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EXISTENCE AND STABILIZATION RESULTS FOR A SINGULAR PARABOLIC EQUATION INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. In this article, we study the following parabolic equation involving the fractional Laplacian with singular nonlinearity

$$(P_t^s)\begin{cases} u_t + (-\Delta)^s u = u^{-q} + f(x,u),\ u > 0 \text{ in } (0,T)\times\Omega,\\ u = 0 \text{ in } (0,T)\times(\mathbb{R}^n\setminus\Omega),\\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^n,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s,\ s \in (0,1),\ q > 0,\ q(2s-1) < (2s+1),\ u_0 \in L^\infty(\Omega)\cap X_0(\Omega)$ and $T > 0$. We suppose that the map $(x,y) \in \Omega\times\mathbb{R}^+ \mapsto f(x,y)$ is a bounded from below Carathéodory function, locally Lipschitz with respect to the second variable and uniformly for $x \in \Omega$ and it satisfies

$$\limsup_{y\to+\infty}\frac{f(x,y)}{y} < \lambda_1^s(\Omega), \tag{0.1}$$

where $\lambda_1^s(\Omega)$ is the first eigenvalue of $(-\Delta)^s$ in Ω with homogeneous Dirichlet boundary condition in $\mathbb{R}^n\setminus\Omega$. We prove the existence and uniqueness of a weak solution to (P_t^s) on assuming u_0 satisfies an appropriate cone condition. We use the semi-discretization in time with implicit Euler method and study the stationary problem to prove our results. We also show additional regularity on the solution of (P_t^s) when we regularize our initial function u_0.

1. **Introduction.** In this paper, we study the existence and uniqueness of weak solution for the following fractional parabolic equation with singular nonlinearity

$$(P_t^s)\begin{cases} u_t + (-\Delta)^s u = u^{-q} + f(x,u),\ u > 0 \text{ in } \Lambda_T,\\ u = 0 \text{ in } \Gamma_T,\\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^n,\end{cases}$$

where $\Lambda_T = (0,T)\times\Omega$, $\Gamma_T = (0,T)\times(\mathbb{R}^n\setminus\Omega)$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ (atleast C^2), $n > 2s,\ s \in (0,1),\ q > 0,\ q(2s-1) < (2s+1)$

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and $T > 0$. The map $(x, y) \in \Omega \times \mathbb{R} \mapsto f(x, y)$ is assumed to be a bounded from below Carathéodary function, locally Lipschitz with respect to the second variable and uniformly for $x \in \Omega$ and it satisfies

$$\limsup_{y \to +\infty} \frac{f(x, y)}{y} < \lambda_1^s(\Omega),$$

where $\lambda_1^s(\Omega)$ is the first eigenvalue of $(-\Delta)^s$ in Ω with (homogeneous) Dirichlet boundary condition in $\mathbb{R}^n \setminus \Omega$. The fractional Laplace operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = 2C_n^s \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, \mathrm{d}y$$

where P.V. denotes the Cauchy principal value and $C_n^s = \pi^{-\frac{n}{2}} 2^{2s-1} s \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}$, Γ being the Gamma function.

In this article, we will be concerned with the nonlocal problem (P_t^s) that involves the fractional Laplacian. A large variety of diffusive problems in Physics are satisfactorily described by the classical Heat equation. However, the anomalous diffusion that follows non-Brownian scaling is nowadays intensively studied with a wide range of applications in physics, finance, biology and many others. The governing equations of such mathematical models involve the fractional Laplacian. For a detailed survey on this, we refer to [25, 26] and the references therein. It is natural to study the local and global existence and stabilization results for such problems.

Singular parabolic problems in the local case have been studied by authors in [5, 11, 14]. The inspiring point for us was the work of M. Badra et al. [6], here the existence and stabilization results for parabolic problem where the principal part of the equation is the p-Laplacian operator, has been studied when $0 < q < 2 + \frac{1}{p-1}$. In [9], Bougherara and Giacomoni proved the existence of unique mild solution to the problem for all $q > 0$ when $u_0 \in (C_0(\overline{\Omega}))^+$. In the present work, we extend the results obtained in [6] to the non-local case. However, there is a substantial difference between local and nonlocal operators. This difference is reflected in the way of construction of sub and super solutions of stationary problems associated to (P_t^s) as well as the validity of the weak comparison principle. Nonetheless, we will show that the semi-discretization in time method used in [6] can still be effective in this case.

