{w}_n\|_{L^\infty(B_1(0))} \geqslant \left|\hat{w}_n\left(\frac{x_n}{|x_n|^2}\right)\right| \geqslant \frac{1}{2}$$

for all large n. This proves the theorem. □

7. Exact multiplicity result: Proof of Theorem 1.4

Proof of Theorem 1.4. Since the stable zeroes of $\mathcal{V}_0$ are isolated there exists an $R > 0$ such that zeroes of $\mathcal{V}_0$ are contained in the interior of a closed ball $K = \overline{B}_R(0) \subset \mathbb{R}^+ \times \mathbb{R}^4$. Let m be the number of zeroes of $\mathcal{V}_0$. By Theorems 1.1, 1.2 and 1.3 we conclude that there exists $\varepsilon_1 = \varepsilon_1(K) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ the problem (1.6) has at least m solutions u^i_ε and m points $(\delta_i, y_i) \in K$ such that $u^i_\varepsilon - U_{\delta_i, y_i} \to 0$ in H_{δ_i, y_i}, $i = 1, \ldots, m$. Let

$$\mathcal{S}_\mu = \left\{u \text{ solves (1.6) for } \varepsilon \in (0, \mu),\ u - U_{1,0} \in H_{1,0}\right\} \setminus \left\{u^i_\varepsilon\right\}_{0<\varepsilon<\mu,\ 1\leqslant i\leqslant m}.$$

Define now the quantity

$$\theta_\mu = \inf_{u \in \mathcal{S}_\mu} d_{H_{1,0}}(u, \mathcal{M}_K).$$

We now claim that

$$\theta_0 = \liminf_{\mu \to 0} \theta_\mu > 0.$$

If possible let $\theta_0 = 0$. Then we find sequences $\{u_n\} \subset \mathcal{S}_\mu$ and $\{(\delta_n, y_n)\} \subset K$ such that $\|u_n - U_{\delta_n, y_n}\|_{H_{1,0}} \to 0$ as $n \to \infty$. Let $(\delta_n, y_n) \to (\delta, y) \in K$. This means that $\{u_n\}$ is a sequence of solutions bifurcating from (δ, y). By Theorem 1.2, we have that $\mathcal{V}_0(\delta, y) = 0$. But the uniqueness result in Theorem 1.3 contradicts the fact that $\{u_n\} \subset \mathcal{S}_\mu$. This proves the claim.

Therefore, we can choose $\mu_0 > 0$ small such that $\theta_\mu \geqslant \frac{\theta_0}{2}$ for all $\mu < \mu_0$. By Theorem 1.2, there exists some $C > 0$ and $\varepsilon_2 > 0$,

$$d\big(u_\varepsilon^i, \mathcal{M}_K\big) \leqslant C\varepsilon, \quad i = 1, \dots, m, \ \varepsilon \in (0, \varepsilon_2).$$

The conclusion of the theorem now follows by taking $\rho_0 = \frac{\theta_0}{2}$ and $\varepsilon_0 = \min\{\frac{\theta_0}{2C}, \mu_0, \varepsilon_2\}$. □

8. A concrete approach to finding stable zeroes of $\mathcal{V}_0$

Throughout this section we assume

(f1) $$f \in C^1\big(\mathbb{R}^4\big) \cap L^\infty\big(\mathbb{R}^4\big).$$

By a change of variable J can be written as

$$J(\delta, \xi) = 16 \int_{\mathbb{R}^4} \frac{f(\delta x + \xi)}{(1 + |x|^2)^4}\, dx. \tag{8.1}$$

Let $Crit(f)$, $Crit(J)$ denote respectively the set of critical points of f and J. We have

$$J(0, \xi) = 16 f(\xi) \int_{\mathbb{R}^4} \frac{1}{(1 + |x|^2)^4}\, dx. \tag{8.2}$$

Since $\langle \nabla f(\xi), x\rangle$ is an odd function,

$$D_\delta J(0, \xi) = \lim_{\delta \to 0} (D_\delta J)(\delta, \xi) = 16 \int_{\mathbb{R}^4} \frac{\langle \nabla f(\xi), x\rangle}{(1 + |x|^2)^4}\, dx = 0. \tag{8.3}$$

Therefore we can extend J as an even function of δ to $\mathbb{R} \times \mathbb{R}^4$. Without loss of generality we denote this function by J. Also

$$\xi \in Crit(f) \quad \Leftrightarrow \quad (0, \xi) \in Crit(J).$$

Lemma 8.1. *Assume the following conditions on f:*

(f2) *there exists $\rho > 0$ such that $\langle \nabla f(x), x\rangle < 0$ for any $|x| \geqslant \rho$,*
(f3) $\langle \nabla f(x), x\rangle \in L^1(\mathbb{R}^4)$, $\int_{\mathbb{R}^4} \langle \nabla f(x), x\rangle\, dx < 0$.

