Computers and Mathematics with Applications 71 (2016) 1745–1771

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

h-p spectral element methods for three dimensional elliptic problems on non-smooth domains, Part-III: Error estimates, preconditioners, computational techniques and numerical results ☆

P. Dutt [a], A. Husain [b,*], A.S. Vasudeva Murthy [c], C.S. Upadhyay [d]

[a] *Department of Mathematics & Statistics, Indian Institute of Technology, Kanpur 208016, UP, India*

[b] *Department of Mathematics, The LNM Institute of Information Technology, Jaipur 302031, Rajasthan, India*

[c] *TIFR Centre for Applicable Mathematics, Tata Institute of Fundamental Research, Bangalore 560065, Karnataka, India*

[d] *Department of Aerospace Engineering, Indian Institute of Technology, Kanpur 208016, UP, India*

ARTICLE INFO

Article history:
Received 5 February 2015
Received in revised form 11 December 2015
Accepted 30 December 2015
Available online 2 April 2016

Keywords:
Spectral element method
Geometric mesh
Numerical scheme
Preconditioner
Error estimates
Exponential accuracy

ABSTRACT

The present paper is the third of a series of papers devoted to the study of *h-p* spectral element methods for three dimensional elliptic problems on non-smooth domains using parallel computers. In this paper we provide error estimates, preconditioners and numerical results. The spectral element functions are fully non-conforming. We propose preconditioners on non-smooth domains which can be diagonalized using separation of variables technique. Optimal error estimates in terms of number of layers in the geometrical mesh and in terms of number of degrees of freedom are obtained. The method is easy to implement on a parallel computer and we briefly outline computational techniques. We give results of numerical simulations to confirm the theoretical estimates. Theoretical results have been also validated by computational experiments which are published independently in Dutt et al. (2014).

1. Introduction

It is well known that the regularity of solutions of elliptic boundary value problems on polyhedral domains is severely affected due to the presence of singularities in the form of corners and edges in the domain. There are three type of singularities caused by non-smoothness of domains in R^3: the vertex, the edge, and the vertex-edge combined singularities. The solutions of many practical problems on polyhedral domains may be analytic except at the vertices and edges, and their derivative may grow rapidly towards the vertices or edges with increasing order. The regularity results on non-smooth domains described in terms of usual Sobolev spaces and classical weighted Sobolev spaces (see [1–7] and references therein) were unable to reflect the natures of singularities, and qualitative features of the growth of the derivatives of the solutions

☆ This work was carried out at the Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur-208016, India.

* Corresponding author.
E-mail addresses: pravir@iitk.ac.in (P. Dutt), akhlaq@lnmiit.ac.in (A. Husain), vasu@math.tifrbng.res.in (A.S. Vasudeva Murthy), shekar@iitk.ac.in (C.S. Upadhyay).

http://dx.doi.org/10.1016/j.camwa.2015.12.046

were missing. The regularity results of Babuška and Guo [8,9] in terms of countably weighted Sobolev spaces and countably normed spaces with dynamical weights in the neighbourhoods of vertices, edges and vertex-edges over polyhedral domains address these issues and proved to be the foundation stones for comprehensive study of the regularity theory for solving elliptic problems in three-dimensional non-smooth domain arising from mechanics and engineering. Recently, Costabel and coworkers settled the proof of the analytic regularity estimates [10,11] using anisotropic weighted Sobolev spaces introduced in [8,9] by filling the gap which was left over by Babuška and Guo.

The *h-p* version of the finite element method (FEM) for elliptic problems was proposed in the mid 80ies by Babuška and his coworkers for solving problems in structural mechanics. The *h-p* version simultaneously refines the mesh and increases the polynomial degree to solve problems on non-smooth domains and achieve optimal convergence. The *h-p* version of Spectral Element Method (SEM) is a new development of the FEM which employ global polynomials of higher degree in order to recover the so called *spectral/exponential convergence*. It is well established that SEM deliver exponential convergence for elliptic problems on smooth domains and have been successfully implemented in practical problems (see [12–14] and references therein). However, in many engineering and scientific applications we require the numerical solutions of elliptic boundary value problems in non-smooth domains which give rise to singularities in the solution. In such cases the accuracy of the solution obtained by SEM deteriorates and we need to devise an efficient numerical scheme to capture the spectral accuracy.

In [15–18] we proposed a non-conforming *h-p* spectral element method to solve elliptic boundary value problems on non-smooth domains in R^3. To formulate the numerical scheme we minimize a functional over the space of spectral element functions which is the sum of a weighted squared norm of the residuals in the partial differential equations and the squared norm of the residuals in the boundary conditions in fractional Sobolev spaces and enforce continuity by adding a term which measures the jump in the function and its derivatives at inter-element boundaries in fractional Sobolev norms suitably weighted, to the functional being minimized.

The method is essentially a least-squares collocation method as formulated in [19,20] in two dimensions and to obtain the solution we need to solve the normal equations corresponding to the least-squares formulation. We apply Preconditioned Conjugate Gradient Method (PCGM) to solve normal equations. In this paper we show that the residual in the normal equations can be computed inexpensively without having to compute and store mass and stiffness matrices. Moreover, we show that a preconditioner can be defined for the quadratic form corresponding to the minimization problem. The preconditioner is obtained in the same way as the residuals in the normal equations, but with homogeneous boundary data and the homogeneous form of the partial differential equation. Hence, the algorithm for the preconditioner is quite simple and easy to implement. We prove that our preconditioners are spectrally equivalent to new quadratic forms which can be diagonalized using separation of variables and therefore easy to invert.

This paper is the third of a series of papers devoted to the study of *h-p* spectral element methods for three dimensional elliptic problems on non-smooth domains using parallel computers. In this paper we use differentiability (regularity) estimates and stability estimates of [15,16] to describe parallel preconditioners, computational complexity and prove optimal error estimates for *h-p* version of the spectral element method for elliptic problems on polyhedral domains containing singularities. Numerical results for problems with analytic and singular solutions are presented to verify the theory and analyse the performance of our method.

The first paper [15] deals with the regularity of the solution in the neighbourhoods of vertices, edges and vertex-edges and describe the stability theorem. The second paper [16] addresses proof of the stability theorem. Results of numerical experiments that have been performed to validate the theoretical estimates are presented in [17].

Throughout this paper (x_1, x_2, x_3), (ρ, ϕ, θ) and (r, θ, x_3) denote the Cartesian, the spherical and the cylindrical coordinates respectively. The scope of this paper is as follows. In Section 2, we shall quote the notations and definitions introduced in [15] and recall our main stability theorem, proved in [16], for a non-conforming *h-p* spectral element method. Error estimates are obtained in Section 3 and it is shown that the error decays exponentially with respect to the number of layers in the geometric mesh and the number of degrees of freedom in each variable on each element. Preconditioners on regular as well as singular regions are discussed in Section 4, where we show that there exists a new diagonal preconditioner using separation of variables technique. Section 5, gives a brief description of computational techniques and numerical results are presented in Section 6. Concluding remarks are given in Section 7.

2. Preliminaries

Let Ω denote a polyhedron in R^3, as shown in Fig. 1(a). Let Γ_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$, be the faces of the polyhedron. Let $\mathcal{D}$ be a subset of $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \setminus \mathcal{D}$. We impose Dirichlet boundary conditions on the faces Γ_i, $i \in \mathcal{D}$ and Neumann boundary conditions on the faces $\Gamma_j, j \in \mathcal{N}$. Further, let $\partial\Omega = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^{[1]} = \bigcup_{i\in\mathcal{N}} \bar{\Gamma}_i$. Let us consider the elliptic boundary value problem:

$$\begin{aligned} Lw &= F \quad \text{in } \Omega, \\ w &= g^{[0]} \quad \text{for } x \in \Gamma^{[0]}, \\ \left(\frac{\partial w}{\partial n}\right)_A &= g^{[1]} \quad \text{for } x \in \Gamma^{[1]}, \end{aligned} \tag{2.1}$$

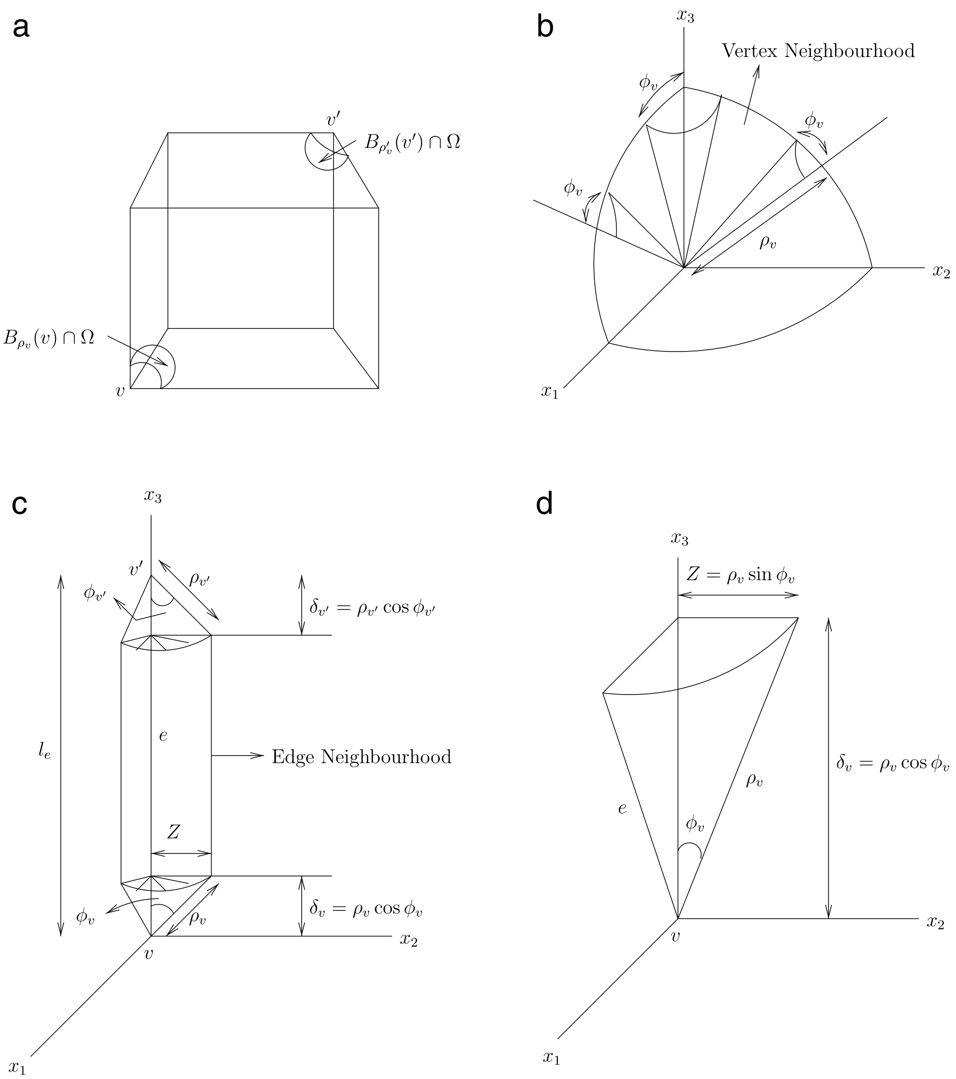

Fig. 1. (a) Polyhedral domain Ω, (b) Vertex neighbourhood Ω^v, (c) Edge neighbourhood Ω^e, (d) Vertex-edge neighbourhood Ω^{v-e}.

where n denotes the outward normal and $\left(\frac{\partial w}{\partial n}\right)_A$ is the usual conormal derivative. Here, the differential operator

$$Lw(x) = \sum_{i,j=1}^{3} -\frac{\partial}{\partial x_i}(a_{i,j} w_{x_j}) + \sum_{i=1}^{3} b_i w_{x_i} + cw$$

is a strongly elliptic differential operator which satisfies the Lax–Milgram conditions. Moreover, $A = a_{ij} = a_{ji}$ for all i, j and the coefficients of the differential operator are analytic. The data F, $g^{[0]}$ and $g^{[1]}$ are analytic on each open face and $g^{[0]}$ is continuous on $\bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$.

In [15,16] we had decomposed the domain Ω into a regular region, a set of vertex neighbourhoods, a set of edge neighbourhoods and a set of vertex-edge neighbourhoods. To overcome the singularities which arise in the neighbourhoods of the vertices, vertex-edges and edges we use local systems of coordinates introduced in [15]. These local coordinates are modified versions of spherical and cylindrical coordinate systems in their respective neighbourhoods. Away from these neighbourhoods standard Cartesian coordinates are used in the regular region of the polyhedron. Table 1 summarizes the system of coordinates used in various regions of the polyhedron Ω. For details we refer to [15].

We now briefly recall the notations, definitions and description of various neighbourhoods of vertices and edges of the polyhedron Ω (see [15] for more details). Let Γ_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$, be the faces (open), S_j, $j \in \mathcal{J} = \{1, 2, \ldots, J\}$, be the edges and A_k, $k \in \mathcal{K} = \{1, 2, \ldots, K\}$, be the vertices of the polyhedron. We shall also denote an edge by e, where $e \in \mathcal{E} = \{S_1, S_2, \ldots, S_J\}$, the set of edges, and a vertex by v where $v \in \mathcal{V} = \{A_1, A_2, \ldots, A_K\}$, the set of vertices. Now consider a vertex v and let e denote one of the edges passing through it, which we assume to coincide with x_3 axis. Let ϕ denote the angle which $x = (x_1, x_2, x_3)$ makes with the x_3 axis. By Ω^v, we denote the vertex neighbourhood of the vertex v

Table 1
System of coordinates used in various parts of Ω.

Region	Coordinates	Type
Regular	x_1, x_2, x_3	Standard Cartesian
Vertex neighbourhood	$x_1^v = \phi, x_2^v = \theta, x_3^v = \chi = \ln\rho$	Modified spherical
Edge neighbourhood	$x_1^e = \tau = \ln r, x_2^e = \theta, x_3^e = x_3$	Modified cylindrical
Vertex-edge neighbourhood	$x_1^{v-e} = \psi = \ln(\tan\phi), x_2^{v-e} = \theta,$ $x_3^{v-e} = \zeta = \ln x_3$	Hybrid

defined by

$$\Omega^v = \left(B_{\rho_v}(v) \setminus \bigcup_{e\in\mathscr{E}^v} \overline{\mathscr{V}_{\rho_v,\phi_v}(v,e)} \right) \bigcap \Omega,$$

where $B_{\rho_v}(v) = \{x : dist(x,v) < \rho_v\}$ and $\mathscr{V}_{\rho_v,\phi_v}(v,e) = \{x \in \Omega : 0 < dist(x,v) < \rho_v,\ 0 < \phi < \phi_v\}$, where ϕ_v is a constant. For every vertex v, ρ_v and ϕ_v are chosen so small that $B_{\rho_v}(v) \cap B_{\rho_{v'}}(v') = \emptyset$ if the vertices v and v' are distinct and $\mathscr{V}_{\rho_v,\phi_v}(v,e') \bigcap \mathscr{V}_{\rho_v,\phi_v}(v,e'') = \emptyset$ if $e'$and e'' are distinct edges having v as a common vertex. Moreover, ρ_v and ϕ_v are chosen so that $\rho_v \sin(\phi_v) = Z$, a constant for all $v \in V$, the set of vertices.

Next, let e denote an edge, which we assume to coincide with the x_3 axis, whose end points are the vertices v and v'. Then we define the edge neighbourhood of the edge e denoted as Ω^e shown in Fig. 1(c) by

$$\Omega^e = \{x \in \Omega :\ \delta_v < x_3 < l_e - \delta_{v'},\ 0 < r < Z\},$$

where l_e is the length of the edge e, $\delta_v = \rho_v \cos(\phi_v)$, $\delta_{v'} = \rho_{v'} \cos(\phi_{v'})$ and $r = \sqrt{{x_1}^2 + {x_2}^2}$.

Now, by Ω^{v-e} we denote the vertex-edge neighbourhood of the vertex v and the edge e shown in Fig. 1(d) defined by

$$\Omega^{v-e} = \{x \in \Omega : 0 < \phi < \phi_v, 0 < x_3 < \delta_v = \rho_v \cos\phi_v\}.$$

Finally, Ω^r denote the portion of the polyhedron Ω obtained after the closure of the vertex-neighbourhoods, edge neighbourhoods and vertex-edge neighbourhoods have been removed from it. Thus let

$$\Delta = \left\{ \bigcup_{v\in\mathscr{V}} \overline{\Omega}^v \right\} \cup \left\{ \bigcup_{e\in\mathscr{E}} \overline{\Omega}^e \right\} \cup \left\{ \bigcup_{v-e\in\mathscr{V}-\mathscr{E}} \overline{\Omega}^{v-e} \right\}.$$

Then

$$\Omega^r = \Omega \setminus \triangle.$$

Unless otherwise stated, as in Babuška and Guo [8,9] we let $w(x^v)$, $w(x^{v-e})$, $w(x^e)$ denote $w(x(x^v))$, $w(x(x^{v-e}))$, $w(x(x^e))$ respectively. Similar notation is being used for the spectral element functions $u(x^v)$, $u(x^{v-e})$, $u(x^e)$ etc. in the ensuing sections.

2.1. Spectral element functions

A set of non-conforming spectral element functions which are a sum of tensor products of polynomials in their respective coordinates are defined on the elements in the regular and singular regions of the domain Ω. Let N denote the number of refinements in the geometrical mesh and W denote an upper bound on the polynomial degree. We shall assume that N is proportional to W. We remark that throughout the paper $\frac{1}{N}$ and W refers to h and p respectively for notational uniformity.