Coming to the non-local case, singular elliptic equations involving fractional Laplacian has been studied by Barios et al. in [8] and Giacomoni et al. in [16]. More specifically, existence and multiplicity results for the equation

$$(-\Delta)^s u = \lambda u^{-q} + u^p \text{ in } \Omega, \; u = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

have been shown for $0 < q \leq 1$ and $0 < p < 2_s^* - 1$ where $2_s^* = \dfrac{2n}{n - 2s}$ in [8] and $p = 2_s^* - 1$ in [19]. Whereas the case $q > 0$ and $p = 2_s^* - 1$ has been studied in [16]. Concerning the parabolic problems involving the fractional Laplacian, we cite [3, 13, 25, 26] and the references therein. Caffarelli and Figalli studied the regularity of solutions to fractional parabolic obstacle problem in [10]. In [17], authors studied the Hölder estimates for singular problems of the type $(-\Delta)^s u^m + u_t = 0$ where $\frac{n-2s}{n+2s} < m < 1$. In [18], the summability of solutions with respect to the summability of the data is studied. In [1], the authors studied the influence of Hardy potential on the existence and nonexistence of positive solutions for the fractional heat equation. To the best of our knowledge, there are no works on parabolic equations with fractional Laplacian and singular nonlinearity.

In this work, we first define the positive cone motivated from the work of [2] and obtain the existence of solutions in this cone for the elliptic problem (S) in section 2 associated to the semi-discretization of (P_t^s). Using this, we proved the existence and uniqueness of solution and its regularity for the parabolic problem (see (G_t^s) in section 2 with bounded source term $h(x,t)$ and principal diffusion operator $(-\Delta)^s - u^{-q}$ in section 4). Finally using the new uniqueness results for the stationary problem proved in section 5, we prove the existence and uniqueness of solutions to the problem (P_t^s) in section 6. Thanks to the nonlinear accretive operators theory, we also find that these solutions are more regular when the regularity assumption is refined on the initial condition. We end our paper by showing that the solution to (P_t^s) converges to the unique solution of its stationary problem as $t \to \infty$ in section 7. In this aim, we extend existence and regularity results about the stationary problem proved in [2].

2. **Functional setting and main results.**

We denote the usual fractional Sobolev space by $H^s(\Omega)$ endowed with the Gagliardo norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

Then we consider the following space

$$X(\Omega) = \left\{ u |\ u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x) - u(y))}{|x-y|^{\frac{n+2s}{2}}} \in L^2(Q) \right\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space $X(\Omega)$ is endowed with the norm defined as

$$\|u\|_{X(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

Now we define the space $X_0(\Omega) = \{u \in X(\Omega) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ equipped with the norm

$$\|u\|_{X_0(\Omega)} = \left(C_n^s \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}$$

where C_n^s is defined in section 1 and it is well known that $X_0(\Omega)$ forms a Hilbert space with this norm (see [21]). From the embedding results, we know that $X_0(\Omega)$ is continuously and compactly embedded in $L^r(\Omega)$ when $1 \leq r < 2_s^* = \frac{2n}{n-2s}$ and the embedding is continuous but not compact if $r = 2_s^*$. For each $\alpha \geq 0$, we set

$$C_\alpha = \sup \left\{ \int_\Omega |u|^\alpha dx : \|u\|_{X_0(\Omega)} = 1 \right\}.$$

Then $C_0 = |\Omega| =$ Lebesgue measure of Ω and $\int_\Omega |u|^\alpha dx \leq C_\alpha \|u\|^\alpha$, for all $u \in X_0(\Omega)$. Let us consider a more general problem

$$(G_t^s) \begin{cases} u_t + (-\Delta)^s u = u^{-q} + h(t,x), \ u > 0 \text{ in } \Lambda_T, \\ u = 0 \text{ in } \Gamma_T, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases}$$

where $T > 0$, $s \in (0,1)$, $h \in L^\infty(\Lambda_T)$, $q > 0$, $q(2s-1) < (2s+1)$ and $u_0 \in L^\infty(\Omega) \cap X_0(\Omega)$. In order to define weak solution for the problem (G_t^s), we need to introduce the following space

$$\mathcal{A}(\Lambda_T) := \{u : \ u \in L^\infty(\Lambda_T), \ u_t \in L^2(\Lambda_T), \ u \in L^\infty(0,T; X_0(\Omega))\}.$$

We have the following result as a direct consequence of Aubin-Lions-Simon Lemma (see [24]).