Then, there exists $R > 0$ such that

$$\big\langle \nabla J(\delta, \xi), (\delta, \xi)\big\rangle < 0 \quad \textit{whenever } \big|(\delta, \xi)\big| \geqslant R. \tag{8.4}$$

Proof. See Lemma 3.3 in [1]. □

We make the following assumption about the "shape" of f near a critical point.

(f4) *Given $\xi \in Crit(f)$, suppose that there exists $\beta_\xi = \beta > 1$ such that:*
  (i) *If $\beta \leqslant 4$, there exist $\mu > 0$ and a map $Q_\xi : \mathbb{R}^4 \to \mathbb{R}$ homogeneous of degree β, that is $Q_\xi(\lambda y) = \lambda^\beta Q_\xi(y)$ for all $y \in \mathbb{R}^4$, such that*

$$f(y) = f(\xi) + Q_\xi(y-\xi) + O\big(|y-\xi|^{\beta+\mu}\big) \quad \text{as } y \to \xi.$$

  (ii) *If $\beta > 4$, we assume that $f(y) = f(\xi) + O(|y-\xi|^\beta)$ as $y \to \xi$.*

Lemma 8.2. *Let* (f4) *hold. Then, as $\delta \to 0^+$,*

$$J(\delta,\xi) - J(0,\xi) = 16 \begin{cases} \delta^\beta (C_{\beta,\xi} + o_\delta(1)) & \text{if } \beta < 4, \\ \delta^4 \log \frac{1}{\delta} (C_{4,\xi} + o_\delta(1)) & \text{if } \beta = 4, \\ \delta^4 (C_{\beta,\xi} + o_\delta(1)) & \text{if } \beta > 4, \end{cases} \tag{8.5}$$

where

$$C_{\beta,\xi} = \begin{cases} \int_0^\infty \frac{r^\beta\, dr}{(1+|x|^2)^4} \int_{\mathbb{S}^3} Q_\xi(\sigma)\, d\sigma & \text{if } \beta < 4, \\ \int_{\mathbb{S}^3} Q_\xi(\sigma)\, d\sigma & \text{if } \beta = 4, \\ \int_{\mathbb{R}^4} |y|^{-8} [f(y+\xi) - f(\xi)]\, dy & \text{if } \beta > 4. \end{cases} \tag{8.6}$$

Proof. *Case* $1 < \beta \leqslant 4$: From (f4)(i) we can find a $C > 0$ and $0 < R < 1$ such that

$$\left| f(\delta x + \xi) - f(\xi) - \delta^\beta |x|^\beta Q_\xi\left(\frac{x}{|x|}\right) \right| \leqslant C\big(\delta|x|\big)^{\beta+\mu}, \quad \forall |x| \leqslant \frac{R}{\delta}. \tag{8.7}$$

We remark that if $\beta < 4$ we can choose $0 < \tilde{\mu} < \mu$ small so that $\beta + \tilde{\mu} < 4$. Since $R < 1$, we see that (8.7) still holds with $\tilde{\mu}$, which we continue to denote by μ. We now compute

$$\begin{aligned} J(\delta,\xi) - J(0,\xi) &= 16 \int_{\mathbb{R}^4} \frac{f(\delta x + \xi) - f(\xi)}{(1+|x|^2)^4}\, dx \\ &= 16 \int_{B_{\frac{R}{\delta}}(0)} \frac{f(\delta x + \xi) - f(\xi)}{(1+|x|^2)^4}\, dx \\ &\quad + 16 \int_{\mathbb{R}^4 \setminus B_{\frac{R}{\delta}}(0)} \frac{f(\delta x + \xi) - f(\xi)}{(1+|x|^2)^4}\, dx \\ &= I^{(1)}(\delta) + I^{(2)}(\delta). \end{aligned} \tag{8.8}$$

We simply estimate

$$\big|I^{(2)}(\delta)\big| \leqslant 16\|f\|_\infty \int_{\mathbb{R}^4 \setminus B_{\frac{R}{\delta}}(0)} \frac{1}{(1+|x|^2)^4}\, dx = O\big(\delta^4\big). \tag{8.9}$$

Using (8.7) in the first integral $I^{(1)}(\delta)$ we get

$$\left| I^{(1)}(\delta) - 16\delta^{\beta} \int\limits_{B_{\frac{R}{\delta}}(0)} \frac{|x|^{\beta} Q_{\xi}(\frac{x}{|x|})}{(1+|x|^2)^4}\,dx \right| \leqslant C\delta^{\beta+\mu} \int\limits_{B_{\frac{R}{\delta}}(0)} \frac{|x|^{\beta+\mu}}{(1+|x|^2)^4}\,dx. \tag{8.10}$$

