ON SOME GENERALIZED AGEING ORDERINGS

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Abstract

Some partial orderings which compare probability distributions with the exponential distribution, are found to be very useful to understand the phenomenon of ageing. Here, we introduce some new generalized partial orderings which describe the same kind of characterization of some generalized ageing classes. We give some equivalent conditions for each of the orderings. Inter-relations among the generalized orderings have also been discussed.

Key words: Generalized ageing classes, Lorenz curve, partial ordering, TTT transform.

1 Introduction

Ageing and partial ordering are two very well known concepts in reliability theory. Positive ageing describes the situation where an older system has shorter remaining lifetime in some stochastic sense than a younger one. Many classes of lifetime distributions

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are characterized by their ageing properties. Exponential distribution is exceptional one which has no ageing property due to its memory less property. It will not be out of the way to mention here that there are some orderings for which Weibull distribution is the borderline distribution, namely, ageing intensity ordering, see, for example, Nanda et al. [16], Righter et al. [17]. Many different types of ageing notions have been studied in the literature, for instance, see Bryson and Siddiqui [5], Barlow and Proschan [4], Klefsjö [9], Deshpande et al. ([6], [7]), Loh [13], Lai and Xie [11] and the references there in. On the other hand, partial orderings are used to compare two different distributions. Shaked and Shanthikumar [19] is a very good reference for this purpose. It has been observed that among all the partial orderings, there are two special kinds of partial orderings which describe the phenomenon of ageing: Firstly, the partial orderings which compare probability distributions with the exponential distribution; secondly, those which compare residual lifetimes at different ages. In our paper we concentrate our discussion particularly on the first case. The significant works in the direction of our work have been developed by Kochar and Wiens [10], Sengupta and Deshpande [18] and many other researchers.

For an absolutely continuous nonnegative random variable X, the probability density function is denoted by $f_X(\cdot)$ and the distribution function by $F_X(\cdot)$. We write $\bar{F}_X(\cdot) \equiv 1 - F_X(\cdot)$ to denote the survival function of the random variable X. Let us write

$$\overline{T}_{X,0}(x) = f_X(x),$$

and

$$\overline{T}_{X,s}(x) = \frac{\int_x^\infty \overline{T}_{X,s-1}(t)dt}{\widetilde{\mu}_{X,s-1}}, \tag{1.1}$$

for $s = 1, 2, \ldots$, where

$$\widetilde{\mu}_{X,s} = \int_0^\infty \overline{T}_{X,s}(t)dt,$$

$s = 0, 1, 2, \ldots, \cdot$ We assume $\widetilde{\mu}_{X,s}$ to be finite. Note that $\overline{T}_{X,2}(\cdot)$ is the survival function of the equilibrium distribution of X, which plays an important role in ageing concepts (Deshpande et al. [6]), whereas $\overline{T}_{X,s}(\cdot)$ is the survival function of the equilibrium distribution of a distribution with survival function $\overline{T}_{X,s-1}(\cdot)$, $s = 1, 2, \ldots \cdot$ We further define, for $s = 1, 2, \ldots$,

$$\begin{aligned} r_{X,s}(x) &= \frac{\overline{T}_{X,s-1}(x)}{\int_x^\infty \overline{T}_{X,s-1}(t)dt} \\ &= \frac{\overline{T}_{X,s-1}(x)}{\widetilde{\mu}_{X,s-1}\overline{T}_{X,s}(x)}, \end{aligned}$$

and

$$\mu_{X,s}(x) = \frac{\int_x^\infty \overline{T}_{X,s}(t)dt}{\overline{T}_{X,s}(x)},$$

where $r_{X_s}(\cdot)$ and $\mu_{X,s}(\cdot)$, respectively, represent the failure rate and the mean residual life functions corresponding to $\overline{T}_{X,s}(\cdot)$. Note that, for $s = 1, 2, \ldots,$

$$\mu_{X,s}(0) = \widetilde{\mu}_{X,s},$$

and, for $s = 2, 3, \ldots,$

$$r_{X,s}(x) = \frac{1}{\mu_{X,s-1}(x)}. \tag{1.2}$$

Let $\mathcal{F}$ be the class of distribution functions $F : [0, \infty) \longrightarrow [0, 1]$ with $F(0) = 0$. We assume that all $F(\in \mathcal{F})$ have their finite generalized means $\widetilde{\mu}_{X,s}$, and are strictly increasing on their support. If F is not strictly increasing, we take the inverse as

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}.$$

Throughout the paper, increasing and decreasing properties are not used in strict sense. For any differentiable function $k(\cdot)$, we write $k'(t)$ to denote the first derivative of $k(t)$ with respect to t.

The scaled total time on test (TTT) transform is a very useful tool to analyze the statistical lifetime data. It was first introduced by Barlow and Campo [3]. To know more about TTT transform, readers may refer to Barlow [2] and the references there in. The TTT transform corresponding to $\overline{T}_{X,s}(\cdot)$ is denoted by $\mathcal{H}^{-1}_{X,s}(\cdot)$, and is defined as

$$\begin{aligned}\mathcal{H}^{-1}_{X,s}(u) &= \frac{1}{\widetilde{\mu}_{X,s}} \int_0^{T^{-1}_{X,s}(u)} \overline{T}_{X,s}(y)dy \\ &= T_{X,s+1}\left(T^{-1}_{X,s}(u)\right),\end{aligned}$$

for $u \in [0, 1]$ and $s = 1, 2, \ldots,$ where $\overline{T}_{X,s}(\cdot) \equiv 1 - T_{X,s}(\cdot)$. Define, for $s = 1, 2, \ldots,$

$$\begin{aligned}\mathcal{R}^{-1}_{X,s}(u) &= 1 - \mathcal{H}^{-1}_{X,s}(1-u) \\ &= \overline{T}_{X,s+1}\left(\overline{T}^{-1}_{X,s}(u)\right).\end{aligned} \tag{1.3}$$

Note that, for $s = 1, 2, \ldots,$

$$\begin{aligned}\mathcal{R}_{X,s}(u) &= 1 - \mathcal{H}_{X,s}(1-u) \\ &= \overline{T}_{X,s}\left(\overline{T}^{-1}_{X,s+1}(u)\right).\end{aligned} \tag{1.4}$$

The Lorenz curve introduced by Lorenz [14], is basically used to understand the concept of income inequalities in Economics. A brief discussion about Lorenz curve may be found in Aaberge [1]. The Lorenz curve of $\overline{T}_{X,s}(\cdot)$, denoted by $\mathcal{L}_{X,s}(\cdot)$, is defined as

$$\mathcal{L}_{X,s}(u) = \frac{1}{\widetilde{\mu}_{X,s}} \int_0^u T_{X,s}^{-1}(y)dy \text{ for } u \in [0,1] \text{ and } s = 1, 2 \ldots.$$

In the literature, the partial orderings with respect to different ageing properties, namely, IFR (Increasing in Failure Rate), IFRA (Increasing in Failure Rate Average), NBU (New Better than Used), DMRL (Decreasing in Mean Residual Life), NBUE (New Better than Used in Expectation) and HNBUE (Harmonically New Better than Used in Expectation) have been defined and discussed in Bryson and Siddiqui [5], Barlow and Proschan [4], Klefsjö [9] and others.

For the sake of completeness, we reproduce the following definitions of generalized ageing classes from Fagiuoli and Pellerey [8].

