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Nonconforming spectral/*hp* element methods for elliptic systems

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Abstract — We propose a nonconforming spectral/*hp* element method for solving elliptic systems on non smooth domains using parallel computers. A geometric mesh is used in a neighbourhood of the corners and a modified set of polar coordinates, as defined by Kondratiev [7], is introduced in these neighbourhoods. In the remaining part of the domain Cartesian coordinates are used. With this mesh we seek a solution which minimizes the sum of a weighted squared norm of the residuals in the partial differential equation and the squared norm of the residuals in the boundary conditions in fractional Sobolev spaces and enforce continuity by adding a term which measures the jump in the function and its derivatives at inter-element boundaries, in fractional Sobolev norms, to the functional being minimized. The set of common boundary values consists only of the values of the spectral element functions at the vertices of the polygonal domain. Since the cardinality of the set of common boundary values is so small, a nearly exact Schur complement matrix can be computed. The method is exponentially accurate and asymptotically faster than the *h-p* finite element method. The normal equations obtained from the least-squares formulation can be solved by the preconditioned conjugate gradient method using a parallel preconditioner. The algorithm is implemented on a distributed memory parallel computer with small inter- processor communication. Numerical results for scalar problems and the equations of elasticity are provided to validate the error estimates and estimates of computational complexity that have been obtained.

Keywords: geometric mesh, stability estimate, least-squares solution, preconditioners, condition numbers, exponential accuracy.

1. Introduction

A method for obtaining a numerical solution to exponential accuracy for elliptic problems with analytic coefficients posed on a curvilinear polygon whose boundary is piecewise analytic with mixed Neumann and Dirichlet boundary conditions was first proposed by Babuska and Guo [1,2] within the framework of the finite element method. They were able to resolve the singularities which arise at the corners by

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using a geometric mesh. In [5], they extended the method to two dimensional elliptic systems also.

In [3], Babuska and H. S. Oh have introduced the method of auxiliary mapping (MAM). With this method exponential rate of convergence was recovered for the Laplace equation with corner singularities, in the context of p version of the finite element method. In [8], Lucas and H. S. Oh extended this method to Helmholtz equations. In [9], Oh and Babuska extended the method so that it can handle plane elasticity problems.

This problem has also been examined by Pathria and Karniadakis in [10] and Karniadakis and Spencer in [6] in the framework of spectral/*hp* element methods. In [13,14], spectral/*hp* element methods for solving elliptic boundary value problems on polygonal domains using parallel computers were proposed. For problems with Dirichlet boundary conditions the spectral element functions were nonconforming. For problems with Neumann and mixed boundary conditions the spectral element functions had to be continuous at the vertices of the elements.

In this paper, we will present a fully nonconforming spectral element method which is exponentially accurate and applicable for general elliptic systems such as the equations of elasticity [4]. To keep the presentation simple and nontechnical, the polygonal domains are restricted to have straight sides although the method works for curvilinear polygons.

We now seek a solution which minimizes the sum of the squares of a weighted squared norm of the residuals in the partial differential equation and the sum of the squares of the residuals in the boundary conditions in fractional Sobolev norms and enforce continuity by adding a term which measures the sum of the squares of the jump in the function and its derivatives in fractional Sobolev norms to the functional being minimized. These computations are done using modified polar coordinates in sectoral neighbourhoods of the corners and a global coordinate system elsewhere in the domain. The spectral element functions are nonconforming.

To obtain a solution, we now need to solve the normal equations for the least-squares problem. To compute the residuals in the normal equations the mass and stiffness matrices do not have to be computed [13,14]. The set of common boundary values for the numerical scheme consists of the values of the function at the vertices of the polygonal domain. Since the cardinality of the set of common boundary values is so small, we can compute a nearly exact approximation to the Schur complement matrix. Let N denote the number of layers in the geometric mesh and W the number of degrees of freedom in each independent variable of the spectral element functions, which are a tensor product of polynomials, and let W be proportional to N. Then the method requires $O(W \ln W)$ iterations of the preconditioned conjugate gradient method (PCGM) to obtain the solution to exponential accuracy whereas the *h-p* finite element method requires $O(W^2 \ln W)$ iterations. Thus the method is asymptotically faster than the spectral/*hp* element method in [14] by a factor of $O(W^{1/2})$ and asymptotically faster than the *h-p* finite element method by a factor of $O(W)$. The method works for non self adjoint problems too where the classical *h-p* finite element method may face difficulties [15].

The outline of this paper is as follows. In Section 2, the problem is defined and the stability estimate is presented. In Section 3, the numerical scheme, which is based on these estimates, is described. In Section 4, we examine the issues of parallelization and preconditioning. Finally in Section 5 computational results are provided.

2. Stability estimates

Let Ω be a polygonal domain with boundary $\partial\Omega = \Gamma$ as shown in Fig. 1. Let the vertices of Ω be given by $E_1, E_2, \ldots, E_p$. The boundary Γ is given by segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$, where Γ_i joins the points E_{i-1} and E_i. Let the angle subtended at E_j be ω_j. Further, let $\Gamma = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i\in\mathscr{D}} \overline{\Gamma}_i$, $\Gamma^{[1]} = \bigcup_{i\in\mathscr{N}} \overline{\Gamma}_i$ where $\mathscr{D}$ is a subset of the set $\{i \mid i = 1, \ldots, p\}$ and $\mathscr{N} = \{i \mid i = 1, \ldots, p\} \setminus \mathscr{D}$.

To keep the presentation simple, we restrict ourselves to a polygonal domain Ω with straight sides.

Let u be a vector of dimension d and $\mathscr{L}$ be a strongly elliptic operator

$$\mathscr{L}(u) = -\sum_{r,s=1}^{2} \left(a_{r,s}(x)\, u_{x_s}\right)_{x_r} + \sum_{r=1}^{2} b_r(x)\, u_{x_r} + c(x)\, u \tag{2.1}$$

where $a_{s,r}(x) = (a_{r,s}(x))^T$, $b_r(x)$, and $c(x)$ are analytic matrices on $\overline{\Omega}$.

Consider the boundary value problem

$$\begin{aligned} \mathscr{L}u &= f && \text{on } \Omega \\ u &= g^{[0]} && \text{on } \Gamma^{[0]} \\ \left(\frac{\partial u}{\partial N}\right)_A &= g^{[1]} && \text{on } \Gamma^{[1]} \end{aligned} \tag{2.2}$$

where $(\partial u/\partial N)_A$ denotes the usual conormal derivative which is now defined. Let $N = (N_1, N_2)$ denote the outward normal to the curve Γ_i for $i \in \mathscr{N}$. Then

$$\left(\frac{\partial u}{\partial N}\right)_A (x) = \sum_{r,s=1}^{2} N_r a_{r,s} \frac{\partial u}{\partial x_s}. \tag{2.3}$$

Moreover, let the bilinear form induced by the operator $\mathscr{L}$ satisfy the inf-sup conditions. It shall be assumed that the given data f is analytic on $\overline{\Omega}$ and $g^{[l]}, l = 0, 1$ is analytic on every closed arc $\overline{\Gamma}_i$ and $g^{[0]}$ is continuous on $\Gamma^{[0]}$.

Now, at the vertex E_k the leading order singularity is of the form

$$S_{k,1}(r_k, \vartheta_k) = v_{k,1}\, r_k^{\alpha_{k,1}} \left\{\psi_{k,1}(\vartheta_k)\right\}. \tag{2.4}$$

Here, (r_k, ϑ_k) denote polar coordinates with origin at the vertex E_k and $v_{k,1}$ denotes a vector. Define

$$\lambda_k^* = \min\left(\mathrm{Re}\left(\alpha_{k,1}\right), 1\right). \tag{2.5}$$

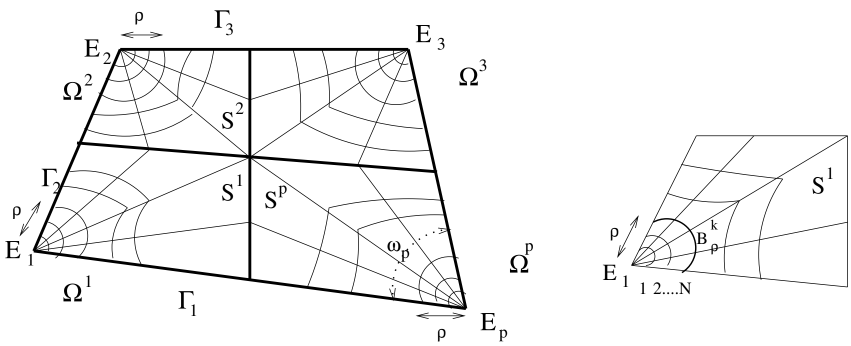


Figure 1. Geometric mesh with N layers.

