Induced dynamics on the hyperspaces

PUNEET SHARMA

Department of Mathematics, Indian Institute of Technology Jodhpur, India (`puneet.iitd@yahoo.com`)

ABSTRACT

In this paper, we study the dynamics induced by finite commutative relation. We prove that the dynamics generated by such a non-trivial collection cannot be transitive/super-transitive and hence cannot exhibit higher degrees of mixing. As a consequence we establish that the dynamics induced by such a collection on the hyperspace endowed with any admissible hit and miss topology cannot be transitive and hence cannot exhibit any form of mixing. We also prove that if the system is generated by such a commutative collection, under suitable conditions the induced system cannot have dense set of periodic points. In the end we give example to show that the induced dynamics in this case may or may not be sensitive.

1. INTRODUCTION

1.1. **Motivation:** Dynamical systems were introduced to investigate different physical and natural phenomenon occurring in nature. Using the theory of dynamical systems, mathematical models for various physical/natural phenomenon were developed and long term behavior of the natural phenomenon/systems were investigated. While logistic maps were used to develop the population model for any species, Lorentz system of differential equations was used for developing mathematical models for weather predictions. Since then, dynamical systems (both discrete and continuous) have found applications in various branches of science and engineering and various phenomenon occurring in a variety of disciplines have been investigated. In some of the recent

studies, it has been observed that many systems observed in different branches of science and engineering can be investigated using set-valued dynamics (c.f. [6, 7, 15, 17]). While [15] used set-valued dynamics to study handwheel force feedback for lanekeeping assistance, [17] used set-valued dynamics to investigate the collective dynamics of an electron and a nuclei. These examples suggest that the dynamics of different systems evolving in various disciplines of science and engineering can be modeled using set-valued dynamics. Thus, it is important to study the set-valued dynamics induced by a continuous self map which inturn can help characterizing the dynamics of a general dynamical system. Many researchers have addressed the problem and many of the questions in this direction have been answered [1, 8, 11, 13, 14, 16]. In the process, the dynamical behavior of a system and its corresponding set-valued counterpart has been investigated and several interesting results have been obtained. In [1, 16], authors proved that while weakly mixing and topological mixing on the two spaces are equivalent, transitivity on the base space need not imply transitivity on the hyperspace. Interesting results relating the topological entropy of the two spaces have been obtained [8]. In [13], Sharma and Nagar investigated some of the natural questions arising from this setting. They investigated the influence of each of the individual units and the role of the underlying hyperspace topology in determining the dynamics induced by a continuous self map on a general topological space. In the process they investigated properties like dense periodicity, transitivity, weakly mixing and topological mixing. They also investigated notions like sensitive dependence on initial conditions, topological entropy, Li-Yorke chaos, existence of Li-Yorke pairs, existence of horseshoe and the corresponding results were established. Investigating the inverse of the problem stated, they also investigated the behavior of an individual component of the system, given the dynamical behavior of the induced system on the hyperspace [13, 14].

Generalizing the stated problem, Nagar and Sharma [11] investigated the dynamics induced by a finite collection of continuous self maps. They derived necessary and sufficient conditions for the induced collective dynamics to exhibit various dynamical notions. They introduced the notion of super-transitivity, super-weakly mixing and super-topological mixing for investigating the dynamics induced on the hyperspace. They proved that super-transitivity of a relation is necessary to induce transitivity on the hyperspace. They proved that for any finite relation F on the space X, the induced map on the hyperspace $\mathcal{K}(X)$ is weakly mixing (topologically mixing) if and only if the relation F is super-weakly mixing (super-topologically mixing). However, the existence of such systems was left open and some natural questions were raised. When does a system induced by a non-trivial family (family of more than one map) exhibit transitivity? When does the dynamics induced by such a family exhibit stronger forms of mixing? Can the dynamics induced by such a system exhibit dense set of periodic points? In this paper, we try to answer some of the questions raised in [11]. We now give some of the preliminaries needed to establish our results.