Lemma 2.1. *Suppose $u \in L^\infty(0,T;X_0(\Omega))$ and $u_t \in L^2(\Lambda_T)$. Then $u \in C([0,T]; L^2(\Omega))$ and the embedding is compact.*

We now define the notion of weak solution for the problem (G_t^s).

Definition 2.2. We say that $u \in \mathcal{A}(\Lambda_T)$ is a weak solution of (G_t^s) if

1. for any compact subset $K \subset \Lambda_T$, $ess\inf_K u > 0$,
2. for every $\phi \in \mathcal{A}(\Lambda_T)$,

$$\int_{\Lambda_T} \frac{\partial u}{\partial t}\phi \ dxdt + C_n^s \int_0^T \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} dydxdt$$
$$= \int_{\Lambda_T} (u^{-q} + h(t,x))\phi dxdt,$$

3. $u(0,x) = u_0(x)$ a.e. in Ω.

We remark that because of Lemma 2.1, we get $\mathcal{A}(\Lambda_T) \subset C([0.T]; L^2(\Omega))$ which means that the third point of the above definition makes sense.

Now, we define a conical shell $\mathcal{C}$ as the set of functions $v \in L^\infty(\Omega)$ such that there exist constants $k_1, k_2 > 0$ such that

$$\begin{cases} k_1\delta^s(x) \le v \le k_2\delta^s(x) & \text{if } q < 1, \\ k_1\delta^s(x)\left(\ln\left(\frac{r}{\delta^s(x)}\right)\right)^{\frac{1}{2}} \le v \le k_2\delta^s(x)\left(\ln\left(\frac{r}{\delta^s(x)}\right)\right)^{\frac{1}{2}} & \text{if } q = 1, \\ k_1\delta^{\frac{2s}{q+1}}(x) \le v \le k_2\delta^{\frac{2s}{q+1}}(x) & \text{if } q > 1, \end{cases}$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $r > \text{diam}(\Omega)$. We set

$$C_0(\overline{\Omega}) := \left\{u \in C(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\right\}.$$

We begin by considering the stationary problem (S):

$$(S) \begin{cases} u + \lambda\left((-\Delta)^s u - u^{-q}\right) = g, \quad u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $g \in L^\infty(\Omega)$ and $\lambda > 0$ is a real parameter. The notion of weak solution is defined as follows.

Definition 2.3. We say $u \in X_0(\Omega)$ is a weak solution of (S) if

1. for any compact subset $K \subset \Omega$, $ess\inf_K u > 0$,
2. for every $\phi \in X_0(\Omega)$,

$$\int_\Omega u\phi \ dx + \lambda\left(C_n^s \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} \ dxdy - \int_\Omega u^{-q}\phi \ dx\right) = \int_\Omega g\phi \ dx.$$

We prove the following theorem considering the problem (S).

Theorem 2.4. *If $g \in L^\infty(\Omega)$, $q > 0$ and $q(2s-1) < (2s+1)$, then for any $\lambda > 0$, problem (S) has a unique weak solution $u_\lambda \in X_0(\Omega) \cap \mathcal{C} \cap C^\alpha(\mathbb{R}^n)$ where $\alpha = s$ if $q < 1$, $\alpha = s - \epsilon$ if $q = 1$, for any $\epsilon > 0$ small enough and $\alpha = \dfrac{2s}{q+1}$ if $q > 1$.*

In the case $q(2s-1) \geq (2s+1)$, we get less regularity on the solution of (S). So we will have a weaker notion of the solution in this case for which we define the set

$$\Theta := \{\phi : \ \phi : \mathbb{R}^n \to \mathbb{R} \text{ measurable and } (-\Delta)^s \phi \in L^\infty(\Omega), \ \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega', \ \Omega' \Subset \Omega\}.$$

Theorem 2.5. *Let $g \in L^\infty(\Omega)$, $q > 1$ and $q(2s-1) \geq (2s+1)$ then for any $\lambda > 0$, there exists a $u_\lambda \in L^1(\mathbb{R}^n)$ satisfying $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, $\inf\limits_K u_\lambda > 0$ for every $K \Subset \Omega$ and*

$$\int_\Omega u_\lambda \phi \ dx + \lambda \left(C_n^s \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+2s}} \ dxdy - \int_\Omega u_\lambda^{-q} \phi \ dx \right) = \int_\Omega g\phi \ dx$$

for any $\phi \in \Theta$. Moreover $u_\lambda^\beta \in X_0(\Omega)$ where $\beta > \max\left\{1, \left(1 - \frac{1}{2s}\right)\left(\frac{q+1}{2}\right)\right\}$ but $u_\lambda \notin X_0(\Omega)$.