Definition 1.1 *For $s = 1, 2, \ldots$, X is said to be*

(i) s-IFR if $r_{X,s}(x)$ is increasing in $x \geq 0$;

(ii) s-IFRA if $\frac{1}{x}\int_0^x r_{X,s}(t)dt$ is increasing in $x > 0$;

(iii) s-NBU if $\overline{T}_{X,s}(x+t) \leq \overline{T}_{X,s}(x).\overline{T}_{X,s}(t)$ for all $x, t \geq 0$;

(iv) s-NBUFR if $r_{X,s}(0) \leq r_{X,s}(x)$ for all $x \geq 0$;

(v) s-NBAFR if $r_{X,s}(0) \leq \frac{1}{x}\int_0^x r_{X,s}(x)$ for all $x > 0$. □

One can easily verify that each of the following equivalence relations holds:

1-IFR ⇔ IFR,	2-IFR ⇔ DMRL,	3-IFR ⇔ DVRL,
1-IFRA ⇔ IFRA,	2-IFRA ⇔ DMRLHA,	1-NBU ⇔ NBU,
1-NBUFR ⇔ NBUFR,	2-NBUFR ⇔ NBUE,	3-NBUFR ⇔ NDVRL,
1-NBAFR⇔NBAFR,	2-NBAFR⇔HNBUE.	

For the definitions of DVRL (Decreasing in Variance Residual Life) and NDVRL (Net DVRL) classes one may refer to Launer [12], DMRLHA (Decreasing Mean Residual Life in Harmonic Average) and NBUFR (New Better than Used in Failure Rate) classes are discussed in Deshpande et al. [6], whereas NBAFR (New Better Than Used in Failure Rate Average) is due to Loh [13].

A function $f(\cdot)$ is called star-shaped (resp. antistar-shaped) if $f(x)/x$ is increasing (resp. decreasing) in x. On the other hand, it is called super-additive (resp. sub-additive) if, for all x, y, $f(x+y) \geq (resp. \leq\ ) f(x) + f(y)$.

Let an absolutely continuous nonnegative random variable Y have the respective generalized functions (analogous to the one defined above for X) $\overline{T}_{Y,s}(\cdot)$, $\widetilde{\mu}_{Y,s}$, $r_{Y,s}(\cdot)$, $\mu_{Y,s}(\cdot)$, $\mathcal{H}_{Y,s}(\cdot)$, $\mathcal{R}_{Y,s}(\cdot)$ and $\mathcal{L}_{Y,s}(\cdot)$. For the sake of simplicity we write, for $x \geq 0$ and $s = 1, 2, \ldots,$

$$\alpha_s(x) = \overline{T}_{Y,s}^{-1}\left(\overline{T}_{X,s}(x)\right) = T_{Y,s}^{-1}\left(T_{X,s}(x)\right).$$

Here, we define and study some more general partial orderings using the generalized ageing properties. These extend the concepts of the generalized ageing, given in Definition 1.1, to compare the ageing properties of two life distributions. In Sections 2, 3, 4, 5 and 6, we discuss s-IFR, s-IFRA, s-NBU, s-NBUFR and s-NBAFR orderings, respectively. We give some equivalent representations for each ordering. We prove that these are all partial orderings. Inter-relations among these orderings are also discussed. We make a bridge by which one can go from these orderings to generalized ageings, and vice versa.

2 s-IFR Ordering

In this section we define s-IFR ordering and study different properties of this ordering.

Definition 2.1 *For any positive integer s, X (or its distribution F_X) is said to be more s-IFR than Y (or its distribution F_Y) (written as $F_X \leq_{s-IFR} F_Y$) if $\alpha_s(x)$ is convex.* □

Remark 2.1 *For $s = 1$, Definition 2.1 gives $F_X \leq_{IFR} F_Y$, for $s = 2$, $F_X \leq_{DMRL} F_Y$, and for $s = 3$, we get $F_X \leq_{DVRL} F_Y$.*

The following lemma may be obtained in Marshall and Olkin ([15], Section 21(f), pp. 699-700).

Lemma 2.1 *Let $f(\cdot)$ and $g(\cdot)$ be two real-valued continuous functions, and $\zeta(\cdot)$ be a strictly increasing (resp. decreasing) and continuous function defined on the range of f and g. Then, for any real number $c > 0$, $f(x) - cg(x)$ and $\zeta(f(x)) - \zeta(cg(x))$ have sign change property in the same (resp. reverse) order, as x traverses from left to right.* □

In the following two propositions, we give some equivalent representations of the s-IFR ordering. The proof of the first proposition can easily be done by using Lemma 2.1, or Proposition 2.C.8 of Marshall and Olkin [15].

Proposition 2.1 *Definition 2.1 can equivalently be written in one of the following forms:*

(i) *For any real numbers a and b, $\overline{T}^{-1}_{Y,s}\overline{T}_{X,s}(x) - (ax+b)$ changes sign at most twice, and if the change of signs occurs twice, they are in the order $+, -, +$, as x traverses from 0 to ∞.*

(ii) *For any real numbers a and b, $\overline{T}_{X,s}(x) - \overline{T}_{Y,s}(ax+b)$ changes sign at most twice, and if the change of signs occurs twice, they are in the order $-, +, -$, as x traverses from 0 to ∞.*

(iii) *For any real numbers a and b, $\overline{T}_{X,s}(ax+b) - \overline{T}_{Y,s}(x)$ changes sign at most twice, and if the change of signs occurs twice, they are in the order $-, +, -$, as x traverses from 0 to ∞.*

(iv) *For any real numbers a and b, $\overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax+b)$ changes sign at most twice, and if the change of signs occurs twice, they are in the order $+, -, +$, as x traverses from 0 to ∞.*

(v) *For any real numbers a and b, $\overline{T}^{-1}_{X,s}\overline{T}_{Y,s}(x) - (ax+b)$ changes sign at most twice, and if the change of signs occurs twice, they are in the order $-, +, -$, as x traverses from 0 to ∞.*

(vi) *$\alpha_s^{-1}(x)$ is concave in $x > 0$.* □

Proposition 2.2 *For $s = 2, 3, \ldots$, Definition 2.1 can equivalently be written in one of the following forms:*

(i) $\frac{r_{X,s}\left(T^{-1}_{X,s}(u)\right)}{r_{Y,s}\left(T^{-1}_{Y,s}(u)\right)}$ *is increasing in $u \in [0,1]$.*

(ii) $\frac{\mu_{X,s-1}\left(T^{-1}_{X,s}(u)\right)}{\mu_{Y,s-1}\left(T^{-1}_{Y,s}(u)\right)}$ *is decreasing in $u \in [0,1]$.*

(iii) $\frac{\overline{T}_{Y,s-1}(\alpha_s(x))}{\overline{T}_{Y,s-1}(\alpha_{s-1}(x))}$ *is decreasing in $x \geq 0$.*

(iv) *$\mathcal{R}_{X,s-1}\mathcal{R}^{-1}_{Y,s-1}(u)$ is antistar-shaped in $u \in [0,1]$.*

(v) $\frac{1-\mathcal{H}_{X,s-1}(u)}{1-\mathcal{H}_{Y,s-1}(u)}$ *is increasing in $u \in [0,1]$.*

(vi) $\frac{\mathcal{R}_{X,s-1}(u)}{\mathcal{R}_{Y,s-1}(u)}$ *is decreasing in $u \in [0,1]$.*

(vii) *$\mathcal{R}^{-1}_{Y,s}\mathcal{R}_{X,s}(u)$ is concave in $u \in [0,1]$.*