1.2. **Dynamics of a Relation.** Let (X, d) be a compact metric space and let $F = \{f_1, f_2, \ldots, f_k\}$ be a finite collection of continuous self maps on X. The pair (X, F) generates a multi-valued dynamical system via the rule $F(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\}$. For convenience, we denote such systems by (X, F). Such a system generalizes the concept of the dynamical system generated by a single map f. The objective of a study of dynamical system is to study the orbit $\{F^n(x) : n \in \mathbb{N}\}$ of an arbitrary point x, where $F^n(x) = \{f_{i_1}(x) \circ f_{i_2}(x) \circ \ldots \circ f_{i_n}(x) : 1 \leq i_1, i_2, \ldots, i_n \leq k\}$ is the n-fold composition of F. We now define some of the basic dynamical notions for such a system.

A point x is called periodic for if there exists $n \in \mathbb{N}$ such that $x \in F^n(x)$. The least such n is known as the period of the point x. The relation F is transitive if for any pair of non-empty open sets U, V in X, there exists $n \in \mathbb{N}$ such that $F^n(U) \cap V \neq \phi$. The relation F is super-transitive if for any pair of non-empty open sets U, V in X, there exists $x \in U, n \in \mathbb{N}$ such that $F^n(x) \subset V$. The relation F is said to be *weakly mixing* if for every two pair of non-empty open sets U_1, U_2 and V_1, V_2, there exists a natural number n such that $F^n(U_i) \bigcap V_i \neq \phi$, $i = 1, 2$. The relation F is said to be *super-weakly mixing* if for every two pair of non-empty open sets U_1, U_2 and V_1, V_2, there exists $x_i \in U_i$ and a natural number n such that $F^n(x_i) \subseteq V_i$, $i = 1, 2$. The relation F is said to be *topologically mixing* if for every pair of non-empty open sets U, V there exists a natural number K such that $F^n(U) \bigcap V \neq \phi$ for all $n \geq K$. The relation F is said to be *super-topologically mixing* if for every pair of non-empty open sets U, V there exists $K \in \mathbb{N}$ such that for each natural number $n \geq K$, there exists $x_n \in U$ such that $F^n(x_n) \subseteq V$. A relation F is sensitive if there exists a $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there exists $n \in \mathbb{N}$ and $y \in X$ such that $d(x, y) < \epsilon$ but $d_H(F^n(x), F^n(y)) > \delta$. It may be noted that as $F^n(x)$ and $F^n(y)$ are subsets (and need not be elements) of X, metric d cannot be used to measure the distance between them. As $F^n(x), F^n(y)$ are elements in the hyperspace and the metric d_H is a natural extension of the metric d, d_H is used to compute the distance between any $F^n(x)$ and $F^n(y)$ (refer to sec. 1.3 for the details). Incase the relation F is map, the above definitions coincide with the known dynamical notions of a system. See [3, 4, 5, 11] for details.

1.3. **Some hyperspace topologies.** Let (X, τ) be a Hausdorff topological space and let Ψ be a subfamily of all non-empty closed subsets of X. Let Ψ be endowed with topology Δ, where the topology Δ is generated using the topology τ of X. Then the pair (Ψ, Δ) is called the hyperspace associated with (X, τ). A hyperspace topology is called admissible if the map $x \to \{x\}$ is continuous. In this paper, we are interested in only admissible hyperspaces (Ψ, Δ) associated with (X, τ). We now give some of the notations and terminologies used in the article.

$CL(X) = \{A \subset X : A \text{ is non-empty and closed}\}$
$\mathcal{K}(X) = \{A \in CL(X) : A \text{ is compact}\}$
$\mathcal{F}(X) = \{A \in CL(X) : A \text{ is finite}\}$

$\mathcal{F}_n(X) = \{A \in CL(X) : |A| = n, \text{ where } |A| \text{ denotes number of elements in A}\}$

$E^- = \{A \in \Psi : A \cap E \neq \phi\}$

$E^+ = \{A \in \Psi : A \subset E\}$

$E^{++} = \{A \in \Psi : \exists \ \epsilon > 0 \text{ such that } S_\epsilon(A) \cap E\},$

where $S_\epsilon(A) = \bigcup\limits_{a \in A} S(a, \epsilon)$, where $S_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$

We now give some of the standard hyperspace topologies.