Definition 2.6. We say that $u(t) \in \mathcal{C}$ uniformly for each $t \in [0,T]$ when there exist $\psi_1, \psi_2 \in \mathcal{C}$ such that $\psi_1(x) \leq u(t,x) \leq \psi_2(x)$ a.e. $(t,x) \in [0,T] \times \Omega$.

We prove the following existence and uniqueness result for the problem (G_t^s) using semi-discretization in time with implicit Euler method, Theorem 2.4, energy estimates and the weak comparison principle.

Theorem 2.7. *If $h(t,x) \in L^\infty(\Lambda_T)$, $u_0 \in X_0(\Omega) \cap \mathcal{C}$, $q > 0$ and $q(2s-1) < (2s+1)$, then there exists a unique weak solution $u \in C([0,T]; X_0(\Omega))$ for the problem (G_t^s) such that $u(t) \in \mathcal{C}$ uniformly for each $t \in [0,T]$. Also, u satisfies*

$$\begin{aligned} &\int_0^t \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dxd\tau + \frac{1}{2}\|u(t,x)\|_{X_0(\Omega)}^2 - \frac{1}{1-q}\int_\Omega u^{1-q}(t,x)dx \\ &\quad = \int_0^t \int_\Omega h(\tau,x)\frac{\partial u}{\partial t} dxd\tau + \frac{1}{2}\|u_0(x)\|_{X_0(\Omega)}^2 - \frac{1}{1-q}\int_\Omega u_0^{1-q}(x)dx \end{aligned} \tag{2.1}$$

for any $t \in [0,T]$.

The solution obtained in above theorem can be shown to be more regular under some extra assumptions as can be seen in the next result.

Proposition 1. *Under the hypothesis of Theorem 2.7, if $u_0 \in \overline{\mathcal{D}(L)}^{L^\infty(\Omega)}$, where*

$$\mathcal{D}(L) := \{v \in \mathcal{C} \cap X_0(\Omega) : \ L(v) := (-\Delta)^s v - v^{-q} \in L^\infty(\Omega)\}$$

then the solution of (G_t^s) obtained in Theorem 2.7 belongs to $C([0,T]; C_0(\overline{\Omega}))$. Also u satisfies:

1. *If v is another solution of (G_t^s) with initial condition $v_0 \in \overline{\mathcal{D}(L)}^{L^\infty(\Omega)}$ and nonhomogenous term $b \in L^\infty(\Lambda_T)$, then for any $t \in [0,T]$,*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(\Omega)} \leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \|h(\tau,\cdot) - b(\tau,\cdot)\|_{L^\infty(\Omega)} d\tau.$$

2. *If $u_0 \in \mathcal{D}(L)$ and $h \in W^{1,1}([0,T]; L^\infty(\Omega))$, then $u \in W^{1,\infty}([0,T]; L^\infty(\Omega))$, $(-\Delta)^s u + u^{-q} \in L^\infty(\Lambda_T)$ and the following holds true for any $t \in [0,T]$,*

$$\left\|\frac{du(t,\cdot)}{dt}\right\|_{L^\infty(\Omega)} \leq \|(-\Delta)^s u_0 + u_0^{-q} + h(0,\cdot)\|_{L^\infty(\Omega)} + \int_0^T \left\|\frac{dh(\tau,\cdot)}{dt}\right\|_{L^\infty(\Omega)} d\tau.$$

In order to establish Theorem 2.9, we need the following result.