Proof: $F_X \leq_{s-IFR} F_Y$ is equivalent to the fact that

$$\alpha_s'(x) \text{ is increasing in } x \geq 0. \tag{2.5}$$

Note that, for $x \geq 0$,

$$\begin{aligned} \alpha_s'(x) &= \left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{Y,s-1}\overline{T}_{Y,s}^{-1}\left(\overline{T}_{X,s}(x)\right)}\right) \\ &= \left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{X,s}(x)}\right)\left(\frac{\overline{T}_{X,s}(x)}{\overline{T}_{Y,s-1}\overline{T}_{Y,s}^{-1}\left(\overline{T}_{X,s}(x)\right)}\right) \\ &= r_{X,s}(x)\left(\frac{\widetilde{\mu}_{Y,s-1}\overline{T}_{Y,s}\left(\overline{T}_{Y,s}^{-1}\overline{T}_{X,s}(x)\right)}{\overline{T}_{Y,s-1}\left(\overline{T}_{Y,s}^{-1}\overline{T}_{X,s}(x)\right)}\right) \\ &= \frac{r_{X,s}(x)}{r_{Y,s}\left(T_{Y,s}^{-1}T_{X,s}(x)\right)}, \end{aligned} \tag{2.6}$$

which can equivalently be written as

$$\alpha_s'\left(T_{X,s}^{-1}(u)\right) = \frac{r_{X,s}\left(T_{X,s}^{-1}(u)\right)}{r_{Y,s}\left(T_{Y,s}^{-1}(u)\right)} \quad \text{for all } u \in [0,1]. \tag{2.7}$$

Thus, the result follows from (2.5). This proves (i). Equivalence of (i) and (ii) follows by using (1.2) in (2.7). By noting the fact that

$$\frac{\overline{T}_{Y,s-1}\left(\alpha_s(x)\right)}{\overline{T}_{Y,s-1}\left(\alpha_{s-1}(x)\right)} = \left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}\right)\frac{1}{\alpha_s'(x)},$$

the equivalence of (i) and (iii) follows from (2.5). Note that

$$\mathcal{R}_{X,s-1}\mathcal{R}_{Y,s-1}^{-1}(u) \text{ is antistar-shaped in } u \in [0,1],$$

if, and only if,

$$\frac{\overline{T}_{X,s-1}\overline{T}_{X,s}^{-1}\overline{T}_{Y,s}\overline{T}_{Y,s-1}^{-1}(u)}{u} \text{ is decreasing in } u \in [0,1],$$

or equivalently,

$$\frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{Y,s-1}\overline{T}_{Y,s}^{-1}\left(\overline{T}_{X,s}(x)\right)} \text{ is increasing in } x \geq 0.$$

Thus, the equivalence of (i) and (iv) follows from (2.5) and (2.6). The equivalence of (i) and (v) follows from (2.6) and using the fact that

$$\frac{1-\mathcal{H}_{X,s-1}(u)}{1-\mathcal{H}_{Y,s-1}(u)} = \frac{\overline{T}_{X,s-1}\left(T_{X,s}^{-1}(u)\right)}{\overline{T}_{Y,s-1}\left(T_{Y,s}^{-1}(u)\right)}.$$

Equivalence of (v) and (vi) follows from (1.4). We write $\Upsilon_{X,s}(u) = \mathcal{R}'_{X,s}(u)$ and $\Upsilon_{Y,s}(u) = \mathcal{R}'_{Y,s}(u)$ for $u \in [0,1]$. Then, we have, for all $u \in [0,1]$,

$$\Upsilon_{X,s}(u) = \left(\frac{\widetilde{\mu}_{X,s}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{X,s-1}\left(\overline{T}^{-1}_{X,s+1}(u)\right)}{\overline{T}_{X,s}\left(\overline{T}^{-1}_{X,s+1}(u)\right)}\right),$$

which gives

$$\Upsilon_{X,s}\left(\mathcal{R}^{-1}_{X,s}(u)\right) = \left(\frac{\widetilde{\mu}_{X,s}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{X,s-1}\left(\overline{T}^{-1}_{X,s}(u)\right)}{u}\right).$$

So, on using (2.6) we have, for all $u \in [0,1]$,

$$\frac{\Upsilon_{X,s}\left(\mathcal{R}^{-1}_{X,s}(u)\right)}{\Upsilon_{Y,s}\left(\mathcal{R}^{-1}_{Y,s}(u)\right)} = \left(\frac{\widetilde{\mu}_{X,s}}{\widetilde{\mu}_{Y,s}}\right)\alpha'_s\left(\overline{T}^{-1}_{X,s}(u)\right). \tag{2.8}$$

Thus, (2.5) can equivalently be written as

$$\frac{\Upsilon_{X,s}\left(\mathcal{R}^{-1}_{X,s}(u)\right)}{\Upsilon_{Y,s}\left(\mathcal{R}^{-1}_{Y,s}(u)\right)} \text{ is decreasing in } u \in [0,1],$$

or equivalently,

$$\frac{\Upsilon_{X,s}(u)}{\Upsilon_{Y,s}\left(\mathcal{R}^{-1}_{Y,s}\mathcal{R}_{X,s}(u)\right)} \text{ is decreasing in } u \in [0,1].$$

This means that

$$\frac{d}{du}\left(\mathcal{R}^{-1}_{Y,s}\mathcal{R}_{X,s}(u)\right) \text{ is decreasing in } u \in [0,1],$$

or equivalently,

$$\mathcal{R}^{-1}_{Y,s}\mathcal{R}_{X,s}(u) \text{ is concave in } u \in [0,1].$$

This gives the equivalence of (i) and (vii). □

Remark 2.2 *For $s=1$, Definition 2.1 can equivalently be written in one of the following forms:*

(i) $\frac{r_{X,1}\left(F_X^{-1}(u)\right)}{r_{Y,1}\left(F_Y^{-1}(u)\right)}$ *is increasing in $u \in [0,1]$.*

(ii) $\mathcal{R}^{-1}_{Y,1}\mathcal{R}_{X,1}(u)$ *is concave in $u \in [0,1]$.* □

Definition 2.2 *Two distribution functions $F_X, F_Y (\in \mathcal{F})$ are said to be equivalent ($F_X \sim F_Y$) if there exists a $\theta > 0$ such that $F_X(x) = F_Y(\theta x)$ for all $x \geq 0$.* □

Following are a few lemmas to be used in proving the upcoming theorems.

Lemma 2.2 *If $F_X \sim F_Y$, then $\overline{T}_{X,s}(x) = \overline{T}_{Y,s}(\theta x)$ for some $\theta > 0$ and all $x \geq 0$, and $s = 1, 2, \ldots$.*

Proof: $F_X \sim F_Y$ if, and only if $\overline{F}_X(x) = \overline{F}_Y(\theta x)$ for some $\theta > 0$ and for all $x \geq 0$. Thus the result is true for $s = 1$. Suppose the result holds for s. Then

$$\overline{T}_{X,s+1}(x) = \frac{1}{\widetilde{\mu}_{X,s}} \int_x^\infty \overline{T}_{X,s}(u)du.$$

Further

$$\begin{aligned}\widetilde{\mu}_{X,s} &= \int_0^\infty \overline{T}_{X,s}(u)du \\ &= \int_0^\infty \overline{T}_{Y,s}(\theta u)du \\ &= \frac{\widetilde{\mu}_{Y,s}}{\theta}.\end{aligned}$$

The second equality follows from the hypothesis. Hence

$$\begin{aligned}\overline{T}_{X,s+1}(x) &= \frac{\theta}{\widetilde{\mu}_{Y,s}} \int_x^\infty \overline{T}_{Y,s}(\theta u)du \\ &= \overline{T}_{Y,s+1}(\theta x).\end{aligned}$$

Hence, by induction, the result is established. □

Following lemma follows from the definition of $r_{X,s}(\cdot)$ and Lemma 2.2.