We now describe the discretization of the domain and the local transformation near each vertex of the domain Ω.

2.1. Discretization

The domain Ω is divided into p non-overlapping polygonal subdomains $S^1, S^2, \ldots, S^p$, where S^k denotes a domain which contains the vertex E_k only as shown in Fig. 1. On each S^k define the geometric mesh in a neighbourhood of the vertex E_k. Let $\mathfrak{S}^k = \{\Omega^k_{i,j}, j = 1, \ldots, J_k, i = 1, \ldots, I_k\}$ be a partition of S^k, where J_k and I_k are integers. I_k is bounded for all k. Let (r_k, ϑ_k) denote polar coordinates with center at E_k. Choose ρ so that the sector Ω^k with sides Γ_k and Γ_{k+1} bounded by the circular arc B^k_ρ, center at E_k and radius ρ, satisfies $\Omega^k \subseteq \bigcup_{\Omega^k_{i,j} \in \mathfrak{S}^k} \overline{\Omega}^k_{i,j}$, where Ω^k may be represented as

$$\Omega^k = \left\{ (x_1, x_2) \in \Omega : 0 < r_k < \rho \right\}. \tag{2.6}$$

Let $\{\psi^k_i\}_{i=1,\ldots,I_k+1}$ be an increasing sequence of points such that $\psi^k_1 = \psi^k_l$ and $\psi^k_{I_k+1} = \psi^k_u$. Let $\Delta\psi^k_i = \psi^k_{i+1} - \psi^k_i$. Choose these points so that

$$\max_k \left(\max_i \Delta\psi^k_i \right) \leqslant \lambda \min_k \left(\min_i \Delta\psi^k_i \right) \tag{2.7a}$$

for some constant λ. Now choose a geometric mesh with N layers in Ω^k with ratio $0 < \mu_k < 1$. Let

$$\sigma^k_1 = 0 \tag{2.7b}$$
$$\sigma^k_j = \rho\,(\mu_k)^{N+1-j}, \qquad j = 2, \ldots, N+1. \tag{2.7c}$$

Let

$$\Omega^k_{i,j} = \left\{ (x_1, x_2) : \ \sigma^k_j < r_k < \sigma^k_{j+1}, \ \ \psi^k_i < \vartheta_k < \psi^k_{i+1} \right\} \tag{2.8}$$
$$i = 1, \ldots, I_k, \quad j = 1, \ldots, N.$$

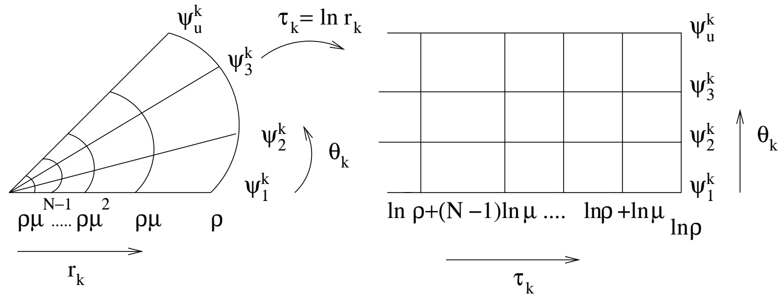


Figure 2. Quasi-uniform mesh in τ_k and ϑ_k coordinates.

In the remaining part of S^k for $k=1,\ldots,p$, we retain the Cartesian coordinate system (x_1,x_2). Let

$$\Omega^{p+1} = \left\{\Omega^k_{i,j} :\ i=1,\ldots,I_k,\ \ j=N+1,\ldots,J_k,\ \ k=1,\ldots,p\right\}. \tag{2.9}$$

Relabel the elements of Ω^{p+1} and write

$$\Omega^{p+1} = \left\{\Omega^{p+1}_l,\ \ l=1,\ldots,L\right\} \tag{2.10}$$

where L denotes the cardinality of Ω^{p+1}. All the elements except the corner elements can be chosen to be rectangles.

Since the solution of (2.2) is singular in the vicinity of the vertices E_k, $k=1,\ldots,p$, we use the auxiliary mapping of the form $z=\ln\xi$ in the sector Ω^k to remove the singularity. It was first introduced by Kondratiev [7]. The approach taken in this paper is described below.

2.2. Local transformation

Now, let $\tau_k = \ln r_k$ in the sector Ω^k for $k=1,\ldots,p$. Define $\zeta^k_j = \ln\sigma^k_j$ for $j=1,\ldots,N+1$. Here $\zeta^k_1 = -\infty$. Define

$$\widetilde{\Omega}^k_{i,j} = \left\{(\tau_k,\vartheta_k) :\ \zeta^k_j < \tau_k < \zeta^k_{j+1},\ \ \psi^k_i < \vartheta_k < \psi^k_{i+1}\right\} \tag{2.11}$$

for $i=1,\ldots,I_k$, $j=1,\ldots,N$. Hence, the geometric mesh $\Omega^k_{i,j}$, $j=2,\ldots,N$, becomes a quasi-uniform mesh (as shown in Fig. 2) in modified polar coordinates. However, $\widetilde{\Omega}^k_{i,1}$ is a semi-infinite strip.

We now describe the spectral element functions which are used to represent the numerical solution. Let $u^k_{i,1}(\tau_k,\vartheta_k) = h_k$, a constant, on $\widetilde{\Omega}^k_{i,1}$. Define the spectral element function

$$u^k_{i,j}(\tau_k,\vartheta_k) = \sum_{r=0}^{W_j}\sum_{s=0}^{W_j} g_{r,s}\,\tau_k^r\,\vartheta_k^s$$

on $\widetilde{\Omega}^k_{i,j}$ for $i = 1,\ldots,I_k$, $j = 2,\ldots,N$, $k = 1,\ldots,p$. Here $1 \leqslant W_j \leqslant W$.

Moreover, there is an analytic mapping M_l^{p+1} from the master square $S = (-1,1)^2$ to Ω_l^{p+1}. We define

$$u_l^{p+1}(M_l^{p+1}(\xi,\eta)) = \sum_{r=0}^{W}\sum_{s=0}^{W} g_{r,s}\,\xi^r\,\eta^s.$$

As described earlier, $\widetilde{\Omega}^k_{i,j}$ is the image of $\Omega^k_{i,j}$ in (τ_k,ϑ_k) coordinates. Let $\mathscr{L}^k$ be the operator defined by $\mathscr{L}^k u = r_k^2\mathscr{L}u$. It has been shown in [13] that if we let $y_1 = \tau_k$ and $y_2 = \vartheta_k$ then

$$\mathscr{L}^k u = -\sum_{i,j=1}^{2}\frac{\partial}{\partial y_i}\left(\tilde{a}^k_{i,j}\frac{\partial u}{\partial y_j}\right) + \sum_{i=1}^{2}\tilde{b}^k_i\,u_{y_i} + \tilde{c}^k u\,. \tag{2.12}$$

Let O^k denote the matrix

$$O^k = \begin{bmatrix} (\cos\vartheta_k)I & (-\sin\vartheta_k)I \\ (\sin\vartheta_k)I & (\cos\vartheta_k)I \end{bmatrix}$$

and

$$\tilde{A}^k = \begin{bmatrix} \tilde{a}^k_{1,1} & \tilde{a}^k_{1,2} \\ \tilde{a}^k_{2,1} & \tilde{a}^k_{2,2} \end{bmatrix}.$$

Then $\tilde{A}^k = (O^k)^T A O^k$. Here I denotes the $d\times d$ identity matrix.

Now in Ω_l^{p+1} for $l = 1,\ldots,L$ we have

$$\int_{\Omega_l^{p+1}}\left|\mathscr{L}w_l^{p+1}\right|^2 \mathrm{d}x_1\,\mathrm{d}x_2 = \int_S\left|\mathscr{L}w_l^{p+1}\right|^2 J_l^{p+1}\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

Here J_l^{p+1} is the Jacobian of the mapping M_l^{p+1} from S to Ω_l^{p+1}. Define $\mathscr{L}_l^{p+1} = \sqrt{J_l^{p+1}}\,\mathscr{L}$. Let

$$\|u(x,y)\|^2_{q,\Omega} = \int_\Omega \sum_{\alpha_1+\alpha_2\leqslant q}\left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}u(x,y)\right|^2 \mathrm{d}x\,\mathrm{d}y.$$

By $H^q(\Omega)$ we denote the usual Sobolev space of integer order q with the norm $\|.\|_{q,\Omega}$ as given above. Further, let

$$\|u\|^2_{s,J} = \int_J u^2(x)\,\mathrm{d}x + \int_J\int_J \frac{|u(x)-u(x')|^2}{|x-x'|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}x'$$

denote the fractional Sobolev norm of order s, where $0 < s < 1$. Here J denotes an interval contained in $\mathbb{R}$.