Vietoris Topology: For any $n \in \mathbb{N}$ and any finite collection of non-empty open sets $\{U_1, U_2, \ldots, U_n\}$, define

$< U_1, U_2, \ldots, U_n > = \{A \in \Psi : A \subset \bigcup\limits_{i=1}^{n} U_i, \quad A \bigcap U_i \neq \phi \quad \forall i\}$

Varying $n \in \mathbb{N}$ and U_i over the collection of all non-empty open sets of X generates a basis for a topology on the hyperspace known as the Vietoris topology.

Hausdorff Metric Topology: Let (X, d) be a metric space. For any $A, B \in \Psi$, define $d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq S_\epsilon(B) \text{ and } B \subseteq S_\epsilon(A)\}$. Then d_H defines a metric on Ψ and the topology generated is known as the Hausdorff metric topology. It may be noted that $d_H(\{x\}, \{y\}) = d(x, y)$ and hence the metric d_H preserves the metric d on X. It is known that the Hausdorff metric topology and the Vietoris topology coincide, incase X is a compact metric space. See [2, 12] for details.

Hit and Miss Topology: Let $\Phi \subseteq CL(X)$ be a subfamily of all non-empty closed subsets of X. The *Hit and Miss topology* generated by the family Φ is the topology generated by sets of the form U^- where U is open in X, and $(E^c)^+$ with $E \in \Phi$, where E^c denotes the complement of E. As a terminology, U is called the *hit set* and any member E of Φ is referred as the *miss set*.

Hit and Far-Miss Topology: Let (X, d) be a metric space and let Φ be a given collection of closed subsets of X. The *Hit and Far Miss topology* generated by the collection Φ is the topology generated by the sets of the form U^- where U is open in X and $(E^c)^{++}$ with $E \in \Phi$.

Here a sub-basic open set in the hyperspace hits an open set $U \subset X$ or far misses the complement of a member of Φ and hence forms a Hit and Far Miss topology. It is known that any topology on the hyperspace is of Hit and Miss or Hit and Far-Miss type [12].

Lower and Upper Vietoris Topology: Consider the collection of sets of the form U^- in the hyperspace, where U is non-empty open set in X. The smallest topology on the hyperspace in which all the sets of the form U^- considered are open is known as the Lower Vietoris topology.

Consider the collection of sets of the form U^+ in the hyperspace, where U is non-empty open set in X. The smallest topology on the hyperspace in which all the sets of the form U^+ considered are open is known as the Upper Vietoris topology. It can be seen that the Vietoris topology equals the join of Upper Vietoris and Lower Vietoris topology, and is an example of a Hit and Miss topology.

A detailed survey on the hyperspace topologies may be found in [2, 9, 10, 12].

1.4. **Dynamics induced by a relation.** Let (X, F) be a dynamical system generated by a finite family of continuous self maps on X, say $\{f_1, f_2, \ldots, f_k\}$. For any $\Psi \subset CL(X)$, the collection Ψ is said to be admissible with respect to F if $F(A) \ (= \bigcup\limits_{i=1}^{k} f_i(A)) \in \Psi$ for all $A \in \Psi$. It may be noted that any collection Ψ admissible to F generates a map $\overline{F}$ on Ψ via the rule $\overline{F}(A) = F(A)$. Consequently, endowing Ψ with any suitable hyperspace topology (such that the map $\overline{F}$ is continuous) generates a dynamical system on the hyperspace.

It is interesting to investigate the relation between the dynamical behavior of (X, F) and the induced system $(\Psi, \overline{F})$. A special case when the family F is a singleton has been investigated by several authors and a lot of work in this direction has already been done[1, 8, 11, 13, 14, 16]. It was proved that while weakly mixing and topological mixing on the two spaces are equivalent, transitivity on the base space need not imply transitivity on the hyperspace[1, 16]. While [8] investigated the topological entropy of induced system of all non-empty compact subsets of X, [13] investigated the problem for a general hyperspace endowed with a general hyperspace topology. They discussed various dynamical notions like dense periodicity, transitivity, weakly mixing, topological mixing and topological entropy. They also investigated metric related dynamical notions like equicontinuity, sensitivity, strong sensitivity and Li-Yorke chaoticity[13, 14]. Authors extended their studies to the general case when the dynamics on X is generated by a finite family and derived the necessary and sufficient conditions for the induced system to exhibit various dynamical notions[11]. In the process, they discussed properties like dense periodicity, transitivity, weakly mixing and topological mixing. They introduced the notion of super-weakly mixing and super-topological mixing for relations to study the induced maps on the hyperspace. Authors proved that for any finite relation F on the space X, the induced map on the hyperspace $\mathcal{K}(X)$ is weakly mixing (resp. topologically mixing) if and only if the relation F is super-weakly mixing (resp. super-topologically mixing). However, the existence of any such collection of maps was not confirmed and the problem was left open. For the sake of completion, we mention some of their results below.