Lemma 2.3 *If $F_X \sim F_Y$, then there exists a $\theta > 0$ such that, for all $x \geq 0$ and $s = 1, 2, \ldots$,*

$$r_{X,s}(x) = \theta r_{Y,s}(\theta x).$$

□

The following lemma gives the converse of Lemma 2.2.

Lemma 2.4 *If $\overline{T}_{X,s}(x) = \overline{T}_{Y,s}(\theta x)$ for some $\theta > 0$, some $s = 1, 2, \ldots$, and all $x \geq 0$, then $\overline{F}_X(x) = \overline{F}_Y(\theta x)$ for all x.*

Proof: Let us fix $s \geq 2$ because for $s = 1$, it is trivial. Then $\overline{T}_{X,s}(x) = \overline{T}_{Y,s}(\theta x)$ for all $x \geq 0$ gives, by (1.1),

$$\frac{\int_x^\infty \overline{T}_{X,s-1}(u)du}{\widetilde{\mu}_{X,s-1}} = \frac{\int_{\theta x}^\infty \overline{T}_{Y,s-1}(u)du}{\widetilde{\mu}_{Y,s-1}} \quad \text{for all } x \geq 0.$$

Taking derivative with respect to x on both sides of the above expression, we get, for all $x \geq 0$,

$$\frac{\overline{T}_{X,s-1}(x)}{\widetilde{\mu}_{X,s-1}} = \theta \frac{\overline{T}_{Y,s-1}(\theta x)}{\widetilde{\mu}_{Y,s-1}} \quad \text{for all } x \geq 0. \tag{2.9}$$

Putting $x = 0$ in (2.9), we get $\widetilde{\mu}_{Y,s-1}/\widetilde{\mu}_{X,s-1} = \theta$. Hence (2.9) becomes

$$\overline{T}_{X,s-1}(x) = \overline{T}_{Y,s-1}(\theta x) \quad \text{for all } x \geq 0.$$

Proceeding in this line, we get $\overline{F}_X(x) = \overline{F}_Y(\theta x)$ for all $x \geq 0$. □

The following two lemmas are easy to prove.

Lemma 2.5 *Let $f(\cdot)$ and $g(\cdot)$ be two nonnegative, increasing, and convex functions. Then $f(g(\cdot))$ is convex.* □

Lemma 2.6 *Let $f(\cdot)$ be a nonnegative, increasing and convex function. Then $f^{-1}(\cdot)$ is concave.* □

The following theorem shows that s-IFR ordering is a partial ordering.

Theorem 2.1 *The relationship $F_X \leq_{s-IFR} F_Y$ is a partial ordering of the equivalence classes of $\mathcal{F}$.*

Proof: (i) That s-IFR ordering is reflexive, is trivial.
(ii) $F_X \leq_{s-IFR} F_Y$ gives that $T^{-1}_{Y,s}(T_{X,s}(x))$ is convex, which, by Lemma 2.6, reduces to the fact that

$$T^{-1}_{X,s}(T_{Y,s}(x)) \text{ is concave.} \tag{2.10}$$

Further, $F_Y \leq_{s-IFR} F_X$ gives that

$$T^{-1}_{X,s}(T_{Y,s}(x)) \text{ is convex.} \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$T^{-1}_{X,s}(T_{Y,s}(x)) = \alpha + \beta x,$$

for some constants α and β. Now, by evaluating the above expression at $x = 0$, we get $\alpha = 0$. Hence, we have

$$T^{-1}_{X,s}(T_{Y,s}(x)) = \beta x,$$

which, by Lemma 2.4, gives $F_X \sim F_Y$.
(iii) On using Lemma 2.5, one can easily see that s-IFR ordering is transitive. □

The following lemma can be easily verified.

Lemma 2.7 *Let $X \sim \overline{F}_X(x) = e^{-\lambda x}$. Then, for $s = 1, 2, \ldots,$*

(i) $r_{X,s}(x) = \lambda$;

(ii) $\overline{T}_{X,s}(x) = e^{-\lambda x}$. $\square$

The following theorem shows that a random variable X is smaller than exponential distribution in s-IFR ordering if, and only if, X has s-IFR distribution.

Theorem 2.2 *If $\overline{F}_Y(x) = \exp(-\lambda x)$, then $F_X \leq_{s-IFR} F_Y$ if, and only if, F_X is s-IFR.*

Proof: By Lemma 2.7, $F_X \leq_{s-IFR} F_Y$ is equivalent to saying that

$$\ln\left(\overline{T}_{X,s}(x)\right) \text{ is concave,}$$

or equivalently,

$$r_{X,s}(x) \text{ is increasing in } x \geq 0,$$

giving that X is s-IFR. $\square$

3 s-IFRA Ordering

We start this section with the following definition.

Definition 3.1 *For any positive integer s, X (or its distribution F_X) is said to be more s-IFRA than Y (or its distribution F_Y) (written as $F_X \leq_{s-IFRA} F_Y$) if $\alpha_s(x)$ is star-shaped.*

$\square$

Remark 3.1 *For $s = 1$, the above definition gives $F_X \leq_{IFRA} F_Y$.*

Below we give some equivalent representations of s-IFRA ordering. The first proposition can easily be proved by using Lemma 2.1.

Proposition 3.1 *Definition 3.1 can equivalently be written in one of the following forms:*

(i) *For any real number a, $\overline{T}_{Y,s}^{-1}\overline{T}_{X,s}(x) - ax$ changes sign at most once, and if the change of sign does occur, it is in the order $-, +$, as x traverses from 0 to ∞.*

(ii) *For any real number a, $\overline{T}_{X,s}(x) - \overline{T}_{Y,s}(ax)$ changes sign at most once, and if the change of sign does occur, it is in the order $+, -$, as x traverses from 0 to ∞.*

(iii) *For any real number a, $\overline{T}_{X,s}(ax) - \overline{T}_{Y,s}(x)$ changes sign at most once, and if the change of sign does occur, it is in the order $+, -$, as x traverses from 0 to ∞.*

(iv) *For any real number a, $\overline{T}_{Y,s}(x) - \overline{T}_{X,s}(ax)$ changes sign at most once, and if the change of sign does occur, it is in the order $-, +$, as x traverses from 0 to ∞.*

(v) *For any real number* a, $\overline{T}^{-1}_{X,s}\overline{T}_{Y,s}(x) - ax$ *changes sign at most once, and if the change of sign does occur, it is in the order* $+, -$, *as* x *traverses from* 0 *to* ∞.

(vi) $\alpha_s^{-1}(x)$ *is antistar-shaped in* $x > 0$. $\square$

Proposition 3.2 *For* $s = 2, 3, \ldots,$ *Definition 3.1 can equivalently be written in one of the following forms:*

(i) $\frac{\overline{T}^{-1}_{Y,s}(u)}{\overline{T}^{-1}_{X,s}(u)}$ *is decreasing in* $u \in [0, 1]$.

(ii) $\frac{r_{X,s}\left(T^{-1}_{X,s}(u)\right)}{r_{Y,s}\left(T^{-1}_{Y,s}(u)\right)} \geq \frac{T^{-1}_{Y,s}(u)}{T^{-1}_{X,s}(u)}$ *for all* $u \in [0, 1]$.