In [15,16] we had further divided each of the elements in the regular region, vertex neighbourhoods, edge neighbourhoods and vertex-edge neighbourhoods into still smaller elements as curvilinear hexahedrons, tetrahedrons and prisms using a geometric mesh (Fig. 2) and by virtue of the fact that a tetrahedron can be split into four hexahedrons [21,22] and a prism can be split into three hexahedrons we can assume that all our elements are hexahedrons to keep the presentation simple.

Let us first consider the regular region Ω^r. The regular region Ω^r is divided into N_r curvilinear hexahedrons, tetrahedrons and prisms. Let Ω_l^r be one of the elements into which Ω^r is divided (Fig. 2(a)), which we shall assume is a curvilinear hexahedron. Let Q denote the standard cube $Q = (-1, 1)^3$. Then there is an analytic map M_l^r from Q to Ω_l^r which has an analytic inverse. Let $\{\Gamma_{l,i}^r\}_{1\le i\le n_l^r}$ be the faces of Ω_l^r. The map M_l^r is of the form

$$x = M_l^r(\lambda_1, \lambda_2, \lambda_3)$$

where $(\lambda_1, \lambda_2, \lambda_3) \in Q$, the master cube. Define the spectral element function u_l^r on Ω_l^r by

$$u_l^r(\lambda) = \sum_{i=0}^{W}\sum_{j=0}^{W}\sum_{k=0}^{W} \alpha_{i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k.$$

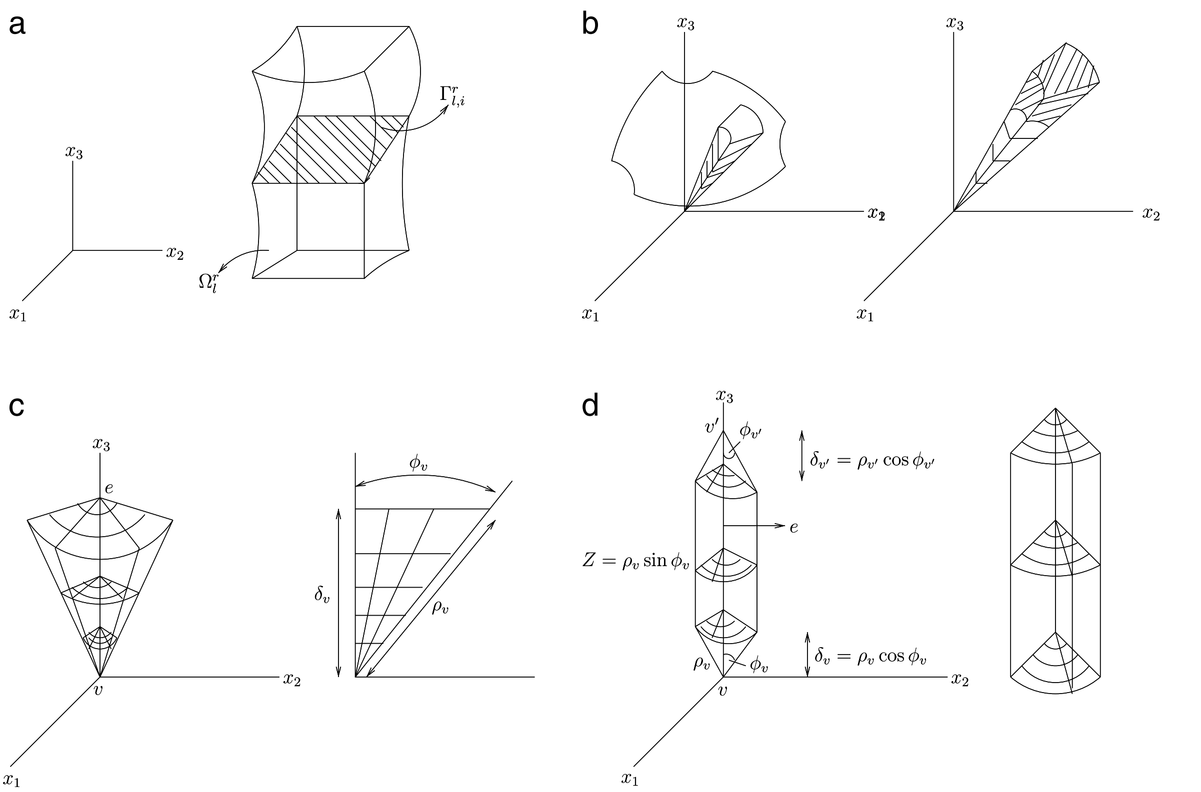

Fig. 2. (a) Elements in Ω^r, Geometric mesh and elements in the (b) vertex neighbourhood Ω^v, (c) vertex-edge neighbourhood Ω^{v-e} and (d) edge neighbourhood Ω^e.

Let v be one of the vertices of Ω. In Fig. 1(a) the vertex neighbourhood Ω^v is shown. Let S^v denote the intersection of the surface of the sphere $B_{\rho_v}(v)$ with $\bar{\Omega}^v$, i.e.

$$S^v = \left\{x \in \bar{\Omega}^v : dist(x, v) = \rho_v\right\}.$$

We divide the surface S^v into a set of triangular and quadrilateral elements as shown in Fig. 3. Let S^v_j denote these elements where $1 \leq j \leq I_v$. Here, I_v denotes a fixed constant. We now divide Ω^v into $N_v = I_v(N+1)$ curvilinear hexahedrons and prisms $\{\Omega^v_l\}_{1\leq l\leq N_v}$ (Fig. 2(b)), where Ω^v_l is of the form

$$\Omega^v_l = \left\{x : \ (\phi, \theta) \in S^v_j, \ \rho^v_k < \rho < \rho^v_{k+1}\right\}$$

for $1 \leq j \leq I_v$ and $0 \leq k \leq N$. Here, $\rho^v_k = \rho_v(\mu_v)^{N+1-k}$ and $0 < \mu_v < 1$ for $1 \leq k \leq N+1$. Moreover, $\rho^v_0 = 0$.

Let $\tilde{\Omega}^v_l$ denote the image of the element Ω^v_l in x^v coordinates. Then the geometric mesh $\{\Omega^v_l\}_{1\leq l\leq N_v}$, is mapped to a quasi-uniform mesh $\{\tilde{\Omega}^v_l\}_{1\leq l\leq N_v}$, except that the corner elements

$$\Omega^v_l = \left\{x : \ (\phi, \theta) \in S^v_j, \ 0 < \rho < \rho^v_1\right\}$$

are mapped to the semi-infinite elements

$$\tilde{\Omega}^v_l = \left\{x^v : \ (\phi, \theta) \in S^v_j, -\infty < \chi < \ln \rho^v_1\right\}.$$

If $\tilde{\Omega}^v_l$ is a corner element of the form

$$\tilde{\Omega}^v_l = \left\{x^v : \ (\phi, \theta) \in S^v_j, -\infty < \chi < \ln \rho^v_1\right\}$$

then we define $u^v_l(x^v) = u^v_0$, where u^v_0 is a constant.

Now there is an analytic map M^v_l from Q, the master cube to $\tilde{\Omega}^v_l$, which has an analytic inverse. Here, the map M^v_l is of the form

$$x^v = M^v_l(\lambda_1, \lambda_2, \lambda_3).$$

We define the spectral element function u^v_l on $\tilde{\Omega}^v_l$ (which is not a corner element) by

$$u^v_l(\lambda) = \sum_{t=0}^{W_l}\sum_{s=0}^{W_l}\sum_{r=0}^{W_l} \beta_{r,s,t}\lambda_1^r\lambda_2^s\lambda_3^t.$$

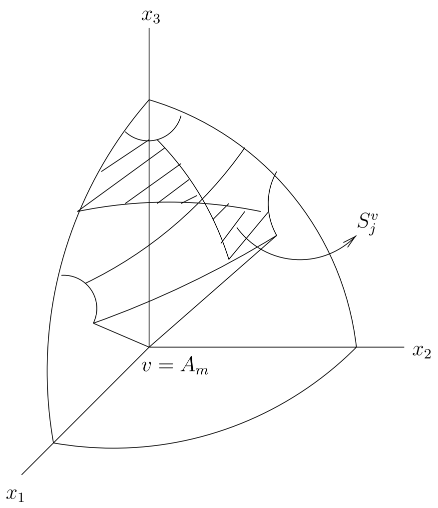

Fig. 3. Mesh imposed on the spherical boundary S^v.

Here, $1 \le W_l \le W$. Moreover, as in [23], $W_l = [\mu_1 i]$ for $1 \le i \le N$, where $\mu_1 > 0$ is a degree factor. Hereafter, $[a]$ denotes the greatest positive integer $\le a$.

Next, we define the spectral element function in the vertex-edge neighbourhoods. Let $v-e$ denote one of the vertex-edges of Ω. Here, $v-e \in \mathcal{V}-\mathcal{E}$, the set of vertex-edges of Ω. Let Ω^{v-e} denote the vertex-edge neighbourhood corresponding to the vertex-edge $v-e$ shown in Fig. 1(d) defined by

$$\Omega^{v-e} = \{x \in \Omega : \ 0 < x_3 < \delta_v, 0 < \phi < \phi_v\}.$$

Here, $\delta_v = \rho_v \cos\phi_v$. We divide Ω^{v-e} into N_{v-e} elements Ω_l^{v-e}, $l = 1, 2, \dots, N_{v-e}$ as follows:

We impose a geometrical mesh on Ω^{v-e} (Fig. 2(c)) by defining

$$(x_3)_0 = 0 \quad \text{and} \quad (x_3)_i = \delta_v(\mu_v)^{N+1-i}$$

for $1 \le i \le N+1$. Let $\zeta_i^{v-e} = \ln((x_3)_i)$ for $0 \le i \le N+1$.

Let us introduce points $\phi_0^{v-e}, \dots, \phi_{N+1}^{v-e}$ such that $\phi_0^{v-e} = 0$ and $\tan\phi_i^{v-e} = \mu_e^{N+1-i}\tan(\phi_v)$, for $1 \le i \le N+1$, $0 < \mu_e < 1$. Thus, we impose a geometrical mesh on ϕ with mesh ratio μ_e.

Finally, $\theta_l^{v-e} < \theta < \theta_u^{v-e}$. A quasi-uniform mesh

$$\theta_l^{v-e} = \theta_0^{v-e} < \theta_1^{v-e} < \dots < \theta_{I_{v-e}}^{v-e} = \theta_u^{v-e}$$

is imposed in θ.

Let $\tilde{\Omega}^{v-e}$ be the image of Ω^{v-e} in x^{v-e} coordinates. Thus, $\tilde{\Omega}^{v-e}$ is divided into $N_{v-e} = I_{v-e}(N+1)^2$ hexahedrons $\{\tilde{\Omega}_l^{v-e}\}_{l=1,\dots,N_{v-e}}$, where

$$\tilde{\Omega}_l^{v-e} = \left\{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e},\ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e},\ \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\right\}.$$

We now define the spectral element functions on the elements in $\tilde{\Omega}^{v-e}$. Consider an element

$$\tilde{\Omega}_l^{v-e} = \left\{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e},\ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e},\ -\infty < \zeta < \zeta_1^{v-e}\right\}.$$

Then on $\tilde{\Omega}_l^{v-e}$ we define

$$u_l^{v-e} = u_0^{v-e} = u_0^v$$

where u_0^v is the same constant as for the spectral element function u_l^v defined on the corner element

$$\tilde{\Omega}_l^v = \left\{x^v : \ (\phi, \theta) \in S_j^v,\ -\infty < \chi < \ln(\rho_1^v)\right\}.$$

Next, we consider the element

$$\tilde{\Omega}_l^{v-e} = \left\{x^{v-e} : -\infty < \psi < \psi_1^{v-e},\ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e},\ \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\right\}, \quad k \ge 1.$$

Then on $\tilde{\Omega}_l^{v-e}$ we define

$$u_l^{v-e}(x^{v-e}) = \sum_{t=0}^{W_l} \beta_t \zeta^t.$$

Here, $1 \le W_l \le W$. Moreover, $W_l = [\mu_2 k]$ for $1 \le k \le N$, where $\mu_2 > 0$ is a degree factor.

Now consider an element of the form (which is a non-corner element away from edges and vertices)

$$\tilde{\Omega}_l^{v-e} = \left\{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\right\}$$

for $1 \le i \le N$, $1 \le k \le N$. Then on $\tilde{\Omega}_l^{v-e}$ we define

$$u_l^{v-e}(x^{v-e}) = \sum_{r=0}^{W_l}\sum_{s=0}^{W_l}\sum_{t=0}^{V_l} \gamma_{r,s,t} \ \psi^r\theta^s\zeta^t.$$

Here, $1 \le W_l \le W$ and $1 \le V_l \le W$. Moreover, $W_l = [\mu_1 i]$, $V_l = [\mu_2 k]$ for $1 \le i, k \le N$, where $\mu_1, \mu_2 > 0$ are degree factors [23].

Finally, we consider the edge e whose end points are v and v'. The edge e coincides with the x_3 axis and the vertex v with the origin. Let the length of the edge e be l_e. Now the edge neighbourhood Ω^e shown in Fig. 1(c) is defined as

$$\Omega^e = \left\{x \in \Omega : 0 < r < Z = \rho_v \sin\phi_v, \theta_{v-e}^l < \theta < \theta_{v-e}^u, \delta_v < x_3 < l_e - \delta_v'\right\}.$$

A geometrical mesh in r (Fig. 2(d)) is imposed by defining $r_0^e = 0$ and $r_j^e = Z(\mu_e)^{N+1-j}$ for $j = 1, 2, \ldots, N+1$. We impose the same quasi-uniform mesh on θ as we did in the vertex-edge neighbourhood, viz.

$$\theta_l^e = \theta_0^e < \theta_1^e < \cdots < \theta_{I^e}^e = \theta_u^e.$$

Here, $I_e = I_{v-e}$ and $\theta_k^e = \theta_k^{v-e}$ for $0 \le k \le I_e$. A quasi-uniform mesh is defined in x_3, by choosing

$$\delta_v = Z_0^e < Z_1^e < \cdots < Z_{J_e}^e = l_e - \delta_v'.$$

Let $\tilde{\Omega}^e$ be the image of Ω^e in x^e coordinates. Thus, $\tilde{\Omega}^e$ is divided into $N_e = I_e J_e\,(N+1)$ hexahedrons $\{\tilde{\Omega}_l^e\}_{l=1,\ldots,N_e}$ where

$$\tilde{\Omega}_l^e = \left\{x^e : \ln(r_i^e) < x_1^e < \ln(r_{i+1}^e), \theta_j^e < x_2^e < \theta_{j+1}^e, Z_k^e < x_3^e < Z_{k+1}^e\right\}.$$

We now define the spectral element functions on the elements in $\tilde{\Omega}^e$. Consider an element of the form

$$\tilde{\Omega}_l^e = \left\{x^e : -\infty < x_1^e < \ln(r_1^e), \theta_j^e < x_2^e < \theta_{j+1}^e, Z_k^e < x_3^e < Z_{k+1}^e\right\}.$$

Then we define

$$u_l^e(x^e) = \sum_{t=0}^{W}\alpha_t(x_3^e)^t.$$

This representation is valid for all j for fixed k.

Next, consider the element (away from edges)

$$\tilde{\Omega}_l^e = \left\{x^e : \ln(r_i^e) < x_1^e < \ln(r_{i+1}^e), \theta_j^e < x_2^e < \theta_{j+1}^e, Z_k^e < x_3^e < Z_{k+1}^e\right\}$$

for $1 \le i \le N$. Then we define

$$u_l^e(x^e) = \sum_{r=0}^{W_l}\sum_{s=0}^{W_l}\sum_{t=0}^{W}\alpha_{r,s,t}(x_1^e)^r(x_2^e)^s(x_3^e)^t.$$

Here, $1 \le W_l \le W$. Moreover, $W_l = [\mu_1 i]$ for all $1 \le i \le N$, where $\mu_1 > 0$ is a degree factor [23].

Let $\{\mathcal{F}_u\}$ denote the spectral element representation of the function u on the whole domain Ω. We now define the functional $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ in brief (for details see [15]) which is used to formulate our numerical scheme, as follows:

$$\mathcal{R}^{N,W}(\{\mathcal{F}_u\}) = \mathcal{R}^{N,W}_{regular}(\{\mathcal{F}_u\}) + \mathcal{R}^{N,W}_{vertices}(\{\mathcal{F}_u\}) + \mathcal{R}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) + \mathcal{R}^{N,W}_{edges}(\{\mathcal{F}_u\}). \tag{2.2}$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and let $f_l^r(\lambda) = f(M_l^r(\lambda_1, \lambda_2, \lambda_3))$ where $\lambda \in Q$ for $l = 1, 2, \ldots, N_r$ and let $J_l^r(\lambda)$ denote the Jacobian of the mapping M_l^r. Define $F_l^r(\lambda) = f_l^r(\lambda)\sqrt{J_l^r(\lambda)}$, and $L_l^r u_l^r(\lambda) = Lu_l^r(M_l^r(\lambda))\sqrt{J_l^r(\lambda)}$.

Now consider the boundary conditions $w = g_k$ on Γ_k for $k \in \mathcal{D} = \Gamma^{[0]}$ and $\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)_A = h_k$ on Γ_k for $k \in \mathcal{N} = \Gamma^{[1]}$. Let $\Gamma_{i,k}^r = \Gamma_k \cap \partial\Omega_i^r$ be the image of the mapping M_i^r corresponding to $\lambda_1 = -1$. Let $g_{i,k}^r = g_k(M_i^r(-1, \lambda_2, \lambda_3))$ and $h_{i,k}^r = h_k(M_i^r(-1, \lambda_2, \lambda_3))$ where $-1 \le \lambda_2, \lambda_3 \le 1$.