Theorem 1.1 ([11]). *Let β be any base for the topology on X and Δ be the topology on $\Psi \subseteq CL(X)$ such that U^+ is non-empty and $U^+ \in \Delta$ for every*

$U \in \beta$. *Then, the system* $(\Psi, \overline{F})$ *is transitive implies that the relation* (X, F) *is super-transitive.*

Theorem 1.2 ([11])**.** *Let* $\mathcal{F}(X) \subseteq \Psi$. *If* F *is super-weakly mixing, then* $\overline{F}$ *is weakly mixing. The converse holds if there exists a base* β *for topology on* X *such that* $U^+ \in \Delta$ *for every* $U \in \beta$.

Theorem 1.3 ([11])**.** *Let* $\mathcal{F}(X) \subseteq \Psi$. *If* F *is super-topologically mixing, then* $\overline{F}$ *is topologically mixing. The converse holds if there exists a base* β *for topology on* X *such that* $U^+ \in \Delta$ *for every* $U \in \beta$.

In this paper, we answer some of the questions raised in [11] when X is a compact metric space. We prove that the dynamics generated by any such commutative family cannot be transitive/super-transitive and hence cannot generate any of the complex mixing notions on the hyperspace. We also prove that dynamics induced by such a collection on the hyperspace cannot exhibit dense set of periodic points. In the end we give example to prove that the dynamics generated by such family may be sensitive.

2. Main Results

Lemma 2.1. *Let* (X, F) *be a dynamical system generated by a finite commutative family of continuous self maps on* X. *Then,* F *is super-transitive if and only if for each non-empty open set* U *and any point* $x \in X$, *there exists* $u \in U$ *and a sequence* (n_i) *in* $\mathbb{N}$ *such that* $F^{n_i}(u) \to \{x\}$ *in* $(\mathcal{K}(X), d_H)$.

Proof. Let $F = \{f_1, f_2, \ldots, f_k\}$ and let d be a compatible metric on X. Let U be a non-empty open subset of X and let $V_1 = S_1(x)$. As F is super-transitive, there exists $x_1 \in U$ and $n_1 \in \mathbb{N}$ such that $F^{n_1}(x_1) \subset V_1$. As $F^{n_1}(x) = \{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_1}} : 1 \leq i_1, i_2, \ldots, i_{n_1} \leq k\}$ and each $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_1}}$ is continuous (and are finitely many maps), there exists a neighborhood U_1 of x_1 such that $U_1 \subset U$ and $F^{n_1}(\overline{U_1}) \subset V_1$.

Let $V_2 = S_{\frac{1}{2}}(x)$. As F is super-transitive (applying transitivity to U_1 and V_2), there exists $x_2 \in U_1$ and $n_2 \in \mathbb{N}$ such that $F^{n_2}(x_2) \subset V_2$. As $F^{n_2}(x) = \{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_2}} : 1 \leq i_1, i_2, \ldots, i_{n_2} \leq k\}$ and each $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_2}}$ is continuous (and number of maps are finite), there exists a neighborhood U_2 of x_2 such that $U_2 \subset U_1$ and $F^{n_2}(\overline{U_2}) \subset V_2$.