(iii) $\frac{\mu_{Y,s-1}\left(T^{-1}_{Y,s}(u)\right)}{\mu_{X,s-1}\left(T^{-1}_{X,s}(u)\right)} \geq \frac{T^{-1}_{Y,s}(u)}{T^{-1}_{X,s}(u)}$ *for all* $u \in [0, 1]$.

(iv) $\frac{\overline{T}_{Y,s-1}(\alpha_{s-1}(x))}{\overline{T}_{Y,s-1}(\alpha_s(x))} \geq \left(\frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{\alpha_s(x)}{x}\right)$ *for all* $x \geq 0$.

(v) $\frac{\mathcal{R}_{X,s-1}(u)}{\mathcal{R}_{Y,s-1}(u)} \geq \left(\frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{\overline{T}^{-1}_{Y,s}(u)}{\overline{T}^{-1}_{X,s}(u)}\right)$ *for all* $u \in [0, 1]$.

(vi) $\frac{1-\mathcal{H}_{X,s-1}(u)}{1-\mathcal{H}_{Y,s-1}(u)} \geq \left(\frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{T^{-1}_{Y,s}(u)}{T^{-1}_{X,s}(u)}\right)$ *for all* $u \in [0, 1]$.

Proof: The proof of (i) follows from definition. Again, (i) can equivalently be written as

$$\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{Y,s-1}\overline{T}^{-1}_{Y,s}\left(\overline{T}_{X,s}(x)\right)}\right) \geq \frac{\overline{T}^{-1}_{Y,s}\overline{T}_{X,s}(x)}{x}. \tag{3.12}$$

The above inequality holds if, and only if, for all $x \geq 0$,

$$\frac{r_{X,s}\left(T^{-1}_{X,s}(u)\right)}{r_{Y,s}\left(T^{-1}_{Y,s}(u)\right)} \geq \frac{T^{-1}_{Y,s}(u)}{T^{-1}_{X,s}(u)} \text{ for all } u \in [0, 1],$$

which is (ii). Equivalence of (ii) and (iii) follows from (1.2). Note that

$$\begin{aligned}\frac{\overline{T}_{Y,s-1}\left(\alpha_{s-1}(x)\right)}{\overline{T}_{Y,s-1}\left(\alpha_s(x)\right)} &= \frac{\overline{T}_{X,s-1}(x)}{\overline{T}_{Y,s-1}\overline{T}^{-1}_{Y,s}\left(\overline{T}_{X,s}(x)\right)} \\ &\geq \left(\frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{\alpha_s(x)}{x}\right),\end{aligned}$$

where the inequality follows from (3.12). This gives the equivalence of (i) and (iv). On using (3.12), the equivalence of (iv) and (v) follows. For $u \in [0,1]$,

$$\begin{aligned}\frac{1-\mathcal{H}_{X,s-1}(u)}{1-\mathcal{H}_{Y,s-1}(u)} &= \frac{\overline{T}_{X,s-1}\left(T_{X,s}^{-1}(u)\right)}{\overline{T}_{Y,s-1}\left(T_{Y,s}^{-1}(u)\right)} \\ &\geq \left(\frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{T_{Y,s}^{-1}(u)}{T_{X,s}^{-1}(u)}\right),\end{aligned}$$

where the inequality follows from (3.12). Hence, the equivalence of (i) and (vi) follows. □

Remark 3.2 *For $s=1$, Definition 3.1 can equivalently be written in one of the following forms:*

(i) $\frac{\bar{F}_Y^{-1}(u)}{\bar{F}_X^{-1}(u)}$ *is decreasing in $u \in [0,1]$.*

(ii) $\frac{r_{X,1}\left(F_X^{-1}(u)\right)}{r_{Y,1}\left(F_Y^{-1}(u)\right)} \geq \frac{F_Y^{-1}(u)}{F_X^{-1}(u)}$ *for all $u \in [0,1]$.* □

The following theorem gives some equivalent characterization of s-IFRA ordering.

Theorem 3.1 *The following statements are equivalent:*

(i) $F_X \leq_{s-IFRA} F_Y$.

(ii) *For all functions $\alpha(\cdot)$ and $\beta(\cdot)$, such that $\alpha(\cdot)$ is nonnegative and $\alpha(\cdot)$ and $\alpha(\cdot)/\beta(\cdot)$ are decreasing, and such that $\int_0^1 \alpha(u)dT_{X,s}^{-1}(u) < \infty$, and $\int_0^1 \alpha(u)dT_{Y,s}^{-1}(u) < \infty$, $0 \neq \int_0^1 \beta(u)dT_{X,s}^{-1}(u) < \infty$, and $0 \neq \int_0^1 \beta(u)dT_{Y,s}^{-1}(u) < \infty$, we have*

$$\frac{\int_0^1 \alpha(u)dT_{Y,s}^{-1}(u)}{\int_0^1 \alpha(u)dT_{Y,s}^{-1}(u)} \leq \frac{\int_0^1 \beta(u)dT_{X,s}^{-1}(u)}{\int_0^1 \beta(u)dT_{X,s}^{-1}(u)}.$$

(iii) *For any increasing functions $a(\cdot)$ and $b(\cdot)$ such that $b(\cdot)$ is nonnegative, if $\int_0^1 a(u)b(u)dT_{X,s}^{-1}(u) = 0$, then $\int_0^1 a(u)b(u)dT_{Y,s}^{-1}(u) \leq 0$.*

Proof: The proof follows from Theorem 4.B.10 of Shaked and Shanthikumar [19] by noting the fact that $T_{X,s}$ and $T_{Y,s}$ are playing the role of F and G, respectively. □

Below we give two lemmas to be used in the upcoming theorem. The proofs are omitted.

Lemma 3.1 *Let $f(\cdot)$ and $g(\cdot)$ be two nonnegative, increasing, and star-shaped functions. Then $f\left(g(\cdot)\right)$ is star-shaped.* $\square$

Lemma 3.2 *Let $f(\cdot)$ be a nonnegative, increasing, and star-shaped function. Then $f^{-1}(\cdot)$ is antistar-shaped.* $\square$

Below we show that s-IFRA ordering is a partial ordering.

Theorem 3.2 *The relationship $F_X \leq_{s-IFRA} F_Y$ is a partial ordering of the equivalence classes of $\mathcal{F}$.*

Proof: (i) It is trivial to show that s-IFRA ordering is reflexive.
(ii) $F_X \leq_{s-IFRA} F_Y$ gives that $T_{Y,s}^{-1}(T_{X,s}(x))$ is star-shaped, which, by Lemma 3.2, reduces to the fact that

$$T_{X,s}^{-1}(T_{Y,s}(x)) \text{ is antistar-shaped.} \tag{3.13}$$

Further, $F_Y \leq_{s-IFRA} F_X$ gives that

$$T_{X,s}^{-1}(T_{Y,s}(x)) \text{ is star-shaped.} \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$T_{X,s}^{-1}(T_{Y,s}(x)) = \theta x,$$

for some constant θ. This, by Lemma 2.4, gives $F_X \sim F_Y$.
(iii) By Lemms 3.1, we have that the s-IFRA ordering is transitive. $\square$

The following theorem is a bridge between s-IFRA ordering and s-IFRA ageing.

