

EXISTENCE OF THREE POSITIVE SOLUTIONS FOR A NONLOCAL SINGULAR DIRICHLET BOUNDARY PROBLEM

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Abstract

In this article, we prove the existence of at least three positive solutions for the following nonlocal singular problem

$$(P_\lambda) \begin{cases} (-\Delta)^s u = \lambda \frac{f(u)}{u^q}, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplace operator for $s \in (0, 1)$, $n > 2s$, $q \in (0, 1)$, $\lambda > 0$ and Ω is smooth bounded domain in \mathbb{R}^n . Here $f : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing map satisfying $\lim_{u \rightarrow \infty} \frac{f(u)}{u^{q+1}} = 0$. We show that under certain additional assumptions on f , (P_λ) possesses at least three distinct solutions for a certain range of λ . We use the method of sub-supersolutions and a critical point theorem by Amann [3] to prove our results. Moreover, we prove a new existence result for a suitable infinite semipositone nonlocal problem which played a crucial role to obtain our main result and is of independent interest.

Key words: Fractional Laplacian, singular nonlinearity, infinite semipositone problem, sub-supersolutions, three positive solutions.

2010 Mathematics Subject Classification: 35R11, 35R09, 35A15.

1 Introduction

In the present paper, we consider the following nonlocal singular problem

$$(P_\lambda) \begin{cases} (-\Delta)^s u = \lambda \frac{f(u)}{u^q}, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where $s \in (0, 1)$, $q \in (0, 1)$, $\lambda > 0$ and Ω is smooth bounded domain in \mathbb{R}^n . We have the following assumptions on $f \in C^1([0, \infty))$:

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$$(f1) \quad f(0) > 0,$$

$$(f2) \quad \lim_{u \rightarrow \infty} \frac{f(u)}{u^{q+1}} = 0,$$

$$(f3) \quad u \mapsto f(u) \text{ is non decreasing in } \mathbb{R}^+,$$

$$(f4) \quad \text{There exist a } 0 < \sigma_1 < \sigma_2 \text{ such that } \frac{f(u)}{u^q} \text{ is non decreasing on } (\sigma_1, \sigma_2).$$

Note that from (f1) we have $\lim_{u \rightarrow 0} \frac{f(u)}{u^q} = \infty$.

Remark 1.1 *It is easy to see that the function f defined as $f(t) = e^{\frac{\alpha t}{\alpha+t}}$ for any $t \geq 0$ with $\alpha > 4q$ satisfy assumptions (f1)-(f4).*

The fractional Laplace operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = 2C_s^n \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$$

where P.V. denotes the Cauchy principal value and $C_s^n = \pi^{-\frac{n}{2}} 2^{2s-1} s \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}$, Γ being the Gamma function. The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion processes and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [5] for instance. Fractional Sobolev spaces were introduced mainly in the framework of harmonic analysis in the middle part of last century and are the natural setting to study weak solutions to problems involving the fractional Laplacian. In this regard, the paper of Caffarelli and Silvestre [8] on the harmonic extension problem have subsequently motivated many works on equations and systems involving the fractional Laplacian $(-\Delta)^s$, $s \in (0, 1)$. We also refer [29] to readers for a detailed study on variational methods for fractional elliptic problems and for additional references.

In the local case, i.e. $s = 1$, the study of elliptic singular problems starts mainly with the pioneering work of Crandal, Rabinowitz and Tartar [11]. This seminal work inspired a huge list of articles where authors have investigated many different issues (existence/nonexistence, uniqueness/multiplicity, regularity of solutions, etc.) about singular problems in the local and more recently in the nonlocal set up. We cite here some related works with no intent to furnish an exhaustive list. The multiplicity of solutions for singular problem with critical nonlinearity has been studied in [23, 25, 26] while the exponential critical nonlinearity has been dealt with in [14]. Semilinear elliptic and singular problems with convection term was first studied in [19] whereas [15] brought existence results to elliptic equations involving a singular absorption term. We refer the surveys [18] and [24] for further details on singular elliptic equations in the local setting. In the nonlocal case, singular problem with critical nonlinearity has been studied in [6, 20, 30]. Recently, Adimurthi, Giacomoni and Santra [2] studied the following nonlocal singular problem:

$$(-\Delta)^s u = \lambda(K(x)u^{-\delta} + f(u)), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \quad (1.1)$$

where $\delta, \lambda > 0$, $K : \Omega \rightarrow \mathbb{R}^+$ is a Hölder continuous function in Ω behaves like $\text{dist}(x, \partial\Omega)^{-\beta}$, $\beta \in [0, 2s)$ and f is a positive real valued C^2 function. They established existence, regularity and bifurcation results using the framework of weighted spaces.

Recently, [16] has established the existence of three non-zero solutions for a Dirichlet type boundary value problem involving the fractional Laplacian. But the study of three solutions for singular nonlocal problems was completely open till now. Our work brings new results in this regard. We use the method of sub and supersolution combined with a fix point theorem due to Amann to achieve the objective. For the construction of the barrier functions, we have taken some ideas from [28].

The salient feature of this work is the presence of the singular term u^{-q} which is a primary hindrance in making the operator monotone. We slightly transform the problem to a new one and show that the operator associated with it becomes monotone increasing and compact. This idea had been formerly used in [13] in the local case. But here itself we remark that their approach can not be directly applied to the problem (P_λ) due to the presence of the nonlocal operator ' $(-\Delta)^s$ ' instead of Δ . Most substantially, Theorem 3.6 of [13] can not be adapted here due to the lack of an explicit form of $(-\Delta)^s \delta^s(x)$ where $\delta(x) = \text{dist}(x, \partial\Omega)$ denotes the distance function up to the boundary. To overcome this difficulty we construct a subsolution v satisfying

$$(-\Delta)^s v + \frac{cv}{\delta^q(x)} \leq 0 \text{ in } \Omega_r$$

where $c > 0$ is constant and $\Omega_r \subset \Omega$. To obtain this, we separately study, by bifurcation arguments, a nonlocal infinite semipositone problem (I_θ) in section 6. This leads naturally to a solution of the required problem. In the local setting, we refer to readers [31, 27, 22] concerning infinite semipositone problems. But we indicate that 'nonlocal' infinite semipositone problem has not been studied in the past. So our results are completely new in this regard. The main result of our paper is accomplished by using a well known critical point theorem by Amann [3]. Last but not the least, we additionally prove the uniqueness of solutions to (P_λ) when λ becomes sufficiently large under appropriate condition of ' f '. This result is motivated by the paper [9] of Castro, Eunkyung and Shivaji. Nevertheless, we point out that their approach can not be exactly applied here due to the presence of the nonlocal operator ' $(-\Delta)^s$ '. We still succeed to obtain the result by an appropriate application of the Hardy's inequality for fractional Laplacian (refer section 5). Now we state the main results of our paper as follows.

Theorem 1.2 *There exists constants $\lambda_1, \lambda_2 > 0$ such that if $\lambda \in [\lambda_1, \lambda_2]$ then the problem (P_λ) has at least three solutions in $C_{\phi_{1,s}}^+(\Omega)$.*

Remark 1.3 *Theorem 1.2 still holds if (f3) is replaced by the weakened assumption (f3'): There exists $k > 0$ such that $u \mapsto f(u) + ku$ is non decreasing in \mathbb{R}^+ .*

Theorem 1.4 *There exist a $\lambda^* > 0$ such that (P_λ) has a unique solution when $\lambda > \lambda^*$.*

Remark 1.5 *Since $f(0) > 0$, it is not difficult to show that for $\lambda > 0$ small enough, there exists a unique solution with small norm. Then Theorem 1.2 and 1.4 entail that the bifurcation curve of solutions to (P_λ) emanating from $(0, 0)$ is S-shaped.*

The outline of this paper is as follows: In the second Section we give some useful preliminaries about the main equation in (P_λ) . In the third Section, we construct the sub and supersolutions used to apply the fixed point theorem of Amann. In Section 4, we prove the main result of our paper: Theorem 1.2. In Section 5, we prove our main uniqueness result: Theorem 1.4 and finally in Section 6, we investigate the fractional and singular semipositone problem (I_θ) used crucially for proving the strong increasingness of the operator T in Section 4.

2 Preliminaries

We start with defining the function spaces. Given any $\phi \in C_0(\overline{\Omega})$ such that $\phi > 0$ in Ω we define

$$C_\phi(\Omega) := \{u \in C_0(\Omega) \mid \exists c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \forall x \in \Omega\}$$

with the usual norm $\left\| \frac{u}{\phi} \right\|_{L^\infty(\Omega)}$ and the associated positive cone. We define the following open convex subset of $C_\phi(\Omega)$ as

$$C_\phi^+(\Omega) := \left\{ u \in C_\phi(\Omega) \mid \inf_{x \in \Omega} \frac{u(x)}{\phi(x)} > 0 \right\}.$$

In particular, C_ϕ^+ contains all those functions $u \in C_0(\Omega)$ with $k_1\phi \leq u \leq k_2\phi$ in Ω for some $k_1, k_2 > 0$. We consider the following fractional Sobolev space

$$\tilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

equipped with the norm

$$\|u\| = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}, \text{ where } Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega).$$

Definition 2.1 *We say that $u \in \tilde{H}^s(\Omega)$ is a weak solution to (P_λ) if $\inf_K u > 0$ for every compact subset $K \subset \Omega$ and for any $\varphi \in \tilde{H}^s(\Omega)$,*

$$\int_\Omega (-\Delta)^s u \varphi = C_s^n \int_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy = \lambda \int_\Omega \frac{f(u)}{u^q} \varphi dx. \quad (2.1)$$

Definition 2.2 *By a subsolution of the problem (P_λ) , we mean a function $v \in \tilde{H}^s(\Omega)$ which satisfies (weakly)*

$$(-\Delta)^s v \leq \lambda \frac{f(v)}{v^q}, \quad v > 0 \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \quad (2.2)$$

Whereas if the reverse inequality holds in (2.2), we call v to be a supersolution of (P_λ) . Also we call them strict sub and supersolution if the inequality in (2.2) is strict.

We define the distance function as $\delta(x) := \text{dist}(x, \partial\Omega)$, $x \in \Omega$. Let $\phi_{1,s}$ denotes the first positive eigenfunction of $(-\Delta)^s$ in $\tilde{H}^s(\Omega)$ corresponding to its principal eigenvalue $\lambda_{1,s}$ such that $\|\phi_{1,s}\|_{L^\infty(\Omega)} = 1$. We recall that $\phi_{1,s} \in C^s(\mathbb{R}^n)$ and also $\phi_{1,s} \in C_{\delta^s}^+(\Omega)$ (see for instance Proposition 1.1 and Theorem 1.2 of [32]).