We now define

$$\begin{aligned}\mathcal{R}^{N,W}_{regular}(\{\mathcal{F}_u\}) = &\sum_{l=1}^{N_r}\int_{Q=(M_l^r)^{-1}(\Omega_l^r)}\left|L_l^r u_l^r(\lambda) - F_l^r(\lambda)\right|^2 d\lambda \\ &+ \sum_{\Gamma_{l,i}^r\subseteq\bar{\Omega}^r\setminus\partial\Omega}\left(\|[u]\|_{0,\Gamma_{l,i}^r}^2 + \sum_{k=1}^{3}\left\|[u_{x_k}]\right\|_{1/2,\Gamma_{l,i}^r}^2\right) \\ &+ \sum_{\Gamma_{l,i}^r\subseteq\Gamma^{[0]}}\left\|u_l^r - g_{l,i}^r\right\|_{3/2,\Gamma_{l,i}^r}^2 + \sum_{\Gamma_{l,i}^r\subseteq\Gamma^{[1]}}\left\|\left(\frac{\partial u_l^r}{\partial \boldsymbol{\nu}}\right)_A - h_{l,i}^r\right\|_{1/2,\Gamma_{l,i}^r}^2.\end{aligned} \tag{2.3}$$

Here, $[\cdot]$ denotes the jump in the function and its derivatives at the inter element boundaries.

Now we define $\mathcal{R}^{N,W}_{vertices}(\{\mathcal{F}_u\})$. Consider the vertex neighbourhood Ω^v of the vertex $v \in \mathcal{V}$, the set of vertices. Define $F^v_l(x^v) = e^{5/2\chi}\sqrt{\sin\phi} f(x(x^v))$ for $x^v_l \in \tilde{\Omega}^v_l$, $1 \le l \le N_v$ and we define $L^v u^v_l(x^v)$ so that

$$\int_{\tilde{\Omega}^v_l} \left|L^v u^v_l(x^v)\right|^2 dx^v = \int_{\Omega^v_l} \rho^2 \left|Lu^v_l(x)\right|^2 dx.$$

Here, dx denotes a volume element in x coordinates.

We now consider the boundary conditions $w = g_i$ on Γ_i for $i \in \mathcal{D} = \Gamma^{[0]}$ and $\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)_A = h_i$ on Γ_i for $i \in \mathcal{N} = \Gamma^{[1]}$. Let $\Gamma^v_{l,i} = \Gamma_i \cap \partial\Omega^v_l$ and suppose Ω^v_l is not a corner element. Moreover, it is assumed that $\Gamma^v_{l,i}$ lies on the $x_2 - x_3$ plane for simplicity. Define

$$g^v_{l,i}(x^v) = w = g_i(x(x^v)) \quad \text{for } \Gamma^v_{l,i} \subseteq \Gamma^{[0]},$$
$$h^v_{l,i}(x^v) = \left(\frac{\partial w}{\partial \boldsymbol{\nu}^v}\right)_{A^v} = \frac{e^\chi}{\sin\phi} h_i(x(x^v)) \quad \text{for } \Gamma^v_{l,i} \subseteq \Gamma^{[1]}.$$

Let $R^v_{l,i} = \sup_{x^v \in \tilde{\Gamma}^v_{l,i}}(e^{x^v_3})$. We now define the functional

$$\mathcal{R}^{N,W}_v(\{\mathcal{F}_u\}) = \sum_{l=1,\mu(\tilde{\Omega}^v_l)<\infty}^{N_v} \int_{\tilde{\Omega}^v_l} \left|L^v u^v_l(x^v) - F^v_l(x^v)\right|^2 dx^v + \sum_{\substack{\Gamma^v_{l,i}\subseteq\bar{\Omega}^v\setminus\partial\Omega \\ \mu(\tilde{\Gamma}^v_{l,i})<\infty}} \left(\left\|\sqrt{R^v_{l,i}}[u]\right\|^2_{0,\tilde{\Gamma}^v_{l,i}} + \sum_{k=1}^{3} \left\|\sqrt{R^v_{l,i}}[u_{x^v_k}]\right\|^2_{1/2,\tilde{\Gamma}^v_{l,i}} \right)$$
$$+ \sum_{\substack{\Gamma^v_{l,i}\subseteq\Gamma^{[0]}, \\ \mu(\tilde{\Gamma}^v_{l,i})<\infty}} \left\|\sqrt{R^v_{l,i}}\left(u^v_l - g^v_{l,i}\right)\right\|^2_{3/2,\tilde{\Gamma}^v_{l,i}} + \sum_{\substack{\Gamma^v_{l,i}\subseteq\Gamma^{[1]}, \\ \mu(\tilde{\Gamma}^v_{l,i})<\infty}} \left\|\sqrt{R^v_{l,i}}\left(\left(\frac{\partial u^v_l}{\partial \boldsymbol{\nu}^v}\right)_{A^v} - h^v_{l,i}\right)\right\|^2_{1/2,\tilde{\Gamma}^v_{l,i}}. \tag{2.4}$$

The functional $\mathcal{R}^{N,W}_{vertices}(\{\mathcal{F}_u\})$ is then given by

$$\mathcal{R}^{N,W}_{vertices}(\{\mathcal{F}_u\}) = \sum_{v\in\mathcal{V}} \mathcal{R}^{N,W}_v(\{\mathcal{F}_u\}). \tag{2.5}$$

Next, we define $\mathcal{R}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\})$. Consider the vertex-edge neighbourhood Ω^{v-e} of the vertex-edge $v - e \in \mathcal{V} - \mathcal{E}$. Let $F^{v-e}_l(x^{v-e}) = e^{2x^{v-e}_1} e^{\frac{5}{2}x^{v-e}_3} f(x(x^{v-e}))$ for $x^{v-e} \in \tilde{\Omega}^{v-e}_l$, $1 \le l \le N_{v-e}$ and we define $L^{v-e}u^{v-e}_l(x^{v-e})$ such that

$$\int_{\tilde{\Omega}^{v-e}_n} \left|L^{v-e}u^{v-e}_l(x^{v-e})\right|^2 dx^{v-e} = \int_{\Omega^v_l} \rho^2 \sin^2\phi |Lu^v_l(x)|^2 dx.$$

We now consider the boundary conditions $w = g_k$ on Γ_k for $k \in \mathcal{D} = \Gamma^{[0]}$ and $\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)_A = h_k$ on Γ_k for $k \in \mathcal{N} = \Gamma^{[1]}$. Then $\left(\frac{\partial w}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} = e^{x^{v-e}_3} e^{x^{v-e}_1} h_k(x(x^{v-e}))$. Let $\Gamma^{v-e}_{l,k} = \Gamma_k \cap \partial\Omega^{v-e}_l$ and suppose Ω^{v-e}_l is not a corner element. Moreover, it is assumed that $\Gamma^{v-e}_{l,k}$ lies on the $x_2 - x_3$ plane for simplicity and $\mu(\tilde{\Gamma}^{v-e}_{l,k}) < \infty$. Define

$$g^{v-e}_{l,k}(x^{v-e}) = w = g_k(x(x^{v-e})) \quad \text{for } \Gamma^{v-e}_{l,k} \subseteq \Gamma^{[0]},$$
$$h^{v-e}_{l,k}(x^{v-e}) = \left(\frac{\partial w}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} = e^{x^{v-e}_3} e^{x^{v-e}_1} h_k(x(x^{v-e})) \quad \text{for } \Gamma^{v-e}_{l,k} \subseteq \Gamma^{[1]}.$$

We define the functional

$$\mathcal{R}^{N,W}_{v-e}(\{\mathcal{F}_u\}) = \sum_{l=1,\mu(\tilde{\Omega}^{v-e}_l)<\infty}^{N_{v-e}} \int_{\tilde{\Omega}^{v-e}_l} \left|L^{v-e}u^{v-e}_l(x^{v-e}) - F^{v-e}_l(x^{v-e})\right|^2 dx^{v-e}$$
$$+ \sum_{\substack{\Gamma^{v-e}_{l,k}\subseteq\bar{\Omega}^{v-e}\setminus\partial\Omega, \\ \mu(\tilde{\Gamma}^{v-e}_{l,k})<\infty}} \left(\left\|\sqrt{F^{v-e}_{l,k}G^{v-e}_{l,k}}\,[u]\right\|^2_{0,\tilde{\Gamma}^{v-e}_{l,k}} + \left\|\left|\,[u_{x^{v-e}_1}]\,\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}} + \left\|\left|[u_{x^{v-e}_2}]\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}} + \left\|\left|E^{v-e}_{l,k}\,[u_{x^{v-e}_3}]\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}} \right)$$
$$+ \sum_{\substack{\Gamma^{v-e}_{l,k}\subseteq\Gamma^{[0]}, \\ \mu(\tilde{\Gamma}^{v-e}_{l,k})<\infty}} \left(\left\|\sqrt{F^{v-e}_{l,k}}\,\left(u^{v-e}_l - g^{v-e}_{l,k}\right)\right\|^2_{0,\tilde{\Gamma}^{v-e}_{l,k}} + \left\|\left|\left(u_{x^{v-e}_1} - (g^{v-e}_{l,k})_{x^{v-e}_1}\right)\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}} \right.$$
$$\left. + \left\|\left|E^{v-e}_{l,k}\,\left(u_{x^{v-e}_3} - (g^{v-e}_{l,k})_{x^{v-e}_3}\right)\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}} \right) + \sum_{\substack{\Gamma^{v-e}_{l,k}\subseteq\Gamma^{[1]}, \\ \mu(\tilde{\Gamma}^{v-e}_{l,k})<\infty}} \left\|\left|\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} - h^{v-e}_{l,k}\right|\right\|^2_{\tilde{\Gamma}^{v-e}_{l,k}}. \tag{2.6}$$

Here, $E^{v-e}_{l,k} = \sup_{x^{v-e}\in\tilde{\Gamma}^{v-e}_{l,k}}(\sin\phi)$ and $F^{v-e}_{l,k} = \sup_{x^{v-e}\in\tilde{\Gamma}^{v-e}_{l,k}}(e^{x_3^{v-e}})$. Moreover, $G^{v-e}_{l,k}$ and the anisotropic Sobolev norms $|||(.)|||$ are as defined in Chapter 2 of [18].

Then the functional $\mathcal{R}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\})$ is defined as follows:

$$\mathcal{R}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) = \sum_{v-e\in\mathcal{V}-E}\mathcal{R}^{N,W}_{v-e}(\{\mathcal{F}_u\}). \tag{2.7}$$

Finally, we define the functional $\mathcal{R}^{N,W}_{edges}(\{\mathcal{F}_u\})$. Consider the edge neighbourhood Ω^e of the edge $e \in \mathcal{E}$. Let $F^e_l(x^e) = e^{2x^e_1}f(x(x^e))$ for $x^e \in \tilde{\Omega}^e_l$, $1 \le l \le N_e$ and we define $L^e u^e_l(x^e)$ such that

$$\int_{\tilde{\Omega}^e_l}\left|L^e u^e_l(x^e)\right|^2 dx^e = \int_{\Omega^e_l} r^2|Lu^e_l(x)|^2 dx.$$

We now consider the boundary conditions $w = g_k$ on Γ_k for $k \in \mathcal{D} = \Gamma^{[0]}$ and $\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)_A = h_k$ on Γ_k for $k \in \mathcal{N} = \Gamma^{[1]}$. Then $\left(\frac{\partial w}{\partial \boldsymbol{\nu}^e}\right)_{A^e} = e^{x^e_1}h_k(x(x^e))$. Let $\Gamma^e_{m,k} = \Gamma_k \cap \partial\Omega^e_m$ and suppose Ω^e_m is not a corner element. Moreover, it is assumed that $\Gamma^e_{m,k}$ lies on the $x_2 - x_3$ plane for simplicity and $\mu(\tilde{\Gamma}^e_{m,k}) < \infty$. Define

$$g^e_{m,k}(x^e) = w = g_k(x(x^e)) \quad \text{for } \Gamma^e_{m,k} \subseteq \Gamma^{[0]},$$
$$h^e_{m,k}(x^e) = \left(\frac{\partial w}{\partial \boldsymbol{\nu}^e}\right)_{A^e} = e^{x^e_1}h_k(x(x^e)) \quad \text{for } \Gamma^e_{m,k} \subseteq \Gamma^{[1]}.$$

We define

$$\begin{aligned}\mathcal{R}^{N,W}_e(\{\mathcal{F}_u\}) = &\sum_{l=1,\mu(\tilde{\Omega}^e_l)<\infty}^{N_e}\int_{\tilde{\Omega}^e_l}\left|L^e u^e_l(x^e) - F^e_l(x^e)\right|^2 dx^e \\ &+ \sum_{\substack{\Gamma^e_{m,k}\subseteq\bar{\Omega}^e\setminus\partial\Omega,\\ \mu(\tilde{\Gamma}^e_{m,k})<\infty}}\left(\left\|\sqrt{H^e_{m,k}}\,[u]\right\|^2_{0,\tilde{\Gamma}^e_{m,k}} + \left|\left|\left|\,[u_{x^e_1}]\,\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}} + \left|\left|\left|[u_{x^e_2}]\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}} + \left|\left|\left|G^e_{m,k}\,[u_{x^e_3}]\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}}\right) \\ &+ \sum_{\substack{\Gamma^e_{m,k}\subseteq\Gamma^{[0]},\\ \mu(\tilde{\Gamma}^e_{m,k})<\infty}}\left(\left\|\,(u^e_m - g^e_{m,k})\,\right\|^2_{0,\tilde{\Gamma}^e_{m,k}} + \left|\left|\left|\left(u_{x^e_1} - (g^e_{m,k})_{x^e_1}\right)\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}}\right. \\ &\left. + \left|\left|\left|G^e_{m,k}\left(u_{x^e_3} - (g^e_{m,k})_{x^e_3}\right)\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}}\right) + \sum_{\substack{\Gamma^e_{m,k}\subseteq\Gamma^{[1]},\\ \mu(\tilde{\Gamma}^e_{m,k})<\infty}}\left|\left|\left|\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} - h^e_{m,k}\right|\right|\right|^2_{\tilde{\Gamma}^e_{m,k}}.\end{aligned} \tag{2.8}$$

Here, $G^e_{m,k} = \sup_{x^e\in\tilde{\Gamma}^e_{m,k}}(e^{\tau})$ and $H^e_{m,k}$ and the anisotropic Sobolev norms $|||(.)|||$ are as defined in Chapter 2 of [18].

The functional $\mathcal{R}^{N,W}_{edges}(\{\mathcal{F}_u\})$ is now defined as

$$\mathcal{R}^{N,W}_{edges}(\{\mathcal{F}_u\}) = \sum_{e\in\mathcal{E}}\mathcal{R}^{N,W}_e(\{\mathcal{F}_u\}). \tag{2.9}$$

Finally, using (2.3), (2.5), (2.7) and (2.9) in (2.2) we can define $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$.

Define the quadratic forms $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}(\{\mathcal{F}_u\})$ by

$$\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) = \mathcal{V}^{N,W}_{regular}(\{\mathcal{F}_u\}) + \mathcal{V}^{N,W}_{vertices}(\{\mathcal{F}_u\}) + \mathcal{V}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) + \mathcal{V}^{N,W}_{edges}(\{\mathcal{F}_u\}) \tag{2.10}$$

and

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) = \mathcal{U}^{N,W}_{regular}(\{\mathcal{F}_u\}) + \mathcal{U}^{N,W}_{vertices}(\{\mathcal{F}_u\}) + \mathcal{U}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) + \mathcal{U}^{N,W}_{edges}(\{\mathcal{F}_u\}). \tag{2.11}$$

Here, $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ is the functional $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ as defined in (2.2) with zero data for F, $g^{[0]}$ and $g^{[1]}$ in (2.1) and

$$\mathcal{U}^{N,W}_{regular}(\{\mathcal{F}_u\}) = \sum_{l=1}^{N_r}\int_{Q=(M^r_l)^{-1}(\Omega^r_l)}\sum_{|\alpha|\le 2}\left|D^\alpha_\lambda u^r_l\right|^2 d\lambda$$
$$\mathcal{U}^{N,W}_{vertices}(\{\mathcal{F}_u\}) = \sum_{v\in\mathcal{V}}\mathcal{U}^{N,W}_v(\{\mathcal{F}_u\})$$

$$\mathcal{U}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) = \sum_{v-e\in\mathcal{V}-\mathcal{E}} \mathcal{U}^{N,W}_{v-e}(\{\mathcal{F}_u\})$$
$$\mathcal{U}^{N,W}_{edges}(\{\mathcal{F}_u\}) = \sum_{e\in\mathcal{E}} \mathcal{U}^{N,W}_{e}(\{\mathcal{F}_u\})$$

with

$$\mathcal{U}^{N,W}_{v}(\{\mathcal{F}_u\}) = \sum_{l=1}^{N_v}\int_{\tilde{\Omega}^v_l} e^{x_3^v}\sum_{|\alpha|\le 2}\left|D^\alpha_{x^v}u^v_l(x^v)\right|^2\, dx^v,$$
$$\mathcal{U}^{N,W}_{v-e}(\{\mathcal{F}_u\}) = \sum_{l=1,\mu(\tilde{\Omega}^{v-e}_l)<\infty}^{N_{v-e}}\int_{\tilde{\Omega}^{v-e}_l} e^{x_3^{v-e}}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u^{v-e}_l}{\partial x^{v-e}_i\partial x^{v-e}_j}\right)^2 + \sum_{i=1}^{2}\sin^2\phi\left(\frac{\partial^2 u^{v-e}_l}{\partial x^{v-e}_i\partial x^{v-e}_3}\right)^2\right.$$
$$\left. + \sin^4\phi\left(\frac{\partial^2 u^{v-e}_l}{(\partial x^{v-e}_3)^2}\right)^2 + \sum_{i=1}^{2}\left(\frac{\partial u^{v-e}_l}{\partial x^{v-e}_i}\right)^2 + \sin^2\phi\left(\frac{\partial u^{v-e}_l}{\partial x^{v-e}_3}\right)^2 + (u^{v-e}_l)^2\right)dx^{v-e}$$
$$+ \sum_{\substack{l=1,\\ \mu(\tilde{\Omega}^{v-e}_l)=\infty}}^{N_{v-e}}\int_{\tilde{\Omega}^{v-e}_l}(u^{v-e}_l)^2 e^{x_3^{v-e}} w^{v-e}(x_1^{v-e})dx^{v-e},$$

and

$$\mathcal{U}^{N,W}_{e}(\{\mathcal{F}_u\}) = \sum_{l=1,\mu(\tilde{\Omega}^{e}_l)<\infty}^{N_e}\int_{\tilde{\Omega}^e_l}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u^e_l}{\partial x^e_i\partial x^e_j}\right)^2 + e^{2\tau}\sum_{i=1}^{2}\left(\frac{\partial^2 u^e_l}{\partial x^e_i\partial x^e_3}\right)^2 + e^{4\tau}\left(\frac{\partial^2 u^e_l}{(\partial x^e_3)^2}\right)^2\right.$$
$$\left.+\sum_{i=1}^{2}\left(\frac{\partial u^e_l}{\partial x^e_i}\right)^2 + e^{2\tau}\left(\frac{\partial u^e_l}{\partial x^e_3}\right)^2 + (u^e_l)^2\right)dx^e + \sum_{\substack{l=1,\\ \mu(\tilde{\Omega}^{e}_l)=\infty}}^{N_e}\int_{\tilde{\Omega}^e_l}(u^e_l)^2 w^e(x^e_1)dx^e.$$

Here, dx^v, dx^{v-e} and dx^e denote volume elements in x^v, x^{v-e} and x^e coordinates respectively. Moreover, μ denotes measure and $w^{v-e}(x^{v-e}_1)$, $w^e(x^e_1)$ are properly chosen weight functions [18]. Note that the Sobolev norms defined above are weighted and are anisotropic in the edge and vertex-edge neighbourhoods.