Inductively, let U_r, V_r of non-empty open sets in X such that $U_r \subset U_{r-1}$, $V_r = S_{\frac{1}{r}}(x)$ and $F^{n_r}(\overline{U_r}) \subset V_r$. Let $V_{r+1} = S_{\frac{1}{r+1}}(x)$. As F is super-transitive (applying super-transitivity to the pair (U_r, V_{r+1})), there exists $x_{r+1} \in U_r$ and $n_{r+1} \in \mathbb{N}$ such that $F^{n_{r+1}}(x_{r+1}) \subset V_{r+1}$. One again, as $F^{n_{r+1}}(x) = \{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_{r+1}}} : 1 \leq i_1, i_2, \ldots, i_{n_{r+1}} \leq k\}$ and each $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{n_{r+1}}}$ is continuous (and number of maps are finite), there exists a neighborhood U_{r+1} of x_{r+1} such that $U_{r+1} \subset U_r$ and $F^{n_{r+1}}(\overline{U_{r+1}}) \subset V_{r+1}$.

Consequently, we obtain a nested sequence of open sets (U_r) contained in U such that $d_H(F^{n_r}(u), x) < \frac{1}{r}$ for any $u \in U_r$. As $\overline{U_i}$ is a decreasing sequence of non-empty compact subsets of X, $A = \cap \overline{U_i} \subset U$ is non-empty. Let $u \in A$, then

$u \in U_r$ and hence $F^{n_r}(u) \subset V_r$ for all r. Consequently, $d_H(F^{n_r}(u), x) < \frac{1}{r}$ for all r and hence $F^{n_i}(u) \to \{x\}$.

Conversely, let U, V be a pair of non-empty open sets ion X. For any $\{v\} \in <$ $V >$, there exists $u \in U$ and a sequence (n_i) in $\mathbb{N}$ such that $F^{n_i}(u) \to \{v\}$. Consequently there exists $r \in \mathbb{N}$ such that $F^{n_k}(u) \subset V \; \forall \; k \geq r$ and hence F is super-transitive. $\square$

Remark 2.2. It may be noted that transitivity of a system generated by single map f is equivalent to the existence of a dense orbit. Consequently, the above result is trivially true when the family F is a singleton. Thus the result above is a generalization of the known result to the case when the system is generated using more than one map. It may be noted that F^n is a union of repeated application of the maps $\{f_1, f_2, \ldots, f_k\}$ (n times) in all possible orders and continuity of each component map of F^n guarantees extension of a behavior at a point to a similar behavior in the neighborhood of the point. Further, as the number of maps constituting $F^n(x)$ is finite at each iteration, a neighborhood exhibiting similar behavior for all components of F^n is ensured and hence the existence of U_i is guaranteed. It is worth mentioning that the commutativity of the family F is not used for establishing the result. Hence the result is true even when the generating family F is non-commutative.

Proposition 2.3. *Let (X, F) be a dynamical system generated by a finite commutative family of continuous self maps on X. If F is super-transitive, then F is a singleton.*

Proof. Let U be an non-empty open set in X and $\epsilon > 0$ be a real number. Let V be a non-empty open set and let $v \in V$. By Lemma, there exists $u \in U$ and a sequence (n_i) such that $F^{n_i}(u) \to v$. As each f_i is continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f_i(x), f_i(y)) < \frac{\epsilon}{2}$ for all $i = 1, 2, \ldots, n$. As $F^{n_i}(u) \to v$, for any two distinct elements f, g of F, $f^{n_i}(u) \to v$ and $g^{n_i}(u) \to v$. Further, as $d(F^{n_i}(u)) \to 0$ (where $d(A)$ denotes the diameter of the set A), there exists $r \geq 1$ such that the relation $d(f^{n_i}(u), g^{n_i}(u)) < \delta$ and $d(f \circ g^{n_i-1}(u), g^{n_i}(u)) < \delta$ is true for all $i \geq r$.