By γ_s we shall denote a side common to the elements Ω_m^{p+1} and Ω_n^{p+1}. It may be assumed that γ_s is the image of $\eta = -1$ under the mapping M_m^{p+1} which maps S to Ω_m^{p+1} and also the image of $\eta = 1$ under the mapping M_n^{p+1} which maps S to Ω_n^{p+1}. By the chain rule

$$\begin{aligned}(u_m^{p+1})_{x_1} &= (u_m^{p+1})_\xi\, \xi_{x_1} + (u_m^{p+1})_\eta\, \eta_{x_1}\\ (u_m^{p+1})_{x_2} &= (u_m^{p+1})_\xi\, \xi_{x_2} + (u_m^{p+1})_\eta\, \eta_{x_2}.\end{aligned}$$

Then

$$\begin{aligned}\left\|[u^{p+1}]\right\|^2_{0,\gamma_s} &= \left\|u_m^{p+1}(\xi,-1) - u_n^{p+1}(\xi,1)\right\|^2_{0,I}\\ \left\|[(u_{x_1}^{p+1})]\right\|^2_{1/2,\gamma_s} &= \left\|(u_m^{p+1})_{x_1}(\xi,-1) - (u_n^{p+1})_{x_1}(\xi,1)\right\|^2_{1/2,I}\\ \left\|[(u_{x_2}^{p+1})]\right\|^2_{1/2,\gamma_s} &= \left\|(u_m^{p+1})_{x_2}(\xi,-1) - (u_n^{p+1})_{x_2}(\xi,1)\right\|^2_{1/2,I}.\end{aligned}$$

Here $I = (-1,1)$.

Next, let $\gamma_s \subseteq \Gamma^{[0]} \cap \partial\Omega^{p+1}$ and let γ_s be the image of $\eta = -1$ under the mapping M_m^{p+1} which maps S to Ω_m^{p+1}. Then

$$\left\|u^{p+1}\right\|^2_{0,\gamma_s} + \left\|\left(\frac{\partial u^{p+1}}{\partial T}\right)\right\|^2_{1/2,\gamma_s} = \left\|u_m^{p+1}(\xi,-1)\right\|^2_{0,I} + \left\|\left(\frac{\partial u_m^{p+1}}{\partial T}\right)(\xi,-1)\right\|^2_{1/2,I}.$$

In the same way, if $\gamma_s \subseteq \Gamma^{[1]} \cap \partial\Omega^{p+1}$, $\left\|\left(\partial u^{p+1}/\partial N\right)_A\right\|^2_{1/2,\gamma_s}$ can be defined.

Let $\gamma_s \subseteq \Gamma^{[1]} \cap \partial\Omega^k$ for $k = 1,\ldots,p$ and $\tilde{\gamma}_s$ denote the image of γ_s in (τ_k, ϑ_k) coordinates. Now the normal n at a point $\tilde{P}$ on $\tilde{\gamma}_s$ can be written as $n = (n_1, n_2)$. Then

$$\left(\frac{\partial u^k}{\partial n}\right)_{\tilde{A}^k} = \sum_{i,j=1}^{2} n_i\, \tilde{a}_{i,j}^k\, \frac{\partial u^k}{\partial y_j}.$$

Using this $\left\|\left(\partial u^k/\partial n\right)_{\tilde{A}^k}\right\|^2_{1/2,\tilde{\gamma}_s}$ can be defined.

Let $\gamma_s \subseteq \bar{\Omega}^k$ and

$$d(A_k, \gamma_s) = \inf_{x\in\gamma_s}\{\text{distance}(E_k, x)\}.$$

Choose $0 < \lambda_k < \lambda_k^*$, where λ_k^* is as in (2.5).

Let

$$
\begin{aligned}
&\mathscr{V}^{N,W}_{\text{vertices}}\left(\left\{u^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u^{p+1}_l(\xi,\eta)\right\}_l\right) \qquad (2.13)\\
&= \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|(\mathscr{L}^k)u^k_{i,j}(\tau_k,\vartheta_k)\right\|^2_{0,\tilde{\Omega}^k_{i,j}}\\
&\quad+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k\cup B^k_\rho,\mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left(\left\|[u^k]\right\|^2_{0,\tilde{\gamma}_s}+\left\|[(u^k_{\tau_k})]\right\|^2_{1/2,\tilde{\gamma}_s}+\left\|[(u^k_{\vartheta_k})]\right\|^2_{1/2,\tilde{\gamma}_s}\right)\\
&\quad+\sum_{l\in\mathscr{D}}\sum_{k=l-1}^{l}\left(|h_k|^2+\sum_{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left(\left\|u^k-h_k\right\|^2_{0,\tilde{\gamma}_s}+\left\|u^k_{\tau_k}\right\|^2_{1/2,\tilde{\gamma}_s}\right)\right)\\
&\quad+\sum_{l\in\mathscr{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left\|\left(\frac{\partial u^k}{\partial n}\right)_{\tilde{A}^k}\right\|^2_{1/2,\tilde{\gamma}_s}.
\end{aligned}
$$

Here $\left\{\{u^k_{i,j}(\tau_k,\vartheta_k)\}_{i,j,k},\{u^{p+1}_l(\xi,\eta)\}_l\right\}\in\Pi^{N,W}$, the space of spectral element functions and $u^k_{i,1}=h_k$ for $i=1,\ldots,I_k$. Moreover $\mu(\tilde{\gamma}_s)$ denotes the measure of $\tilde{\gamma}_s$.

Next, define

$$
\begin{aligned}
&\mathscr{V}^{N,W}_{\text{interior}}\left(\left\{u^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u^{p+1}_l(\xi,\eta)\right\}_l\right)\\
&= \sum_{l=1}^{L}\left\|(\mathscr{L}^{p+1}_l)u^{p+1}_l(\xi,\eta)\right\|^2_{0,S}\\
&\quad+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|[u^{p+1}]\right\|^2_{0,\gamma_s}+\left\|[(u^{p+1}_{x_1})]\right\|^2_{1/2,\gamma_s}+\left\|[(u^{p+1}_{x_2})]\right\|^2_{1/2,\gamma_s}\right)\\
&\quad+\sum_{l\in\mathscr{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\left\|u^{p+1}\right\|^2_{0,\gamma_s}+\left\|\left(\frac{\partial u^{p+1}}{\partial T}\right)\right\|^2_{1/2,\gamma_s}\right)\\
&\quad+\sum_{l\in\mathscr{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left\|\left(\frac{\partial u^{p+1}}{\partial N}\right)_A\right\|^2_{1/2,\gamma_s}.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\mathscr{V}^{N,W}\left(\left\{u^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u^{p+1}_l(\xi,\eta)\right\}_l\right)\\
&= \mathscr{V}^{N,W}_{\text{vertices}}\left(\left\{u^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u^{p+1}_l(\xi,\eta)\right\}_l\right) \qquad (2.14)\\
&\quad+\mathscr{V}^{N,W}_{\text{interior}}\left(\left\{u^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u^{p+1}_l(\xi,\eta)\right\}_l\right).
\end{aligned}
$$

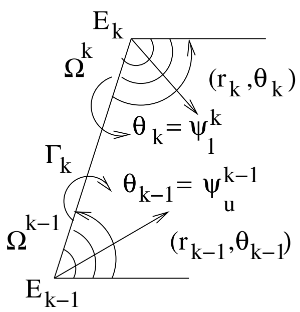


Figure 3. Edge Γ_k common to Ω^{k-1} and Ω^k.

Now, we will state the stability estimate.

Theorem 2.1. *For N and W large enough the estimate*

$$\sum_{k=1}^{p}\left(|h_k|^2+\sum_{i=1}^{I_k}\sum_{j=2}^{N}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|u_{i,j}^k(\tau_k,\vartheta_k)-h_k\right\|_{2,\tilde{\Omega}_{i,j}^k}^2\right)+\sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{2,S}^2$$
$$\leqslant C(\ln W)^2\,\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) \tag{2.15}$$

holds. Here C is a constant.