3 Sub and Supersolutions of (P_λ)

In this section we show the existence of two pairs of sub-supersolutions (ζ_1, ϑ_1) and (ζ_2, ϑ_2) such that $\zeta_1 \leq \zeta_2 \leq \vartheta_1$, $\zeta_1 \leq \vartheta_2 \leq \vartheta_1$ and $\zeta_2 \not\leq \vartheta_2$. Moreover it holds that ζ_2, ϑ_2 are strict sub and supersolutions of (P_λ) . Let w denotes the unique solution of the problem

$$(-\Delta)^s w = \frac{1}{w^q}, \quad w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \quad (3.1)$$

Then from the proof of Theorem 1.1 of [2], we know that $w \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}^+(\Omega)$ and $w \in C^s(\mathbb{R}^n)$. We construct our supersolution ϑ_1 first. Since (f2) holds, we get $\lim_{u \rightarrow \infty} \frac{f(u)}{u^{q+1}} = 0$. This implies that if we choose a constant $M_\lambda \gg 1$ sufficiently large such that

$$\frac{f(M_\lambda \|w\|_{L^\infty(\Omega)})}{(M_\lambda \|w\|_{L^\infty(\Omega)})^{q+1}} \leq \frac{1}{\lambda \|w\|_{L^\infty(\Omega)}^{q+1}} \quad \text{i.e.} \quad M_\lambda^{q+1} \geq \lambda f(M_\lambda \|w\|_{L^\infty(\Omega)})$$

then $\vartheta_1 = M_\lambda w \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}^+(\Omega)$ forms a supersolution of (P_λ) . Indeed using non decreasing nature of f we get

$$(-\Delta)^s \vartheta_1 = \frac{M_\lambda^{q+1}}{(M_\lambda w)^q} \geq \lambda \frac{f(M_\lambda \|w\|_{L^\infty(\Omega)})}{(M_\lambda w)^q} \geq \lambda \frac{f(M_\lambda w)}{(M_\lambda w)^q} = \lambda \frac{f(\vartheta_1)}{(\vartheta_1)^q}.$$

Now since $\lim_{u \rightarrow 0} \frac{f(u)}{u^q} = \infty$, we can choose $m_\lambda > 0$ sufficiently small so that

$$\lambda_{1,s} m_\lambda \phi_{1,s} \leq \lambda \frac{f(m_\lambda \phi_{1,s})}{(m_\lambda \phi_{1,s})^q}, \quad \text{for each } \lambda > 0.$$

Now we define $\zeta_1 = m_\lambda \phi_{1,s} \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}^+(\Omega)$ and it is easy to see that

$$(-\Delta)^s \zeta_1 = m_\lambda \lambda_{1,s} \phi_{1,s} \leq \lambda \frac{f(m_\lambda \phi_{1,s})}{(m_\lambda \phi_{1,s})^q} = \lambda \frac{f(\zeta_1)}{\zeta_1^q}.$$

Therefore ζ_1 is a subsolution of (P_λ) . It is not hard to see that we always choose m_λ small enough so that $\zeta_1 \leq \vartheta_1$. This completes our construction of first pair of sub-supersolution.

Our next step is to construct the second pair of sub-supersolution of (P_λ) . We first construct our positive supersolution ϑ_2 such that $\|\vartheta_2\|_{L^\infty(\Omega)} = \sigma_1$ (see (f4)). Let us define $\vartheta_2 = \frac{\sigma_1 w}{\|w\|_{L^\infty(\Omega)}} \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}^+(\Omega)$ and assume that

$$0 < \lambda \leq \frac{\sigma_1^{q+1}}{f(\sigma_1) \|w\|_{L^\infty(\Omega)}^{q+1}}.$$

Then using the non decreasing nature of f we find that it satisfies

$$(-\Delta)^s \vartheta_2 = \frac{\sigma_1}{\|w\|_{L^\infty(\Omega)} w^q} \geq \lambda \frac{f(\sigma_1) \|w\|_{L^\infty(\Omega)}^q}{(\sigma_1 w)^q} \geq \lambda \frac{f\left(\frac{\sigma_1 w}{\|w\|_{L^\infty(\Omega)}}\right)}{\left(\frac{\sigma_1 w}{\|w\|_{L^\infty(\Omega)}}\right)^q} = \lambda \frac{f(\vartheta_2)}{\vartheta_2^q}.$$

Now we construct our second positive supersolution of (P_λ) which is one of the crucial part of our paper. For this, we let $\sigma \in (0, \sigma_1]$ be such that $f^*(\sigma) = \min_{0 < x \leq \sigma} \frac{f(x)}{x^q}$ and also define $h \in C([0, \infty))$ such that

$$h(u) = \begin{cases} f^*(\sigma), & \text{if } u \leq \sigma \\ \frac{f(u)}{u^q}, & \text{if } u \geq \sigma_1 \end{cases}$$

so that h is a non decreasing function on $(0, \sigma_1]$ and $h(u) \leq \frac{f(u)}{u^q}$ for all $u \geq 0$. With this definition of h , we consider a non singular problem

$$(Q_\lambda) \begin{cases} (-\Delta)^s u = \lambda h(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $G_s(x, y)$ denote the Green function associated to $(-\Delta)^s$ with homogeneous Dirichlet boundary condition in Ω . Then we have

$$u(x) = \begin{cases} \lambda \int_{\Omega} G_s(x, y) h(y) \, dy, & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $B_{\hat{R}}(0)$ denotes the ball (centered at 0 where w.l.o.g. we assume that $0 \in \Omega$) of largest radius \hat{R} that is inscribed in Ω and also let $R < \hat{R}$. Suppose $K : L^2(\Omega) \rightarrow L^2(\Omega)$ be the linear map defined as

$$K(g)(x) = \int_{\Omega} G_s(x, y) g(y) \, dy.$$

Let $\chi_R : \Omega \rightarrow \mathbb{R}$ be the characteristic function defined as

$$\chi_R(x) = \begin{cases} 1, & \text{if } x \in B_R(0), \\ 0, & \text{if } x \in \Omega \setminus B_R(0). \end{cases}$$

Then from Theorem 1.1 of [10], for each $(x, y) \in \Omega \times \Omega$ we have that

$$0 \leq G_s(x, y) \leq C \min \left\{ \frac{\delta^s(x) \delta^s(y)}{|x - y|^n}, \frac{\delta^s(x)}{|x - y|^{n-s}} \right\}. \quad (3.2)$$

Therefore there exists a constant $C_1 > 0$ depending in R such that

$$K(\chi_R)(x) = \int_{\Omega} G_s(x, y) \, dy \leq C \delta^s(x) \int_{B_R(0)} \frac{dy}{|x - y|^{n-s}} \leq C_1$$

uniformly in $x \in \Omega$. Let $M_2 = \left(\min_{x \in \Omega} K(\chi_R)(x) \right)^{-1} > 0$ and $a \in (\sigma_1, \sigma_2]$. Also we define $v = a \chi_R$ in Ω . Then if v_1 denotes a solution to the problem

$$(-\Delta)^s v_1 = h(v) \text{ in } \Omega, \quad v_1 = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \quad (3.3)$$

then for each $x \in \Omega$ we get

$$v_1(x) = \lambda \int_{\Omega} G_s(x, y) h(a\chi_R)(y) \, dy \leq \lambda h(a) \int_{\Omega} G_s(x, y) \, dy$$

where we used the fact that h is non decreasing and $\chi_R \leq 1$ in Ω . Therefore $v_1 \leq \sigma_2$ in $B_R(0)$ if

$$\lambda \leq \frac{M_3 \sigma_2}{h(a)}, \text{ where } M_3 = \left(\max_{x \in \Omega} \int_{\Omega} G_s(x, y) \, dy \right)^{-1} < +\infty.$$

Claim: $v_1 \geq v$ in Ω for certain range of λ .

Let $x \in \Omega \setminus B_R(0)$ then since $G_s(x, y) \geq 0$ for all $x, y \in \Omega$ and $h(u) > 0$ for all $u \geq 0$ we get $v(x) = 0 \leq v_1(x)$. Now let $x \in B_R(0)$ then

$$\begin{aligned} v_1(x) &= \lambda \int_{\Omega} G_s(x, y) h(a\chi_R)(y) \, dy \\ &= \lambda \left(\int_{B_R(0)} G_s(x, y) h(a) \, dy + \int_{\Omega \setminus B_R(0)} G_s(x, y) h(0) \, dy \right) \\ &\geq \lambda h(a) \int_{B_R(0)} G_s(x, y) \, dy, \text{ since } h(0) = \frac{f(\sigma)}{\sigma^q} > 0 \text{ and } G_s(x, y) \geq 0 \\ &\geq \lambda \frac{h(a)}{M_2}. \end{aligned}$$

So if we assume that $\lambda \geq \frac{M_2 a}{h(a)}$ then by definition of v we get

$$v_1(x) \geq a \geq v(x), \text{ for all } x \in \Omega. \quad (3.4)$$

Hence we finally get that $0 \leq v \leq v_1 \leq \sigma_2$ in Ω if

$$\frac{M_2 a}{h(a)} \leq \lambda \leq \frac{M_3 \sigma_2}{h(a)}.$$

Since we assumed h to be non decreasing in $(0, \sigma_2)$ (because of (f4)) we get $h(v(x)) \leq h(v_1(x))$, for $x \in \Omega$. So v_1 weakly satisfies the problem

$$(-\Delta)^s v_1 \leq \lambda h(v_1) \text{ in } \Omega, \quad v_1 = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

Since $h(v) \in L^\infty(\Omega)$, using Theorem 1.2 of [32] we get that $v_1 \in C^s(\mathbb{R}^n)$. This gives us that v_1 forms a subsolution of (Q_λ) . Moreover (3.3) and the strong maximum principle implies that $v_1 > 0$ in Ω . Therefore, using the fact that $h(u) \leq \frac{f(u)}{u^q}$ for all $u \geq 0$ implies that $\zeta_2 = v_1$ forms a positive subsolution of (P_λ) .

Therefore we got that if

$$\lambda_1 := \frac{M_2 a}{h(a)} \leq \lambda \leq \min \left\{ \frac{\sigma_1^{q+1}}{f(\sigma_1) \|w\|_{L^\infty(\Omega)}^{q+1}}, \frac{M_3 \sigma_2}{h(a)} \right\} =: \lambda_2$$

then we obtain a positive subsolution ζ_2 and a positive supersolution ϑ_2 of (P_λ) such that $\zeta_2 \not\leq \vartheta_2$. Indeed, $\|\vartheta_2\|_{L^\infty(\Omega)} = \sigma_1$ and $\|\zeta_2\|_{L^\infty(\Omega)} \geq a > \sigma_1$.

4 Proof of main result

In this section we prove our main result after establishing some necessary results. We begin by noticing that our problem (P_λ) can be rewritten as

$$(\tilde{P}_\lambda) \left\{ \begin{array}{l} (-\Delta)^s u - \lambda \frac{f(0)}{u^q} = \tilde{f}(u), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{array} \right.$$

where $\tilde{f}(u) = \lambda \left(\frac{f(u) - f(0)}{u^q} \right)$. We fix that $\lambda \in [\lambda_1, \lambda_2]$. Since $f \in C^1([0, \infty))$, by Mean value theorem we get $\tilde{f}(u) = \lambda f'(v)u^{1-q}$ for some $v \in (0, u)$. Also this implies $\tilde{f}(0) = 0$ because $\lim_{t \rightarrow 0} |f'(t)| < \infty$ and $q \in (0, 1)$. Therefore \tilde{f} can be considered as a continuous function on $[0, \infty)$ such that $\tilde{f}(0) = 0$. We assume also the following-

(h1) There exists a constant $\tilde{k} > 0$ such that $\tilde{f}(t) + \tilde{k}t$ is increasing in $[0, \infty)$.