Consequently, $d(f^{n_i+1}(u), f \circ g^{n_i}(u)) < \frac{\epsilon}{2}$ and $d(f \circ g^{n_i}(u), g^{n_i+1}(u)) < \frac{\epsilon}{2}$ for all $i \geq r$. Using triangle inequality we get $d(f^{n_i+1}(u), g^{n_i+1}(u)) < \epsilon$ for all $i \geq r$. Consequently, $(f^{n_i+1}(u)), (g^{n_i+1}(u))$ are parallel sequences and hence have the same limit(say y). Also $f^{n_i}(u) \to v$ and $g^{n_i}(u) \to v$ implies $f^{n_i+1}(u) \to f(v)$ and $g^{n_i+1}(u) \to g(v)$. Consequently, $f(v) = g(v)$. As the argument holds for any open set V, the points at which f and g coincide is dense in X. Hence $f = g$. $\square$

Remark 2.4. The above proof establishes that the system induced by more than one map cannot be super-transitive. The proof establishes that under stated conditions, if $(f^{n_i}(x))$ and $(g^{n_i}(x))$ are parallel then $(f^{n_i+1}(x))$ and $(g^{n_i+1}(x))$

are also parallel and hence have the same limit. Further as $f^{n_i}(x), g^{n_i}(x)$ converge to v, $f(v)$ and $g(v)$ are unique limit points of $(f^{n_i+1}(x))$ and $(g^{n_i+1}(x))$ respectively. Consequently f and g coincide on a dense set and hence are equal.

Remark 2.5. The above result proves that if the dynamics on X is generated by more than one map then the system cannot be super-transitive and hence cannot exhibit any stronger forms of mixing. Further as super-transitivity of the system (X, F) is necessary for any induced admissible system $(\Psi, \overline{F})$ to be transitive [11], the system induced on the hyperspace by a family of more than one map cannot be transitive and hence cannot exhibit stronger notions of mixing. In light of the remark stated, we obtain the following corollary.

Corollary 2.6. *Let (X, F) be a dynamical system generated by a finite commutative family of continuous self maps on X. Let (Ψ, Δ) be the associated hyperspace and let $\overline{F}$ be the corresponding induced map. If the family F contains more than one map then $\overline{F}$ cannot be transitive.*

We now show that the dynamics induced by a commutative family on the hyperspace cannot exhibit dense set of periodic points.

Proposition 2.7. *Let (X, F) be a dynamical system generated by a finite commutative family of continuous self maps on X and let $(\mathcal{K}(X), \overline{F})$ be the induced system endowed with the Vietoris topology on the hyperspace. If $(\mathcal{K}(X), \overline{F})$ exhibits dense set of periodic points then F is a singleton.*

Proof. Let if possible, the induced map $\overline{F}$ exhibit dense set of periodic points. Let $x \in X$ be arbitrary and let $\epsilon > 0$ be given. For any two members f and g of F, as f and g are continuous on a compact set, they are uniformly continuous, i.e. there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, we have $d(f(x), f(y)) < \frac{\epsilon}{3}$ and $d(g(x), g(y)) < \frac{\epsilon}{3}$.

Let $U = S_{\frac{\delta}{2}}(x)$. As $< U >$ is open in the hyperspace and $\overline{F}$ has dense set of periodic points, there exists $A \in < U >$ and $n \in \mathbb{N}$ such that $\overline{F}^n(A) = A$. Let $a \in A$. As $\overline{F}^n(A) = A \subset U$ we have $d(x, f^n(a)) < \frac{\delta}{2}$. Consequently, we have $d(f(x), f^{n+1}(a)) < \frac{\epsilon}{3}$ and $d(g(x), g \circ f^n(a)) < \frac{\epsilon}{3}$. Also $\overline{F}^n(A) = A \subset U$, implies $d(f^n(a), g \circ f^{n-1}(a)) < \delta$ and hence $d(f^{n+1}(a), g \circ f^n(a)) < \frac{\epsilon}{3}$ (as F is commutative). Using triangle inequality we have, $d(f(x), g(x)) \leq d(f(x), f^{n+1}(a)) + d(f^{n+1}(a), g \circ f^n(a)) + d(g \circ f^n(a), g(x)) < \epsilon$.

As $\epsilon > 0$ was arbitrary, $d(f(x), g(x)) = 0$ which implies $f(x) = g(x)$. As the proof holds for any $x \in X$, any two members of the family F coincide and hence F is a singleton. □

Remark 2.8. The result establishes that if the dynamics on the space X is generated by more than one map then the induced dynamics on the hyperspace cannot exhibit dense set of periodic points. The proof uses the openness of the sets of the form $< U >$, where U is non-empty open in X and does utilize the complete structure of the Vietoris topology. Further, the proof does not

utilize the structure of the hyperspace $\mathcal{K}(X)$ and holds good for any general admissible hyperspace (Ψ, Δ). Hence the induced map cannot have dense set of periodic points in this case and the result is true for the induced system when the admissible hyperspace is endowed with topology finer than the upper Vietoris topology. In light of the remark stated, we get the following corollary.