We now state the main stability estimate theorem of [15].

Theorem 2.1 (*Theorem 4.1 of [15]*). *Consider the elliptic boundary value problem (2.1). Suppose the boundary conditions are Dirichlet. Then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \le C(\ln W)^2\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

Next, we state the corresponding result for general boundary conditions.

Theorem 2.2 (*Theorem 4.2 of [15]*). *If the boundary conditions for the elliptic boundary value problem (2.1) are mixed then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \le CN^4\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

For proof of the stability theorem we refer to [15,16].

In [17], we presented a numerical scheme based on the stability estimate theorem where we minimize a functional over the space of spectral element functions which is the sum of a weighted squared norm of the residuals in the partial differential equations and the squared norm of the residuals in the boundary conditions in fractional Sobolev spaces and enforce continuity by adding a term which measures the jump in the function and its derivatives at inter-element boundaries in fractional Sobolev norms suitably weighted, to the functional being minimized. Thus, our numerical scheme reads as:

Find $\mathcal{F}_s \in \mathcal{S}^{N,W}(\Omega)$ *which minimizes the functional* $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ *over all* $\mathcal{F}_u \in \mathcal{S}^{N,W}$. *Here,* $\mathcal{S}^{N,W}(\Omega)$ *denotes the space of spectral element functions* $\mathcal{F}_u$ *on* Ω *and* $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ *is as defined in (2.2).*

3. Error estimates

It is well known that for three dimensional elliptic problems containing singularities in the form of vertices and edges, the geometric mesh and a proper choice of element degree distribution leads to exponential convergence and efficiency of computations (see [24,23,25] and references therein).

In this section we show that the error obtained from the proposed method is exponentially small in N. The optimal rate of convergence with respect to N_{dof}, the number of degrees of freedom is also provided. Our analysis of error estimates is similar to that in two dimensions (see [26,24,27,20] and references therein). Here, we briefly describe the main steps of the proof and leave the details which are similar to those in [27].

Let $\mathcal{S}^{N,W}(\Omega^v)$, $\mathcal{S}^{N,W}(\Omega^{v-e})$, $\mathcal{S}^{N,W}(\Omega^e)$, and $\mathcal{S}^{N,W}(\Omega^r)$ denote the restrictions of $S^{N,W}(\Omega)$ to the set of vertex neighbourhoods Ω^v, vertex-edge neighbourhoods Ω^{v-e}, edge neighbourhoods Ω^e and regular region Ω^r respectively. Let $\{\mathcal{F}_z\}$ minimize $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ over all $\{\mathcal{F}_u\} \in \mathcal{S}^{N,W}(\Omega)$, the space of spectral element functions. We write one more representation for $\{\mathcal{F}_z\}$ as follows:

$$\{\mathcal{F}_z\} = \left\{\left\{z_l^r(\lambda_1, \lambda_2, \lambda_3)\right\}_{l=1}^{N_r}, \left\{z_l^v(\phi, \theta, \chi)\right\}_{l=1}^{N_v}, \left\{z_l^{v-e}(\psi, \theta, \zeta)\right\}_{l=1}^{N_{v-e}}, \left\{z_l^e(\tau, \theta, x_3)\right\}_{l=1}^{N_e}\right\}.$$

Here, $z_l^r(\lambda_1, \lambda_2, \lambda_3)$ is a polynomial of degree W in each of its variables.

On corner elements $\tilde{\Omega}_l^v$ with $\mu(\tilde{\Omega}_l^v) = \infty$, we let $z_l^v = a_v$, where a_v is a constant. In all other elements in the vertex neighbourhoods, $z_l^v(\phi, \theta, \chi)$ is a polynomial of degree W_l, $1 \le W_l \le W$, $W_l = [\mu_1 i]$ for all $1 \le i \le N+1$, in ϕ, θ and χ variables separately, where $\mu_1 > 0$ is a degree factor.

$z_l^{v-e} = a_{v-e} = a_v$, on corner elements $\tilde{\Omega}_l^{v-e}$ of the form

$$\tilde{\Omega}_l^{v-e} = \{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ -\infty < \zeta < \zeta_1^{v-e}\}$$

and z_l^{v-e} is a polynomial of degree V_l in ζ, $1 \le V_l \le W$, $V_l = [\mu_2 n]$ for all $1 \le n \le N$, $\mu_2 > 0$, on corner elements $\tilde{\Omega}_l^{v-e}$ of the form

$$\tilde{\Omega}_l^{v-e} = \left\{x^{v-e} : -\infty < \psi < \psi_1^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_n^{v-e} < \zeta < \zeta_{n+1}^{v-e}\right\}$$

with $n \ge 1$.

On the remaining elements in the vertex-edge neighbourhoods, $z_l^{v-e}(\psi, \theta, \zeta)$ is a polynomial of degree W_l, $1 \le W_l \le W$, $W_l = [\mu_1 i]$ for all $1 \le i \le N$, in ψ, θ variables and of degree V_l, $1 \le V_l \le W$, $V_l = [\mu_2 n]$ for all $1 \le n \le N$, in ζ variable with $\mu_1 > 0$, $\mu_2 > 0$.

Finally, on corner elements $\tilde{\Omega}_l^e$ with $\mu(\tilde{\Omega}_l^e) = \infty$, z_l^e is a polynomial of degree W in x_3 and on the remaining elements $\tilde{\Omega}_l^e$ away from edges $z_l^e(\tau, \theta, x_3)$ is a polynomial of degree W_l, $1 \le W_l \le W$, $W_l = [\mu_1 i]$, $1 \le i \le N$, $\mu_1 > 0$ in τ, θ variables and of degree W in the x_3 variable.

Approximation in the regular region:

Let us first consider the regular region Ω^r of Ω. Ω^r has been divided into Ω_l^r, $l = 1, \ldots, N_r$ curvilinear hexahedrons, tetrahedrons and prisms. Let M_l^r be the analytic map from Q to Ω_l^r.

Let $\Pi^{W,W,W}(w(M_l^r(\lambda)))$ denote the projection of the solution w into the space of polynomials of degree N in each of its variables with respect to the usual inner product in $H^2(Q)$. Then on Ω_l^r we define

$$s_l^r(\lambda) = \Pi^{W,W,W}(w(M_l^r(\lambda))) = \Pi^{W,W,W}(w(\lambda)), \quad \text{for } \lambda \in Q.$$

Approximation in vertex neighbourhoods:

Let us now consider the vertex neighbourhood Ω^v of the vertex $v \in \mathcal{V}$, where $\mathcal{V}$ denotes the set of vertices of Ω (see Fig. 2(b)). We had divided Ω^v into Ω_l^v, $l = 1, \ldots, N_v$ elements [15]. If $\tilde{\Omega}_l^v$ is a corner element of the form

$$\tilde{\Omega}_l^v = \{x^v : (\phi, \theta) \in S_j^v, -\infty < \chi < \ln(\rho_1^v)\}$$

then on $\tilde{\Omega}_l^v$ we define

$$s_l^v = w_v,$$

where $w_v = w(v)$ denotes the value of w at the vertex v.

If $\tilde{\Omega}_l^v$ is of the form

$$\tilde{\Omega}_l^v = \{x^v : (\phi, \theta) \in S_j^v, \ln(\rho_i^v) < \chi < \ln(\rho_{i+1}^v)\}$$

then on $\tilde{\Omega}_l^v$ we approximate $(w(x^v) - w_v)$ by its projection, denoted by Π^{W_l,W_l,W_l}, into the space of polynomials of degree N in each of its variables separately with respect to the usual inner product in $H^2(\tilde{\Omega}_l^v)$ and define

$$s_l^v(x^v) = \Pi^{W_l,W_l,W_l}(w(x^v) - w_v) + w_v.$$

Here, $1 \le W_l \le W$, $W_l = [\mu_1 i]$ for all $1 \le i \le N$, where $\mu_1 > 0$ is a degree factor [23].

Approximation in vertex-edge neighbourhoods:

We now consider the vertex-edge neighbourhood Ω^{v-e} of the vertex-edge $v-e \in \mathcal{V}-\mathcal{E}$ (see Fig. 2(c)). Here, as earlier, $\mathcal{V}-\mathcal{E}$ denotes the set of vertex-edges of the domain Ω. Ω^{v-e} is divided into Ω_q^{v-e}, $q = 1, \ldots, N_{v-e}$ elements using a geometric mesh in ϕ, x_3 variables and a quasi-uniform mesh in θ variable.

Let $\tilde{\Omega}_q^{v-e}$ be the image of Ω_q^{v-e} in x^{v-e} coordinates. If $\tilde{\Omega}_q^{v-e}$ is a corner element of the form

$$\tilde{\Omega}_q^{v-e} = \{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ -\infty < \zeta < \zeta_1^{v-e}\}$$

then on $\tilde{\Omega}_q^{v-e}$ we define

$$s_l^{v-e} = w_{v-e} = w_v.$$

Here, w_v is the value of w at the vertex v.

Next, suppose $\tilde{\Omega}_q^{v-e}$ is a corner element of the form

$$\tilde{\Omega}_q^{v-e} = \left\{x^{v-e} : -\infty < \psi < \psi_1^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_n^{v-e} < \zeta < \zeta_{n+1}^{v-e}\right\}$$

with $n \geq 1$. Let $s(x_3^{v-e}) = w(x_1, x_2, x_3)|_{(x_1=0, x_2=0)}$ be the value of w along the edge e. Define

$$\sigma(x_3^{v-e}) = s(x_3^{v-e}) - w_v.$$

Let $\Pi^{V_q}(\sigma(x_3^{v-e}))$ be the orthogonal projection of $\sigma(x_3^{v-e})$ into the space of polynomials in $H^2(I)$. Then we define

$$s_l^{v-e}(x_3^{v-e}) = \Pi^{V_q}(\sigma(x_3^{v-e})) + w_v = \Pi^{V_q} s(x_3^{v-e}).$$

Here, $1 \leq V_q \leq W$. Moreover, $W_q = [\mu_2 n]$ for all $1 \leq n \leq N$, where $\mu_2 > 0$ is a degree factor [23].

The remaining elements $\tilde{\Omega}_q^{v-e}$ in $\tilde{\Omega}^{v-e}$ are of the form

$$\tilde{\Omega}_q^{v-e} = \left\{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_n^{v-e} < \zeta < \zeta_{n+1}^{v-e}\right\}$$

with $i \geq 1, k \geq 1$. Let us write $\alpha(x^{v-e}) = w(x^{v-e}) - s(x_3^{v-e})$. Then on $\tilde{\Omega}_q^{v-e}$ we approximate $\alpha(x^{v-e})$ by its projection, denoted by Π^{W_q, W_q, V_q}, into the space of polynomials with respect to the usual inner product in $H^2(\tilde{\Omega}_q^{v-e})$. We now define

$$s_l^{v-e}(x^{v-e}) = \Pi^{W_q, W_q, V_q}(\alpha(x^{v-e})) + \Pi^{V_q}(s(x_3^{v-e})).$$

Here, $1 \leq W_q \leq W$ and $1 \leq V_q \leq W$. Moreover, $W_q = [\mu_1 i]$, $V_q = [\mu_2 n]$ for all $1 \leq i, n \leq N$, where $\mu_1, \mu_2 > 0$ are degree factors [23].

Approximation in edge neighbourhoods:

Finally, we discuss approximation in the edge neighbourhood elements and define comparison functions there. Consider the edge neighbourhood Ω^e of the edge $e \in \mathcal{E}$ (see Fig. 2(d)). Here, as before, $\mathcal{E}$ denotes the set of edges of the domain Ω. We had divided Ω^e into Ω_p^e, $p = 1, \ldots, N_e$ elements.

Let $\tilde{\Omega}_p^e$ be the image of Ω_p^e in x^e coordinates. Let $\tilde{\Omega}_p^e$ be a corner element of the form

$$\tilde{\Omega}_p^e = \left\{x^e : -\infty < x_1^e < \ln(r_1^e), \theta_j^e < x_2^e < \theta_{j+1}^e, Z_n^e < x_3^e < Z_{n+1}^e\right\}.$$

Let $s(x_3^e) = w(x_1, x_2, x_3)|_{(x_1=0, x_2=0)}$. Then on $\tilde{\Omega}_p^e$ we approximate $s(x_3^e)$ by its projection onto the space of polynomials with respect to the usual inner product in $H^2(I)$. Let $\Pi^W(s(x_3^e))$ denote this projection, then we define

$$s_l^e(x_3^e) = \Pi^W(s(x_3^e)).$$

Next, let $\tilde{\Omega}_p^e$ be of the form

$$\tilde{\Omega}_p^e = \left\{x^e : \ln(r_i^e) < x_1^e < \ln(r_{i+1}^e), \theta_j^e < x_2^e < \theta_{j+1}^e, Z_n^e < x_3^e < Z_{n+1}^e\right\}$$

with $1 \leq i \leq N$, $0 \leq j \leq I_e$, $0 \leq n \leq J_e$. Let us write $\beta(x^e) = w(x^e) - s(x_3^e)$. Then on $\tilde{\Omega}_p^e$ we approximate $\beta(x^e)$ by its projection, denoted by $\Pi^{W_p, W_p, W}$, into the space of polynomials with respect to the usual inner product in $H^2(\tilde{\Omega}_p^e)$. Define

$$s_l^e(x^e) = \Pi^{W_p, W_p, W}(\beta(x^e)) + \Pi^W(s(x_3^e)).$$

Here, $1 \leq W_p \leq W$. Moreover, $W_p = [\mu_1 i]$ for all $1 \leq i \leq N$, where $\mu_1 > 0$ is a degree factor [23].

Now consider the set of functions $\left\{\{s_l^r\}_{l=1}^{N_r}, \{s_l^v\}_{l=1}^{N_v}, \{s_l^{v-e}\}_{l=1}^{N_{v-e}}, \{s_l^e\}_{l=1}^{N_e}\right\}$ and denote it by $\{\mathcal{F}_s\}$. We will show that the functional $\mathcal{R}^{N,W}(\{\mathcal{F}_s\})$ is exponentially small in N.

Using results on approximation theory in [24,27] it follows that there exist constants C and $b > 0$ such that the estimate

$$\mathcal{R}^{N,W}(\{\mathcal{F}_s\}) \leq Ce^{-bN} \tag{3.1}$$

holds.

Now $\{\mathcal{F}_z\}$ minimizes $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ over all $\{\mathcal{F}_u\} \in \mathcal{S}^{N,W}(\Omega)$, the space of spectral element functions. Then from (3.1), we have

$$\mathcal{R}^{N,W}(\{\mathcal{F}_z\}) \leq Ce^{-bN}. \tag{3.2}$$

Let $\mathcal{V}^{N,W}$ be the quadratic form as defined in [15]. Then from (3.1) and (3.2) we can conclude that

$$\mathcal{V}^{N,W}\left(\{\mathcal{F}_{(s-z)}\}\right) \le Ce^{-bN}, \tag{3.3}$$

where C and b are generic constants.

Hence, using the Stability Theorem 2.1 we obtain

$$\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(s-z)}\}\right) \le Ce^{-bN}. \tag{3.4}$$

Here, the quadratic form $\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(s-z)}\}\right)$ is defined similar to the quadratic form $\mathcal{U}^{N,W}(\{\mathcal{F}_u\})$ as in (2.10).

Let $U_l^r(\lambda) = w(X_l^r(\lambda_1,\lambda_2,\lambda_3)) \equiv w(M_l^r(\lambda))$ for $\lambda \in Q$, $U_l^v(x^v) = w(x^v)$ for $x^v \in \tilde{\Omega}_l^v$, $U_l^{v-e}(x^{v-e}) = w(x^{v-e})$ for $x^{v-e} \in \tilde{\Omega}_l^{v-e}$ and $U_l^e(x^e) = w(x^e)$ for $x^e \in \tilde{\Omega}_l^e$. Here, w is the solution of the boundary value problem (2.1).