This follows from the Theorem 3.1 in [4]. The proof of this Theorem is based on Theorem 5.2 (the regularity estimate) in [5], Lemma 7.1 and Lemma 7.2 in [4].

3. The numerical scheme

Consider the data in the boundary value problem (2.2). Now in the element Ω_l^{p+1}, let $f_l^{p+1}(\xi,\eta)=f(M_l^{p+1}(\xi,\eta))$ for $l=1,\dots,L$ and $J_l^{p+1}(\xi,\eta)$ denote the Jacobian of the mapping $M_l^{p+1}(\xi,\eta)$ (which is from master square $S=(-1,1)^2$ to Ω_l^{p+1}). Define

$$F_l^{p+1}(\xi,\eta)=f_l^{p+1}(\xi,\eta)\sqrt{J_l^{p+1}(\xi,\eta)}.$$

Next, let the vertex $E_k=\left(x_1^k,x_2^k\right)$ and $F_{i,j}^k(\tau_k,\vartheta_k)=e^{2\tau_k}f\left(x_1^k+e^{\tau_k}\cos\vartheta_k,x_2^k+e^{\tau_k}\sin\vartheta_k\right)$ in $\tilde{\Omega}_{i,j}^k$ for $k=1,\dots,p$, $j=2,\dots,N$, $i=1,\dots,I_k$.

Consider the boundary conditions $u=g_k$ on $\Gamma_k\cap\partial\Omega^k$ (as shown in Fig. 3) for $k\in\mathscr{D}$, and $(\partial u/\partial N)_A=g_k$ on $\Gamma_k\cap\partial\Omega^k$ for $k\in\mathscr{N}$.

Let

$$l_1^k(\tau_k) = \begin{cases} u = g_k(x_1^k + e^{\tau_k}\cos(\psi_l^k), x_2^k + e^{\tau_k}\sin(\psi_l^k)), & k \in \mathscr{D} \\ \left(\dfrac{\partial u}{\partial n}\right)_{\tilde{A}^k} = e^{\tau_k}\, g_k(x_1^k + e^{\tau_k}\cos(\psi_l^k),\ x_2^k + e^{\tau_k}\sin(\psi_l^k)), & k \in \mathscr{N}. \end{cases}$$

Consider the boundary condition $u = g_k$ for $k \in \mathscr{D}$, and $(\partial u/\partial N)_A = g_k$ for $k \in \mathscr{N}$ on $\Gamma_k \cap \partial\Omega^{k-1}$. Define

$$l_2^k(\tau_{k-1}) = \begin{cases} u = g_k(x_1^{k-1} + e^{\tau_{k-1}}\cos(\psi_u^{k-1}),\ x_2^{k-1} + e^{\tau_{k-1}}\sin(\psi_u^{k-1})), & k \in \mathscr{D} \\ \left(\dfrac{\partial u}{\partial n}\right)_{\tilde{A}^k} = e^{\tau_{k-1}} g_k\big(x_1^{k-1} + e^{\tau_{k-1}}\cos(\psi_u^{k-1}),\ x_2^{k-1} + e^{\tau_{k-1}}\sin(\psi_u^{k-1})\big), & \\ & k \in \mathscr{N}. \end{cases}$$

Finally, let $\Gamma_k \cap \partial\Omega_t^{p+1} = C_t^k$ be the image of the mapping M_t^{p+1} of $\bar{S}$ onto $\overline{\Omega}_t^{p+1}$ corresponding to the side $\xi = -1$, and $o_t^k(\eta) = g_k\big(M_t^{p+1}(-1,\eta)\big)$, where $-1 \leqslant \eta \leqslant 1$. Define $a_k = u(E_k)$.

Now we will formulate the numerical scheme based on the stability estimate.

Let $\big\{\{v_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k}, \{v_l^{p+1}(\xi,\eta)\}_l\big\} \in \Pi^{N,W}$, the space of spectral element functions. Define the functional

$$\begin{aligned} &\mathfrak{r}_{\text{vertices}}^{N,W}\left(\left\{v_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k}, \left\{v_l^{p+1}(\xi,\eta)\right\}_l\right) \\ =\ & \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k} (\rho\mu_k^{N+1-j})^{-2\lambda_k} \left\|(\mathscr{L}^k) v_{i,j}^k(\tau_k,\vartheta_k) - F_{i,j}^k(\tau_k,\vartheta_k)\right\|_{0,\tilde{\Omega}_{i,j}^k}^2 \\ & + \sum_{k=1}^{p}\sum_{\gamma_s \subseteq \Omega^k \cup B_p^k, \mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left(\left\|[v^k]\right\|_{0,\tilde{\gamma}_s}^2 + \left\|[(v_{\tau_k}^k)]\right\|_{1/2,\tilde{\gamma}_s}^2 + \left\|[(v_{\vartheta_k}^k)]\right\|_{1/2,\tilde{\gamma}_s}^2\right) \\ & + \sum_{m\in\mathscr{D}}\sum_{k=m-1}^{m}\sum_{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_m,\mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left(\left\|(v^k - h_k) - (l_{m-k+1}^m - a_k)\right\|_{0,\tilde{\gamma}_s}^2\right. \\ & \left. + \left\|v_{\tau_k}^k - (l_{m-k+1}^m)_{\tau_k}\right\|_{1/2,\tilde{\gamma}_s}^2\right) + \sum_{m\in\mathscr{D}}\sum_{k=m-1}^{m}(h_k - a_k)^2 \\ & + \sum_{m\in\mathscr{N}}\sum_{k=m-1}^{m}\sum_{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_m,\mu(\tilde{\gamma}_s)<\infty} d(E_k,\gamma_s)^{-2\lambda_k}\left\|\left(\frac{\partial v^k}{\partial n}\right)_{\tilde{A}^k} - l_{m-k+1}^m\right\|_{1/2,\tilde{\gamma}_s}^2. \quad (3.1) \end{aligned}$$

In the above $\mu(\tilde{\gamma}_s)$ denotes the measure of $\tilde{\gamma}_s$.

Next, define

$$
\begin{aligned}
&\mathfrak{r}^{N,W}_{\text{interior}}\left(\left\{v^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{v^{p+1}_l(\xi,\eta)\right\}_l\right)\\
&= \sum_{l=1}^{L}\left\|(\mathscr{L}^{p+1}_l)v^{p+1}_l(\xi,\eta)-F^{p+1}_l(\xi,\eta)\right\|^2_{0,S}\\
&\quad+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|[v^{p+1}]\right\|^2_{0,\gamma_s}+\left\|[(v^{p+1}_{x_1})]\right\|^2_{1/2,\gamma_s}+\left\|[(v^{p+1}_{x_2})]\right\|^2_{1/2,\gamma_s}\right) \qquad (3.2)\\
&\quad+\sum_{l\in\mathscr{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\left\|v^{p+1}-o^l\right\|^2_{0,\gamma_s}+\left\|\left(\frac{\partial v^{p+1}}{\partial T}\right)-\left(\frac{\partial o^l}{\partial T}\right)\right\|^2_{1/2,\gamma_s}\right)\\
&\quad+\sum_{l\in\mathscr{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left\|\left(\frac{\partial v^{p+1}}{\partial N}\right)_A-o^l\right\|^2_{1/2,\gamma_s}.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\mathfrak{r}^{N,W}\left(\left\{v^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{v^{p+1}_l(\xi,\eta)\right\}_l\right)\\
&= \mathfrak{r}^{N,W}_{\text{vertices}}\left(\left\{v^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{v^{p+1}_l(\xi,\eta)\right\}_l\right) \qquad (3.3)\\
&\quad+\mathfrak{r}^{N,W}_{\text{interior}}\left(\left\{v^k_{i,j}(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{v^{p+1}_l(\xi,\eta)\right\}_l\right).
\end{aligned}
$$

We choose as our approximate solution the unique $\left\{\{z^k_{i,j}(\tau_k,\vartheta_k)\}_{i,j,k},\{z^{p+1}_l(\xi,\eta)\}_l\right\}$ $\in\Pi^{N,W}$, the space of spectral element functions, which minimizes the functional $\mathfrak{r}^{N,W}\left(\{v^k_{i,j}(\tau_k,\vartheta_k)\}_{i,j,k},\{v^{p+1}_l(\xi,\eta)\}_l\right)$ over all $\left\{\{v^k_{i,j}(\tau_k,\vartheta_k)\}_{i,j,k},\{v^{p+1}_l(\xi,\eta)\}_l\right\}$. Here $z^k_{i,1}=b_k$ for all i and k, $z^k_{i,j}(\tau_k,\vartheta_k)$ is a polynomial in τ_k and ϑ_k of degree W_j, $W_j\leqslant W$, and $z^{p+1}_l(\xi,\eta)$ is a polynomial in ξ and η of degree W. Choose W proportional to N. Then we have the following error estimate.