Corollary 2.9. *Let (X, F) be a dynamical system generated by a finite commutative family of continuous self maps on X and let $(\Psi, \overline{F})$ be the induced system endowed with any hyperspace topology finer than the upper Vietoris topology. If the family F contains more than one map then $\overline{F}$ cannot exhibit dense set of periodic points.*

Remark 2.10. The above corollary establishes the generalization of the proved result to the system induced on a more general admissible hyperspace. Further, it is worth mentioning that if any of the members of F exhibit dense set of periodic points, then the system (X, F) exhibits dense set of periodic points as $f^n(x) \in F^n(x)$ for any member f of F. Hence if the dynamics on X is induced by more than one function, the system may exhibit dense set of periodic points but the induced system can not exhibit dense set of periodic points which is contrary to the case when F is a singleton. In the light of the remark stated, we get the following corollary.

Corollary 2.11. *Let (X, F) be a dynamical system and let $(\Psi, \overline{F})$ be the corresponding induced system. Then, (X, F) has dense set of periodic points $\nRightarrow$ $(\Psi, \overline{F})$ has dense set of periodic points.*

Remark 2.12. The above corollary shows that there can exist dynamical systems (X, F) with the dense set of periodic points such that the induced system does not have dense set of periodic points. Such a scenario happens when the dynamics on X is induced by more than one map. Such a behavior for the induced system is due to the fact that although the dynamics on the space X is generated by a finite commutative relation F, the dynamics on the hyperspace is generated by a function and hence conventional methodology for investigating the dynamics on the hyperspace are to be used. In particular, x is periodic for F if there exists $n \in \mathbb{N}$ such that $x \in F^n(x)$ but A is periodic for $\overline{F}$ if there exists $n \in \mathbb{N}$ such that $A = \overline{F}^n(A) \; (= F^n(A))$. Consequently, there is a significant difference in the basic dynamical notions of the system which induces such a contrasting behavior on the hyperspace. We now give an example in support of our argument.

Example 2.13. Let $X = S^1$ be the unit circle and let f_1 and f_2 be the rational rotations on unit circle defined as $f_1(\theta) = \theta + p$ and $f_2(\theta) = \theta + q$ respectively.

Let $F = \{f_1, f_2\}$ be the commutative relation and let (X, F) be the corresponding dynamical system generated. As f_i is rotation on the unit circle by a rational multiple of 2π, f_i exhibits dense set of periodic points. Further as $f_i^n(x) \in F^n(x)$ for all i and n, F exhibits dense set of periodic points. However, as each f_i is a rotation on the unit circle (by different angles), for any set $A \neq X$ $F^n(A) = A$ never holds good and hence the hyperspace has a unique

periodic point X. Consequently, the example shows that there exists dynamical systems (X, F) with the dense set of periodic points such that the induced system does not have dense set of periodic points.

Remark 2.14. The above example proves that the induced dynamics cannot exhibit dense set of periodic points when induced by a commutative family of more than one map. On similar lines, considering f_i $(i = 1, 2)$ to be irrational rotations on the unit circle gives an example of a relation where each component is transitive but the relation is not super-transitive. The relation is not super-transitive as f_1 and f_2 are isometries and hence for each natural number $2n$ (or $2n + 1$) $f_1^n \circ f_2^n$ and $f_1 \circ f_2^{2n-1}$ (or $f_1^n \circ f_2^n$ and $f_1 \circ f_2^{2n}$) cannot be pushed inside an open set of arbitrary arc length. Consequently, the dynamics induced on the hyperspace by the family considered is not super-transitive.