Definition 4.1 We say that $z \in \tilde{H}^s(\Omega)$ is a weak solution of (\tilde{P}_λ) if $\inf_K z > 0$ for every compact subset $K \subset \Omega$ and for any $\varphi \in C_c^\infty(\Omega)$,

$$C_s^m \int_Q \frac{(z(x) - z(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy - \lambda f(0) \int_\Omega \frac{\varphi}{z^q} dx = \int_\Omega \tilde{f}(z) \varphi dx. \quad (4.1)$$

We remark that for any $\varphi \in \tilde{H}^s(\Omega)$ and $z(x) \geq k_1 \delta^s(x)$ in Ω , Hardy's inequality gives that

$$\left| \int_\Omega \frac{\varphi}{z^q} dx \right| \leq k_1 \int_\Omega \frac{|\varphi(x)|}{\delta^s(x)} \delta^{s(1-q)}(x) dx \leq k_1 \left(\int_\Omega \frac{|\varphi(x)|^2}{\delta^{2s}(x)} \right)^{\frac{1}{2}} \left(\int_\Omega \delta^{2s(1-q)}(x) dx \right)^{\frac{1}{2}} \leq C \|\varphi\|$$

where $k_1, C > 0$ are constants. So now following the arguments of Lemma 3.2 of [20] we can prove that a weak solution $z \in \tilde{H}^s(\Omega)$ of (\tilde{P}_λ) satisfies (4.1) for every $\varphi \in \tilde{H}^s(\Omega)$ if $z \geq k_1 \delta^s(x)$ in Ω .

We extend the function f and \tilde{f} naturally as $f(t) = f(0)$ and $\tilde{f}(t) = \tilde{f}(0)$ for $t \leq 0$. Because of the assumption in (h1), w.l.o.g. we can assume that \tilde{f} is increasing in \mathbb{R}^+ . Now we define a map $T : C_0(\bar{\Omega}) \rightarrow C_{\phi_{1,s}}(\Omega)$ as $T(u) = z$ if and only if z is a weak solution of

$$(S_\lambda) \left\{ \begin{array}{l} (-\Delta)^s z - \lambda \frac{f(0)}{z^q} = \tilde{f}(u), \quad z > 0 \text{ in } \Omega, \quad z = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{array} \right.$$

By saying that $z \in \tilde{H}^s(\Omega)$ is a weak solution of (S_λ) , we mean that it satisfies

$$C_s^m \int_Q \frac{(z(x) - z(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy - \lambda f(0) \int_\Omega \frac{\varphi}{z^q} dx = \int_\Omega \tilde{f}(u) \varphi dx. \quad (4.2)$$

for all $\varphi \in C_c^\infty(\Omega)$. But repeating the arguments as above for (\tilde{P}_λ) we can show that $z \in \tilde{H}^s(\Omega)$ of (S_λ) satisfies (4) for every $\varphi \in \tilde{H}^s(\Omega)$ if $z \geq k_1 \delta^s(x)$ in Ω .

Proposition 4.2 A function $z \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}(\Omega)$ is a weak solution of (S_λ) if and only if z is a fixed point of T .

Proof. Suppose $z \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}(\Omega)$ is a weak solution of (S_λ) then it is clear that z forms a fixed point of the map T . Conversely assume $T(z) = z$ for some $z \in C_0(\overline{\Omega})$. Then it satisfies (4) but it remains to show that $z \in C_{\phi_{1,s}}^+(\Omega)$. Since $z > 0$ in Ω and $\tilde{f}(z)$ is locally Hölder continuous in Ω , we can follow the arguments of Theorem 1.2 of [2] to obtain that $z \in C_{\phi_{1,s}}^+(\Omega)$. \blacksquare

Proposition 4.3 *The map T is well defined from $C_0(\overline{\Omega})$ to $C_{\phi_{1,s}}^+(\Omega)$.*

Proof. Let $u \in C_0(\overline{\Omega})$ and define $v = \tilde{f} \circ u$ then $v \in C_0(\overline{\Omega})$ and $v \geq 0$ in Ω . To show that T is a well defined map we need to show that (S_λ) has a unique solution corresponding to the above u . We introduce the following approximated problem for $\epsilon > 0$

$$(S_\lambda^\epsilon) \left\{ \begin{array}{l} (-\Delta)^s z - \lambda \frac{f(0)}{(z + \epsilon)^q} = \tilde{f}(u), \quad z > 0 \text{ in } \Omega, \quad z = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{array} \right.$$

Then (S_λ^ϵ) has a unique solution in $\tilde{H}^s(\Omega)$. Indeed, let $\tilde{H}^s(\Omega)^+$ denote the positive cone of $\tilde{H}^s(\Omega)$ and define the energy functional $E_\epsilon : \tilde{H}^s(\Omega)^+ \rightarrow \mathbb{R}$ as

$$E_\epsilon(z) = \frac{\|z\|^2}{2} - \lambda \frac{f(0)}{1-q} \int_{\Omega} (z + \epsilon)^{1-q} dx - \int_{\Omega} \tilde{f}(u) z dx$$

where $z \in \tilde{H}^s(\Omega)^+$. Then E_ϵ is weakly lower semi-continuous, strictly convex and coercive in $\tilde{H}^s(\Omega)^+$. Therefore, E_ϵ admits a unique minimizer, say $z_\epsilon \neq 0$ in $\tilde{H}^s(\Omega)^+$. Since for small $t > 0$ the term $t^{1-q} \int_{\Omega} (z + \epsilon)^{1-q} dx$ dominates so $E_\epsilon(tz)$ can be made small enough and we get $\inf_{\tilde{H}^s(\Omega)^+} E_\epsilon < 0$. We choose $m > 0$ (independent of ϵ) sufficiently small such that

$$m\lambda_{1,s}\phi_{1,s} \leq v + \lambda \frac{f(0)}{(m\phi_{1,s} + 1)^q} \text{ in } \Omega.$$

Then we get that

$$(-\Delta)^s(m\phi_{1,s}) = m\lambda_{1,s}\phi_{1,s} \leq v + \lambda \frac{f(0)}{(m\phi_{1,s} + 1)^q} \leq v + \lambda \frac{f(0)}{(m\phi_{1,s} + \epsilon)^q} \text{ in } \Omega. \quad (4.3)$$

Claim (1): $m\phi_{1,s} \leq z_\epsilon$, for each $\epsilon > 0$.

We define $\tilde{z}_\epsilon := (m\phi_{1,s} - z_\epsilon)^+$ and assume that $\text{meas}(\text{Supp } \tilde{z}_\epsilon)$ is non zero. Then $\eta : [0, 1] \rightarrow \mathbb{R}$ defined by $\eta(t) = E_\epsilon(z_\epsilon + t\tilde{z}_\epsilon)$ is a convex function since $E_\epsilon|_{\tilde{H}^s(\Omega)^+}$ is convex. Also

$$\eta'(1) = C_s^n \int_Q \frac{((z_\epsilon + \tilde{z}_\epsilon)(x) - (z_\epsilon + \tilde{z}_\epsilon)(y))(\tilde{z}_\epsilon(x) - \tilde{z}_\epsilon(y))}{|x - y|^{n+2s}} dx dy - \lambda \int_{\Omega} \frac{f(0)\tilde{z}_\epsilon}{(z_\epsilon + \tilde{z}_\epsilon + \epsilon)^q} - \int_{\Omega} \tilde{f}(u)\tilde{z}_\epsilon$$

in $(0, 1]$. The fact that z_ϵ is a minimizer of E_ϵ gives that $\lim_{t \rightarrow 0^+} \eta'(t) \geq 0$ and $0 \leq \eta'(0) \leq \eta'(1)$.

Let us recall the following inequality for any ψ being a convex Lipschitz function:

$$(-\Delta)^s \psi(u) \leq \psi'(u)(-\Delta)^s u.$$

Therefore using this with $\psi(x) = \max\{x, 0\}$ and (4.3), we get $\eta'(1) \leq \langle E'_\epsilon(m\phi_{1,s}), \tilde{z}_\epsilon \rangle < 0$ which is a contradiction. Hence $\text{supp}(\tilde{z}_\epsilon)$ must have measure zero which establishes the claim (1). Thus E_ϵ is Gâteaux differentiable at z_ϵ and z_ϵ satisfies (S_λ^ϵ) weakly. Since

$$\tilde{f}(u) + \lambda \frac{f(0)}{(z_\epsilon + \epsilon)^q} \in L^\infty(\Omega), \text{ for each } \epsilon > 0,$$

from Proposition 1.1 and Theorem 1.2 of [32] and claim (1) we get that $z_\epsilon \in C^s(\mathbb{R}^n) \cap C_{\phi_{1,s}}^+(\Omega)$ for each $\epsilon > 0$. Thus following the arguments in proof of Theorem 1.1(p.7) of [2] we can show that $\{z_\epsilon\}_{\epsilon>0}$ is a monotone increasing sequence as $\epsilon \downarrow 0^+$ that is for $0 < \epsilon < \epsilon'$, it must be $z_{\epsilon'} < z_\epsilon$ in Ω . Thus we infer that $z = \lim_{\epsilon \downarrow 0^+} z_\epsilon \geq m\phi_{1,s}$. From z_ϵ satisfying (S_λ^ϵ) we obtain

$$\|z_\epsilon\|^2 = \lambda \int_\Omega \frac{f(0)z_\epsilon}{(z_\epsilon + \epsilon)^q} dx + \int_\Omega \tilde{f}(u)z_\epsilon dx. \quad (4.4)$$

We recall the function $w \in \tilde{H}^s(\Omega) \cap C_{\phi_{1,s}}^+(\Omega)$ satisfying (3.1). Let $\bar{z} = Mw$ for $M \gg 1$ (independent of ϵ) sufficiently large so that

$$M \left(\frac{1}{w^q} - \frac{\lambda f(0)}{(Mw)^q} \right) > \tilde{f}(u) \text{ in } \Omega.$$

Then \bar{z} satisfies

$$(-\Delta)^s \bar{z} - \lambda \frac{f(0)}{(\bar{z} + \epsilon)^q} = \frac{M}{w^q} - \frac{\lambda f(0)}{(Mw + \epsilon)^q} > M \left(\frac{1}{w^q} - \frac{\lambda f(0)}{(Mw)^q} \right) > \tilde{f}(u) \text{ in } \Omega.$$

Now we prove that $z_\epsilon \leq \bar{z}$ by using a comparison argument, which we will refer as comparison principle in future. We know that $h = (z_\epsilon - \bar{z}) \in \tilde{H}^s(\Omega)$ satisfies the equation

$$(\Delta)^s(z_\epsilon - \bar{z}) \leq \lambda f(0) \left(\frac{1}{(z_\epsilon + \epsilon)^q} - \frac{1}{(\bar{z} + \epsilon)^q} \right) \text{ in } \Omega. \quad (4.5)$$

If we denote $h^+ = \max\{h, 0\}$ and $h^- = -\min\{h, 0\}$ then $h = h^+ - h^-$. Let $\Omega_h^+ = \{x \in \Omega : z_\epsilon > \bar{z}\}$ and $\Omega_h^- = \Omega \setminus \Omega_h^+$ then testing (4.5) with h^+ gives

$$C_s^n \int_Q \frac{(h(x) - h(y))(h^+(x) - h^+(y))}{|x - y|^{n+2s}} dx dy \leq \lambda f(0) \int_{\Omega_h^+} \left(\frac{1}{(z_\epsilon + \epsilon)^q} - \frac{1}{(\bar{z} + \epsilon)^q} \right) h^+ dx. \quad (4.6)$$