Theorem 2.8. *Suppose $q > 0$, $q(2s-1) < (2s+1)$ and $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be bounded from below Carathéodory function satisfying (0.1). Assume that f is locally Lipschitz with respect to the second variable uniformly in Ω and $\frac{f(x,y)}{y}$ is decreasing in $\mathbb{R}^+$ for a.e. $x \in \Omega$. Then the following problem (Q^s) has a unique solution $\hat{u} \in X_0(\Omega) \cap \mathcal{C} \cap C^\alpha(\mathbb{R}^n)$ where $\alpha = s$ if $q < 1$, $\alpha = s - \epsilon$ if $q = 1$, for any $\epsilon > 0$ small enough and $\alpha = \dfrac{2s}{q+1}$ if $q > 1$:*

$$(Q^s) \begin{cases} (-\Delta)^s \hat{u} - \hat{u}^{-q} = f(x, \hat{u}) \text{ in } \Omega, \\ \hat{u} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Coming back to our original problem (P_t^s), we have the following theorem :

Theorem 2.9. *Assume $q > 0$, $q(2s-1) < (2s+1)$ and $f(t,x)$ to be a bounded from below Carathéodory function, locally Lipschitz with respect to the second variable uniformly in $x \in \Omega$ and satisfies (0.1). If $u_0 \in X_0(\Omega) \cap \mathcal{C}$, then for any $T > 0$, there exists a unique weak solution u to (P_t^s) such that $u(t) \in \mathcal{C}$ uniformly for $t \in [0,T]$ and $u \in C([0,T]; X_0(\Omega))$. Moreover for any $t \in [0,T]$,*

$$\int_0^t \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx d\tau + \frac{1}{2}\|u(t,x)\|^2_{X_0(\Omega)} - \frac{1}{1-q}\int_\Omega u^{1-q}(t,x)dx$$
$$= \int_\Omega F(x, u(t))dx + \frac{1}{2}\|u_0(x)\|^2_{X_0(\Omega)} - \frac{1}{1-q}\int_\Omega u_0^{1-q}(x)dx - \int_\Omega F(x, u_0)dx,$$

where $F(x,z) := \int_0^z f(x,z)dz$.

Using Proposition 1, on a similar note we have the following proposition regarding the solution of problem (P_t^s).

Proposition 2. *Assume that the hypothesis of Theorem 2.9 are true. If $u_0 \in \overline{\mathcal{D}(L)}^{L^\infty(\Omega)}$, then the solution of (P_t^s) belongs to $C([0,T]; C_0(\overline{\Omega}))$. Let $\alpha \geq 0$ denotes the Lipschitz constant of $f(\cdot, x)$ in $[\underline{u}, \overline{u}]$, where $\underline{u}$ and $\overline{u}$ denotes the sub and super solution respectively of (Q^s), then the following holds:*

1. *If v is another weak solution of (P_t^s) with initial condition $v_0 \in \overline{\mathcal{D}(L)}^{L^\infty(\Omega)}$, then*
$$\|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(\Omega)} \leq \exp(\alpha t)\|u_0 - v_0\|_{L^\infty(\Omega)}, \ 0 \leq t \leq T.$$
2. *If $u_0 \in \mathcal{D}(L)$, then $u \in W^{1,\infty}([0,T]; L^\infty(\Omega))$ and $(-\Delta)^s u + u^{-q} \in L^\infty(\Lambda_T)$. Also the following holds:*
$$\left\|\frac{du(t,\cdot)}{dt}\right\|_{L^\infty(\Omega)} \leq \exp(\alpha t)\|(-\Delta)^s u_0 + u_0^{-q} + f(x, u_0)\|_{L^\infty(\Omega)}.$$

Finally, we can show the following asymptotic behavior of solutions of (P_t^s).

Theorem 2.10. *Under the hypothesis of Theorem 2.9 and the assumption that $y \mapsto \frac{f(x,y)}{y}$ is decreasing in $(0,\infty)$ a.e. $x \in \Omega$, the solutions of (P_t^s) is defined in $(0,\infty) \times \Omega$ and it satisfies*

$$u(t) \to \hat{u} \text{ in } L^\infty(\Omega) \text{ as } t \to \infty,$$

where $\hat{u}$ is defined in Theorem 2.8.

Remark 1. We can conclude the results for the problem (P_t^s) in a similar manner when $q > -1$ and $q(2s-1) < (2s+1)$ holds.

3. **Existence of solution to** (S)**.** Basically we prove Theorem 2.4 in this section. Before proving this, we give a Lemma that will be recalled in our work several times as the weak comparison principle.