Remark 2.15. The above results establish that when the dynamics on X is generated by a commutative family of more than one function, the dynamics on the hyperspace cannot be super-transitive and cannot have dense set of periodic points. The commutativity of the family F plays an important role in proving the result and hence cannot be dropped. In absence of commutativity, the order of f and g in any member of $F^n(x)$ cannot be altered and the proof of the result does not hold good. However under non-commutativity, $g \circ f^n(x)$ and $f \circ g \circ f^{n-1}(x)$ (and others where f_i's are applied same number of times but in different order) do not coincide and hence $F^n(x)$ contains more elements. Consequently, it is expected that as super-transitivity (or dense periodicity) does not hold under commutativity, it will not hold in the non-commutative case (as $F^n(x)$ contains more elements). However, any proof for the belief is not available and hence is left open.

We now give an example to show, that the induced dynamics, when induced by more than one function, might be sensitive.

Example 2.16. Let Σ_2 be the sequence space of all bi-finite sequences of two symbols 0 and 1. For any two sequences $x = (x_i)$ and $y = (y_i)$, define
$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$$
It is easily seen that the metric d generates the product topology on $\sum_2$.

Let $\sigma : \Sigma_2 \to \Sigma_2$ be defined as
$$\sigma(\ldots x_{-2} x_{-1}.x_0 x_1 \ldots) = \ldots x_{-2} x_{-1} x_0.x_1 x_2 x_3 \ldots$$

The map σ is known as the shift map and is continuous with respect to the metric d defined.

Let $\mathbb{F} = \{\sigma, \sigma^2\}$ and let $\overline{F}$ be the corresponding induced map on the hyperspace. We claim that the induced map is sensitive on $\mathcal{K}(X)$. Equivalently, it is sufficient to show that the map is sensitive on a dense subset of $\mathcal{K}(X)$.

Let $A = \{x_1, x_2, \ldots, x_k\}$ be a finite set where each $x_i = (\ldots x_i^{-2} x_i^{-1}.x_i^0 x_i^1 \ldots x_i^n \ldots)$ is an element in Σ and let $\epsilon > 0$ be given. Let $r \in \mathbb{N}$ such that $\frac{1}{2^r} < \epsilon$.

Let $y_i = (\ldots x_i^{-2} x_i^{-1}.x_i^0 \ldots x_i^{r+1} 000 \ldots)$ and $z_i = (\ldots x_i^{-2} x_i^{-1} x_i^0 \ldots x_i^{r+1} 111 \ldots)$. Then $B = \{y_1, y_2, \ldots, y_k\}$ and $C = \{z_1, z_2, \ldots, z_k\}$ are elements in $S(A, \epsilon)$ such that $F^{r+2}(y_i) = (\ldots x_i^{-2} x_i^{-1} \ldots x_i^r.0000 \ldots)$ and $F^{r+2}(z_i) = (\ldots x_i^{-2} x_i^{-1} \ldots x_i^r.11111 \ldots)$.

Consequently, $d_H(\overline{F}^{r+2}(B), \overline{F}^{r+2}(C)) \geq 1$ and hence $\overline{F}$ is sensitive at A. As the proof holds for any finite subset S of $Sigma_2$, $\overline{F}$ is sensitive at finite subsets of Σ. Consequently, $\overline{F}$ is sensitive on Σ.

3. Conclusion

The paper discusses the dynamics of the induced function on the hyperspace, when the function is induced by a non-trivial family of commuting continuous self maps on X. It is observed that the dynamics is contrary to the case when the map on the hyperspace is induced using a single function. While the map induced by single function can exhibit complex dynamical behavior (for example weakly mixing or topological mixing), the dynamics induced by a collection of two or more commuting maps cannot even be transitive and hence cannot exhibit any of the higher notions of mixing. Further, it is established that the dynamics induced by such a family cannot have dense set of periodic points. This once again is a contrary to the case when the map is induced by a single function, as the map induced in that case always has dense set of periodic points, if the original system has dense set of periodic points. We also give an example to show that the dynamics induced by a commutative family may be sensitive. It is worth mentioning that although the problem has been solved when the dynamics on the hyperspace is induced by a commutative family, the non-commutative case is still open for investigation. As $F^n(x)$ contains more points in the non-commutative case, similar results are expected to hold. However as the proof derived does not work for non-commutative case, we leave it open for further investigation.

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