We now define another quadratic form $\mathcal{E}^{N,W}(\{z-U\})$ by

$$\mathcal{E}^{N,W}(\{z-U\}) = \mathcal{E}^{N,W}_{regular}(\{z_l^r - U_l^r\}) + \mathcal{E}^{N,W}_{vertices}(\{z_l^v - U_l^v\}) + \mathcal{E}^{N,W}_{vertex-edges}(\{z_l^{v-e} - U_l^{v-e}\}) + \mathcal{E}^{N,W}_{edges}(\{z_l^e - U_l^e\}), \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{E}^{N,W}_{regular}(\{z_l^r - U_l^r\}) &= \sum_{l=1}^{N_r} \int_{Q=(M_l^r)^{-1}(\Omega_l^r)} \sum_{|\alpha|\le 2} \left|D_\lambda^\alpha \left(z_l^r - U_l^r\right)(\lambda)\right|^2 d\lambda, \\
\mathcal{E}^{N,W}_{vertices}(\{z_l^v - U_l^v\}) &= \sum_{v\in\mathcal{V}} \mathcal{E}_v^{N,W}\left(z_l^v - U_l^v\right), \\
\mathcal{E}^{N,W}_{v}(\{z_l^v - U_l^v\}) &= \sum_{l=1}^{N_v} \int_{\tilde{\Omega}_l^v} e^{x_3^v} \sum_{|\alpha|\le 2} \left|D_{x^v}^\alpha \left(z_l^v - U_l^v\right)(x^v)\right|^2 dx^v, \\
\mathcal{E}^{N,W}_{vertex-edges}(\{z_l^{v-e} - U_l^{v-e}\}) &= \sum_{v-e\in\mathcal{V}-\mathcal{E}} \mathcal{E}^{N,W}_{v-e}(z_l^{v-e} - U_l^{v-e}), \\
\mathcal{E}^{N,W}_{v-e}(\{z_l^{v-e} - U_l^{v-e}\}) &= \sum_{l=1}^{N_{v-e}} \int_{\tilde{\Omega}_l^{v-e}} e^{x_3^{v-e}} \sum_{|\alpha|\le 2} \left|D_{x^{v-e}}^\alpha \left(z_l^{v-e} - U_l^{v-e}\right)(x^{v-e})\right|^2 dx^{v-e}, \\
\mathcal{E}^{N,W}_{edges}(\{z_l^e - U_l^e\}) &= \sum_{e\in\mathcal{E}} \mathcal{E}_e^{N,W}(z_l^e - U_l^e), \\
\mathcal{E}^{N,W}_{e}(\{z_l^e - U_l^e\}) &= \sum_{l=1}^{N_e} \int_{\tilde{\Omega}_l^e} \sum_{|\alpha|\le 2} \left|D_{x^e}^\alpha \left(z_l^e - U_l^e\right)(x^e)\right|^2 dx^e.
\end{aligned}$$

Using (3.4) it is easy to verify that

$$\begin{aligned}
\mathcal{E}^{N,W}_{regular}(\{s_l^r - U_l^r\}) &\le Ce^{-bN}, \\
\mathcal{E}^{N,W}_{vertices}(\{s_l^v - U_l^v\}) &\le Ce^{-bN}, \\
\mathcal{E}^{N,W}_{vertex-edges}(\{s_l^{v-e} - U_l^{v-e}\}) &\le Ce^{-bN}, \\
\mathcal{E}^{N,W}_{edges}(\{s_l^e - U_l^e\}) &\le Ce^{-bN},
\end{aligned} \tag{3.6}$$

where the quadratic forms $\mathcal{E}^{N,W}_{regular}\left(\{s_l^r - U_l^r\}\right)$, $\mathcal{E}^{N,W}_{vertices}\left(\{s_l^v - U_l^v\}\right)$ etc. are defined similar to those in (2.10). Now define

$$\mathcal{E}^{N,W}(\{s-U\}) = \mathcal{E}^{N,W}_{regular}(\{s_l^r - U_l^r\}) + \mathcal{E}^{N,W}_{vertices}(\{s_l^v - U_l^v\}) + \mathcal{E}^{N,W}_{vertex-edges}(\{s_l^{v-e} - U_l^{v-e}\}) + \mathcal{E}^{N,W}_{edges}(\{s_l^e - U_l^e\}).$$

Then from (3.6) it follows that

$$\mathcal{E}^{N,W}(\{s-U\}) \le Ce^{-bN}. \tag{3.7}$$

Finally, using estimates (3.4) and (3.7), we obtain

$$\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(z-U)}\}\right) \le Ce^{-bN}.$$

Our main theorem on error estimates is now stated

Theorem 3.1. *Let $\{\mathcal{F}_z\}$ minimize $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ over all $\{\mathcal{F}_u\} \in \mathcal{S}^{N,W}(\Omega)$. Then there exist constants C and b (independent of N) such that*

$$\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(z-U)}\}\right) \le Ce^{-bN}. \tag{3.8}$$

Here, $\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(z-U)}\}\right)$ is as defined in (2.11).

Remark 3.1. After having obtained the non-conforming spectral element solution we can make a correction to it so that the corrected solution is conforming and the error in the H^1 norm is exponentially small in N. These corrections are explained in Appendix-C of [18].

To end this section, let us estimate the error in terms of number of degrees of freedom in various subregions of the domain Ω.

The regular region Ω^r:

The regular region Ω^r contains no vertices and edges of the domain Ω. Here, the solution w has no singularity and is analytic.

There are $O(1)$ number of elements in this region and each element has $O(W^3)$ degrees of freedom. Let $N_{dof}(\Omega^r)$ denotes the number of degrees of freedom in Ω^r. Then

$$N_{dof}(\Omega^r) = O(W^3) = O(N^3).$$

The vertex neighbourhoods Ω^v:

In a vertex neighbourhood Ω^v there are $O(N)$ elements with $O(W^3)$ degrees of freedom in each element. If $N_{dof}(\Omega^v)$ denotes the number of degrees of freedom in Ω^v. Then

$$N_{dof}(\Omega^v) = O(NW^3) = O(N^4).$$

The vertex-edge neighbourhoods Ω^{v-e}:

There are $O(N^2)$ number of elements in each of the vertex-edge neighbourhoods Ω^{v-e} and each element has $O(W^3)$ degrees of freedom. Then

$$N_{dof}(\Omega^{v-e}) = O(N^2W^3) = O(N^5).$$

Here, $N_{dof}(\Omega^{v-e})$ denotes the number of degrees of freedom in Ω^{v-e}.

The edge neighbourhoods Ω^e:

An edge neighbourhood Ω^e has $O(N)$ elements with $O(W^3)$ degrees of freedom within each element. Let $N_{dof}(\Omega^e)$ be the number of degrees of freedom in Ω^e. Then

$$N_{dof}(\Omega^e) = O(NW^3) = O(N^4).$$

Hence, the error estimate Theorem 3.1 in terms of number of degrees of freedom assumes the form

Theorem 3.2. *Let $\{\mathcal{F}_z\}$ minimizes $\mathcal{R}^{N,W}(\{\mathcal{F}_u\})$ over all $\{\mathcal{F}_u\} \in \mathcal{S}^{N,W}(\Omega)$. Then there exist constants C and b (independent of N) such that*

$$\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(z-U)}\}\right) \leq Ce^{-bN_{dof}^{1/5}}. \tag{3.9}$$

Here, $\mathcal{U}^{N,W}\left(\{\mathcal{F}_{(z-U)}\}\right)$ is as defined in (2.11) and $N_{dof} = dim(\mathcal{S}^{N,W}(\Omega))$ is the number of degrees of freedom.

Proof. Follows from Theorem 3.1. $\square$

Remark 3.2. From the above theorem it is clear that the exponential rate of convergence will be visible only for a large value of N_{dof}, as a result we need to sufficiently refine the geometric mesh both in the direction of edges and in the direction perpendicular to the edges.

Remark 3.3. It follows that with a fewer number of layers in the geometric mesh, we may expect the convergence rate to be $e^{-bN_{dof}^{\beta}}$ with $\frac{1}{4} < \beta < \frac{1}{5}$.

Remark 3.4. Since the majority of degrees of freedom is present in the vertex-edge neighbourhoods the factor $N_{dof}^{1/5}$ in the theorem is due to the vertex-edge singularity in the solution. Hence the optimal convergence rate will be $e^{-bN_{dof}^{1/5}}$.

Remark 3.5. It was conjectured in [24,23] that for $h-p$ version of the finite element method in $\mathbb{R}^3$ the optimal convergence rate will be $e^{-bN_{dof}^{1/5}}$, and it cannot be improved further with any mesh and any anisotropic polynomial order within the elements.

Remark 3.6. It can be argued as in [24,23] that computationally, the optimal convergence rate $e^{-bN_{dof}^{1/5}}$ may be improved further by properly selecting the geometric mesh factors and degree factors μ_v, μ_e, μ_1, μ_2 etc.

4. Preconditioners

Our construction of preconditioners is similar to that for elliptic problems in two dimensions (see [19,20]). As mentioned in earlier sections, we had divided the polyhedral domain Ω into a regular region Ω^r, a set of vertex neighbourhoods Ω^v, a set of edge neighbourhoods Ω^e and a set of vertex-edge neighbourhoods Ω^{v-e}. Ω^r is divided into a set of curvilinear hexahedrons, tetrahedrons and prisms and the elements in the singular regions in the neighbourhoods of vertices, edges and vertex-edges are divided into hexahedrons and prisms using a geometric mesh. The elements in the regular region and the vertex neighbourhoods are mapped to the unit cube $Q = (-1, 1)^3$. The numerical solution is approximated by a constant on the corner most elements in vertex and vertex-edge neighbourhoods and it is a function of only one variable on the corner elements in edge and vertex-edge neighbourhoods that are in the direction of the edges away from the vertices. In the regular region and vertex neighbourhoods we approximate the solution by a polynomial of degree N in λ_1, λ_2 and λ_3 variables separately, where λ_1, λ_2 and λ_3 denote the transformed variables on the unit cube Q. In the remaining elements in edge neighbourhoods and vertex-edge neighbourhoods, the solution is approximated by a polynomial of degree W in each of the variables in the modified coordinates in their respective neighbourhoods. Then the proposed method gives exponentially accurate solution in N provided the data satisfy usual conditions [23,28,15,20].

We construct a preconditioner $\mathcal{B}(u)$ on each of these element in the neighbourhoods of vertices, edges, vertex-edges and the regular region. We shall prove as in [29] that there is another quadratic form $\mathcal{C}(u)$ which is spectrally equivalent to $\mathcal{B}(u)$ and which can be easily diagonalized using the separation of variables. Then the matrix corresponding to the quadratic form $\mathcal{C}(u)$ will be easy to invert.

4.1. Preconditioners on the regular region

In the regular region the preconditioner which needs to be examined corresponds to the quadratic form

$$\mathcal{B}(u) = \|u\|^2_{H^2(Q)} \tag{4.1}$$

where $Q = (-1, 1)^3 =$ master cube, $u = u(\lambda) = u(\lambda_1, \lambda_2, \lambda_3)$ is a polynomial of degree W in λ_1, λ_2 and λ_3 separately.

Let $u(\lambda_1, \lambda_2, \lambda_3)$ be the spectral element function, defined on $Q = (-1, 1)^3$, as

$$u(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=0}^{W}\sum_{j=0}^{W}\sum_{k=0}^{W} a_{i,j,k} L_i(\lambda_1) L_j(\lambda_2) L_k(\lambda_3). \tag{4.2}$$

Here, $L_i(\cdot)$ denotes the Legendre polynomial of degree i.

The quadratic form $\mathcal{B}(u)$ can be written as

$$\mathcal{B}(u) = \int_Q \sum_{|\alpha|\le 2} |D^\alpha_\lambda u|^2 d\lambda. \tag{4.3}$$

Let I denote the interval $(-1, 1)$ and

$$v(\lambda_1) = \sum_{i=0}^{W} \beta_i L_i(\lambda_1). \tag{4.4}$$

Moreover, $b = (\beta_0, \beta_1, \dots, \beta_W)^T$. We now define the quadratic form

$$\mathcal{G}(v) = \int_I (v^2_{\lambda_1\lambda_1} + v^2_{\lambda_1}) d\lambda_1 \tag{4.5}$$

and

$$\mathcal{H}(v) = \int_I v^2 d\lambda_1. \tag{4.6}$$

Clearly there exist $(W + 1) \times (W + 1)$ matrices G and H such that

$$\mathcal{G}(v) = b^T G b \tag{4.7}$$

and

$$\mathcal{H}(v) = b^T H b. \tag{4.8}$$

Here, the matrices G and H are symmetric and H is positive definite.

Hence, there exist $W+1$ eigenvalues $0 \le \mu_0 \le \mu_1 \le \cdots \le \mu_W$ and $W+1$ eigenvectors $b_0, b_1, \dots, b_W$ of the symmetric eigenvalue problem

$$(G - \mu H)b = 0. \tag{4.9}$$

Here,

$$(G - \mu_i H)b_i = 0.$$

The eigenvectors b_i are normalized so that

$$b_i^T H b_j = \delta_j^i. \tag{4.10a}$$

Moreover, the relations

$$b_i^T G b_j = \mu_i \delta_j^i \tag{4.10b}$$

hold. Let $b_i = (b_{i,0}, b_{i,1}, \dots, b_{i,W})$. We now define the polynomial

$$\phi_i(\lambda_1) = \sum_{j=0}^{W} b_{i,j} L_j(\lambda_1) \quad \text{for } 0 \le i \le W. \tag{4.11}$$

Next, let $\psi_{i,j,k}$ denote the polynomial

$$\psi_{i,j,k}(\lambda_1, \lambda_2, \lambda_3) = \phi_i(\lambda_1)\phi_j(\lambda_2)\phi_k(\lambda_3) \tag{4.12}$$

for $0 \le i \le W,\ 0 \le j \le W,\ 0 \le k \le W$.

Let $u(\lambda_1, \lambda_2, \lambda_3)$ be a polynomial as in (4.2). Define the quadratic form

$$\mathcal{C}(u) = \int_Q (u^2_{\lambda_1\lambda_1} + u^2_{\lambda_2\lambda_2} + u^2_{\lambda_3\lambda_3} + u^2_{\lambda_1} + u^2_{\lambda_2} + u^2_{\lambda_3} + u^2) d\lambda_1 d\lambda_2 d\lambda_3. \tag{4.13}$$

Then the quadratic form $\mathcal{C}(u)$ is spectrally equivalent to the quadratic form $\mathcal{B}(u)$, defined in (4.1). Moreover, the quadratic form $\mathcal{C}(u)$ can be diagonalized in the basis $\psi_{i,j,k}(\lambda_1, \lambda_2, \lambda_3)$. Note that $\{\psi_{i,j,k}(\lambda_1, \lambda_2, \lambda_3)\}_{i,j,k}$ is the tensor product of the polynomials $\phi_i(\lambda_1)$, $\phi_j(\lambda_2)$ and $\phi_k(\lambda_3)$. The eigenvalue $\mu_{i,j,k}$ corresponding to the eigenvector $\psi_{i,j,k}$ is given by the relation

$$\mu_{i,j,k} = \mu_i + \mu_j + \mu_k + 1. \tag{4.14}$$

Hence, the matrix corresponding to the quadratic form $\mathcal{C}(u)$ is easy to invert.

Using the extension theorems in [30] and Lemma 2.1 in [29] we can extend $u(\lambda_1, \lambda_2, \lambda_3)$ defined in (4.2) to $U(\lambda_1, \lambda_2, \lambda_3)$ by the method of reflection (see Theorem 4.26 of [30]). This extension $U(\lambda_1, \lambda_2, \lambda_3)$ of $u(\lambda_1, \lambda_2, \lambda_3)$ is such that $U(\lambda_1, \lambda_2, \lambda_3) \in H^2(\mathbb{R}^3)$ and satisfies the estimate

$$\int_{\mathbb{R}^3} \left(U^2_{\lambda_1\lambda_1} + U^2_{\lambda_2\lambda_2} + U^2_{\lambda_3\lambda_3} + U^2\right) d\lambda \le K \int_Q \left(u^2_{\lambda_1\lambda_1} + u^2_{\lambda_2\lambda_2} + u^2_{\lambda_3\lambda_3} + u^2\right) d\lambda.$$

Here, K is a constant independent of W. Now making use of Theorem 2.1 of [29] and extending it to three dimensions it follows that there exists a constant L (independent of W) such that

$$\begin{aligned}\frac{1}{L}\|u\|^2_{H^2(Q)} &\le \int_Q \left(|u_{\lambda_1\lambda_1}|^2 + |u_{\lambda_2\lambda_2}|^2 + |u_{\lambda_3\lambda_3}|^2 + |u_{\lambda_1}|^2 + |u_{\lambda_2}|^2 + |u_{\lambda_3}|^2 + |u|^2\right) d\lambda \\ &\le \|u\|^2_{H^2(Q)}.\end{aligned}$$

i.e. the quadratic forms $\mathcal{B}(u)$ and $\mathcal{C}(u)$ are spectrally equivalent.