Lemma 3.1. *Assume* $\lambda > 0$ *and* $u, v \in X_0(\Omega)$ *are weak solutions of*

$$A_\lambda u = g_1 \text{ in } \Omega, \tag{3.1}$$

$$A_\lambda v = g_2 \text{ in } \Omega \tag{3.2}$$

with $g_1, g_2 \in L^2(\Omega)$ *such that* $g_1 \leq g_2$*, where* $A_\lambda : X_0(\Omega) \cap \mathcal{C} \to (X_0(\Omega))^*$ *(dual space of* $X_0(\Omega)$*) is defined as* $A_\lambda(u) := u + \lambda((-\Delta)^s u - u^{-q})$*, with* $\lambda > 0$ *fixed. Then* $u \leq v$ *a.e. in* Ω*. Moreover, for* $g \in L^\infty(\Omega)$ *the problem*

$$A_\lambda u = g \text{ in } \Omega, \; u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \tag{3.3}$$

has a unique solution in $X_0(\Omega)$*.*

Proof. Let $w = (u - v)$, then $w = w^+ - w^-$ where $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Let $\Omega^+ := \{x \in \Omega : u(x) > v(x)\}$ and $\Omega^- := \Omega \setminus \Omega^+$, then $\Omega = \Omega^+ \cup \Omega^-$. Multiplying (3.1) and (3.2) by w^+, integrating over $\mathbb{R}^n$ on both sides and subtracting, we get

$$\int_{\Omega^+} (u-v)^2 dx + \lambda \left(C_n^s \int_Q \frac{((u-v)(x) - (u-v)(y))(w^+(x) - w^+(y))}{|x-y|^{n+2s}} dxdy \right.$$
$$\left. - \int_{\Omega^+} \left(\frac{1}{v^q} - \frac{1}{u^q} \right) (u-v) dx \right) = \int_{\Omega^+} (g_1 - g_2) w^+ dx.$$

Since for $(x, y) \in \Omega \times \mathcal{C}\Omega$, $((u - v)(x) - (u - v)(y))(w^+(x) - w^+(y)) = (u - v)(x)w^+(x) \geq 0$ and for $(x, y) \in \Omega^+ \times \Omega^-$, $((u - v)(x) - (u - v)(y))w^+(x) \geq 0$ we get

$$\begin{aligned} &\int_{\Omega^+} (u-v)^2 dx + \lambda \left(C_n^s \int_{\Omega^+} \int_{\Omega^+} \frac{((u-v)(x) - (u-v)(y))^2}{|x-y|^{n+2s}} dxdy \right. \\ &\qquad \left. - \int_{\Omega^+} \left(\frac{1}{v^q} - \frac{1}{u^q} \right) (u-v) dx \right) \leq \int_{\Omega^+} (g_1 - g_2) w^+ dx. \end{aligned} \tag{3.4}$$

We can also prove that A_λ is a strictly monotone operator (for definition refer [7]). So left-hand side of (3.4) is positive whereas $\int_{\Omega^+} (g_1 - g_2) w^+ dx \leq 0$. Therefore we arrive at a contradiction which implies $u \leq v$ a.e. in Ω. Then the uniqueness of (3.3) follows directly. □

Proof of Theorem 2.4. For $\epsilon > 0$, we consider the following approximated problem corresponding to (S) as

$$(S_\epsilon) \begin{cases} u + \lambda \left((-\Delta)^s u - (u+\epsilon)^{-q} \right) = g, \quad u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $X_0^+(\Omega) = \{u \in X_0(\Omega) : \; u \geq 0\}$. The energy functional associated to (S_ϵ) is $E_\lambda : X_0^+(\Omega) \to \mathbb{R}$ given by

$$E_\lambda(u) = \frac{1}{2} \int_\Omega u^2 \, dx + \frac{\lambda}{2} \|u\|^2_{X_0(\Omega)} - \frac{\lambda}{1-q} \int_\Omega (u+\epsilon)^{1-q} \, dx - \int_\Omega gu \, dx$$

which can be shown to be weakly lower semicontinuous, coercive and strictly convex in $X_0^+(\Omega)$. Since $X_0(\Omega)$ is reflexive and $X_0^+(\Omega)$ is a closed convex subset of $X_0(\Omega)$, E_λ has a unique global minimizer $u_{\lambda,\epsilon} \in X_0^+(\Omega)$ i.e. $u_{\lambda,\epsilon} \geq 0$ a.e. in Ω. Let $\phi_{1,s}$

denotes the normalized first eigenfunction associated with the first eigenvalue $\lambda_{1,s}$ of $(-\Delta)^s$ with Dirichlet boundary condition in $\mathbb{R}^n \setminus \Omega$ i.e.