Theorem 3.1. *Let $a_k=u(E_k)$ and $U^k_{i,j}(\tau_k,\vartheta_k)=u(x(\tau_k,\vartheta_k))$ for $(\tau_k,\vartheta_k)\in\tilde{\Omega}^k_{i,j}$ and $U^{p+1}_l(\xi,\eta)=u(M^{p+1}_l(\xi,\eta))$ for $(\xi,\eta)\in S$. Choose $\alpha j\leqslant W_j\leqslant W$ for some positive α for $j>2$. Then there exist positive constants C and b such that for W large enough the estimate*

$$
\begin{aligned}
&\sum_{k=1}^{p}|b_k-a_k|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|(z^k_{i,j}-U^k_{i,j})(\tau_k,\vartheta_k)-(b_k-a_k)\right\|^2_{2,\tilde{\Omega}^k_{i,j}}\\
&\quad+\sum_{l=1}^{L}\left\|(z^{p+1}_l-U^{p+1}_l)(\xi,\eta)\right\|^2_{2,S}\leqslant Ce^{-bW} \qquad (3.4)
\end{aligned}
$$

holds.

Proof. The proof is very similar to the analysis in [2,14]. Here, we briefly describe the main steps of the proof. Using the results on approximation theory given in [2], there exists a polynomial $\Phi_l^{p+1}(\xi,\eta)$ of degree W in each variable separately such that

$$\left\|U_l^{p+1}(\xi,\eta)-\Phi_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \leqslant C_s W^{-2s+8}(Cd^s s!)^2 \tag{3.5}$$

for $l=0,\dots,L$, $W>s$, where $C_s=C_1e^{2s}$.

Define $\tilde{U}_{i,j}^k(\tau_k,\vartheta_k)=u(\tau_k,\vartheta_k)-u(A_k)$ for $(\tau_k,\vartheta_k)\in\tilde{\Omega}_{i,j}^k$. Then there exists a polynomial $\Phi_{i,j}^k(\tau_k,\vartheta_k)$ of degree W_j in τ_k and ϑ_k separately such that

$$\left\|\tilde{U}_{i,j}^k(\tau_k,\vartheta_k)-\Phi_{i,j}^k(\tau_k,\vartheta_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \leqslant C_{s_j}(W_j)^{-2s_j+8}(\chi_k)^{2s_j}\left\|\tilde{U}_{i,j}^k\right\|_{s_j,\tilde{\Omega}_{i,j}^k}^2 \tag{3.6}$$

where $\chi_k=\max\{\max_i(\Delta\psi_i^k)/2,|\ln\mu_k|/2,1\}$.

Define $\Psi_{i,1}^k(\tau_k,\vartheta_k)=a_k$ for $i=1,\dots,I_k$, $k=1,\dots,p$, as well as $\Psi_{i,j}^k(\tau_k,\vartheta_k)=\Phi_{i,j}^k(\tau_k,\vartheta_k)+a_k$ for $(\tau_k,\vartheta_k)\in\tilde{\Omega}_{i,j}^k$ and $\Psi_l^{p+1}(\xi,\eta)=\Phi_l^{p+1}(\xi,\eta)$ for $(\xi,\eta)\in S$.

Choose $\alpha j\leqslant W_j\leqslant\beta W$, where $0<\alpha$ and $\beta\leqslant 1$. With proper choice of s_j, t_j, s, and t we can prove that there exists a constant $b>0$ such that the estimate

$$\mathfrak{r}^{M,W}\left(\left\{\Psi_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{\Psi_l^{p+1}(\xi,\eta)\right\}_l\right)\leqslant Ce^{-bW} \tag{3.7}$$

holds.

Using the stability Theorem 2.1 we obtain

$$\sum_{k=1}^p\left(|a_k-b_k|^2+\sum_{j=2}^N\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|(\Psi_{i,j}^k-z_{i,j}^k)-(a_k-b_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2\right) + \sum_{l=1}^L\left\|\Psi_l^{p+1}(\xi,\eta)-z_l^{p+1}(\xi,\eta)\right\|_{2,S}^2\leqslant Ce^{-bW}. \tag{3.8}$$

It easy to show that

$$\sum_{k=1}^p\sum_{j=2}^N\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|(\Psi_{i,j}^k-U_{i,j}^k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2+\sum_{l=1}^L\left\|\Psi_l^{p+1}(\xi,\eta)-U_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \leqslant Ce^{-bW}.$$

And using the above estimates we obtain

$$\sum_{k=1}^p|b_k-a_k|^2+\sum_{k=1}^p\sum_{j=2}^N\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|(z_{i,j}^k-U_{i,j}^k)(\tau_k,\vartheta_k)-(b_k-a_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 + \sum_{l=1}^L\left\|(z_l^{p+1}-U_l^{p+1})(\xi,\eta)\right\|_{2,S}^2\leqslant Ce^{-bW}.$$

□

Remark 3.1. After having obtained the solution we can make a correction to it so that the corrected solution is conforming and the error in the $H^1(\Omega)$ norm is exponentially small in W [13]. These corrections are similar to those described in Lemma 4.55 of [11].

4. Parallelization and preconditioning

The method is essentially a least-squares method and the solution can be obtained by using the preconditioned conjugate gradient techniques (PCGM) to solve the normal equations. To be able to do so we must be able to compute the residuals in the normal equations inexpensively. In [13,14] it has been shown how the residual vector can be computed without storing mass and stiffness matrices and also shown that $O(W^3)$ operations are required to compute the residual vector on a parallel computer with $O(W)$ processors. Now we explain the solution technique and the construction of preconditioner for the matrix in the normal equations.

Let U be a vector assembled from $\{g_k\}_{k=1}^{p}$, where $u_{i,1}^k = g_k$ for all i, and the values of $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right\}$ at the Gauss–Lobatto–Legendre points arranged in lexicographic order for $k=1,\ldots,p$, $j=2,\ldots,J_k$, $i=1,\ldots,I_k$. Let $\left\{\{z_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{z_l^{p+1}(\xi,\eta)\}_l\right\}$ minimize $\mathfrak{r}^{N,W}\left(\{v_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{v_l^{p+1}(\xi,\eta)\}_l\right)$ over all $\left\{\{v_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{v_l^{p+1}(\xi,\eta)\}_l\right\}\in\Pi^{N,W}$, the space of spectral element functions.

Let U_B denote the values $\{g_k\}_{k=1}^{p}$ and U_I the remaining values of U. We now define a quadratic form

$$\mathscr{Z}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) \tag{4.1}$$
$$=\sum_{k=1}^{p}|g_k|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|u_{i,j}^k(\xi,\eta)-g_k\right\|_{2,S}^2+\sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{2,S}^2.$$

It should be noted that $u_{i,1}^k(\tau_k,\vartheta_k)=g_k$ for $i=1,\ldots,I_k$. Moreover, for $j\leqslant N$, ξ is a linear function of τ_k and η is a linear function of ϑ_k, such that the linear mapping $M_{i,j}^k(\xi,\eta)$ maps the master square S onto $\tilde{\Omega}_{i,j}^k$.

To solve the minimization problem we have to solve a system of equations of the form

$$A\,Z=h. \tag{4.2}$$

Here A is a symmetric positive definite matrix and

$$\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right)=U^TAU \tag{4.3}$$

where $\mathscr{V}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right)$ is as defined in (2.14) in Section 2.

Now A has the form

$$A = \begin{bmatrix} A_{II} & A_{IB} \\ A_{BI} & A_{BB} \end{bmatrix} \tag{4.4}$$

corresponding to the decomposition of U as

$$U = \begin{bmatrix} U_I \\ U_B \end{bmatrix}$$

and h has the form

$$h = \begin{bmatrix} h_I \\ h_B \end{bmatrix}.$$

To solve the matrix equation (4.2) we use the block LU factorization of A, viz.

$$A = \begin{bmatrix} I & 0 \\ A_{IB}^T A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & 0 \\ 0 & \mathbb{S} \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1} A_{IB} \\ 0 & I \end{bmatrix} \tag{4.5}$$

where the Schur complement matrix $\mathbb{S}$ is defined as

$$\mathbb{S} = A_{BB} - A_{IB}^T A_{II}^{-1} A_{IB}. \tag{4.6}$$

To solve the matrix equation (4.2) based on the LU factorization of A given in (4.5) reduces to solving the system of equations

$$\mathbb{S}\, Z_B = \tilde{h}_B \tag{4.7}$$

Where

$$\tilde{h}_B = h_B - A_{IB}^T A_{II}^{-1} h_I. \tag{4.8}$$

The feasibility of such a process depends on the ability to compute $A_{IB}V_B$, $A_{II}V_I$ and $A_{BB}V_B$ for any V_I, V_B efficiently and this can always be done, since AV can be computed inexpensively as explained in [13,14].