Theorem 4.1. *The quadratic forms $\mathcal{B}(u)$ and $\mathcal{C}(u)$ are spectrally equivalent.*

We now show that the quadratic form $\mathcal{C}(u)$ defined in (4.13) as

$$\mathcal{C}(u) = \int_Q \left(u^2_{\lambda_1\lambda_1} + u^2_{\lambda_2\lambda_2} + u^2_{\lambda_3\lambda_3} + u^2_{\lambda_1} + u^2_{\lambda_2} + u^2_{\lambda_3} + u^2\right) d\lambda_1 d\lambda_2 d\lambda_3$$

can be diagonalized in the basis $\{\psi_{i,j,k}\}_{i,j,k}$. Here, u is a polynomial in λ_1, λ_2 and λ_3 as defined in (4.2). Let $\widetilde{\mathcal{C}}(f, g)$ denote the bilinear form induced by the quadratic form $\mathcal{C}(u)$. Then

$$\widetilde{\mathcal{C}}(f, g) = \int_Q \left(f_{\lambda_1\lambda_1} g_{\lambda_1\lambda_1} + f_{\lambda_2\lambda_2} g_{\lambda_2\lambda_2} + f_{\lambda_3\lambda_3} g_{\lambda_3\lambda_3} + f_{\lambda_1} g_{\lambda_1} + f_{\lambda_2} g_{\lambda_2} + f_{\lambda_3} g_{\lambda_3} + fg\right) d\lambda_1 d\lambda_2 d\lambda_3. \tag{4.15}$$

Let $\mathcal{G}(v)$ and $\mathcal{H}(v)$ be the quadratic forms defined in (4.5) and (4.6) and let $\widetilde{\mathcal{G}}(f, g)$ and $\widetilde{\mathcal{H}}(f, g)$ denote the bilinear forms induced by $\mathcal{G}(v)$ and $\mathcal{H}(v)$ respectively. Then

$$\widetilde{\mathcal{G}}(f, g) = \int_I (f_{\lambda_1\lambda_1} g_{\lambda_1\lambda_1} + f_{\lambda_1} g_{\lambda_1}) d\lambda_1 \tag{4.16a}$$

Table 2
Condition number κ as a function of W.

W	κ
2	3.69999999999999
4	4.90406593328559
6	5.27448215795748
8	5.48239323328901
10	5.62480021244268
12	5.72673215953223
14	5.80192403338903
16	5.85907843805046

and

$$\widetilde{\mathcal{H}}(f,g) = \int_I fg d\lambda_1. \tag{4.16b}$$

Here, I denotes the unit interval and $f(\lambda_1)$, $g(\lambda_1)$ are polynomials of degree W in λ_1.

Finally, let $\phi_i(\lambda_1)$ be the polynomial as defined in (4.11). Then relation (4.10a) may be written as

$$\widetilde{\mathcal{H}}(\phi_i,\phi_j) = \int_I \phi_i(\lambda_1)\phi_j(\lambda_1) d\lambda_1 = \delta_j^i. \tag{4.17a}$$

Moreover, relation (4.10b) may be stated as

$$\widetilde{\mathcal{G}}(\phi_i,\phi_j) = \int_I ((\phi_i)_{\lambda_1\lambda_1}(\phi_j)_{\lambda_1\lambda_1} + (\phi_i)_{\lambda_1}(\phi_j)_{\lambda_1}) d\lambda_1 = \mu_i\delta_j^i. \tag{4.17b}$$

Recalling that $\psi_{i,j,k}(\lambda_1,\lambda_2,\lambda_3) = \phi_i(\lambda_1)\phi_j(\lambda_2)\phi_k(\lambda_3)$ and using (4.17) in (4.15) it is easy to show that

$$\begin{aligned}\widetilde{\mathcal{C}}(\psi_{i,j,k},\psi_{l,m,n}) &= (\mu_i+\mu_j+\mu_k+1)\delta_l^i\delta_m^j\delta_n^k \\ &= \mu_{i,j,k}\delta_l^i\delta_m^j\delta_n^k.\end{aligned}$$

Hence, the eigenvectors of the quadratic form $\mathcal{C}(u)$ are $\{\psi_{i,j,k}\}_{i,j,k}$ and the eigenvalues are $\{\mu_{i,j,k}\}_{i,j,k}$. Moreover, the quadratic form $\mathcal{C}(u)$ can be diagonalized in the basis $\{\psi_{i,j,k}\}_{i,j,k}$ and consequently the matrix corresponding to $\mathcal{C}(u)$ is easy to invert.

Let

$$u(\lambda_1,\lambda_2,\lambda_3) = \sum_{i=0}^{W}\sum_{j=0}^{W}\sum_{k=0}^{W} \beta_{i,j,k} L_i(\lambda_1)L_j(\lambda_2)L_k(\lambda_3)$$

and β denotes the column vector whose components are $\beta_{i,j,k}$ arranged in lexicographic order. Then there is a $(W+1)^3 \times (W+1)^3$ matrix C such that

$$\mathcal{C}(u) = \beta^T C\beta.$$

As in [29] it can be shown that the system of equations

$$C\beta = \rho$$

can be solved in $O(W^4)$ operations. Therefore the quadratic form $\mathcal{C}(u)$ can be inverted in $O(W^4)$ operations.

Let κ denote the condition number of the preconditioned system obtained by using the quadratic form $\mathcal{C}(u)$ as a preconditioner for the quadratic form $\mathcal{B}(u)$. Then the values of κ as a function of W are shown in Table 2.

In Fig. 4, the condition number κ is plotted against the polynomial order W.

4.2. Preconditioners on singular regions

A set of spectral element functions has been defined on all elements in the regular region and various singular regions. We choose our spectral element functions to be fully non-conforming. As earlier, let $\mathcal{F}_u$ denote the spectral element representation of the function u.

We define the quadratic form

$$\mathcal{W}^{N,W}(\{\mathcal{F}_u\}) = \mathcal{W}^{N,W}_{regular}(\{\mathcal{F}_u\}) + \mathcal{W}^{N,W}_{vertices}(\{\mathcal{F}_u\}) + \mathcal{W}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\}) + \mathcal{W}^{N,W}_{edges}(\{\mathcal{F}_u\}). \tag{4.18}$$

Here, $\mathcal{W}^{N,W}_{regular}(\{\mathcal{F}_u\})$, $\mathcal{W}^{N,W}_{vertices}(\{\mathcal{F}_u\})$, $\mathcal{W}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\})$ and $\mathcal{W}^{N,W}_{edges}(\{\mathcal{F}_u\})$ are defined similar to the quadratic forms $\mathcal{U}^{N,W}_{regular}(\{\mathcal{F}_u\})$, $\mathcal{U}^{N,W}_{vertices}(\{\mathcal{F}_u\})$, $\mathcal{U}^{N,W}_{vertex-edges}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}_{edges}(\{\mathcal{F}_u\})$ respectively as in (2.10). Then using Theorem 2.1 it follows that for problems with Dirichlet boundary conditions the estimate

$$\mathcal{W}^{N,W}(\{\mathcal{F}_u\}) \le C(\ln W)^2\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.19}$$

holds, provided $W = O(e^{N^\alpha})$ for $\alpha < 1/2$.

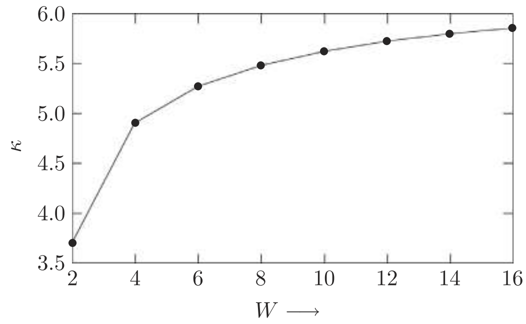

Fig. 4. Condition number κ vs. W.

At the same time using trace theorems for Sobolev spaces there exists a constant k such that

$$\frac{1}{k}\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \le \mathcal{W}^{N,W}(\{\mathcal{F}_u\}). \tag{4.20}$$

Hence, using (4.19) and (4.20) we conclude that the two quadratic forms $\mathcal{W}^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ are spectrally equivalent and there exists a constant K such that

$$\frac{1}{K}\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \le \mathcal{W}^{N,W}(\{\mathcal{F}_u\}) \le K(\ln W)^2\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.21}$$

provided $W = O(e^{N^\alpha})$ for $\alpha < 1/2$.

We can now use the quadratic form $\mathcal{W}^{N,W}(\{\mathcal{F}_u\})$ which consists of a decoupled set of quadratic forms on each element as a preconditioner. It follows that the condition number of the preconditioned system is $O(\ln W)^2$.

The other case is when the boundary conditions are of mixed Neumann and Dirichlet type. In this case, as above, using Theorem 2.2 and trace theorems for Sobolev spaces it follows that for W and N large enough the following estimate holds

$$\frac{1}{K}\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \le \mathcal{W}^{N,W}(\{\mathcal{F}_u\}) \le KN^4\mathcal{V}^{N,W}(\{\mathcal{F}_u\}).$$

Here, K is a constant. It is clear that the quadratic form $\mathcal{W}^{N,W}(\{\mathcal{F}_u\})$ can be used as a preconditioner and the condition number of the preconditioned system is $O(N^4)$.

We will now construct preconditioners on each of the elements in the neighbourhoods of vertices, edges, vertex-edges and the regular region. Here, u denotes the spectral element function which is a polynomial of degree W in each of its variables separately defined in various regions of the polyhedron.

The quadratic forms which need to be examined are

$$\mathcal{B}_{regular}(u) = \| u \|^2_{H^2(Q)} = \int_Q \sum_{|\alpha|\le 2} |D^\alpha_{\lambda_1,\lambda_2,\lambda_3} u|^2 d\lambda_1 d\lambda_2 d\lambda_3, \tag{4.22}$$

$$\mathcal{B}_{vertices}(u) = \| e^{\chi/2} u \|^2_{H^2(\tilde{\Omega}^v_l)} = \int_{\tilde{\Omega}^v_l} e^\chi \sum_{|\alpha|\le 2} |D^\alpha_{\phi,\theta,\chi} u|^2 d\phi d\theta d\chi, \tag{4.23}$$

$$\begin{aligned}\mathcal{B}_{vertex-edges}(u) = \int_{\tilde{\Omega}^{v-e}_l} e^\zeta (&u^2_{\psi\psi} + u^2_{\theta\theta} + u^2_{\psi\theta} + \sin^2\phi u^2_{\phi\zeta} + \sin^2\phi u^2_{\theta\zeta} \\ &+ \sin^4\phi u^2_{\zeta\zeta} + u^2_\psi + u^2_\theta + \sin^2\phi u^2_\zeta + u^2) d\psi d\theta d\zeta,\end{aligned} \tag{4.24}$$

$$\mathcal{B}_{edges}(u) = \int_{\tilde{\Omega}^e_l} (u^2_{\tau\tau} + u^2_{\theta\theta} + u^2_{\tau\theta} + e^{2\tau}u^2_{\tau x_3} + e^{2\tau}u^2_{\theta x_3} + e^{4\tau}u^2_{x_3x_3} + u^2_\tau + u^2_\theta + e^{2\tau}u^2_{x_3} + u^2) d\tau d\theta dx_3. \tag{4.25}$$

Here, (ϕ, θ, χ), (ψ, θ, ζ) and (τ, θ, x_3) denote the modified systems of coordinates introduced in Table 1 in vertex neighbourhoods, vertex-edge neighbourhoods and edge neighbourhoods respectively. Moreover $\tilde{\Omega}^v_l$, $\tilde{\Omega}^{v-e}_l$ and $\tilde{\Omega}^e_l$ denote elements in the vertex neighbourhood, vertex-edge neighbourhood and edge neighbourhood respectively.

The construction of preconditioners corresponding to the quadratic forms $\mathcal{B}_{regular}(u)$ and $\mathcal{B}_{vertices}(u)$ is similar to the case of a smooth domain already discussed so we omit the details. It follows that there exist quadratic forms $\mathcal{C}_{regular}(u)$ and $\mathcal{C}_{vertices}(u)$ which are spectrally equivalent to $\mathcal{B}_{regular}(u)$ and $\mathcal{B}_{vertices}(u)$ respectively and which can be diagonalized using separation of variables technique.

We will now obtain preconditioners for elements in edge and vertex-edge neighbourhood. For this purpose we observe that for quadratic forms in edge and vertex-edge neighbourhoods it is enough to examine the quadratic form

$$\mathcal{B}^\star(u) = \int_{-1}^1\int_{-1}^1\int_{-1}^1 (u^2_{xx} + u^2_{yy} + \eta^2 u^2_{xz} + \eta^2 u^2_{yz} + \eta^4 u^2_{zz} + u^2_x + u^2_y + \eta^2 u^2_z + u^2) dxdydz. \tag{4.26}$$

Here, $\eta = \sin\phi$ and $\eta = e^{\tau}$ for vertex-edge and edge neighbourhood elements respectively. We remark that the factor η becomes smaller towards the vertices and edges of the domain Ω.

Making the transformation $\tilde{z} = \frac{z}{\eta}$, so that $\frac{d}{dz} = \frac{1}{\eta}\frac{d}{d\tilde{z}}$, the quadratic form $\mathcal{B}^{\star}(u)$ assumes the form

$$\mathcal{B}^{\star}(u) = \int_{-\frac{1}{\eta}}^{\frac{1}{\eta}}\int_{-1}^{1}\int_{-1}^{1}(u_{xx}^2 + u_{yy}^2 + u_{xy}^2 + u_{x\tilde{z}}^2 + u_{y\tilde{z}}^2 + u_{\tilde{z}\tilde{z}}^2 + u_x^2 + u_y^2 + u_{\tilde{z}}^2 + u^2)dxdyd\tilde{z}.$$

Let us define the quadratic form

$$\mathcal{C}^{\star}(u) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(u_{xx}^2 + u_{yy}^2 + \eta^4 u_{zz}^2 + u_x^2 + u_y^2 + \eta^2 u_z^2 + u^2)dxdydz. \tag{4.27}$$

We now show that the quadratic form $\mathcal{C}^{\star}(u)$ is spectrally equivalent to the quadratic form $\mathcal{B}^{\star}(u)$, defined in (4.26). Moreover, $\mathcal{C}^{\star}(u)$ can be diagonalized using separation of variables technique.

Let

$$v(x) = \sum_{i=0}^{W}\beta_i L_i(x), \quad \text{and} \quad v(z) = \sum_{i=0}^{W}\gamma_i L_i(z).$$

Moreover $b = (\beta_0, \beta_1, \ldots, \beta_W)^T$ and $d = (\gamma_0, \gamma_1, \ldots, \gamma_W)^T$.

We now define the quadratic forms

$$\mathcal{G}(v) = \int_I (v_{xx}^2 + v_x^2)dx, \quad \mathcal{H}(v) = \int_I v^2 dx, \tag{4.28}$$

and

$$\mathcal{M}(v) = \int_I (\eta^4 v_{zz}^2 + \eta^2 v_z^2)dz, \quad \mathcal{N}(v) = \int_I v^2 dz. \tag{4.29}$$

Here, I denotes the unit interval $(-1, 1)$. Clearly, there exist $(W+1)\times(W+1)$ matrices G, M and H, N such that

$$\mathcal{G}(v) = b^T G b, \qquad \mathcal{M}(v) = d^T M d, \tag{4.30}$$

and

$$\mathcal{H}(v) = b^T H b, \qquad \mathcal{N}(v) = d^T N d. \tag{4.31}$$

Here, the matrices G, M and H, N are symmetric and H, N are positive definite.

Hence, there exist $W+1$ eigenvalues $0 \le \mu_0 \le \mu_1 \le \cdots \le \mu_W$ and $W+1$ eigenvectors $b_0, b_1, \ldots, b_W$ of the symmetric eigenvalue problem

$$(G - \mu H)b = 0. \tag{4.32}$$

Here,

$$(G - \mu_i H)b_i = 0.$$

Similarly, there exist $W+1$ eigenvalues $0 \le \nu_0 \le \nu_1 \le \cdots \le \nu_W$ and $W+1$ eigenvectors $d_0, d_1, \ldots, d_W$ of the symmetric eigenvalue problem

$$(M - \nu N)d = 0. \tag{4.33}$$

Here,

$$(M - \nu_i N)d_i = 0.$$

The eigenvectors b_i and c_i are normalized so that

$$b_i^T H b_j = \delta_j^i, \quad \text{and} \quad d_i^T N d_j = \delta_j^i. \tag{4.34a}$$

Moreover, the relations

$$b_i^T G b_j = \mu_i \delta_j^i \quad \text{and} \quad d_i^T M d_j = \nu_i \delta_j^i \tag{4.34b}$$

hold. Let $b_i = (b_{i,0}, b_{i,1}, \ldots, b_{i,W})$ and $d_i = (d_{i,0}, d_{i,1}, \ldots, d_{i,W})$. We now define the polynomials

$$\phi_i(x) = \sum_{m=0}^{W} b_{i,m}L_m(x), \qquad \phi_j(y) = \sum_{m=0}^{W} b_{j,m}L_m(y), \qquad \theta_k(z) = \sum_{m=0}^{W} d_{k,m}L_m(z) \quad \text{for } 0 \le i, j, k \le W.$$

Next, let $\chi_{i,j,k}$ denote the polynomial

$$\chi_{i,j,k}(x,y,z) = \phi_i(x)\phi_j(y)\theta_k(z) \tag{4.35}$$

for $0 \le i \le W,\ 0 \le j \le W,\ 0 \le k \le W$.