It is easy to see that $(h(x) - h(y))(h^+(x) - h^+(y)) = h(x)h^+(x) \geq 0$ on $\Omega \times \mathcal{C}\Omega$ and $(h(x) - h(y))(h^+(x) - h^+(y)) \geq 0$ on $\Omega_h^+ \times \Omega_h^-$. This gives

$$0 < \int_{\Omega_h^+} \int_{\Omega_h^+} \frac{(h(x) - h(y))(h^+(x) - h^+(y))}{|x - y|^{n+2s}} dx dy \leq \int_Q \frac{(h(x) - h(y))(h^+(x) - h^+(y))}{|x - y|^{n+2s}} dx dy.$$

Therefore from (4.6) we obtain

$$0 < C_s^n \int_{\Omega_h^+} \int_{\Omega_h^+} \frac{(h(x) - h(y))(h^+(x) - h^+(y))}{|x - y|^{n+2s}} dx dy \leq \lambda f(0) \int_{\Omega_h^+} \left(\frac{1}{(z_\epsilon + \epsilon)^q} - \frac{1}{(\bar{z} + \epsilon)^q} \right) h^+ dx \leq 0.$$

Hence it must be that $z_\epsilon \leq \bar{z}$ in Ω for each $\epsilon > 0$. Now we use this in (4.4) and Hölder inequality to get

$$\|z_\epsilon\|^2 \leq \lambda f(0) \int_{\Omega} \bar{z}^{1-q} dx + \|\tilde{f}(u)\|_{L^2(\Omega)} \|\bar{z}\|_{L^2(\Omega)} := m_0 < +\infty$$

which implies $\limsup_{\epsilon > 0} \|z_\epsilon\| < +\infty$. Thus $\{z_\epsilon\}_{\epsilon > 0}$ is a bounded sequence in $\tilde{H}^s(\Omega)$ and so there must exist a $z \in \tilde{H}^s(\Omega)$ such that, up to a subsequence, $z_\epsilon \rightharpoonup z$ weakly in $\tilde{H}^s(\Omega)$ as $\epsilon \rightarrow 0$. We already know that $z_\epsilon \rightarrow z$ pointwise a.e. in Ω . Moreover by Hardy's inequality, for any $\varphi \in \tilde{H}^s(\Omega)$ we get

$$0 < \left| \frac{\varphi}{(z_\epsilon + \epsilon)^q} \right| \leq \left| \frac{\varphi}{(m\phi_{1,s})^q} \right| \in L^1(\Omega).$$

Therefore we can use Lebesgue Dominated convergence theorem to pass through the limit as $\epsilon \rightarrow 0^+$ in (S_λ^ϵ) to obtain

$$C_s^n \int_Q \frac{(z(x) - z(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy - \lambda f(0) \int_{\Omega} \frac{\varphi}{z^q} dx = \int_{\Omega} \tilde{f}(u) \varphi dx.$$

that is z is a weak solution of (S_λ) . Finally it remains to show that $z \in C_{\phi_{1,s}}^+(\Omega)$. But it easily following from $\bar{z} \geq z \geq z_\epsilon \geq m\phi_{1,s}$ in Ω . Thus, T is well defined and this completes the proof. \blacksquare

Before proving our next result, we recall Theorem 1.2 from [12] as follows.

Theorem 4.4 *Let $c \in L_{loc}^1(\Omega)$ be a non positive function and $u \in H^s(\Omega)$ be a weak supersolution of*

$$(-\Delta)^s u = c(x)u \text{ in } \Omega$$

then

1. If Ω is bounded and $u \geq 0$ a.e. in $\mathcal{C}\Omega$ then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in \mathbb{R}^n .
2. If $u \geq 0$ a.e. in \mathbb{R}^n then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in \mathbb{R}^n .

Lemma 4.5 *The map T is strictly monotone increasing from $C_0(\bar{\Omega})$ to $C_{\phi_{1,s}}^+(\Omega)$.*

Proof. First we show that T is monotone increasing. For this, we let $u_1, u_2 \in C_0(\bar{\Omega})$ be such that $u_1 \leq u_2$. Then $\tilde{f}(u_1) \leq \tilde{f}(u_2)$, since \tilde{f} is increasing. Now let $z_i = T(u_i)$ for $i = 1, 2$, then each z_i satisfies

$$(-\Delta)^s z_i - \lambda \frac{f(0)}{z_i^q} = \tilde{f}(u_i), \quad z_i > 0 \text{ in } \Omega, \quad z_i = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

and $\hat{z} := (z_2 - z_1) \in \tilde{H}^s(\Omega)$ satisfies

$$(-\Delta)^s (z_2 - z_1) - \lambda f(0) \left(\frac{1}{z_2^q} - \frac{1}{z_1^q} \right) = \tilde{f}(u_2) - \tilde{f}(u_1) \geq 0 \text{ in } \Omega. \quad (4.7)$$

Then testing (4.7) with \hat{z}^- gives

$$\int_Q \frac{(\hat{z}(x) - \hat{z}(y))(\hat{z}^-(x) - \hat{z}^-(y))}{|x - y|^{n+2s}} dx dy \geq \lambda f(0) \int_{\{z_2 < z_1\}} \left(\frac{1}{z_2^q} - \frac{1}{z_1^q} \right) \hat{z}^- dx. \quad (4.8)$$

It is easy to see that the right hand side of (4.8) is non positive and the left hand side can be estimated as

$$\int_Q \frac{(\hat{z}(x) - \hat{z}(y))(\hat{z}^-(x) - \hat{z}^-(y))}{|x - y|^{n+2s}} dx dy \leq - \int_{\{z_2 < z_1\}} \int_{\{z_2 < z_1\}} \frac{|\hat{z}^-(x) - \hat{z}^-(y)|^2}{|x - y|^{n+2s}} dx dy \leq 0.$$

Therefore it must be that $z^- = 0$ in Ω that is $z_2 \geq z_1$ in Ω . Now we assume that $u_2 \geq u_1$ and $u_1 \not\equiv u_2$ then we show that $z_2 > z_1$ in Ω . We already know that $z_2 \geq z_1$ and by Mean value theorem we get that there exists a $\xi \in (z_1, z_2)$ such that (4.7) can be written as

$$(-\Delta)^s(z_2 - z_1) + \lambda f(0) \left(\frac{q}{\xi^{q+1}} \right) (z_2 - z_1) = \tilde{f}(u_2) - \tilde{f}(u_1) \geq 0 \text{ in } \Omega. \quad (4.9)$$

Let $c(x) = \frac{1}{\xi^{q+1}(x)}$ then since $\xi \in (z_1, z_2)$ and $z_i \in C_{\phi_{1,s}}^+(\Omega)$, for $i = 1, 2$ we easily get that $c \in L_{loc}^1(\Omega)$. Therefore from Theorem 4.4, we obtain that $z_2 - z_1 > 0$ in Ω . That is T is a strictly monotone increasing map. \blacksquare

The proof of our next result is motivated by the proof of Lemma 4.3 in [2].

Proposition 4.6 *The map $T : C_{\phi_{1,s}}(\Omega) \rightarrow C_{\phi_{1,s}}(\Omega)$ is compact.*

Proof. Let $u \in C_{\phi_{1,s}}(\Omega)$ and $T(u) = z \in C_{\phi_{1,s}}(\Omega)$ then z solves (S_λ) . We can write z as

$$z = (\Delta)^{-s} \left(\lambda \frac{f(0)}{z^q} \right) + (-\Delta)^{-s}(\tilde{f}(u)) \text{ in } \Omega.$$

Let $\{u_k\}_{k \in \mathbb{N}} \subset C_{\phi_{1,s}}(\Omega)$ be a bounded sequence that is $\sup_{k \in \mathbb{N}} \left\| \frac{u_k}{\delta^s} \right\|_{L^\infty(\Omega)} < +\infty$ and $T(u_k) = z_k \in C_{\phi_{1,s}}(\Omega)$ for each k . Then we have

$$z_k = (\Delta)^{-s} \left(\lambda \frac{f(0)}{z_k^q} \right) + (-\Delta)^{-s}(\tilde{f}(u_k)) \text{ in } \Omega.$$

From the proof of Proposition 4.3 we infer that $m\phi_{1,s}$ and Mw forms sub and supersolution of (S_λ) respectively for appropriate choice of positive constants m and M (independent of k). Then by weak comparison principle we get that

$$m\phi_{1,s} \leq z_k \leq Mw \text{ that is } k_1\delta^s(x) \leq z_k(x) \leq k_2\delta^s(x) \text{ in } \Omega \quad (4.10)$$

for some constants $k_1, k_2 > 0$. In order to prove compactness of the map T , we need to show that the sequence $\{z_k\}$ is relatively compact in $C_{\phi_{1,s}}^+(\Omega)$. Since $\{u_k\}$ is bounded in $C_{\phi_{1,s}}(\Omega)$

we get $\tilde{f}(u_k) \in L^\infty(\Omega)$ and $\sup_{k \in \mathbb{N}} \|\tilde{f}(u_k)\|_{L^\infty(\Omega)} \leq C_1$ for some constant $C_1 > 0$. Therefore from Theorem 1.2 of [32] we obtain

$$\left\| \frac{(-\Delta)^{-s} \tilde{f}(u_k)}{\delta^s} \right\|_{C^{0,\alpha}(\Omega)} \leq C \|\tilde{f}(u_k)\|_{L^\infty(\Omega)} \leq C_2 \quad (4.11)$$

for some constant $C, C_2 > 0$ (independent of k) and $0 < \alpha < \min\{s, 1-s\}$. Now for fix $\epsilon > 0$ we define the set

$$D_\epsilon := \{x \in \Omega : \delta(x) \geq \epsilon\}$$

and let χ_{D_ϵ} denote the corresponding characteristic function on D_ϵ . We also define the following functions

$$\begin{aligned} z_k^{1,\epsilon} &:= (\Delta)^{-s} \left(\lambda \frac{f(0)\chi_{D_\epsilon}}{z_k^q} \right) + (-\Delta)^{-s}(\tilde{f}(u_k)) \\ z_k^{2,\epsilon} &:= \left((\Delta)^{-s} \left(\lambda \frac{f(0)(1-\chi_{D_\epsilon})}{z_k^q} \right) \right) \chi_{D_{3\epsilon}} \\ z_k^{3,\epsilon} &:= \left((\Delta)^{-s} \left(\lambda \frac{f(0)(1-\chi_{D_\epsilon})}{z_k^q} \right) \right) (1 - \chi_{D_{3\epsilon}}) \end{aligned}$$

then clearly, $z_k = z_k^{1,\epsilon} + z_k^{2,\epsilon} + z_k^{3,\epsilon}$. Therefore it is enough to prove that each $\{z_k^{i,\epsilon}\}$ for $i = 1, 2, 3$ is relatively compact in $C_{\phi_{1,s}}(\Omega)$. Because of (4.10) we have