However in addition to this it is imperative that we should be able to construct effective preconditioners for the matrix A_{II} so that the condition number of the preconditioned system is as small as possible. If this can be done, then it will be possible to compute $A_{II}^{-1}V_I$ efficiently using the preconditioned conjugate gradient method (PCGM) for any vector V_I.

Consider the space of spectral element functions $\Pi_0^{N,W}$, with the property that for $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k}, \{u_l^{p+1}(\xi,\eta)\}_l\right\} \in \Pi_0^{N,W}$ we have $u_{i,1}^k = 0$ for all i and k. Let U be the vector corresponding to the spectral element function $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\right.$ $\left.\{u_l^{p+1}(\xi,\eta)\}_l\right\}$. Then $U_B = 0$ and $U = \begin{bmatrix} U_I \\ 0 \end{bmatrix}$ and so

$$\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k}, \left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) = U_I^T A_{II} U_I\,. \tag{4.9}$$

Now using Theorem 2.1 we have the following result.

Let $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right\}\in\Pi_0^{N,W}$. Then the estimate

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|u_{i,j}^k(\xi,\eta)\right\|_{2,S}^2+\sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \leqslant C(\ln W)^2\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) \tag{4.10}$$

holds for W for large enough. In the above $u_{i,1}^k=0$ for $k=1,\dots,p$ and $i=1,\dots,I_k$.

Let us define the quadratic form

$$\mathscr{U}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) =\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\left\|u_{i,j}^k(\xi,\eta)\right\|_{2,S}^2+\sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \tag{4.11}$$

for all $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right\}\in\Pi_0^{N,W}$.

Now using the trace theorems for Sobolev spaces it can be concluded that there exists a constant K such that

$$\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) \leqslant K\mathscr{U}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right) \tag{4.12}$$

for $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right\}\in\Pi_0^{N,W}$. Hence using (4.10) and (4.12) it follows that there exists a constant C such that

$$\begin{aligned}\frac{1}{C}\mathscr{V}^{N,W}&\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right)\\ &\leqslant \mathscr{U}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right)\\ &\leqslant C(\ln W)^2\mathscr{V}^{N,W}\left(\left\{u_{i,j}^k(\tau_k,\vartheta_k)\right\}_{i,j,k},\left\{u_l^{p+1}(\xi,\eta)\right\}_l\right)\end{aligned} \tag{4.13}$$

for all $\left\{\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right\}\in\Pi_0^{N,W}$.

Thus, the two quadratic forms $\mathscr{V}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right)$ and $\mathscr{U}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right)$ are spectrally equivalent.

We can now use the quadratic form $\mathscr{U}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right)$ which consists of a decoupled set of quadratic forms on each element as a preconditioner for A_{II}. This can be done by inverting the block diagonal matrix representation for $\mathscr{U}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\right)$. The construction of the preconditioner on each element is now described.

The quadratic form $\mathscr{U}^{N,W}\left(\{u_{i,j}^k(\tau_k,\vartheta_k)\}_{i,j,k}\,,\,\{u_l^{p+1}(\xi,\eta)\}_l\right)$ has a decoupled block diagonal matrix representation where each block corresponds to the H^2 norm of the spectral element function representation of each component of the vector (since the vector is of dimension d) on a particular element which is mapped onto the master square S multiplied by a constant. Consider the bilinear form $\mathscr{R}^W(u^W,v^W)$ induced by the H^2 norm on S corresponding to a component of the vector i.e.

$$\mathscr{R}^W(u^W,u^W) = \left\|u^W\right\|_{2,S}^2 .$$

Here u^W and v^W are polynomials of degree W in ξ and η respectively. Now u^W can be represented by the vector $(u_1,u_2,\ldots,u_{(W+1)^2})^T$, where u_i for $i=1,\ldots,(W+1)^2$ are the coefficients of the polynomial u^W in a suitable basis. Then there is a matrix A such that

$$\mathscr{R}^W(u^W,v^W) = \sum_{i=1}^{(W+1)^2}\sum_{j=1}^{(W+1)^2} u_i A_{i,j} v_j .$$

We consider the tensor product space of the following hierarchic shape functions as a basis for the matrix corresponding to the bilinear form $\mathscr{R}^W$ [11]

$$\begin{aligned}
N_1(\xi) &= \frac{1}{4}(1-\xi)^2(1+\xi), \qquad N_2(\xi) = \frac{1}{4}(1-\xi)^2(2+\xi)\\
N_3(\xi) &= \frac{1}{4}(1+\xi)^2(1-\xi), \qquad N_4(\xi) = -\frac{1}{4}(2+\xi)^2(1-\xi)\\
N_i(\xi) &= \sqrt{\frac{2i-5}{2}}\int_{-1}^{\xi}\int_{-1}^{\eta_1} L_{i-3}(\eta_2)\,d\eta_2\,d\eta_1, \qquad i=5,\ldots,W+1
\end{aligned}$$

where $L_i(\xi)$ are Legendre polynomials.

In this basis the matrix corresponding to $\mathscr{R}^W$ is a $(W+1)^2\times(W+1)^2$ sparse and structured matrix with a semi bandwidth of $5W$ and some additional fill in. Since this is almost a banded matrix this can be computed and inverted in $O(W^4)$ operations and the action of the inverse of this matrix on a vector can be performed in $O(W^3)$ operations.

Now from (4.13) we can conclude that if we were to compute $(A_{II})^{-1}U_I$ using the PCGM then the condition number of the preconditioned matrix would be $O((\ln W)^2)$. Hence to compute $(A_{II})^{-1}U_I$ to an accuracy of $O(e^{-bW})$ would require $O(W\ln W)$ iterations of the PCGM.

Let us return to the steps involved in solving the system of equations (4.2). As a first step it would be necessary to solve the much smaller system of equations (4.7). Here the dimension of the vector Z_B is dp, where p is the number of vertices of the domain Ω and d is the dimension of the vector. Now to be able to solve (4.7) to an accuracy of $O(e^{-bW})$ using PCGM the residual

$$R_B = \mathbb{S}U_B - \tilde{h}_B$$

would need to be computed with the same accuracy and in an efficient manner. The bottleneck in computing R_B consists in computing $(A_{II})^{-1}A_{IB}U_B$ to an accuracy of $O(e^{-bW})$ and it has already been seen that this can be done using $O(W \ln W)$ iterations of the PCGM for computing $(A_{II})^{-1}A_{IB}U_B$ for a given vector U_B.

Here $\mathbb{S}$ is a $dp \times dp$ matrix, where d is the dimension of the vector and p is the number of vertices of the polygonal domain. Let e_k be a column vector of dimension dp with a 1 in the k th place and 0 elsewhere. Let $\mathbb{S}_k = \mathbb{S}e_k$.

Then, the Schur complement matrix $\mathbb{S}$ can be written as

$$\mathbb{S} = [\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_{dp}].$$

In [4] it has been shown that an approximation $\mathbb{S}^a$ to $\mathbb{S}$ can be obtained to an accuracy of $O(e^{-bW})$ using $O(W \ln W)$ iterations of PCGM. Hence we can obtain Z_B^a, an approximation to Z_B, to an accuracy of $O(e^{-bW})$ by replacing $\mathbb{S}$ by $\mathbb{S}^a$ in

$$\mathbb{S}Z_B = \tilde{h}_B$$

using $O(W \ln W)$ iterations of PCGM [4].

Having solved for Z_B we obtain Z_I by solving

$$A_{II}Z_I = h_I - A_{IB}Z_B$$

using $O(W \ln W)$ iterations of the PCGM.