Note that $\{\chi_{i,j,k}(x,y,z)\}_{i,j,k}$ is the tensor product of the polynomials $\phi_i(x), \phi_j(y)$ and $\theta_k(z)$. The eigenvalue $\sigma_{i,j,k}$ corresponding to the eigenvector $\chi_{i,j,k}$ is given by the relation

$$\sigma_{i,j,k} = \mu_i + \mu_j + \nu_k + 1. \tag{4.36}$$

Let $\widetilde{\mathcal{C}}^{\star}(f,g)$ be the bilinear form induced by the quadratic form $\mathcal{C}^{\star}(u)$. Then

$$\widetilde{\mathcal{C}}^{\star}(f,g) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(f_{xx}f_{xx} + f_{yy}g_{yy} + \eta^4 f_{zz}g_{zz} + f_x g_x + f_y g_y + \eta^2 f_z g_z + fg)\,dxdydz.$$

It is easy to show that

$$\begin{aligned}\widetilde{\mathcal{C}}^{\star}(\chi_{i,j,k}, \chi_{l,m,n}) &= (\mu_i + \mu_j + \nu_k + 1)\delta_l^i \delta_m^j \delta_n^k \\ &= \sigma_{i,j,k}\delta_l^i \delta_m^j \delta_n^k.\end{aligned}$$

Hence, the eigenvectors of the quadratic form $\mathcal{C}^{\star}(u)$ are $\{\chi_{i,j,k}\}_{i,j,k}$ and the eigenvalues are $\{\sigma_{i,j,k}\}_{i,j,k}$. Thus, the quadratic form $\mathcal{C}^{\star}(u)$ can be diagonalized in the basis $\{\chi_{i,j,k}\}_{i,j,k}$. Therefore, the matrix corresponding to the quadratic form $\mathcal{C}^{\star}(u)$ is easy to invert.

Now proceeding as earlier, it can be shown that the quadratic forms $\mathcal{B}^{\star}(u)$ and $\mathcal{C}^{\star}(u)$ are spectrally equivalent. Moreover, the quadratic form $\mathcal{C}^{\star}(u)$ can be inverted in $O(W^4)$ operations. Thus, it follows that there exist quadratic forms $\mathcal{C}_{vertex-edges}(u)$ and $\mathcal{C}_{edges}(u)$ which are spectrally equivalent to $\mathcal{B}_{vertex-edges}(u)$ and $\mathcal{B}_{edges}(u)$ respectively and which can be diagonalized using separation of variables technique.

5. Computational techniques

In minimizing the functional $\mathcal{R}^{N,W}(\{\mathcal{F}_v\})$ we seek a solution which minimizes the sum of weighted norms of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the sum of squares of the jumps in the function and its derivatives at inter-element boundaries in appropriate anisotropic Sobolev norm, suitably weighted in various regions of the polyhedron.

In this section, we show how to compute the residuals in the normal equations without having to compute and store mass and stiffness matrices and we discuss computational complexity of our method in brief.

In order to obtain a solution using PCGM we must we able to compute residuals in the normal equations inexpensively, since we are minimizing $\mathcal{R}^{N,W}(\{\mathcal{F}_v\})$ over all $\{\mathcal{F}_v\} \in \mathcal{S}^{N,W}$ (space of spectral element functions) we have

$$\mathcal{R}^{N,W}(U+\epsilon V) = \mathcal{R}^{N,W}(U) + 2\epsilon V^t(XU - YG) + O(\epsilon^2)$$

for all V, where U is a vector assembles from the values of

$$\left\{\left\{u_l^r(\lambda)\right\}_{l=1}^{N_r}, \left\{u_l^v(x^v)\right\}_{l=1}^{N_v}, \left\{u_l^{v-e}(x^{v-e})\right\}_{l=1}^{N_{v-e}}, \left\{u_l^e(x^e)\right\}_{l=1}^{N_e}\right\}.$$

V is a vector similarly assembled and G is assembled from the data. Here, X and Y denote matrices. Thus we have to solve $XU - YG = 0$ and so we must be able to compute $XU - YG$ economically during the iterative process. The idea is very similar to the case of two dimensional problems so we refer the reader to [20] for details.

The above minimization amounts to an overdetermined system of equations consisting of collocating the residuals in the partial differential equation, the residuals in the boundary conditions and jumps in the function and its derivatives at inter-element boundaries at an over determined set of collocation points, weighted suitably. In fact we collocate the partial differential equation on a finer grid of Gauss–Lobatto–Legendre (GLL) points and then we apply the adjoint differential operator to these residuals and project these values back to the original grid. Such a treatment obviously involves integration by parts and hence leads to evaluation of terms at the boundaries. These boundary terms can be evaluated by a collocation procedure and the other boundary terms corresponding to jump terms at the inter-element boundaries can be easily calculated (see [18]).

Since the majority of elements i.e. $O(N^2)$ is concentrated in the neighbourhoods of vertices and edges and each element is mapped onto a separate processor therefore, we employ a parallel computer with $O(N^2)$ processors. For problems with Dirichlet boundary conditions the condition number of the preconditioned system is $O((\ln W)^2)$, provided $W = O(e^{N^\alpha})$ for $\alpha < 1/2$. Hence, the method requires $O(N \ln N)$ iterations of the PCGM to obtain solution to an accuracy of $O(e^{-bN})$ and it requires $O(N^5 \ln(N))$ operations on a parallel computer with $O(N^2)$ processors to compute the solution. For mixed problems with Neumann and Dirichlet boundary conditions the condition number of the preconditioned system is $O(N^4)$, provided $W = O(e^{N^\alpha})$ for $\alpha < 1/2$. Hence, it requires $O(N^3)$ iterations of the PCGM to obtain solution to an accuracy of $O(e^{-bN})$ and requires $O(N^7)$ operations on a parallel computer with $O(N^2)$ processors to compute the solution.

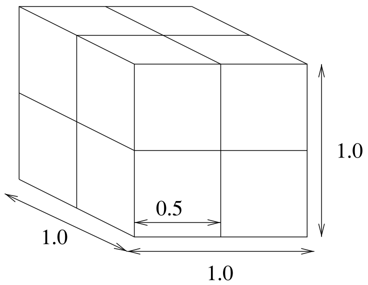

Fig. 5. Mesh imposed on $\Omega = (0, 1)^3$ with mesh size $h = 0.5$.

Table 3
Performance of the p-version for *Laplace* equation with Dirichlet boundary conditions.

$p = W$	N_{dof}	Iterations	$\|E\|_{rel}(\%)$
2	64	51	0.380275E+02
4	512	213	0.204875E+01
6	1,728	303	0.269917E−01
8	4,096	380	0.221613E−03
10	8,000	456	0.106885E−05
12	13,824	523	0.452056E−08

6. Numerical results

We now present results of simulations that have been performed to validate the theory on polyhedral domains. Throughout this section N denote the number of refinements in each direction and W the degree of the polynomials used for approximation. In all our computations we have employed a parallel computer and each element is mapped onto a single processor.

In what follows, by iterations, we always mean the total number of iterations required to compute the solution up to desired accuracy by PCGM. In all our examples the relative error is plotted on a log-scale. Let u_{SE} be the spectral element solution obtained from the minimization problem and w be the exact solution. Then the relative error (in H^1-norm) is defined as

$$\|E\|_{rel} = \frac{\|u_{SE} - w\|_{H^1}}{\|w\|_{H^1}}.$$

6.1. Test problems with smooth solutions

We first analyse performance of our method for various test problems on polyhedral domains on which the solution is smooth. From Section 3, it is clear that the error in the regular (smooth) region obeys

$$\|u_{SE} - w\|_{H^1} \leq Ce^{-bN_{dof}^{1/3}}. \tag{6.1}$$

Here, N_{dof} denotes the number of degrees of freedom (DOF).

Thus, in case the solution is analytic on $\bar{\Omega}$, exponential convergence can be achieved by increasing the polynomial order and keeping the number of elements fixed. Hence, for practical implementation it is enough to compute the error for p-version of the method.

Example 6.1 (*Laplace Equation with Dirichlet Boundary Conditions*)**.** Our first example is the Laplace equation in the unit cube $\Omega = (0, 1)^3$ shown in Fig. 5, with Dirichlet boundary conditions:

$$\begin{aligned} \triangle w &= 0 \quad \text{in } \Omega, \\ w &= g \quad \text{on } \partial\Omega \end{aligned}$$

where the data g is chosen so that the exact solution is

$$w(x, y, z) = \frac{1}{\pi^2 \sinh \sqrt{2}\pi} \sin(\pi x) \sin(\pi y) \sinh(\sqrt{2}\pi z).$$

The results are given in Table 3. The relative error (in %) against polynomial order W and iterations against W are plotted in Fig. 6(a) and (b) respectively. In Fig. 6(c) a graph is drawn for $\|E\|_{rel}$ against degrees of freedom on a log scale. The error curve is a straight line and this shows the exponential rate of convergence in agreement with (6.1).

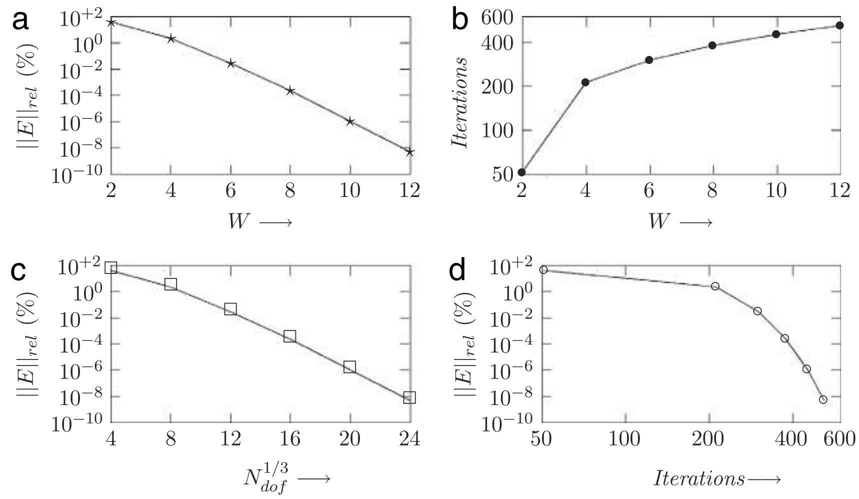

Fig. 6. (a) ln $\|E\|_{\text{rel}}$ vs. p, (b) Iterations vs. N, (c) ln $\|E\|_{\text{rel}}$ vs. $N_{dof}^{1/3}$ and (d) ln $\|E\|_{\text{rel}}$ vs. Iterations for Laplace equation with Dirichlet boundary conditions.

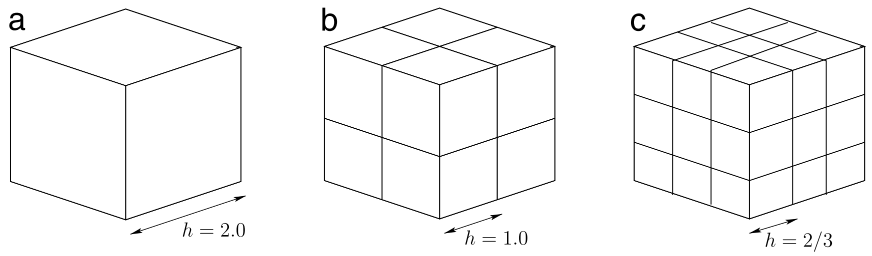

Fig. 7. The domain $\Omega = (-1, 1)^3$ with uniform mesh refinements (a) Mesh 1, (b) Mesh 2 and (c) Mesh 3.

The example presented above deals with a constant coefficient differential operator. However, the method works for a general non self-adjoint elliptic problem too. Let $\Omega = (-1, 1)^3$ denote the standard cube in $\mathbb{R}^3$ with boundary $\partial\Omega$. In our next example we impose three different meshes on Ω with uniform mesh size $h = 2.0$, 1.0 and 0.67 in each direction which corresponds to $N = 1, 2$ and 3 respectively (Fig. 7).

Example 6.2 (*General Elliptic Equation with Variable Coefficients: A Non Self-adjoint Problem*)**.** Let us consider the non self-adjoint general elliptic problem with mixed boundary conditions.

$$\begin{aligned} a(x,y,z)w_{xx} + b(x,y,z)w_{yy} + c(x,y,z)w_{zz} + d(x,y,z)(w_{xy} + w_{yz} + w_{zx}) + e(x,y,z)w &= f \quad \text{in } \Omega, \\ w &= g \quad \text{on } \mathcal{D}, \\ \frac{\partial w}{\partial \nu} &= h \quad \text{on } \mathcal{N}. \end{aligned}$$

Here, $\mathcal{D}$ and $\mathcal{N}$ denote the Dirichlet and Neumann boundary part of $\partial\Omega$ respectively such that $\mathcal{D} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_1, Γ_2 and Γ_3 are the faces corresponding to $x = -1$, $x = 1$ and $y = -1$ respectively. $\mathcal{N} = \Gamma_4 \cup \Gamma_5 \cup \Gamma_6$, where Γ_4, Γ_5 and Γ_6 are the faces corresponding to $y = 1$, $z = -1$ and $z = 1$ respectively. Moreover, ν denotes the outer unit normal to the faces where Neumann boundary conditions are imposed. Further, we choose the coefficients of the problem as follows:

$$\begin{aligned} &a(x,y,z) = -(0.50 + 0.05\exp(xyz)), \qquad b(x,y,z) = -(1.00 + 0.015\cos(x+y)), \\ &c(x,y,z) = -(2.50 + 0.02\exp(y+z)), \qquad d(x,y,z) = -0.001\sin(\pi(x+y+z)) \\ &\text{and} \quad e(x,y,z) = 4.05 + 0.045\cos\left(\frac{\pi(x+y+z)}{2}\right). \end{aligned}$$

Moreover, the right hand side function f and the data g and h are chosen such that the true solution is

$$w(x,y,z) = \left(\sin(\pi x) + \sin\left(\frac{\pi y}{2}\right)\right)\cos(\pi z).$$

We examine the p-version of the method on different meshes in Table 4 for polynomial degree $W = 2, \ldots, 10$. It is clear that the method performs best on Mesh 3 and the error reduces to approximately 10^{-6}%. However, on Mesh 1 the

Table 4
Error as a function of W for different values of h for general elliptic (non self-adjoint) problem.

$p = W$	Mesh 1	Mesh 2	Mesh 3
2	0.498666E+02	0.485737E+02	0.291328E+02
3	0.507328E+02	0.107829E+02	0.586317E+01
4	0.227058E+02	0.517751E+01	0.149092E+01
5	0.143968E+02	0.814482E+00	0.237686E+00
6	0.408230E+01	0.234810E+00	0.292025E−01
7	0.184822E+01	0.279261E−01	0.289156E−02
8	0.311612E+00	0.430208E−02	0.236661E−03
9	0.104765E+00	0.377498E−03	0.168223E−04
10	0.136308E−01	0.431639E−04	0.352480E−05

Table 5
Performance of the p-version for non self-adjoint problem.

$p = W$	N_{dof}	Iterations	Relative error(%)
2	64	207	0.670184E+02
3	216	364	0.741499E+01
4	512	464	0.536971E+01
5	1000	519	0.725120E+00
6	1728	547	0.228276E+00
7	2744	605	0.267290E−01
8	4096	642	0.420000E−02
9	5832	671	0.364194E−03
10	8000	694	0.424524E−04

relative error decays slowly (see Table 5). Fig. 8 shows $\log \|E\|_{rel}$ plotted against W for different meshes. In Fig. 9(a) we plot error against polynomial order W. Error as a function of degrees of freedom is plotted in Fig. 9(c) on a log-scale showing exponential convergence.

6.2. Test problems containing singularities

To show the effectiveness of the proposed method for problems containing singularities we now consider test problems having singularities of various types discussed in Section 2.

As earlier, N will denote the number of layers in the geometric mesh and W, the polynomial order used. In case of examples with vertex and edge singularities all our calculations are based on a parallel computer with $O(N)$ processors and in case of vertex-edge singularities we employ a parallel computer with $O(N^2)$ processors (since there are N^2 elements in the geometric mesh in this case) with each element being mapped onto a single processor. The geometric mesh factors in the neighbourhoods of singularities are chosen as $\mu_v = 0.15$ and $\mu_e = 0.15$ which give optimal results.

Our first example is the Poisson equation containing only a vertex singularity with mixed boundary conditions. For computational simplicity we shall assume that the singularity arises only at one vertex of the domain under consideration. Our example is similar to that of Guo and Oh reported in [28].

Example 6.3 (*Mixed Problem Containing Vertex Singularity*)**.** Consider the axisymmetric Poisson equation with mixed boundary conditions:

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega^{(v)}, \\ u &= g \quad \text{on } \mathcal{D} \subset \partial\Omega^{(v)}, \\ \frac{\partial u}{\partial \nu} &= h \quad \text{on } \mathcal{N} = \partial\Omega^{(v)} \setminus \mathcal{D}, \end{aligned} \tag{6.2}$$

where the domain $\Omega^{(v)}$ is shown in Fig. 10 and

$$\begin{aligned} \mathcal{D} &= \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = \Gamma_{\mathcal{D}}^1 \cup \Gamma_{\mathcal{D}}^2, \\ \Gamma_{\mathcal{D}}^1 &= \{(\phi, \theta, \rho) : \phi = \pi/6, \pi/3, 0 \le \theta \le 3\pi/2, 0 \le \rho \le 1\}, \\ \Gamma_{\mathcal{D}}^2 &= \{(\phi, \theta, \rho) : \pi/6 \le \phi \le \pi/3, \theta = 0, 3\pi/2, 0 \le \rho \le 1\}, \\ \mathcal{N} &= \Gamma_5 = \{(\phi, \theta, \rho) : \pi/6 \le \phi \le \pi/3, 0 \le \theta \le 3\pi/2, \rho = 1\}. \end{aligned}$$

We choose data f, g and h such that the function $w = \rho^{0.1}(1 - \rho) \sin 2\phi$ is the true solution of (6.2) satisfying prescribed boundary conditions. Here, ν denotes the exterior unit normal to the part of the boundary where we impose Neumann boundary conditions.