$$\lambda \frac{f(0)\chi_{D_\epsilon}}{z_k^q} \leq \lambda \frac{f(0)\chi_{D_\epsilon}}{k_1 \delta^{sq}(x)} \leq \lambda \frac{f(0)}{k_1 \epsilon^{sq}}$$

which implies

$$\sup_{k \in \mathbb{N}} \left\| \lambda \frac{f(0)\chi_{D_\epsilon}}{z_k^q} \right\|_{L^\infty(\Omega)} \leq C_3 = C_3(\epsilon)$$

for some constant $C_3 > 0$. So from (4.11) and Theorem 1.2 of [32] we infer that

$$\left\| \frac{z_k^{1,\epsilon}}{\delta^s} \right\|_{C^{0,\alpha}(\Omega)} \leq C \left\| \lambda \frac{f(0)\chi_{D_\epsilon}}{z_k^q} \right\|_{L^\infty(\Omega)} + \left\| \frac{(-\Delta)^{-s} \tilde{f}(u_k)}{\delta^s} \right\|_{C^{0,\alpha}(\Omega)} \leq C_4 = C_4(\epsilon) \quad (4.12)$$

for some constant $C_4 > 0$. Thus for each fixed $\epsilon > 0$, $\{z_k^{1,\epsilon}\}$ is relatively compact in $C_{\phi_{1,s}}(\Omega)$. Considering the sequence $\{z_k^{2,\epsilon}\}$, for any $x, x' \in D_{3\epsilon}$ we get

$$\begin{aligned} \left| \frac{z_k^{2,\epsilon}(x)}{\delta^s(x)} - \frac{z_k^{2,\epsilon}(x')}{\delta^s(x')} \right| &= \lambda \left| \int_{\Omega} \left(\frac{G_s(x,y)}{\delta^s(x)} - \frac{G_s(x',y)}{\delta^s(x')} \right) \frac{f(0)(1-\chi_{D_\epsilon})(y)}{z_k^q(y)} dy \right| \\ &\leq C' \left| \int_{\Omega} \left(\frac{G_s(x,y)}{\delta^s(x)} - \frac{G_s(x',y)}{\delta^s(x')} \right) \frac{(1-\chi_{D_\epsilon})(y)}{k_1 \delta^{sq}(y)} dy \right| \end{aligned}$$

for some constant $C' > 0$. It has been proved in Lemma 4.3 of [2] that the map $x \mapsto \frac{G_s(x,y)}{\delta^s(x)}$ is Hölder continuous in $D_{3\epsilon}$ uniformly with respect to $y \in \Omega \setminus D_\epsilon$ (but still depending on ϵ).

This implies that there exists $C_\epsilon > 0$ constant such that $\|G_s(x, y)\|_{C^s(D_{3\epsilon})} \leq C_\epsilon$ uniformly with respect to $y \in \Omega \setminus D_\epsilon$. Therefore we finally get that

$$\left| \frac{z_k^{2,\epsilon}(x)}{\delta^s(x)} - \frac{z_k^{2,\epsilon}(x')}{\delta^s(x')} \right| \leq \tilde{C}_\epsilon |x - x'|^s \int_{\Omega \setminus D_\epsilon} \frac{1}{\delta^{sq}(y)} dy \leq \hat{C}_\epsilon |x - x'|^s$$

for some constant $\tilde{C}_\epsilon, \hat{C}_\epsilon > 0$. This clearly gives that $\{z_k^{2,\epsilon}\}$ is relatively compact in $C_{\phi_{1,s}}(\Omega)$. Lastly we consider the sequence $\{z_k^{3,\epsilon}\}$ and fix $\beta \in (sq, s)$. Recalling the estimate (3.2) for $G_s(x, y)$, for $x \in \Omega \setminus D_{3\epsilon}$ we get

$$\begin{aligned} \left| \frac{z_k^{3,\epsilon}(x)}{\delta^s(x)} \right| &= \left| \lambda \frac{f(0)}{\delta^s(x)} \int_{\mathbb{R}^n} \frac{G_s(x, y)(1 - \chi_{D_\epsilon})(y)}{\delta^s(x) z_k^q(y)} dy \right| \\ &\leq \left| \frac{\lambda f(0)}{\delta^s(x)} \int_{\mathbb{R}^n \setminus D_\epsilon} \min \left(\frac{\delta^s(x) \delta^s(y)}{|x - y|^n}, \frac{\delta^s(x)}{|x - y|^{n-s}} \right) \frac{1}{k_1 \delta^{sq}(y)} dy \right| \\ &\leq \lambda f(0) \epsilon^{\beta - sq} \int_{\mathbb{R}^n \setminus D_\epsilon} \min \left(\frac{\delta^s(y)}{|x - y|^n}, \frac{1}{|x - y|^{n-s}} \right) \frac{1}{k_1 \delta^\beta(y)} dy \leq O(\epsilon^{\beta - sq}). \end{aligned} \quad (4.13)$$

Now we show that $\left\{ \frac{z_k}{\delta^s} \right\}$ is relatively compact in $L^\infty(\Omega)$. Let $\tau > 0$ be small enough. Then

because of (4.13) we can always choose ϵ small enough such that $\left\| \frac{z_k^{3,\epsilon}}{\delta^s} \right\|_{L^\infty(\Omega)} \leq \tau$. For

each such $\epsilon > 0$ we can get a convergent subsequences $\{z_{k_m}^{1,\epsilon}\}$ and $\{z_{k_m}^{2,\epsilon}\}$ of $\{z_k^{1,\epsilon}\}$ and $\{z_k^{2,\epsilon}\}$ respectively in $L^\infty(\Omega)$, since they are relatively compact in $C_{\phi_{1,s}}(\Omega)$. Hence we have

$$\left\| \frac{z_{k_m}}{\delta^s} - \frac{z_{k_{m'}}}{\delta^s} \right\|_{L^\infty(\Omega)} \leq \left\| \frac{z_{k_m}^{1,\epsilon}}{\delta^s} - \frac{z_{k_{m'}}^{1,\epsilon}}{\delta^s} \right\|_{L^\infty(\Omega)} + \left\| \frac{z_{k_m}^{2,\epsilon}}{\delta^s} - \frac{z_{k_{m'}}^{2,\epsilon}}{\delta^s} \right\|_{L^\infty(\Omega)} + 2\tau \leq 4\tau$$

when $m, m' \geq K$ for some $K \in \mathbb{N}$. This implies that $\{z_{k_m}\}$ is a Cauchy sequence in $C_{\phi_{1,s}}(\Omega)$ and hence convergent too. This proves that the sequence $\{z_k\}$ is relatively compact in $C_{\phi_{1,s}}(\Omega)$. \blacksquare

We seek help of solution to a nonlocal infinite semipositone problem (discussed in later section) for proving our next result that is the map T is strongly increasing. By strongly increasing, we mean that if $u_1 \leq u_2$ and $u_1 \not\equiv u_2$ then $T(u_2) - T(u_1) \in C_{\phi_{1,s}}^+(\Omega)$.

Theorem 4.7 *The map $T : C_{\phi_{1,s}}(\Omega) \rightarrow C_{\phi_{1,s}}(\Omega)$ is strongly increasing.*

Proof. Let $u_1 \leq u_2$ such that $u_1 \not\equiv u_2$ and $T(u_i) = z_i$ for $i = 1, 2$. Then from Lemma 4.5 we already know that $z_1 > z_2$ in Ω and $z_2 - z_1 \in C_{\phi_{1,s}}(\Omega)$. So it remains to prove that there exist a $k_1 > 0$ such that $k_1 \delta^s(x) \leq (z_2 - z_1)(x)$ in Ω . We know that $(z_2 - z_1)$ satisfies (4.9) and since $z_1(x) \leq \xi(x) \leq z_2(x)$ in Ω and each $z_i \in C_{\phi_{1,s}}(\Omega)$, we can get a constant $k > 0$ such that $\lambda f(0) \frac{q}{\xi^{q+1}} \leq \frac{k}{\delta^{s(q+1)}(x)}$ in Ω . Therefore if we set $\tilde{z} = (z_2 - z_1)$ then we obtain

$$(-\Delta)^s \tilde{z} + \frac{k}{\delta^{s(q+1)}(x)} \tilde{z} \geq 0, \quad \tilde{z} > 0 \text{ in } \Omega, \quad \tilde{z} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \quad (4.14)$$

From Theorem 6.2, we know that for sufficiently small $\theta > 0$, there exists a $v \in C_{\phi_{1,s}}^+(\Omega)$ which satisfies weakly

$$(-\Delta)^s v = v^p - \frac{\theta}{v^\gamma}, \quad v > 0 \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

where $\gamma \in (q, 1)$ and $p \in (0, 1)$. So there exist constants $m_1, m_2 > 0$ such that $m_1 \delta^s(x) \leq v(x) \leq m_2 \delta^s(x)$ in Ω . From this we get

$$(-\Delta)^s v + \frac{kv}{\delta^{s(q+1)}(x)} = v^p - \frac{\theta}{v^\gamma(x)} + \frac{kv}{\delta^{s(q+1)}(x)} \leq v^p - \frac{m_2^{-\gamma}\theta}{\delta^{s\gamma}(x)} + \frac{m_2 k}{\delta^{sq}(x)} \text{ in } \Omega. \quad (4.15)$$

Since $\gamma \in (q, 1)$, the term $\frac{m_2\theta}{\delta^{s\gamma}(x)}$ dominates near the boundary of Ω . We define $\Omega_\eta = \{x \in \Omega : \delta(x) < \eta\}$ and choose $\eta > 0$ small enough so that (4.15) gives

$$(-\Delta)^s v + \frac{kv}{\delta^{s(q+1)}(x)} \leq 0, \quad v > 0 \text{ in } \Omega_\eta, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega \quad (4.16)$$

From (4.14) we have

$$(-\Delta)^s \tilde{z} + \frac{k\tilde{z}}{\delta^{s(q+1)}(x)} \geq 0, \quad \tilde{z} > 0 \text{ in } \Omega_\eta, \quad \tilde{z} = 0 \text{ in } \mathbb{R}^n \setminus \Omega \quad (4.17)$$

We choose $m_3 > 0$ small enough such that $m_3 v \leq \tilde{z}$ in $\Omega \setminus \Omega_\eta$. Thus from (4.16) and (4.17) gives

$$(-\Delta)^s (m_3 v - \tilde{z}) + \frac{k(m_3 v - \tilde{z})}{\delta^{s(q+1)}(x)} \leq 0 \text{ in } \Omega_\eta, \quad (m_3 v - \tilde{z}) \leq 0 \text{ in } \mathbb{R}^n \setminus \Omega_\eta.$$

By comparison principle we get $m_1 m_3 \delta^s(x) \leq m_3 v \leq \tilde{z}$ in Ω_η . Since $0 < \tilde{z} \in C_{\phi_{1,s}}(\Omega)$, we get $\inf_{x \in \Omega \setminus \Omega_\eta} \tilde{z} > 0$. Hence there must exist a constant $k_1 > 0$ such that $k_1 \phi_{1,s}(x) \leq \tilde{z}$ in Ω . This proves that $(z_2 - z_1) \in C_{\phi_{1,s}}^+(\Omega)$ and the map T is strongly increasing on $C_{\phi_{1,s}}(\Omega)$. \blacksquare

We recall a fixed point theorem by Amann [3] which will help us to get the desired result.