Theorem 3.3 *If $\overline{F}_Y(x) = e^{-\lambda x}$, $\lambda > 0$, then*

$$F_X \leq_{IFRA} F_Y \textit{ if, and only if, } F_X \textit{ is s-IFRA.}$$

Proof: The proof follows from Definition 3.1 and Lemma 2.7. $\square$

Since every convex function is star-shaped, we have the following theorem.

Theorem 3.4 *If $F_X \leq_{s-IFR} F_Y$, then $F_X \leq_{s-IFRA} F_Y$.* $\square$

4 s-NBU Ordering

In this section we study s-NBU ordering.

Definition 4.1 *For any positive integer s, X (or its distribution F_X) is said to be more s-NBU than Y (or its distribution F_Y) (written as $F_X \leq_{s-NBU} F_Y$) if $\alpha_s(x)$ is super-additive.*

$\square$

Remark 4.1 *For $s = 1$, the above definition gives $F_X \leq_{NBU} F_Y$.*

Proposition 4.1 *Definition 4.1 can equivalently be written as*

$$\overline{T}_{X,s}\left(\overline{T}^{-1}_{X,s}(u) + \overline{T}^{-1}_{X,s}(v)\right) \leq \overline{T}_{Y,s}\left(\overline{T}^{-1}_{Y,s}(u) + \overline{T}^{-1}_{Y,s}(v)\right) \text{ for all } u, v \in [0,1].$$

Proof: $F_X \leq_{s-NBU} F_Y$ holds if, and only if, for all $x, y \geq 0$,

$$\overline{T}^{-1}_{Y,s}\left(\overline{T}_{X,s}(x+y)\right) \geq \overline{T}^{-1}_{Y,s}\left(\overline{T}_{X,s}(x)\right) + \overline{T}^{-1}_{Y,s}\left(\overline{T}_{X,s}(y)\right).$$

Writing $x = \overline{T}^{-1}_{X,s}(u)$ and $y = \overline{T}^{-1}_{X,s}(v)$ in the above inequality, we get the required result. $\square$

To prove the next theorem we use two lemmas which are given below. The proofs are omitted.

Lemma 4.1 *Let $f(\cdot)$ and $g(\cdot)$ be two nonnegative, increasing, and super-additive functions. Then $f(g(\cdot))$ is super-additive.* $\square$

Lemma 4.2 *Let $f(\cdot)$ be a nonnegative, increasing, and super-additive function. Then $f^{-1}(\cdot)$ is sub-additive.* $\square$

The following theorem shows that s-NBU ordering is a partial ordering.

Theorem 4.1 *The relationship $F_X \leq_{s-NBU} F_Y$ is a partial ordering of the equivalence classes of $\mathcal{F}$.*

Proof: (i) The proof of reflexive property of s-NBU ordering is trivial.
(ii) Let $F_X \leq_{s-NBU} F_Y$. Then

$$T^{-1}_{Y,s}\left(T_{X,s}(x)\right) \text{ is super-additive.}$$

By Lemma 4.2, the above statement can equivalently be written as

$$T^{-1}_{X,s}\left(T_{Y,s}(x)\right) \text{ is sub-additive.} \tag{4.15}$$

Further, $F_Y \leq_{s-NBU} F_X$ gives that

$$T_{X,s}^{-1}\left(T_{Y,s}(x)\right) \text{ is super-additive.} \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$T_{X,s}^{-1}\left(T_{Y,s}(x)\right) = \beta x,$$

for some constant β, which, by Lemma 2.4, gives $F_X \sim F_Y$.
(iii) On using Lemma 4.1, one can easily verify that s-NBU ordering is transitive. $\square$

Below Theorem 4.2 shows that, if a probability distribution is smaller than exponential distribution in s-NBU ordering, then it is actually an s-NBU distribution. The proof follows from Lemma 2.7.

Theorem 4.2 *Let $\overline{F}_Y(x) = e^{-\lambda x}$, $\lambda > 0$. Then, for $s = 1, 2, \ldots,$*

$$F_X \leq_{s-NBU} F_Y \textit{ if, and only if, } F_X \textit{ is s-NBU.}$$

$\square$

Since, all star-shaped functions are super-additive, we have the following theorem.

Theorem 4.3 *If $F_X \leq_{s-IFRA} F_Y$, then $F_X \leq_{s-NBU} F_Y$.* $\square$

5 s-NBUFR Ordering

We begin this section with the following definition.

Definition 5.1 *For any positive integer s, X (or its distribution F_X) is said to be more s-NBUFR than Y (or its distribution F_Y) (written as $F_X \leq_{s-NBUFR} F_Y$) if $\alpha_s'(x) \geq \alpha_s'(0)$.* $\square$

Remark 5.1 *For $s = 1$, $s = 2$ and $s = 3$, the above definition gives $F_X \leq_{NBUFR} F_Y$, $F_X \leq_{NBUE} F_Y$ and $F_X \leq_{NDVRL} F_Y$, respectively.*

In the following proposition we discuss some equivalent conditions of the s-NBUFR ordering.

Proposition 5.1 *For $s = 2, 3, \ldots,$ Definition 5.1 can equivalently be written in one of the following forms:*

(i) $\alpha_s(x) \geq \alpha_{s-1}(x)$ *for all $x \geq 0$.*

(ii) $\frac{r_{X,s}\left(T_{X,s}^{-1}(u)\right)}{r_{Y,s}\left(T_{Y,s}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}$ *for all* $u \in [0,1]$.

(iii) $\frac{r_{X,s}\left(T_{X,s-1}^{-1}(u)\right)}{r_{Y,s}\left(T_{Y,s-1}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}$ *for all* $u \in [0,1]$.

(iv) $\frac{\mu_{Y,s-1}\left(T_{Y,s}^{-1}(u)\right)}{\mu_{X,s-1}\left(T_{X,s}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}$ *for all* $u \in [0,1]$.

(v) $\frac{\mu_{Y,s-1}\left(T_{Y,s-1}^{-1}(u)\right)}{\mu_{X,s-1}\left(T_{X,s-1}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}$ *for all* $u \in [0,1]$.

(vi) $\frac{\overline{T}_{Y,s-1}(\alpha_s(x))}{\overline{T}_{Y,s-1}(\alpha_{s-1}(x))} \leq 1$ *for all* $x \geq 0$.

(vii) $\mathcal{R}_{X,s-1}(u) \geq \mathcal{R}_{Y,s-1}(u)$ *for all* $u \in [0,1]$.

(viii) $\mathcal{H}_{X,s-1}(u) \leq \mathcal{H}_{Y,s-1}(u)$ *for all* $u \in [0,1]$.

Proof: $F_X \leq_{NBUFR} F_Y$ if, and only if, for all $x \geq 0$,

$$\overline{T}_{X,s-1}(x) \geq \overline{T}_{Y,s-1}\overline{T}_{Y,s}^{-1}\left(\overline{T}_{X,s}(x)\right), \tag{5.17}$$

or equivalently,

$$\alpha_s(x) \geq \alpha_{s-1}(x),$$

which is (i). Note that, for all $x \geq 0$, (5.17) can equivalently be written as

$$\frac{r_{X,s}(x)}{r_{Y,s}\left(\overline{T}_{Y,s}^{-1}\overline{T}_{X,s}(x)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}},$$

or equivalently,

$$\frac{r_{X,s}\left(T_{X,s}^{-1}(u)\right)}{r_{Y,s}\left(T_{Y,s}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} \quad \text{for all } u \in [0,1].$$

This proves the equivalence of (i) and (ii). Now, for all $u \in [0,1]$,

$$\frac{r_{X,s}\left(T_{X,s-1}^{-1}(u)\right)}{r_{Y,s}\left(T_{Y,s-1}^{-1}(u)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}$$

holds if, and only if,

$$\frac{r_{X,s}(x)}{r_{Y,s}\left(\overline{T}_{Y,s-1}^{-1}\overline{T}_{X,s-1}(x)\right)} \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} \quad \text{for all } x \geq 0.$$

The above inequality can equivalently be written as

$$\overline{T}_{Y,s}\left(\overline{T}_{Y,s-1}^{-1}\overline{T}_{X,s-1}(x)\right) \geq \overline{T}_{X,s}(x),$$

or equivalently,

$$\alpha_s(x) \geq \alpha_{s-1}(x) \quad \text{for all } x \geq 0,$$

giving the equivalence of (i) and (iii). On using (1.2) in (ii) and (iii), we get (iv) and (v), respectively. Equivalence of (i), and (vi) and (vii) follows from (5.17). Equivalence of (vii) and $(viii)$ follows from (1.4). $\square$

The following theorem shows that s-NBUFR ordering is a partial ordering.