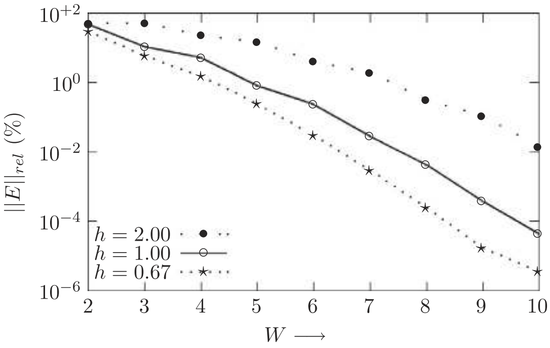

Fig. 8. Error as a function of W for different values of h for general elliptic (non self-adjoint) problem.

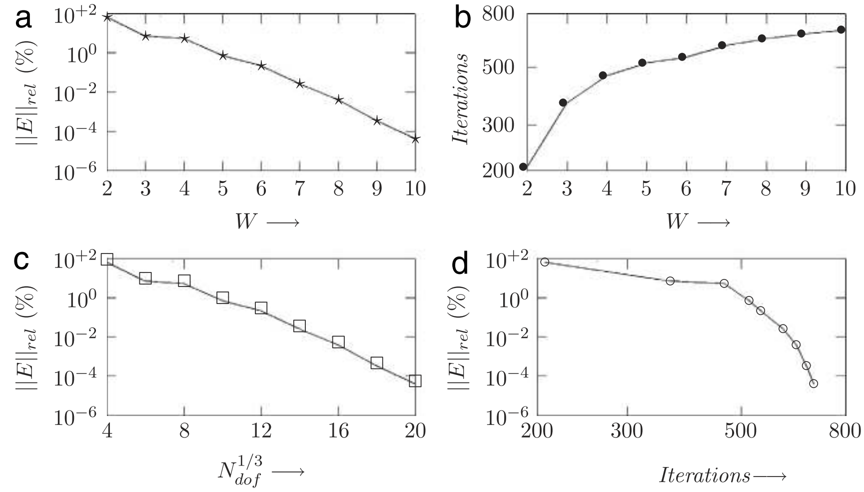

Fig. 9. (a) ln $\|E\|_{\text{rel}}$ vs. p, (b) Iterations vs. N, (c) ln $\|E\|_{\text{rel}}$ vs. $N_{dof}^{1/3}$ and (d) ln $\|E\|_{\text{rel}}$ vs. Iterations for general elliptic (non self-adjoint) equation with variable coefficients.

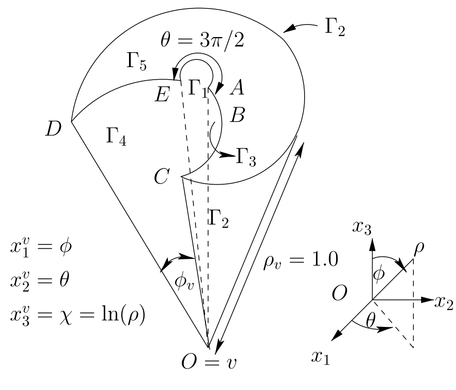

Fig. 10. The domain $\Omega^{(v)}$ containing a vertex singularity.

We know that the error in the neighbourhoods of vertices satisfies

$$\|u_{SE} - w\|_{H^1} \leq Ce^{-bN_{dof}^{1/4}}. \tag{6.3}$$

Table 6 contains the relative error obtained by applying the method on geometrically refined mesh in ρ. Fig. 11 contains plots for various parameters. The relative error vs. $N_{dof}^{1/4}$ (on a log-scale) is depicted in Fig. 11(c). The error profile is almost a straight line. This confirms our theoretical estimate (6.3) on the exponential convergence. Iterations against number of layers are plotted in Fig. 11(b). The relative error against polynomial order and iterations is plotted in Fig. 11(a) and (d) respectively. It is evident from the plots that the method is very effective in dealing with vertex singularities.

Table 6
Performance of the h-p version for mixed problem on $\Omega^{(v)}$ containing a vertex singularity.

$p = W$	N_{dof}	Iterations	Relative error(%)
2	9	16	0.962637E+01
3	55	39	0.252012E+01
4	193	128	0.191490E+00
5	501	176	0.212320E−01
6	1081	314	0.192391E−02
7	2059	409	0.884830E−03
8	3585	743	0.412629E−03
9	5833	814	0.470681E−04

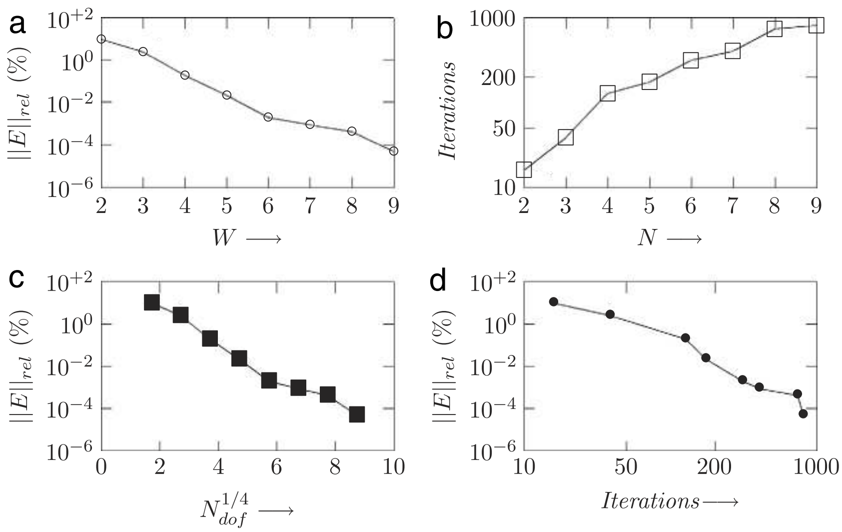

Fig. 11. (a) Error vs. p, (b) Iterations vs. N, (c) Error vs. N_{dof} and (d) Error vs. Iterations for mixed problem containing a vertex singularity.

Next, we apply our method to Laplace equation containing an edge singularity.

Example 6.4 (*Laplace Equation Containing Edge Singularity*)**.** Consider the boundary value problem:

$$\begin{aligned} -\triangle w &= 0 \quad \text{in } \Omega^{(e)}, \\ w &= g \quad \text{on } \partial\Omega^{(e)}, \end{aligned} \tag{6.4}$$

where the domain $\Omega^{(e)}$ (see Fig. 12) is given by

$$\Omega^{(e)} = \{(r, \theta, x_3) : 0 \le r \le 1, 0 \le \theta \le \pi/2, 0 \le x_3 \le 1\}.$$

We impose Dirichlet boundary conditions on all the faces marked as Γ_i, $i = 1, \ldots, 5$.

Let $w(r, \theta, x_3) = r^{\frac{1}{3}} \sin(\frac{\theta}{3})x_3$. Then w is the exact solution of (6.4) satisfying the Dirichlet boundary conditions $u|_{\partial\Omega_1^{(e)}} = w$. Note that w has an edge singularity.

Table 7 contains the numerical results and it shows that $\approx 10^{-5}$(%) of relative error in the H^1-norm is achieved with $W = 10$ and $N_{dof} \approx 9000$. The relative error against polynomial degree for $W = 2, \ldots, 10$ is drawn in Fig. 13(a). In Fig. 13(c) and (d) error as a function of degrees of freedom and iterations is plotted on a log scale. It follows that the error decays exponentially and obeys the theoretical estimate (6.3).

7. Summary and conclusions

We have established error estimates of our method for elliptic problems on three dimensional non-smooth domains, based on the non-conforming hp-version of the spectral element method. The error between the exact and the approximate solution is shown to be exponentially small in N, the number of layers in the geometrical mesh. The method is essentially a least-squares method and we use PCGM to solve normal equations using a block diagonal preconditioner. Moreover, there exists a new preconditioner which can be diagonalized in a new set of basis functions, and hence it is easily inverted on each element. The residuals in the normal equations can be obtained without computing and storing mass and stiffness matrices.

Table 7
Performance of the h-p version for Laplace equation on $\Omega^{(e)}$.

$p = W$	N_{dof}	Iterations	Relative error(%)
2	10	24	0.464542E+00
3	57	34	0.131359E+00
4	196	38	0.402204E−01
5	504	49	0.123974E−01
6	1085	58	0.364617E−02
7	2064	68	0.107525E−02
8	3592	75	0.315249E−03
9	5840	85	0.920349E−04
10	9001	98	0.279848E−04

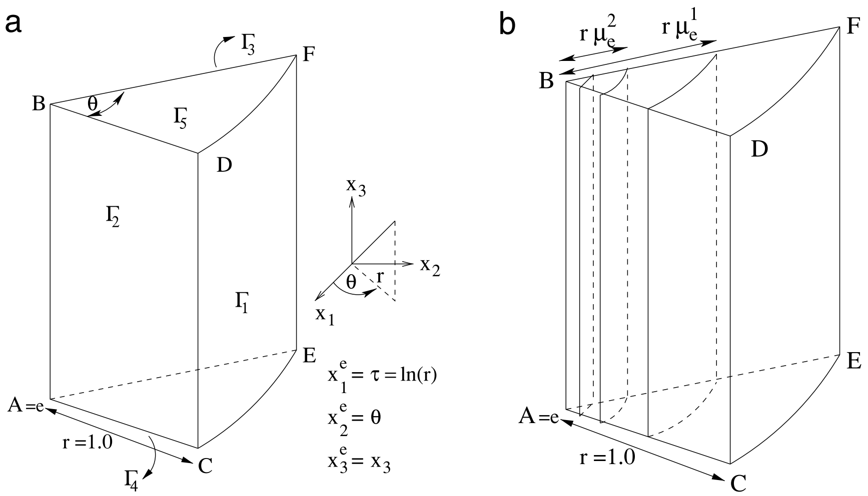

Fig. 12. (a) The domain $\Omega^{(e)}$ containing an edge singularity, (b) Geometrical mesh imposed on $\Omega^{(e)}$.

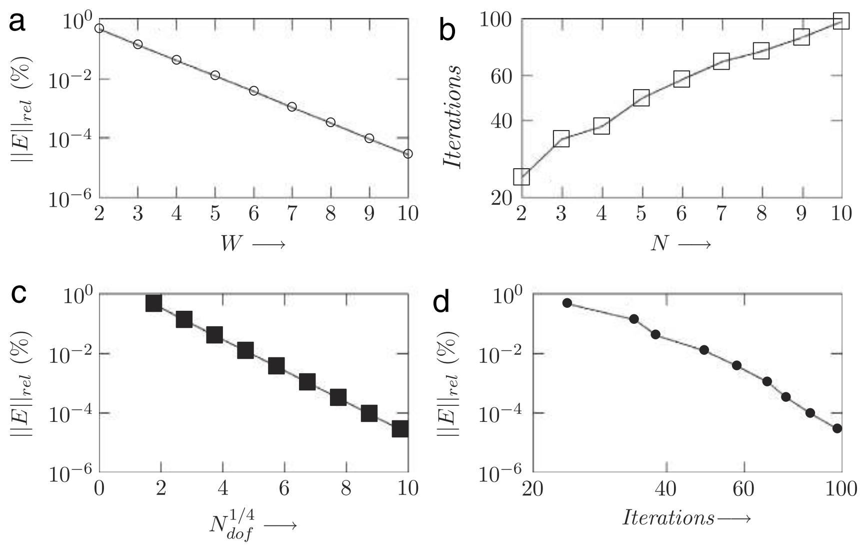

Fig. 13. (a) Error vs. p, (b) Iterations vs. N, (c) Error vs. N_{dof} and (d) Error vs. Iterations for Laplace equation containing an edge singularity.

Numerical experiments on non-smooth domains with analytic and singular solutions confirm our estimates of the error and computational complexity.

The method presented in this series of papers can be applied to the elliptic problems arising from mechanics and engineering such as elasticity problems on polyhedral domains and magnetic–electric problems on smooth and non-smooth domains in three dimensions. We intend to do more rigorous computations on some of these problems in the future work.

References

[1] M. Dauge, Higher order oblique derivative problems on polyhedral domains, Comm. Partial Differential Equations 14 (1989) 1193–1227.
[2] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, in: Lecture Notes in Math., vol. 1341, Springer, New York, 1988.
[3] P. Grisvard, Singularities in Boundary Value Problems, Masson, Springer, Paris, Berlin, 1992.
[4] P. Grisvard, Singularité en élasticitè, Arch. Ration. Mech. Anal. 107 (1989) 157–180.
[5] P. Grisvard, Elliptic Problems in Nonsmooth Domains, in: Monographs and Studies in Mathematics, vol. 24, Pitman Advanced Publishing Program, 1985.
[6] V.G. Mazya, B.A. Plamenevskii, On the coefficients in the asymptotic of solutions of elliptic boundary value problems in domain with conical points, Amer. Math. Soc. Transl. Ser. 2 123 (1984) 57–88.
[7] V.G. Mazya, B.A. Plamenevskii, Estimates in L_p and Hölder class and the Miranda-Agmon maximum principle for solutions of elliptic boundary problems in domains with singular points on the boundary, Amer. Math. Soc. Transl. Ser. 2 123 (1984) 1–56.
[8] I. Babuška, B. Guo, Regularity of the solutions for elliptic problems on non-smooth domains in R^3, Part I: Countably Normed spaces on polyhedral domains, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 77–126.
[9] I. Babuška, B. Guo, Regularity of the solutions for elliptic problems on non-smooth domains in R^3, Part II: Regularity in neighbourhoods of edges; *Tech. Note BN-1182*, IPST, University of Maryland, College Park, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 517–545.
[10] M. Costabel, M. Dauge, S. Nicaise, Weighted analytic regularity in polyhedra, Comput. Math. Appl. 67 (4) (2014) 807–817.
[11] M. Costabel, M. Dauge, S. Nicaise, Analytic regularity for linear elliptic systems in polygons and polyhedra, Math. Models Methods Appl. Sci. 22 (8) (2012).
[12] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods, Fundamentals in Single Domains, in: Scientific Computaion, Springer, 2006.
[13] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral methods, Evolution to Complex Domains and Applications to Fluid Dynamics, in: Scientific Computaion, Springer, 2007.
[14] G. Karniadakis, J. Sherwin, Spectral/*hp* Element Methods for CFD, Oxford University Press, 1999.
[15] P. Dutt, Akhlaq Husain, A.S. Vasudeva Murthy, C.S. Upadhyay, *h-p* spectral element methods for three dimensional elliptic problems on non-smooth domains using parallel computers, Part-I: Regularity estimates and stability theorem, Proc. Math. Sci. 125 (2) (2015) 239–270.
[16] P. Dutt, Akhlaq Husain, A.S. Vasudeva Murthy, C.S. Upadhyay, *h-p* spectral element methods for three dimensional elliptic problems on non-smooth domains, Part-II: Proof of stability theorem, Proc. Math. Sci. 125 (3) (2015) http://dx.doi.org/10.1007/s12044-015-0239-2.
[17] P. Dutt, Akhlaq Husain, A.S. Vasudeva Murthy, C.S. Upadhyay, *h-p* Spectral element methods for three dimensional elliptic problems on non-smooth domains, Appl. Math. Comput. 234 (2014) 13–35.
[18] Husain Akhlaq, *h-p* spectral element methods for three dimensional elliptic problems on non-smooth domains using parallel computers (Ph.D. Thesis), IIT Kanpur India, 2011, Reprint available at http://arxiv.org/abs/1110.2316.
[19] P. Dutt, S. Tomar, Stability estimates for *h-p* spectral element methods for general elliptic problems on curvilinear domains, Proc. Indian Acad. Sci. Math. Sci. 113 (4) (2003) 395–429.
[20] S.K. Tomar, *h-p* spectral element methods for elliptic problems over non-smooth domains using parallel computers, Computing 78 (2006) 117–143.
[21] D. Schötzau, C. Schwab, T.P. Wihler, *hp*-DGFEM for second order elliptic problmes in polyhedra, I: Stability and quasioptimality on geometric meshes, SIAM J. Numer. Anal. 51–53 (2013) 1610–1633.
[22] D. Schötzau, C. Schwab, T.P. Wihler, *hp*-DGFEM for second order elliptic problmes in polyhedra, II: Exponential convergence, SIAM J. Numer. Anal. 51–54 (2013) 2005–2035.
[23] B. Guo, $h-p$ version of the finite element method in R^3: Theory and algorithm, in: A.V. Ilin, L.R. Scott (Eds.), ICOSAHOM 95: Proceedings of the Third International Conference on Spectral and Higher Order Methods, Houston, TX, June 5, 1995, Houston J. Math. (1996), 487–500.
[24] I. Babuška, B. Guo, The *h-p* version of the finite element method on domains with curved boundaries, SIAM J. Numer. Anal. 25 (1988) 837.
[25] B. Guo, in: M. Costabel, M. Dauge, C. Nicaise (Eds.), The *h-p* Version of the Finite Element Method for Solving Boundary Value Problems in Polyhedral Domains: Boundary Value Problems and Integral Equations in Non-smooth Domains, Marcel Dekker Inc., 1994, pp. 101–120.
[26] I. Babuška, B. Guo, The *h-p* Version of the finite element method, Part I: The basic approximation results, Comput. Mech. 1 (1986) 21–41.
[27] P.K. Dutt, N. Kishore Kumar, C.S. Upadhyay, Non-conforming $h-p$ spectral element methods for elliptic problems, Proc. Indian Acad. Sci. Math. Sci. 117 (1) (2007) 109–145.
[28] B. Guo, S.H. Oh, The method of auxiliary mapping for the finite element solutions of elliptic partial differential equations on non-smooth domains in R^3, 1995. Available at http://home.cc.umanitoba.ca/~guo/publication.htm.
[29] P. Dutt, P. Biswas, G.N. Raju, Preconditioners for spectral element methods for elliptic and parabolic problems, J. Comput. Appl. Math. 215 (1) (2008) 152–166.
[30] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.