Theorem 4.8 *Let X be a retract of some Banach space and $f : X \rightarrow X$ be a compact map. Suppose that X_1 and X_2 are disjoint subsets of X and let $U_k, k = 1, 2$ be open subsets of X such that $U_k \subset X_k, k = 1, 2$. Moreover, suppose that $f(X_k) \subset X_k$ and that f has no fixed points on $X_k \setminus U_k, k = 1, 2$. Then f has at least three distinct fixed points x_1, x_2, x_3 with $x_k \in X_k, k = 1, 2$ and $x \in X \setminus (X_1 \cup X_2)$.*

We also recall Corollary 6.2 of [3].

Lemma 4.9 *Let X be an ordered Banach space and $[y_1, y_2]$ be an ordered interval in X . Let $f : [y_1, y_2] \rightarrow X$ is an increasing compact map such that $f(y_1) \geq y_1$ and $f(y_2) \leq y_2$. Then f has a minimal fixed point \underline{x} and a maximal fixed point \bar{x} .*

Now the proof of the main result goes as follows.

Proof of Theorem 1.2: To obtain solutions of (P_λ) or equivalently (\tilde{P}_λ) , it is enough to find fixed points of the map T , thanks to Proposition 4.2. We define the sets $X = [\zeta_1, \vartheta_1]$, $X_1 = [\zeta_1, \vartheta_2]$ and $X_2 = [\zeta_2, \vartheta_1]$. Since X and X_i 's for each $i = 1, 2$ are non empty closed and convex subsets of $C_{\phi_{1,s}}(\Omega)$, they form retracts of $C_{\phi_{1,s}}(\Omega)$. By construction (done in section 2), we know that $X_1 \cap X_2 = \emptyset$ in X . Since ζ_1 and ϑ_1 are ordered sub and supersolutions of (P_λ) respectively and T is strictly increasing (Lemma 4.5), by comparison principle we obtain

$$\zeta_1 \leq T(\zeta_1) \leq T(\vartheta_1) \leq \vartheta_1.$$

This implies that $T(X) \subset X$ and similarly it also holds that $T(X_k) \subset X_k$ for $k = 1, 2$. Because of Proposition 4.6 and Theorem 4.7, we get that $T : X \rightarrow X$ is compact and a strongly increasing map. It has been proved that ϑ_2 is a strict supersolution of (P_λ) and $T(\vartheta_2) \leq \vartheta_2$. So using Theorem 4.4 we infer that $T(\vartheta_2) < \vartheta_2$, $T(\vartheta_1) < \vartheta_1$, $T(\zeta_1) > \zeta_1$ and $T(\zeta_2) < \zeta_2$. Therefore Lemma 4.9 implies that T has a maximal fixed point $u_1 \in X_1$ such that $u_1 \in (\zeta_1, \vartheta_2)$ and a minimal fixed point $u_2 \in X_2$ such that $u_2 \in (\zeta_2, \vartheta_1)$. Now repeating the arguments in Theorem 4.7, we can prove that there exist constants $a_1, a_2 > 0$ such that

$$u_1 \geq a_1\phi_{1,s} + \zeta_1, \quad \vartheta_2 - u_1 \geq a_1\phi_{1,s}, \quad u_2 + a_2\phi_{1,s} \leq \vartheta_1 \quad \text{and} \quad u_2 - \zeta_2 \geq a_2\phi_{1,s} \quad \text{in } \Omega.$$

We define the open ball B in X as

$$B := X \cap \left\{ \varphi \in C_{\phi_{1,s}}(\Omega) : \left\| \frac{\varphi}{\phi_{1,s}} \right\|_{L^\infty(\Omega)} < a \right\} \quad \text{with } a = \min(a_1, a_2).$$

Then for each $i = 1, 2$, $u_i + B \subset X_i$ and thus X_i 's have non empty interior. We construct open balls around each fixed point of T in X_i for each $i = 1, 2$ and take U_i as the largest open set in X_i containing all these open balls and such that $X_i \setminus U_i$ contains no fixed point of T . Now applying Theorem 4.8, we get the existence of third fixed point u_3 of T lying in $X \setminus (X_1 \cap X_2)$. This completes the proof. \blacksquare

5 Uniqueness for large λ

In this section, we prove that (P_λ) admits a unique solution when λ is sufficiently large. We assume only that $f \in C^1([0, \infty))$ satisfies (f1)-(f3) and

(f5) there exists a $\alpha > 0$ such that $\frac{f(u)}{u^q}$ is decreasing for $u > \alpha$.

Theorem 5.1 *The problem (P_λ) admits a solution for each $\lambda > 0$.*

Proof. We recall the first pair of sub-supersolution (ζ_1, ϑ_1) of (P_λ) constructed in section 2. Without loss of generality, we can assume that f is non decreasing in $[\zeta_1, \vartheta_1]$. As in the

proof of Theorem 1.2, we can show that if $X = [\zeta_1, \vartheta_1]$ then $T : X \rightarrow X$ is strictly increasing, compact map and $T(X) \subset X$. Therefore we can apply Lemma 4.9 to conclude that T has a fixed point u_λ in X . Then Proposition 4.2 gives us that $u_\lambda \in C_{\phi_{1,s}}^+(\Omega)$ is a solution of (P_λ) which completes the proof. \blacksquare

Lemma 5.2 *Any positive solution u_λ in $C_{\phi_{1,s}}^+(\Omega)$ of (P_λ) satisfies $u_\lambda \geq \Theta_\lambda w$ in Ω where $\Theta_\lambda = (\lambda f(0))^{\frac{1}{1+q}}$ and w is the solution to (3.1).*

Proof. Let u_λ solves (P_λ) . Since f is nondecreasing, we get

$$(-\Delta)^s u_\lambda(x) = \frac{\lambda f(u_\lambda(x))}{u_\lambda^q(x)} \geq \frac{\lambda f(0)}{u_\lambda^q(x)} \quad \text{in } \Omega$$

and

$$(-\Delta)^s(\Theta_\lambda w)(x) = \frac{\lambda f(0)}{(\Theta_\lambda w)^q(x)} \quad \text{in } \Omega.$$

Therefore by weak comparison principle (see Lemma 4.5), we conclude that $u_\lambda - \Theta_\lambda w \geq 0$ in \mathbb{R}^n . \blacksquare

Corollary 5.3 *There exist a minimal solution of (P_λ) in $C_{\phi_{1,s}}^+(\Omega)$, for each $\lambda > 0$.*

Proof. From Theorem 5.1 we know that (P_λ) has a solution $u_\lambda \in C_{\phi_{1,s}}^+(\Omega)$ such that

$$\zeta_1 \leq u_\lambda \leq \vartheta_1 \quad \text{in } \Omega$$

where both $\zeta_1, \vartheta_1 \in C_{\phi_{1,s}}^+(\Omega)$. Now the result follows from Lemma 5.2 and Lemma 4.9. \blacksquare

Theorem 5.4 *There exist a $\lambda^* > 0$ such that (P_λ) has a unique solution when $\lambda > \lambda^*$.*

Proof. Let u_λ and \bar{u}_λ be two distinct positive solutions of (P_λ) in $C_{\phi_{1,s}}^+(\Omega)$ such that u_λ is the minimal solution as obtained from Corollary 5.3. So it holds that $u_\lambda \leq \bar{u}_\lambda$ in Ω . We have

$$\int_\Omega (-\Delta)^s(u_\lambda - \bar{u}_\lambda)(u_\lambda - \bar{u}_\lambda) \, dx = \lambda \int_\Omega \left(\frac{f(u_\lambda)}{u_\lambda^q} - \frac{f(\bar{u}_\lambda)}{\bar{u}_\lambda^q} \right) (u_\lambda - \bar{u}_\lambda) \, dx \quad (5.1)$$

which gives

$$C_s^n \|(u_\lambda - \bar{u}_\lambda)\|^2 = \lambda \int_\Omega \left(\frac{f(u_\lambda)}{u_\lambda^q} - \frac{f(\bar{u}_\lambda)}{\bar{u}_\lambda^q} \right) (u_\lambda - \bar{u}_\lambda) \, dx = \lambda \int_\Omega \left(\int_0^1 f'_0(u_\lambda + t(\bar{u}_\lambda - u_\lambda)) \, dt \right) (u_\lambda - \bar{u}_\lambda)^2 \, dx \quad (5.2)$$

where $f_0(u) = \frac{f(u)}{u^q}$. From Lemma 5.2 we know that $u_\lambda \geq \Theta_\lambda w \geq \Theta_\lambda k_1 \delta^s(x)$ in Ω for some $k_1 > 0$. So if we define

$$\Omega_0 := \{x \in \Omega : f_1(x) \geq 0\} \subset \left\{ x \in \Omega : \delta^s(x) \leq \frac{\alpha}{\Theta_\lambda k_1} \right\}$$

where $f_1(x) = \int_0^1 f'_0(u_\lambda + t(\bar{u}_\lambda - u_\lambda)) dt$, then $u_\lambda(x) \geq \alpha$ in Ω_0 . From (5.2) we obtain

$$C_s^n \|(u_\lambda - \bar{u}_\lambda)\|^2 = \lambda \left(\int_{\Omega_0} f_1(x)(u_\lambda - \bar{u}_\lambda)^2 dx + \int_{\Omega \setminus \Omega_0} f_1(x)(u_\lambda - \bar{u}_\lambda)^2 dx \right).$$

Since $f_1 \leq 0$ in $\Omega \setminus \Omega_0$,

$$C_s^n \|(u_\lambda - \bar{u}_\lambda)\|^2 \leq \lambda \int_{\Omega_0} f_1(x)(u_\lambda - \bar{u}_\lambda)^2 dx. \quad (5.3)$$

We also notice that $\lim_{\lambda \rightarrow +\infty} \delta(x) = 0$ for $x \in \Omega_0$ and since by (f5) $u_\lambda + t(\bar{u}_\lambda - u_\lambda)(x) \leq \alpha$ for $x \in \Omega_0$, there exists a $M_2 > 0$ such that $|f'((u_\lambda + t(\bar{u}_\lambda - u_\lambda))(x))| \leq M_2$ for all $x \in \Omega_0$. Therefore we also have the following estimate using Hardy's inequality

$$\begin{aligned} \lambda \int_{\Omega_0} f_1(x)(u_\lambda - \bar{u}_\lambda)^2(x) dx &\leq \lambda \int_{\Omega_0} \left(\int_0^1 \frac{f'((u_\lambda + t(\bar{u}_\lambda - u_\lambda))(x))}{(u_\lambda + t(\bar{u}_\lambda - u_\lambda))^q(x)} dt \right) (u_\lambda - \bar{u}_\lambda)^2(x) dx \\ &\leq \lambda M_2 \int_{\Omega_0} \frac{(u_\lambda - \bar{u}_\lambda)^2(x)}{u_\lambda^q(x)} dx \\ &\leq \lambda M_2 (\Theta_\lambda k_1)^{-q} \int_{\Omega_0} \frac{(u_\lambda - \bar{u}_\lambda)^2(x) \delta^{(2-q)s}(x)}{\delta^{2s}(x)} dx, \text{ since } u_\lambda \geq \Theta_\lambda w \\ &\leq \lambda M_2 (\Theta_\lambda k_1)^{-q} \left(\frac{\sigma}{\Theta_\lambda k_1} \right)^{2-q} \int_{\Omega_0} \frac{(u_\lambda - \bar{u}_\lambda)^2(x)}{\delta^{2s}(x)} dx \\ &\leq C(\lambda) \|(u_\lambda - \bar{u}_\lambda)\|^2 \end{aligned}$$

where $C(\lambda) = O(\lambda^{-\frac{(1-q)}{1+q}})$. This gives a contradiction for λ large enough since $q \in (0, 1)$. Therefore we state that $u_\lambda \equiv \bar{u}_\lambda$ when λ is sufficiently large and this completes the proof. ■