If $\beta < 4$ (hence $\beta + \mu < 4$), the above inequality gives

$$I^{(1)}(\delta) = 16\delta^{\beta} \int\limits_0^{\infty} \frac{r^{\beta}\,dr}{(1+|x|^2)^4} \int\limits_{\mathbb{S}^3} Q_{\xi}(\sigma)\,d\sigma \left[1 + O\left(\delta^{\mu}\right)\right]. \tag{8.11}$$

If $\beta = 4$, again from (8.10) we get

$$I^{(1)}(\delta) = 16\delta^4 \log\left(\frac{1}{\delta}\right) \int\limits_{\mathbb{S}^3} Q_{\xi}(\sigma)\,d\sigma \left[1 + o_{\delta}(1)\right]. \tag{8.12}$$

Putting together (8.9), (8.11) and (8.12) we complete the case $\beta \leqslant 4$.

Case $\beta > 4$: Using (f4) and dominated convergence theorem,

$$J(\delta, \xi) - J(0, \xi) = 16\delta^4 \int\limits_{\mathbb{R}^4} |y|^{-8}\big(f(y+\xi) - f(\xi)\big)\,dy + o_{\delta}(1).$$

This shows (8.5)–(8.6) for $\beta > 1$. □

The proof of the following two results is a slight modification of Lemmas 3.6 and Lemma 3.8 respectively in [1].

Corollary 8.1. *Let $\xi \in Crit(f)$ be isolated and assume that f satisfies* (f1)–(f4)*. Suppose that $C_{\beta,\xi} \neq 0$. Then $q = (0, \xi)$ is an isolated critical point of J and*

$$C_{\beta,\xi} > 0 \quad \Rightarrow \quad deg_{loc}(\nabla J, q) = deg_{loc}(\nabla f, \xi),$$
$$C_{\beta,\xi} < 0 \quad \Rightarrow \quad deg_{loc}(\nabla J, q) = -deg_{loc}(\nabla f, \xi).$$

Proposition 8.1. *If f has finitely many critical points and satisfies*

(i) *assumptions* (f1)–(f4) *and at any $\xi \in Crit(f)$,*
(ii) $C_{\beta,\xi} \neq 0$ *(see* (8.6)*), and*
(iii) $\sum_{C_{\beta,\xi}<0} deg_{loc}(\nabla f, \xi) \neq 1$,

then the vector field ∇J has a stable zero.

Remark 8.1. We remark that the expression for $C_{\beta,\xi}$ when $\beta > 4$ depends on global behavior of f, in contrast to the expressions for $C_{\beta,\xi}$ when $\beta \leqslant 4$ which depend of "shape" of f near ξ. It is easy to see that if ξ is a point of global maximum (minimum) for f, $\beta = \beta_{\xi} > 4$, then $C_{\beta,\xi} < 0$ (respectively > 0).

Remark 8.2. In fact, if $Crit(f) \subset B_R(0)$ for some $R > 0$ and for some ε suitably small we have $\max_{x_1,x_2 \in B_R(0)} |f(x_1) - f(x_2)| < \varepsilon$ and $\min_{\xi \in Crit(f)} |f(\xi)| > \frac{1}{\varepsilon}$, then we can ensure that (ii) holds for all $\xi \in Crit(f)$ with $\beta = \beta_{\xi} > 4$ by letting f decay suitably outside the ball $B_R(0)$.

Remark 8.3. In the particular case, when $\beta = 2$, we obtain results similar to Wei and Xu [19,20].

Corollary 8.2. *Let us suppose that f is a $C^{2,\mu}_{loc}(\mathbb{R}^4)$ function satisfying:*

(i) *assumptions* (f1)–(f4) *at any $\xi \in Crit(f)$,*
(ii) *for any $\xi \in Crit(f)$, $\Delta f(\xi) \neq 0$, and*
(iii) $\sum_{\Delta f(\xi)<0} deg_{loc}(\nabla f, \xi) \neq 1$.

Then the vector field ∇J has a stable zero.

Now we state the existence result for the problem (1.6) in more concrete terms.

Theorem 8.1. *Let f satisfy the assumptions* (i)–(iii) *in Proposition* 8.1*. Fix a compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^4$ with a nonempty interior. Then there exists $\varepsilon_0 = \varepsilon_0(K) > 0$ such that* (1.6) *admits a solution u_ε for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, $u_\varepsilon = U_{\delta_\varepsilon, y_\varepsilon} + \phi_\varepsilon$ with $\phi_\varepsilon \to 0$ in $H_{\delta,y}$ and $(\delta_\varepsilon, y_\varepsilon) \to (\delta, y)$ as $\varepsilon \to 0$. Furthermore, local uniqueness and exact multiplicity results as in Theorems* 1.3, 1.4 *hold if (δ, y) is a stable zero of J such that the Hessian $D^2 J(\delta, y)$ is invertible and $\nabla f \in L^\infty(\mathbb{R}^N)$.*

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