$$(-\Delta)^s \phi_{1,s} = \lambda_{1,s}\phi_{1,s} \text{ in } \Omega, \;\; \phi_{1,s} = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where $0 < \phi_{1,s} \in X_0(\Omega) \cap L^\infty(\Omega)$ is normalized by $\|\phi_{1,s}\|_{L^2(\Omega)} = 1$, refer [[22], Proposition 9, p. 8]. Also there exists a $l > 0$ such that $l\delta^s(x) \leq \phi_{1,s}(x)$ for a.e. $x \in \Omega$ (see [20]). Since $g \in L^\infty(\Omega)$, if we choose $m > 0$ (depending on λ, q and g) small enough such that (in the weak sense)

$$m\|\phi_{1,s}\|_\infty + \lambda\lambda_{1,s} m\|\phi_{1,s}\|_\infty - \frac{\lambda}{m^q\|\phi_{1,s}+\epsilon\|_\infty^q} < g,$$

then $m\phi_{1,s}$ forms a strict subsolution of (S_ϵ) (independent of ϵ) i.e.

$$\begin{cases} m\phi_{1,s} + \lambda\left((-\Delta)^s(m\phi_{1,s}) - \dfrac{1}{(m\phi_{1,s}+\epsilon)^q}\right) < g \text{ in } \Omega, \\ m\phi_{1,s} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.5}$$

We define $w_\epsilon := (m\phi_{1,s} - u_{\lambda,\epsilon})^+$ with the assumption that $\text{supp}(w_\epsilon)$ has non zero measure and for $t > 0$, $\zeta(t) := E_\lambda(u_{\lambda,\epsilon} + tw_\epsilon)$, then

$$\begin{aligned} \zeta'(t) = &\int_\Omega (u_{\lambda,\epsilon} + tw_\epsilon)w_\epsilon \\ &+ \lambda C_n^s \int_Q \frac{((u_{\lambda,\epsilon}+tw_\epsilon)(x) - (u_{\lambda,\epsilon}+tw_\epsilon)(y))(w_\epsilon(x) - w_\epsilon(y))}{|x-y|^{n+2s}}\, dxdy \\ &- \lambda \int_\Omega \frac{w_\epsilon}{(u_{\lambda,\epsilon}+tw_\epsilon+\epsilon)^q} - \int_\Omega g w_\epsilon \, dx \end{aligned}$$

in $(0,1]$. Since $u_{\lambda,\epsilon}$ is the minimizer of E_λ, $\lim\limits_{t\to 0^+} \zeta'(t) \geq 0$. Moreover, convexity of E_λ assures that the map $t \mapsto \zeta'(t)$ is non decreasing. This implies $0 \leq \zeta'(0^+) \leq \zeta'(1)$. Let us recall the following inequality for any ψ being a convex Lipschitz function:

$$(-\Delta)^s\psi(u) \leq \psi'(u)(-\Delta)^s u.$$

Therefore using this with $\psi(x) = \max\{x, 0\}$ and (3.5), we get $\zeta'(1) \leq \langle E'_\lambda(m\phi_{1,s}), w_\epsilon\rangle < 0$ which is a contradiction. Hence $\text{supp}(w_\epsilon)$ must have measure zero which implies that

$$m\phi_{1,s} \leq u_{\lambda,\epsilon}. \tag{3.6}$$

Using (3.6), we can show that E_λ is Gâteaux differentiable in $u_{\lambda,\epsilon}$ and as a result, $u_{\lambda,\epsilon}$ satisfies in the sense of distributions

$$u_{\lambda,\epsilon} + \lambda(-\Delta)^s u_{\lambda,\epsilon} = \lambda u_{\lambda,\epsilon}^{-q} + g \text{ in } \Omega.$$