Theorem 5.1 *The relationship $F_X \leq_{s-NBUFR} F_Y$ is a partial ordering of the equivalence classes of $\mathcal{F}$.*

Proof: For $s = 1$, the result follows from Kochar and Wiens [10]. We only prove the result for $s = 2, 3, \ldots$ · Let us fix s.
(i) It is easy to verify that s-NBUFR ordering is reflexive.
(ii) By Proposition 5.1(i), $F_X \leq_{s-NBUFR} F_Y$ holds if, and only if,

$$\overline{T}^{-1}_{Y,s-1}(\overline{T}_{X,s-1}(x)) \leq \overline{T}^{-1}_{Y,s}(\overline{T}_{X,s}(x)). \tag{5.18}$$

Further, $F_Y \leq_{s-NBUFR} F_X$ gives

$$\overline{T}^{-1}_{X,s-1}(\overline{T}_{Y,s-1}(x)) \leq \overline{T}^{-1}_{X,s}(\overline{T}_{Y,s}(x)). \tag{5.19}$$

Replacing x by $\overline{T}^{-1}_{Y,s}(\overline{T}_{X,s}(x))$ in (5.19), we have

$$\overline{T}^{-1}_{Y,s}(\overline{T}_{X,s}(x)) \leq \overline{T}^{-1}_{Y,s-1}(\overline{T}_{X,s-1}(x)). \tag{5.20}$$

Combining (5.18) and (5.20), we get

$$\alpha_s(x) = \alpha_{s-1} \text{ for all } x \geq 0. \tag{5.21}$$

Note that

$$\begin{aligned} \alpha'_s(x) &= \left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}\right)\left(\frac{\overline{T}_{Y,s-1}\left(\alpha_{s-1}(x)\right)}{\overline{T}_{Y,s-1}\left(\alpha_s(x)\right)}\right) \\ &= \theta, \end{aligned} \tag{5.22}$$

where the last equality follows from (5.21) and $\theta = \widetilde{\mu}_{Y,s-1}/\widetilde{\mu}_{X,s-1}$ (constant). Now, integrating (5.22) from 0 to x, and then using $\alpha_s(0) = 0$, we have $\overline{T}_{X,s}(x) = \overline{T}_{Y,s}(\theta x)$. Thus, on using Lemma 2.4, we have $F_X \sim F_Y$.
(iii) $F_X \leq_{s-NBUFR} F_Y$ gives

$$\overline{T}^{-1}_{Y,s-1}(\overline{T}_{X,s-1}(x)) \leq \overline{T}^{-1}_{Y,s}(\overline{T}_{X,s}(x)) \tag{5.23}$$

and $F_Y \leq_{s-NBUFR} F_Z$ gives

$$\overline{T}^{-1}_{Z,s-1}(\overline{T}_{Y,s-1}(x)) \leq \overline{T}^{-1}_{Z,s}(\overline{T}_{Y,s}(x)). \tag{5.24}$$

Now,

$$\begin{aligned}
\overline{T}^{-1}_{Z,s-1}(\overline{T}_{X,s-1}(x)) &= \overline{T}^{-1}_{Z,s-1}\overline{T}_{Y,s-1}\left(\overline{T}^{-1}_{Y,s-1}\overline{T}_{X,s-1}(x)\right) \\
&\leq \overline{T}^{-1}_{Z,s-1}\overline{T}_{Y,s-1}\left(\overline{T}^{-1}_{Y,s}\overline{T}_{X,s}(x)\right) \\
&\leq \overline{T}^{-1}_{Z,s}\overline{T}_{Y,s}\left(\overline{T}^{-1}_{Y,s}\overline{T}_{X,s}(x)\right) \\
&= \overline{T}^{-1}_{Z,s}(\overline{T}_{X,s}(x)),
\end{aligned}$$

where the first inequality follows from (5.23) and using the fact that $\overline{T}^{-1}_{Z,s-1}\overline{T}_{Y,s-1}(\cdot)$ is an increasing function. The second inequality holds from (5.24). Thus, s-NBUFR ordering is transitive. $\square$

The following theorem shows that a random variable X is s-NBUFR if, and only if, X is smaller than exponential distribution in s-NBUFR ordering. The proof follows from Lemma 2.7.

Theorem 5.2 *If $\overline{F}_Y(x) = e^{-\lambda x},\ \lambda > 0$, then $F_X \leq_{s-NBUFR} F_Y$ if, and only if, F_X is s-NBUFR.* $\square$

In the following theorem, we prove that s-NBU ordering implies s-NBUFR ordering.

Theorem 5.3 $F_X \leq_{s-NBU} F_Y \ \Rightarrow F_X \leq_{s-NBUFR} F_Y$.

Proof: $F_X \leq_{s-NBU} F_Y$ gives that, for all $x, y \geq 0$,

$$\alpha_s(x+y) \geq \alpha_s(x) + \alpha_s(y).$$

Taking limit as $y \to 0$ on both sides of the above inequality, and then using $\alpha_s(0) = 0$, we get the required result. $\square$

6 s-NBAFR Ordering

In this section we study s-NBAFR ordering. We start with the following definition.

Definition 6.1 *For any positive integer s, X (or its distribution F_X) is said to be more s-NBAFR than Y (or its distribution F_Y) (written as $F_X \leq_{s-NBAFR} F_Y$) if $\alpha_s(x) \geq x\alpha'_s(0)$.* $\square$

Remark 6.1 *For $s = 1$ and $s = 2$, the above definition gives $F_X \leq_{NBAFR} F_Y$ and $F_X \leq_{HNBUE} F_Y$, respectively.*

$\square$

Below we give some equivalent representations of the s-NBAFR ordering.

Proposition 6.1 *For $s = 2, 3, \ldots$, Definition 6.1 can equivalently be written in one of the following forms:*

(i) $\overline{T}_{X,s}\left(x\widetilde{\mu}_{X,s-1}\right) \leq \overline{T}_{Y,s}\left(x\widetilde{\mu}_{Y,s-1}\right)$ *for all* $x \geq 0$.

(ii) $\mathcal{H}^{-1}_{X,s-1}(u) \geq \mathcal{H}^{-1}_{Y,s-1}\left[T_{Y,s-1}\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}T^{-1}_{X,s-1}(u)\right)\right]$ *for all* $u \in [0,1]$.

(iii) $\mathcal{R}^{-1}_{X,s-1}(u) \leq \mathcal{R}^{-1}_{Y,s-1}\left[T_{Y,s-1}\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}T^{-1}_{X,s-1}(u)\right)\right]$ *for all* $u \in [0,1]$.