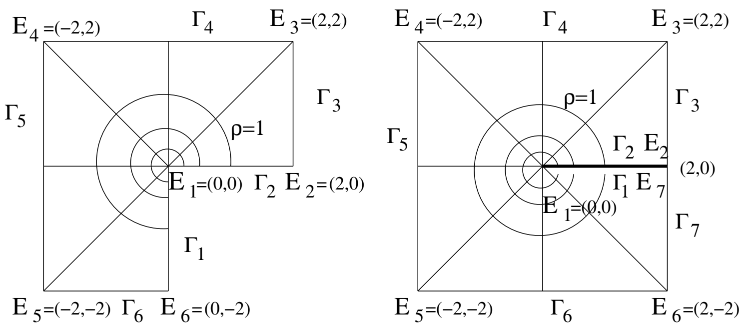
Hence the solution Z can be obtained to exponential accuracy using $O(W \ln W)$ iterations of the PCGM on a parallel machine with $O(W)$ processors and requires $O(W^4 \ln W)$ operations since each iteration requires $O(W^3)$ operations to compute the residuals in the normal equations and the action of inverse.

We shall now briefly examine the complexity of the solution procedure for the *h-p* finite element method. Since finite elements have to be continuous along the sides of the elements, the cardinality of the set of common boundary value is large for the *h-p* finite element method. Let $\mathbb{S}$ denote the Schur complement matrix for the *h-p* finite element method. In [11] it has been shown that an approximation $\mathbb{S}^a$ to $\mathbb{S}$ can be obtained such that the condition number χ of the preconditioned system satisfies

$$\chi \leqslant C(1 + (\ln W)^2)$$

where C denotes a constant. Then to solve $\mathbb{S}U_B = h_B$ to an accuracy of $O(e^{-bW})$ will require $O(W \ln W)$ iterations of the PCGM using $\mathbb{S}^a$ as a preconditioner. Now to compute the residual in the Schur complement system to an accuracy of $O(e^{-bW})$ requires $O(W)$ iterations of the PCGM to compute $A_{II}^{-1}A_{IB}V_B$. Hence we would need to perform $O(W^2 \ln W)$ iterations of the PCGM for computing $A_{II}^{-1}V_I$, where V_I will vary after every sequence of $O(W \ln W)$ steps. So the *h-p* finite element method require $O(W^2 \ln W)$ iterations of the PCGM to obtain the solution. Therefore it requires $O(W^5 \ln W)$ operations to compute the solution.

Hence the proposed method is asymptotically faster than the *h-p* finite element method by a factor of $O(W)$.



(a) Re−entrant corner with angle 270 degrees

(b) Re−entrant corner with angle 360 degrees

Figure 4. Polygonal domains with re-entrant corner at E_1.

5. Computational results

To verify the asymptotic error estimates and estimates of computational complexity, we consider the Laplace's equation with Dirichlet and mixed boundary conditions, a non self adjoint problem and a plane strain elasticity problem on polygonal domains with a re-entrant corner. We choose $W_j = W$ for all j and $N = W$, for simplicity of programming and each element is mapped onto a separate processor of a parallel computer. After having obtained the nonconforming solution a correction is made to it so that the corrected spectral element functions are conforming. We show that the error between the exact solution and the corrected approximation in the $H^1(\Omega)$ norm is exponentially small. Since the total number degrees of freedom M is proportional to W^3 the error E in the $H^1(\Omega)$ norm satisfies the estimate $E \leqslant ke^{-dM^{1/3}}$ for some constants k and d. We now present numerical results for these problems on an L-shaped domain and a panel with a crack as shown in Fig. 4.

P1. Dirichlet boundary conditions on a crack panel

Consider Laplace's equation with Dirichlet boundary conditions on E_1E_2, E_1E_7 and on the other sides of the domain as shown in Fig. 4b. Let r_1 and ϑ_1 denote polar coordinates with origin at the vertex E_1. The internal angle ω_1 at the origin is 2π. Let us choose the data so that the solution u has the form of the leading singularity at the vertex E_1 of the Laplace's equation on the domain Ω with Dirichlet boundary conditions. Thus $u = r_1^{1/2}\sin(\vartheta_1/2)$. Clearly the solution is analytic in Ω and has a singularity at the vertex E_1. Moreover $\triangle u = 0$.

The boundary conditions are as follows:

$$u|_{\Gamma_i} = 0, \quad i = 1,2$$
$$u|_{\Gamma_i} = g_i, \quad i = 3,\ldots,7.$$

We impose a geometric mesh in the sector which has E_1 as its center, radius $\rho = 1$ and sides E_1E_2 and E_1E_7 with the geometric ratio $\mu_1 = 0.15$. Now λ_1^*, as defined in Section 2, is given by $\lambda_1^* = 1/2$. Let us choose $\lambda_1 = 1/5$.

P2. Neumann boundary on an L-shaped domain

We seek a solution to Laplace's equation on an L-shaped domain as shown in Fig. 4a. Neumann boundary conditions are imposed on the sides E_1E_2 and E_1E_6 and Dirichlet boundary conditions on the other sides. Let us choose our data so that the solution u has the form of the leading singularity at the vertex E_1 with the given boundary conditions. Thus $u = r_1^{2/3}\cos((2/3)\,\vartheta_1)$. Clearly u satisfies $\Delta u = 0$.

The boundary conditions are as follows:

$$\begin{aligned} \left.\frac{\partial u}{\partial n}\right|_{\Gamma_i} &= 0, \quad i = 1,2 \\ u|_{\Gamma_i} &= g_i, \quad i = 3,\ldots,6. \end{aligned}$$

Now λ_1^*, as defined in Section 2, is given by $\lambda_1^* = 1/3$. Let us choose $\lambda_1 = 1/4$.

P3. Non self adjoint problem

The proposed method works for non self adjoint problems also. To verify this, let us consider a non self adjoint problem with Dirichlet boundary conditions having an analytic solution on the L-shaped domain shown in Fig. 4a.

Consider

$$\mathscr{L}u = \Delta u + au_x + bu_y = f \quad \text{on } \Omega.$$

Here f is chosen such that solution of the problem is $u = y(y-3x)/2$. Moreover we choose $a = 0.1$ and $b = 0.1$ and Dirichlet boundary conditions are imposed on the boundary. Clearly the solution u is analytic and the bilinear form induced by the operator $\mathscr{L}$ satisfies the inf-sup conditions.

Let us choose $\lambda_1 = 1/4$.

P4. Linear elasticity problem with Dirichlet boundary conditions on L-shaped domain

Consider a plane strain linear elasticity problem on the L-shaped domain as shown in Fig. 4a. Let $u = (u_1, u_2)^T$ be a displacement vector. Consider the equilibrium equations of linear elasticity in two dimensions when body forces are not present with Dirichlet boundary conditions:

$$\begin{aligned} -\frac{\partial}{\partial x_1}\left(c_{11}\frac{\partial u_1}{\partial x_1} + c_{12}\frac{\partial u_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left[c_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right] &= 0 \\ -\frac{\partial}{\partial x_1}\left[c_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right] - \frac{\partial}{\partial x_2}\left(c_{12}\frac{\partial u_1}{\partial x_1} + c_{22}\frac{\partial u_2}{\partial x_2}\right) &= 0. \end{aligned}$$

Let E denote the modulus of elasticity and ν denote Poisson's ratio $(0<\nu<0.5)$. The coefficients c_{11}, c_{12}, c_{22}, and c_{66} are given by $c_{11}=c_{22}=E(1-\nu)/((1+\nu)(1-2\nu))$, $c_{12}=E\nu/((1+\nu)(1-2\nu))$, and $c_{66}=E/(2(1+\nu))$.

Let r_1 and ϑ_1 denote polar coordinates with origin at E_1. Let us choose the data such that the solution is of the form of the leading singularity at E_1. Thus we choose the Mode 1 displacement components (Chapter 10 in [12])

$$
\begin{aligned}
u_1 &= \frac{1}{2c_{66}}\, r_1^{\alpha_1}\left[(\varkappa - Q(\alpha_1+1))\cos\alpha_1\vartheta_1 - \alpha_1\cos((\alpha_1-2)\vartheta_1)\right]\\
u_2 &= \frac{1}{2c_{66}}\, r_1^{\alpha_1}\left[(\varkappa + Q(\alpha_1+1))\sin\alpha_1\vartheta_1 + \alpha_1\sin((\alpha_1-2)\vartheta_1)\right]
\end{aligned}
$$

where $\varkappa = 3-4\nu$, $\alpha_1=0.544484$, and $Q=0.543075579$.

With this data we consider the following problem:

$$
\begin{aligned}
\mathscr{L}u &= 0 \quad \text{on } \Omega\\
u|_{\Gamma_i} &= g_i, \quad i=1,\ldots,6.
\end{aligned}
$$

We choose $E=1$, $\nu=0.3$ and $\lambda_1=1/4$.