6 A fractional and singular semipositone problem

We devote this section to prove the existence of weak solution for the following nonlocal infinite semipositone problem

$$(I_\theta) : \quad (-\Delta)^s v = v^p - \frac{\theta}{v^\gamma}, \quad v > 0, \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

where $p \in (0, 1)$, θ is a positive parameter and $\gamma \in (q, 1)$. Before this we consider the following problem

$$(I_0) : \quad (-\Delta)^s v = v^p, \quad v > 0, \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

for $p \in (0, 1)$. The energy functional $E_0 : \tilde{H}^s(\Omega) \rightarrow \mathbb{R}$ associated to (I_0) is given by

$$E_0(v) := C_s^n \frac{\|v\|^2}{2} - \frac{1}{p+1} \int_{\Omega} |v|^{p+1} dx$$

for $v \in \tilde{H}^s(\Omega)$. Then E_0 is weakly lower semicontinuous and coercive which implies that E_0 possesses a global minimizer say $v_0 \in \tilde{H}^s(\Omega)$. Since $\inf_{\tilde{H}^s(\Omega)} E_0 < 0$ and $E_0(|v_0|) \leq E_0(v_0)$, we

get $v_0 \neq 0$ and we can assume that $v_0 \geq 0$ in Ω . Now it is easy to see that v_0 solves the problem (I_0) weakly. From Proposition 2.2 of [7], we say that $v_0 \in L^\infty(\Omega)$ and then using Theorem 1.2 of [32] we conclude that $v_0 \in C^s(\mathbb{R}^n) \cap C_{\phi_{1,s}}(\Omega)$. Now by strong maximum principle it follows that $v_0 > 0$ in Ω . For $\eta > 0$ small enough, $\eta\phi_{1,s}$ forms a subsolution of (I_0) and then it is easy to show by weak comparison principle that $v_0 \in C_{\phi_{1,s}}^+(\Omega)$. The uniqueness of v_0 as a solution of (I_0) follows by using the Picone identity (Lemma 6.2 of [4]) and following the arguments in Theorem 5.2 in [21].

For fix $\mu > 0$, let us consider the solution operator $G(\theta, v) : \{|\theta| < \mu\} \times B_\epsilon(v_0) \rightarrow C_{\phi_{1,s}}(\Omega)$ defined as

$$G(\theta, v) := v - (-\Delta)^{-s} \left(v^p - \frac{\theta}{v^\gamma} \right)$$

for $(\theta, v) \in \{|\theta| < \mu\} \times B_\epsilon(v_0)$, where $B_\epsilon(v_0)$ denotes the open ball in $C_{\phi_{1,s}}^+(\Omega)$ with center v_0 and radius $\epsilon > 0$. We point out that for $\epsilon > 0$ small enough, $B_\epsilon(v_0) \subset C_{\phi_{1,s}}^+(\Omega)$. Furthermore, $G(\theta, u) = 0$ if and only if u solves (I_θ) . Let $(\theta, v) \in \{|\theta| < \mu\} \times B_\epsilon(v_0)$ then $(-\Delta)^{-s}v^p \in C_{\phi_{1,s}}(\Omega)$, by Theorem 1.2 of [32]. Since $v \in L^\infty(\Omega)$, we have that $(-\Delta)^{-s}v^p \in C_{\phi_{1,s}}(\Omega)$. Moreover $(-\Delta)^{-s} \left(\frac{\theta}{v^\gamma} \right) \in C_{\phi_{1,s}}(\Omega)$ follows from Theorem 1.2 in [2] and the fact that $v \in C_{\phi_{1,s}}^+(\Omega)$. Thus the map $G(\cdot, \cdot)$ is well defined.

Lemma 6.1 *The map G is continuously Fréchet differentiable.*

Proof. We begin with showing that G is continuous. Let $v, v_k \in B_\epsilon(v_0)$, $|\theta| < \mu$ and $\tau \in \mathbb{R}^n$ be such that $(\|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} + |\tau|) \rightarrow 0$ as $k \rightarrow \infty$, then

$$\begin{aligned} |G(\theta + \tau, v_k) - G(\theta, v)| &= \left| (v_k - v) - (-\Delta)^{-s}(v_k^q - v^q) + (-\Delta)^{-s} \left(\frac{\theta + \tau}{v_k^\gamma} - \frac{\theta}{v^\gamma} \right) \right| \\ &\leq \|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} \phi_{1,s} + C_1 \|v_k^q - v^q\|_{C_{\phi_{1,s}}(\Omega)} \phi_{1,s} \\ &\quad + \theta \left| (-\Delta)^{-s} \left(\frac{\gamma(v_k - v)}{(v + \xi(v_k - v))^{\gamma+1}} + \frac{\tau}{v_k^\gamma} \right) \right| \\ &\leq \|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} \phi_{1,s}(x) + C_1 \|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} \phi_{1,s} \\ &\quad + C_3 \theta \left| (-\Delta)^{-s} \left(\frac{\gamma \|v_k - v\|_{C_{\phi_{1,s}}(\Omega)}}{\delta^{s\gamma}(x)} + \frac{\tau}{\delta^{s\gamma}(x)} \right) \right| \end{aligned}$$

for appropriate constants $C_i > 0$, $i = 1, 2, 3$. Now Proposition 1.2.9 of [1] gives that

$$|G(\theta + \tau, v_k) - G(\theta, v)| \leq O(\|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} + |\tau|) \phi_{1,s}$$

which implies that G is continuous on $\{|\theta| < \mu\} \times B_\epsilon(v_0)$. Following similar arguments we can show that

$$\lim_{t \rightarrow 0^+} \frac{G(\theta, v + t\phi) - G(\theta, v)}{t} = \phi - p(-\Delta)^{-s}(v^{p-1}\phi) - \theta(-\Delta)^{-s}(\gamma v^{-\gamma-1}\phi)$$

for $v, \phi \in B_\epsilon(v_0)$. This implies that $G(\theta, \cdot)$ is Gâteaux differentiable and

$$D_v G(\theta, v)(\phi) = \phi - p(-\Delta)^{-s}(v^{p-1}\phi) - \theta(-\Delta)^{-s}(\gamma v^{-\gamma-1}\phi).$$

Next, to prove that G is Fréchet differentiable we first consider

$$\begin{aligned} & |G(\theta, v + \phi) - G(\theta, v) - D_v G(\theta, v)(\phi)| \\ &= \left| -(-\Delta)^{-s}((v + \phi)^p - v^p - pv^{p-1}\phi) + \theta(-\Delta)^{-s} \left(\frac{1}{(v + \phi)^\gamma} - \frac{1}{v^\gamma} + \frac{\gamma\phi}{v^{\gamma+1}} \right) \right|. \end{aligned}$$

Since $v \in C_{\phi_{1,s}}^+(\Omega)$ and $\phi \in C_{\phi_{1,s}}(\Omega)$, we get $\left| \frac{\phi}{v} \right| \leq K$ for some constant $K \geq 0$. This along with Taylor series expansion gives for some $\theta_0 \in (0, 1)$

$$\left| (v + \phi)^p - v^p - pv^{p-1}\phi = \frac{p(p-1)\phi^2}{2(v + \theta_0\phi)^{2-p}} \right| \leq C \|\phi\|_{C_{\phi_{1,s}}(\Omega)}^2.$$

So applying Theorem 1.2 of [32], we get that

$$|(-\Delta)^{-s}((v + \phi)^p - v^p - pv^{p-1}\phi)| \leq O\left(\|\phi\|_{C_{\phi_{1,s}}(\Omega)}^2\right).$$

Also similar idea gives, for some $\xi_1 \in (0, 1)$

$$\begin{aligned} \frac{1}{(v + \phi)^\gamma} - \frac{1}{v^\gamma} + \frac{\gamma\phi}{v^{\gamma+1}} &= -\frac{\gamma\phi}{(v + \xi_1\phi)^{\gamma+1}} + \frac{\gamma\phi}{v^{\gamma+1}} = \gamma\phi \left(\frac{(v + \xi_1\phi)^{\gamma+1} - v^{\gamma+1}}{(v(v + \xi_1\phi))^{\gamma+1}} \right) \\ &\leq C_4 \frac{\|\phi\|_{C_{\phi_{1,s}}(\Omega)}^2}{\delta^{s\gamma}} \end{aligned}$$

for appropriate constant $C_4 > 0$. So again using Proposition 1.2.9 of [1] we get

$$\left| (-\Delta)^{-s} \left(\frac{1}{(v + \phi)^\gamma} - \frac{1}{v^\gamma} + \frac{\gamma\phi}{v^{\gamma+1}} \right) \right| \leq O\left(\|\phi\|_{C_{\phi_{1,s}}(\Omega)}^2\right).$$

Therefore we get

$$\|G(\theta, v + \phi) - G(\theta, v) - D_v G(\theta, v)(\phi)\|_{C_{\phi_{1,s}}(\Omega)} \rightarrow 0 \text{ as } \|\phi\|_{C_{\phi_{1,s}}(\Omega)} \rightarrow 0.$$

Now we prove continuity of $D_v G(\theta, v)$. Consider $\{v_k\}_{k \in \mathbb{N}} \subset B_\epsilon(v_0)$ such that $\|v_k - v\|_{C_{\phi_{1,s}}(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ then

$$\|D_v G(\theta, v_k) - D_v G(\theta, v)\| = \sup_{0 \neq \phi \in C_{\phi_{1,s}}(\Omega)} \frac{\|(D_v G(\theta, v_k) - D_v G(\theta, v))\phi\|_{C_{\phi_{1,s}}(\Omega)}}{\|\phi\|_{C_{\phi_{1,s}}(\Omega)}}.$$

We have that

$$(D_v G(\theta, v_k) - D_v G(\theta, v))\phi = -p(-\Delta)^{-s}((v_k^{p-1} - v^{p-1})\phi) - \gamma\theta(-\Delta)^{-s} \left(\frac{\phi}{v_k^{\gamma+1}} - \frac{\phi}{v^{\gamma+1}} \right)$$

But using $\phi \in C_{\phi_{1,s}}(\Omega)$ and again using similar arguments as before we get

$$|(D_v G(\theta, v_k) - D_v G(\theta, v))\phi| \leq O\left(\|\phi\|_{C_{\phi_{1,s}}(\Omega)}(\|v_k^{p-1} - v^{p-1}\|_{C_{\phi_{1,s}}(\Omega)} + \|v_k - v\|_{C_{\phi_{1,s}}(\Omega)})\right)$$

which implies that $D_v G(\theta, v)$ is continuous. Similarly we can prove that $D_\theta(\theta, v)$ exists and is continuous where $D_\theta(\theta, v) = (-\Delta)^{-s} \left(\frac{1}{v^\gamma} \right)$. \blacksquare