(iv) $\mathcal{L}_{X,s-1}(u) \geq \mathcal{L}_{Y,s-1}(u)$ *for all* $u \in [0,1]$.

Proof: $F_X \leq_{s-NBAFR} F_Y$ holds if, and only, if, for all $x \geq 0$,

$$\overline{T}^{-1}_{Y,s}(\overline{T}_{X,s}(x)) \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}x. \tag{6.25}$$

Replacing x by $x\widetilde{\mu}_{X,s-1}$ in (6.25), we get (i). Note that (6.25) holds if, and only if, for all $x \geq 0$,

$$T_{X,s}(x) \geq T_{Y,s}\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}}x\right). \tag{6.26}$$

Putting $x = T^{-1}_{X,s-1}(u)$ in (6.26), we see that (i) and (ii) are equivalent. On using (1.3), (ii) and (iii) become equivalent. Now, (i) can equivalently be written as

$$\int_x^\infty \overline{T}_{X,s-1}\left(t\widetilde{\mu}_{X,s-1}\right)dt \leq \int_x^\infty \overline{T}_{Y,s-1}\left(t\widetilde{\mu}_{Y,s-1}\right)dt,$$

or equivalently,

$$\int_x^\infty \overline{T}_{X^*,s-1}\left(t\right)dt \leq \int_x^\infty \overline{T}_{Y^*,s-1}\left(t\right)dt, \tag{6.27}$$

where $\overline{T}_{X^*,s-1}(t) = \overline{T}_{X,s-1}\left(t\widetilde{\mu}_{X,s-1}\right)$ and $\overline{T}_{Y^*,s-1}(t) = \overline{T}_{Y,s-1}\left(t\widetilde{\mu}_{Y,s-1}\right)$ be the respective survivals of two random variables X^* and Y^*. Thus, on using Theorem 4 of Taillie [20], (6.27) can equivalently be written as

$$\frac{1}{\widetilde{\mu}_{X,s-1}}\int_0^u T^{-1}_{X,s-1}(t)dt \geq \frac{1}{\widetilde{\mu}_{Y,s-1}}\int_0^u T^{-1}_{Y,s-1}(t)dt \quad \text{for all } u \in [0,1].$$

This proves the equivalence of (i) and (iv). $\square$

The following theorem shows that s-NBAFR ordering is a partial ordering.

Theorem 6.1 *The relationship $F_X \leq_{s-NBAFR} F_Y$ is a partial ordering of the equivalence classes of $\mathcal{F}$.*

Proof: For $s = 1$, the result follows from Kochar and Wiens [10]. We only prove the result for $s = 2, 3, \ldots$ · Let us fix s.
(i) That s-NBAFR ordering is reflexive, is trivial.
(ii) $F_X \leq_{s-NBAFR} F_Y$ gives

$$\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x \tag{6.28}$$

and $F_Y \leq_{s-NBAFR} F_X$ gives

$$\overline{T}_{X,s}^{-1}(\overline{T}_{Y,s}(x)) \geq \frac{\widetilde{\mu}_{X,s-1}}{\widetilde{\mu}_{Y,s-1}} x. \tag{6.29}$$

Replacing x by $\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x))$ in (6.29), we have

$$\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) \leq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x. \tag{6.30}$$

Combining (6.28) and (6.30), we have

$$\begin{aligned} \overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) &= \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x \\ &= \theta x, \end{aligned}$$

where $\theta = \widetilde{\mu}_{Y,s-1}/\widetilde{\mu}_{X,s-1}$ (constant). Hence, $\overline{T}_{X,s}(x) = \overline{T}_{Y,s}(\theta x)$. Thus, on using Lemma 2.4, we have $F_X \sim F_Y$.
(iii) $F_X \leq_{s-NBAFR} F_Y$ gives

$$\overline{T}_{Y,s}^{-1}(\overline{T}_{X,s}(x)) \geq \frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x \tag{6.31}$$

and $F_Y \leq_{s-NBAFR} F_Z$ gives

$$\overline{T}_{Z,s}^{-1}(\overline{T}_{Y,s}(x)) \geq \frac{\widetilde{\mu}_{Z,s-1}}{\widetilde{\mu}_{Y,s-1}} x. \tag{6.32}$$

Now,

$$\begin{aligned} \overline{T}_{Z,s}^{-1}(\overline{T}_{X,s}(x)) &= \overline{T}_{Z,s}^{-1}\overline{T}_{Y,s}\left(\overline{T}_{Y,s}^{-1}\overline{T}_{X,s}(x)\right) \\ &\geq \overline{T}_{Z,s}^{-1}\overline{T}_{Y,s}\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x\right) \\ &\geq \left(\frac{\widetilde{\mu}_{Z,s-1}}{\widetilde{\mu}_{Y,s-1}}\right)\left(\frac{\widetilde{\mu}_{Y,s-1}}{\widetilde{\mu}_{X,s-1}} x\right) \\ &= \frac{\widetilde{\mu}_{Z,s-1}}{\widetilde{\mu}_{X,s-1}} x, \end{aligned}$$

where the first inequality holds from (6.31) and using the fact that $\overline{T}^{-1}_{Z,s}\overline{T}_{Y,s}(\cdot)$ is an increasing function. The second inequality holds from (6.32). Thus, s-NBAFR ordering is transitive. □

In the following theorem we represent the relationship between s-NBAFR ageing and s-NBAFR ordering. The proof follows from Lemma 2.7

Theorem 6.2 *If $\overline{F}_Y(x) = e^{-\lambda x},\ \lambda > 0$, then $F_X \leq_{s-NBAFR} F_Y$ if, and only if, F_X is s-NBAFR.* □

Below we show that s-NBUFR ordering implies s-NBAFR ordering.

Theorem 6.3 $F_X \leq_{s-NBUFR} F_Y \ \Rightarrow F_X \leq_{s-NBAFR} F_Y$.

Proof: $F_X \leq_{s-NBUFR} F_Y$ gives that, for all $x \geq 0$,

$$\alpha'_s(x) \geq \alpha'_s(0).$$

Integrating with limit from 0 to x on both sides of the above inequality, and then using $\alpha_s(0) = 0$, we get the required result. □

7 Concluding Remarks

In this paper we introduce some new generalized partial orderings. We give some equivalent representations of each generalized ordering in terms of failure rate function, mean residual life function, TTT transform, Lorenz curve, etc. We discuss an alternative way out to study the generalized ageings in terms of generalized orderings. These orderings throw new light on the understanding of the phenomenon of generalized ageings. Such a study is meaningful because it summarizes the existing results available in literature in a unified way. Further, the lives of two systems may have same ageing property, but one may age faster than the other. So, one might be interested to know which one is ageing slower to decide on which of the two systems to be chosen. The ageing orderings help one to decide on this. Again, if one group of components are known to have the less rate of ageing compared to the other set, this will help the design engineers to select the former group of components in place of the latter group while designing a system. We conclude our discussion by mentioning the following chain of implications of generalized orderings.

$$\begin{array}{ccc}
F_X \leq_{s-IFR} F_Y \Rightarrow & F_X \leq_{s-IFRA} F_Y & \\
 & \Downarrow & \\
 & F_X \leq_{s-NBU} F_Y & \\
 & \Downarrow & \\
 & F_X \leq_{s-NBUFR} F_Y & \Rightarrow F_X \leq_{s-NBAFR} F_Y.
\end{array}$$

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