P5. Linear elasticity problem with traction boundary conditions on crack panel

Consider a plane strain linear elasticity problem on the crack panel as shown in Fig. 4b. We consider the same operator and the form of the leading singularity as in P4 with traction boundary conditions $Tu=(T_1u, T_2u)^T$ on E_1E_2 and E_1E_7.

Let $n=(n_1,n_2)$ be the unit outward normal on the boundary then the traction components T_1u, T_2u are given by

$$
\begin{aligned}
T_1u &= \left(c_{11}\frac{\partial u_1}{\partial x_1} + c_{12}\frac{\partial u_2}{\partial x_2}\right)n_1 + c_{66}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)n_2 = g_1\\
T_2u &= c_{66}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)n_1 + \left(c_{12}\frac{\partial u_1}{\partial x_1} + c_{22}\frac{\partial u_2}{\partial x_2}\right)n_2 = g_2.
\end{aligned}
$$

We consider the following problem

$$
\begin{aligned}
\mathscr{L}u &= 0 \quad \text{on } \Omega\\
Tu|_{\Gamma_i} &= 0, \quad i=1,2\\
u|_{\Gamma_i} &= g_i, \quad i=3,\ldots,7.
\end{aligned}
$$

In this case $\alpha_1=0.5$ and $Q=0.333$. We choose $E=1$, $\nu=0.3$ and $\lambda_1=1/7$.

The relative error $\|e\|_{ER}$ is defined as $\|e\|_{ER}=\|e\|_E/\|u\|_E$, where $\|\cdot\|_E$ denotes energy norm. Table 1 shows the relative error $\|e\|_{ER}$ in percent against W for the problems which we have considered.

Table 1.
Relation between relative error $||e||_{ER}$ in percent and W.

W	$\|\|e\|\|_{ER}$ % for P1	$\|\|e\|\|_{ER}$ % for P2	$\|\|e\|\|_{ER}$ % for P3	$\|\|e\|\|_{ER}$ % for P4	$\|\|e\|\|_{ER}$ % for P5
2	0.5252E+01	0.3759E+01	0.6376E+01	0.7494E+01	0.1270E+02
3	0.1853E+01	0.7514E+00	0.1281E+01	0.1776E+01	0.3448E+01
4	0.7113E+00	0.2037E+00	0.2816E+00	0.6142E+00	0.1025E+01
5	0.2751E+00	0.5714E−01	0.5098E−01	0.2170E+00	0.3918E+00
6	0.1065E+00	0.1610E−01	0.7690E−02	0.7660E−01	0.1460E+00
7	0.4124E−01	0.4547E−02	0.9987E−03	0.2728E−01	0.5362E−01
8	0.1596E−01	0.1299E−02	0.1153E−03	0.9689E−02	0.2060E−01
9	0.6176E−02	0.4254E−03	0.1202E−04	0.3453E−02	0.8278E−02

Table 2.
Relation between Iterations and W.

W	Iterations for P1	Iterations for P2	Iterations for P3	Iterations for P4	Iterations for P5
2	132	99	59	159	147
3	150	168	104	369	363
4	230	224	147	538	610
5	264	274	194	688	804
6	320	337	227	800	994
7	401	395	267	939	1114
8	439	451	298	1023	1268
9	478	493	333	1142	1606

Figure 5 shows the log of relative error $||e||_{ER}$ against the polynomial degree W. For P1, P2, P4, and P5 the relationship is almost linear, which confirms the exponential convergence. For P3 the error decays very rapidly since the exact solution is analytic.

By Iterations is denoted the total number of iterations required to compute the Schur complement matrix, solve for the common boundary values and finally obtain the solution. Table 2 shows the Iterations against W.

In Fig. 6 we plot the graph of log(Iterations) against $\log(W)$.

In Section 4 it was shown that Iterations $= O(W \ln W)$. To check the asymptotic estimate we fit a straight line to the data consisting of the points from $W = W_0$ to $W = 9$ and compute the slope using the method of least-squares. Table 3 shows the slope against W_0. The results confirm the estimates that have been obtained.

Conclusions

In Section 4 it was shown that the method requires $O(W \ln W)$ iterations of the preconditioned conjugate gradient method to obtain the solution to exponential accuracy. The computational results confirm the estimates that have been obtained. Since the cardinality of the set of common boundary values is so small so the dimension of

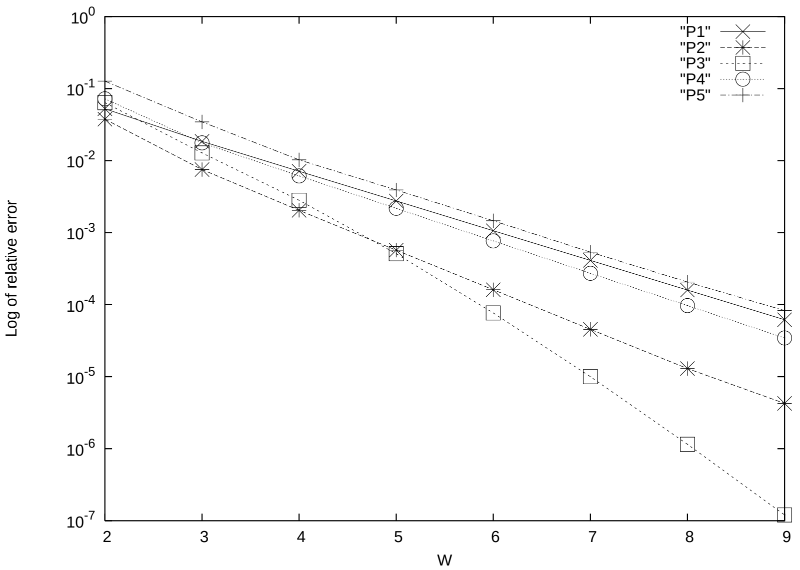


Figure 5. Relation between log of relative error $\|e\|_{ER}$ and W.

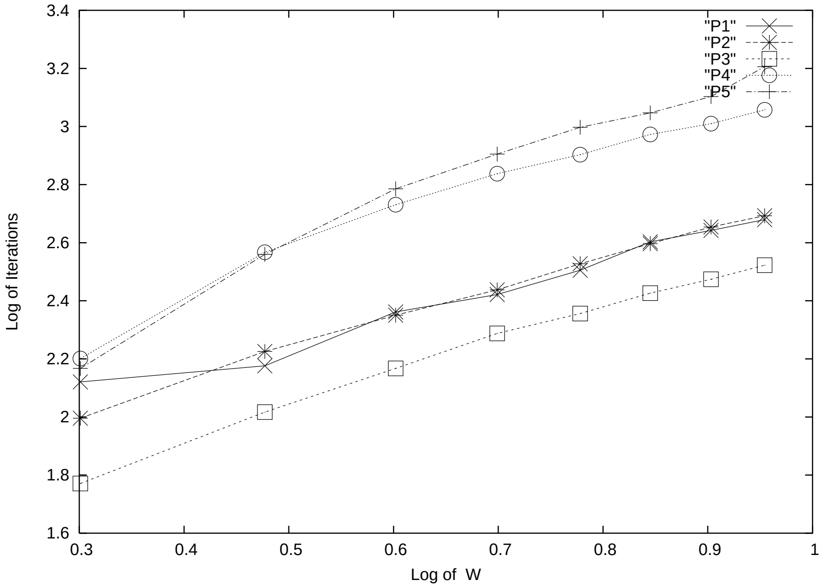


Figure 6. Log of Iterations vs. log of W.

Table 3.
W_0 vs. Slope.

W_0	Slope for P1	Slope for P2	Slope for P3	Slope for P4	Slope for P5
2	0.9241	1.0562	1.1385	1.2472	1.4947
4	0.9604	0.9979	0.9921	0.9158	1.1214
6	0.9697	0.9461	0.9348	0.8566	1.1460

the Schur complement matrix is small. It is very simple to construct a nearly exact approximation to the Schur complement matrix. The algorithm for preconditioner, which is of block diagonal form, easily invertible on each element and with almost optimal condition number, is quite easy to implement. The residuals in the normal equations are obtained efficiently without storing mass and stiffness matrices on a distributed memory parallel computer. The communication among the processors is small. The proposed method works for non self adjoint problems too. The proposed method is asymptotically faster than the *h-p* finite element method by a factor of $O(W)$ and asymptotically faster than the method proposed in [13,14] by a factor of $O(W^{0.5})$.

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