The linearisation of the map G with respect to the second variable at $(\theta, v) \in \{|\theta| < \mu\} \times B_\epsilon(v_0)$ given by $\partial_v G(\theta, v) : C_{\phi_{1,s}}^+(\Omega) \rightarrow C_{\phi_{1,s}}(\Omega)$ is defined as

$$\partial_v G(\theta, v)h = h - (-\Delta)^{-s} \left(pv^{p-1}h + \frac{\theta\gamma h}{u^{\gamma+1}} \right), \text{ for } h \in C_{\phi_{1,s}}(\Omega).$$

Since v_0 solves (I_0) , clearly $G(0, v_0) = 0$. Now we will show that the map $h \mapsto \partial_v G(0, v_0)h$ is invertible. To do this, we will be studying an eigenvalue problem. Let us define

$$\Lambda := \inf_{0 \neq u \in \tilde{H}^s(\Omega)} \left(\frac{\|u\|^2 - p \int_{\Omega} v_0^{p-1} u^2 \, dx}{\int_{\Omega} u^2 \, dx} \right) = \inf_S \left(\|u\|^2 - p \int_{\Omega} v_0^{p-1} u^2 \, dx \right)$$

where $S = \{u \in \tilde{H}^s(\Omega), \int_{\Omega} u^2 \, dx = 1\}$. Then by Hardy's inequality and $v_0 \in C_{\phi_{1,s}}^+(\Omega)$ it follows that

$$\int_{\Omega} v_0^{p-1} u^2 \, dx \leq k_2 \int_{\Omega} \frac{u^2}{\delta^{2s}(x)} \delta^{s(p+1)} \, dx < +\infty.$$

So the functional

$$I_{00}(u) = \|u\|^2 - p \int_{\Omega} v_0^{p-1} u^2 \, dx$$

is well defined on $\tilde{H}^s(\Omega)$. Following standard minimization arguments and using compact embedding of $\tilde{H}^s(\Omega)$ in $L^2(\Omega)$, it is easy to show that $\inf I_{00}(S) = I_{00}(\psi) = \Lambda$ for some $\psi \in S$. Also since $I_{00}(|\psi|) \leq I_{00}(\psi)$, by minimality of ψ we assert that without loss of generality we may assume that $\psi \geq 0$. Then ψ satisfies

$$(-\Delta)^s \psi = \Lambda \psi + pv_0^{p-1} \psi \text{ in } \Omega, \quad \psi = 0 \text{ in } \mathbb{R}^n \setminus \Omega \tag{6.1}$$

which implies that ψ is an eigenfunction corresponding to the eigenvalue Λ for the operator $(-\Delta)^s - pv_0^{p-1}$ with homogenous Dirichlet boundary condition in Ω . We obtain $\psi \in L^\infty(\Omega)$ by following the arguments in Theorem 3.2 (p. 379) of [17]. So local regularity results assert that $\psi \in C_{\text{loc}}^s(\Omega)$. We claim that $\psi > 0$ in Ω because if is not true, then there exist a $x_0 \in K \Subset \Omega$ such that $\psi(x_0) = 0$. But this gives

$$0 > 2C_s^n \int_{\mathbb{R}^n} \frac{(\psi(x_0) - \psi(y))}{|x - y|^{n+2s}} \, dy = \Lambda \psi + pv_0^{p-1}(x_0)\psi(x_0) = 0 \text{ in } K$$

which is a contradiction. Therefore $\psi > 0$ in Ω .

Claim(1): Λ is a principal eigenvalue.

We have to show that any eigenfunction (say ψ) associated to it does not change sign. Assume by contradiction that $\psi^+ \not\equiv 0$ and $\psi^- \not\equiv 0$. Then

$$\begin{aligned} & C_s^n \int_Q \frac{(\psi(x) - \psi(y))(\psi^+(x) - \psi^+(y))}{|x - y|^{n+2s}} \, dy dx \\ &= C_s^n \left(\int_Q \frac{(\psi^+(x) - \psi^+(y))(\psi^+(x) - \psi^+(y))}{|x - y|^{n+2s}} \, dy dx + 2 \int_Q \frac{\psi^-(x)\psi^+(y)}{|x - y|^{n+2s}} \, dy dx \right) \\ &= \Lambda \int_{\Omega} \psi \psi^+ \, dx + p \int_{\Omega} v_0^{p-1} \psi \psi^+ \, dx. \end{aligned}$$

Now if $\psi^+, \psi^- \not\equiv 0$ then

$$\|\psi^+\|^2 - p \int_{\Omega} v_0^{p-1} (\psi^+)^2 dx = \Lambda \int_{\Omega} (\psi^+)^2 dx - 2 \int_Q \frac{\psi^-(x)\psi^+(y)}{|x-y|^{n+2s}} dy dx < \Lambda \int_{\Omega} (\psi^+)^2 dx$$

but this contradicts the definition of Λ . This proves the claim.

Claim(2): Λ is the unique principal eigenvalue.

Suppose not, that is let Λ_0 is another principal eigenvalue of $(-\Delta)^s - pv_0^{p-1}$ and $\psi_0 \in \tilde{H}^s(\Omega)$ denotes the corresponding eigenfunction. Then $\psi_0 > 0$ in Ω and satisfies

$$(-\Delta)^s \psi_0 = \Lambda_0 \psi_0 + pv_0^{p-1} \psi_0 \text{ in } \Omega, \quad \psi_0 = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \quad (6.2)$$

Testing (6.1) with ψ_0 and (6.2) with ψ gives

$$\Lambda \int_{\Omega} \psi \psi_0 dx = \Lambda_0 \int_{\Omega} \psi_0 \psi dx.$$

Since $\psi, \psi_0 > 0$ in Ω , we get $\Lambda = \Lambda_0$.

Claim(3): Any nonnegative eigenfunction $\psi \in C_{\phi_{1,s}}^+(\Omega)$.

For sufficiently small $\eta > 0$, using Theorem 4.4 and Hopf Lemma, it is easy to show that $\psi \geq \eta \phi_{1,s}$. Assume first that $\Lambda \geq 0$ then

$$\psi = \Lambda (-\Delta)^{-s} \psi + p (-\Delta)^{-s} \left(\frac{\psi}{v_0^{1-p}} \right) \text{ in } \Omega.$$

Since $\psi \in L^\infty(\Omega)$ we get $(-\Delta)^{-s} \psi \in C_{\phi_{1,s}}(\Omega)$. Also since $v_0 \in C_{\phi_{1,s}}^+(\Omega)$ we get that

$$\frac{\psi}{v_0^{1-p}} \sim \frac{C}{\delta^{s(1-p)}(x)} \text{ in } \Omega$$

for some $C > 0$. So $(-\Delta)^{-s} \left(\frac{\psi}{v_0^{1-p}} \right) \in C_{\phi_{1,s}}^+(\Omega)$, follows from Proposition 1.2.9 of [1] together with $s(1-p) < s$. Therefore $\psi \in C_{\phi_{1,s}}^+(\Omega)$ when $\Lambda \geq 0$. In the other case, if $\Lambda < 0$ then ψ satisfies

$$(-\Delta)^s \psi + (-\Lambda) \psi = p \left(\frac{\psi}{v_0^{1-p}} \right) \sim \frac{C}{\delta^{s(1-p)}(x)} \geq 0 \text{ in } \Omega.$$

So by using Theorem 1.5(1) of [12] and Theorem 1.2 of [32], we infer that $\psi \in C_{\phi_{1,s}}^+(\Omega)$. This proves the claim.

Claim(4): $\Lambda > 0$.

First we show that Λ is non zero. Suppose it is equal to zero then (6.1) reduces to

$$(-\Delta)^s \psi = pv_0^{p-1} \psi \text{ in } \Omega.$$

Using v_0 as a test function in the expression above gives

$$\int_{\mathbb{R}^n} (-\Delta)^s \psi v_0 dx = p \int_{\Omega} v_0^p \psi dx.$$

But we know that v_0 is a unique solution of (I_0) . Therefore we get

$$p \int_{\Omega} v_0^p \psi \, dx = \int_{\Omega} v_0^p \psi \, dx$$

which gives $p = 1$ since $\psi, v_0 > 0$ in Ω . This gives a contradiction, thus $\Lambda \neq 0$. Now we assume by contradiction that $\Lambda < 0$. For $\epsilon > 0$ small enough, we consider the function $v_0 - \epsilon\psi$. Then since $\Lambda < 0$ and $p - 1 < 0$, we get that in Ω

$$(-\Delta)^s(v_0 - \epsilon\psi) = v_0^p - \epsilon\Lambda\psi - \epsilon p v_0^{p-1}\psi > v_0^p - \epsilon p v_0^{p-1}\psi \geq (v_0 - \epsilon\psi)^p.$$

This implies that $v_0 - \epsilon\psi$ forms a strict supersolution of (I_0) . It is already known that $\eta\phi_{1,s}$ for sufficiently small choice of η forms a subsolution of (I_0) . Therefore there must be a function $\hat{v} \in \tilde{H}^s(\Omega)$ such that $\eta\phi_{1,s} \leq \hat{v} \leq (v_0 - \epsilon\psi)$ which is a solution of (I_0) . But this contradicts the uniqueness of v_0 due to $\epsilon\psi > 0$ in Ω . Hence $\Lambda > 0$.

Theorem 6.2 *For a small range of θ , the problem (I_θ) admits a solution.*

Proof. We have already proved that G is a continuously Fréchet differentiable map. Now we consider the problem

$$(-\Delta)^s u - p v_0^{p-1} u = \psi, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{6.3}$$

By defining the energy functional corresponding to it and minimization arguments, it is easy to show that the above problem has a solution $u \in \tilde{H}^s(\Omega)$. Now suppose $u_1, u_2 \in \tilde{H}^s(\Omega)$ be two distinct solutions of (6.3) then

$$0 = \int_{\mathbb{R}^n} (-\Delta)^s(u_1 - u_2)(u_1 - u_2) \, dx - p \int_{\Omega} v_0^{p-1}(u_1 - u_2)^2 \, dx \geq \Lambda \int_{\Omega} (u_1 - u_2)^2 \, dx > 0$$

since $\Lambda > 0$. This implies that the solution must be unique. Also using similar argument as in Claim(3) gives $u \in C_{\phi_{1,s}}(\Omega)$. All this along with previous claims guarantees that the map $\partial_v G(0, v_0) : C_{\phi_{1,s}}(\Omega) \rightarrow C'_{\phi_{1,s}}(\Omega)$ is invertible. Hence we apply implicit function theorem to get that there exist a subset $\{|\theta| < \mu'\} \times B_{\epsilon'}(v_0) \subset \{|\theta| < \mu\} \times B_{\epsilon}(v_0)$ for $\mu' < \mu$ and $\epsilon' < \epsilon$ and a C^1 -map $h : \{|\theta| < \mu'\} \mapsto B_{\epsilon'}(v_0)$ such that $G(\theta, v) = 0$ in $\{|\theta| < \mu'\} \times B_{\epsilon'}(v_0)$ coincides with the graph of h . This completes the proof. ■

Acknowledgements: The authors were partially funded by IFCAM (Indo-French Centre for Applied Mathematics) UMI CNRS 3494 under the project "Singular phenomena in reaction diffusion equations and in conservation laws".

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