Using Proposition 2.9 of [23], we get $u_{\lambda,\epsilon} \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2\sigma - 1$ where $2\sigma > 1$. Also since $g \in L^\infty(\Omega)$, using Proposition 1.1 (p. 277) of [20] we get $u_{\lambda,\epsilon} \in C^s(\mathbb{R}^n)$. Now we claim that $u_{\lambda,\epsilon}$ is monotone increasing as $\epsilon \downarrow 0^+$. Let $0 < \epsilon_1 < \epsilon_2$, then we show that $u_{\lambda,\epsilon_1} > u_{\lambda,\epsilon_2}$ in Ω. If possible, let $x_0 \in \Omega$ be such that $x_0 := \arg \min\limits_{\overline{\Omega}}(u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2})$ and $u_{\lambda,\epsilon_1}(x_0) \leq u_{\lambda,\epsilon_2}(x_0)$. Then

$$(u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2}) + \lambda(-\Delta)^s(u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2}) = \lambda\left(\frac{1}{(u_{\lambda,\epsilon_1}+\epsilon_1)^q} - \frac{1}{(u_{\lambda,\epsilon_2}+\epsilon_2)^q}\right)$$

which implies that

$$(u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2})(x_0) + \lambda C_n^s \int_{\mathbb{R}^n} \frac{(u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2})(x_0) - (u_{\lambda,\epsilon_1} - u_{\lambda,\epsilon_2})(y)}{|x_0 - y|^{n+2s}}\, dy \tag{3.7}$$

$$= \lambda \left(\frac{1}{(u_{\lambda,\epsilon_1}(x_0) + \epsilon_1)^q} - \frac{1}{(u_{\lambda,\epsilon_2}(x_0) + \epsilon_2)^q} \right). \tag{3.8}$$

But we can see that (3.7) is negative whereas (3.8) is positive which gives a contradiction. Therefore $x_0 \in \partial\Omega$ and $u_{\lambda,\epsilon_1} > u_{\lambda,\epsilon_2}$ in Ω. Thus we get that $u_\lambda := \lim\limits_{\epsilon \downarrow 0^+} u_{\lambda,\epsilon} \geq m\phi_{1,s}$. Let $w \in X_0^+(\Omega)$ solves the problem

$$(-\Delta)^s w = w^{-q} \text{ in } \Omega, \;\; w = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{3.9}$$

Then from the proof of Theorem 1.1 of [2], we know that w satisfies

$$k_1 \phi_{1,s} \ln^{\frac{1}{2}} \left(\frac{2}{\phi_{1,s}} \right) \leq w \leq k_2 \phi_{1,s} \ln^{\frac{1}{2}} \left(\frac{2}{\phi_{1,s}} \right), \text{ if } q = 1 \tag{3.10}$$

$$k_1 \phi_{1,s}^{\frac{2}{q+1}} \leq w \leq k_2 \phi_{1,s}^{\frac{2}{q+1}}, \text{ if } q > 1 \tag{3.11}$$

where $k_1, k_2 > 0$ are appropriate constants. Let $\overline{u} := M_1 w \in \mathcal{C} \cap C_0(\overline{\Omega})$ for $M_1 > 0$. Then we can choose $M_1 >> 1$ (independent of ϵ) large enough such that

$$\begin{aligned} \overline{u} + \lambda \left((-\Delta)^s \overline{u} - \frac{1}{(\overline{u} + \epsilon)^q} \right) &= M_1 w + \lambda \left(\frac{M_1}{w^q} - \frac{1}{(M_1 w + \epsilon)^q} \right) \\ &\geq M_1 w + \lambda \left(\frac{1}{(M_1 w)^q} - \frac{1}{(M_1 w + \epsilon)^q} \right) > g \text{ in } \Omega. \end{aligned}$$

Using Lemma 3.1, we get $u_{\lambda,\epsilon} \leq \overline{u}$ which implies that $u_\lambda \leq \overline{u} = M_1 w$. Now since $m\phi_{1,s} \leq u_\lambda \leq M_1 w$ and both $w, \phi_{1,s} = 0$ in $\mathbb{R}^n \setminus \Omega$, we get $u_\lambda = 0$ in $\mathbb{R}^n \setminus \Omega$. Also, u_λ solves (S) in the sense of distributions. Let $\underline{u} := M_2 w \in \mathcal{C} \cap C_0(\overline{\Omega})$ then $M_2 > 0$ can be chosen small enough so that

$$M_2^{q+1} \left(1 + \frac{w^{q+1}}{\lambda} \right) \leq 1 + \frac{g (M_2 w)^q}{\lambda} \text{ in } \Omega$$

$$\text{i.e. } \underline{u} + \lambda (-\Delta)^s \underline{u} < \frac{\lambda}{\underline{u}^q} + g \text{ in } \Omega.$$

This implies that $\underline{u}$ forms a subsolution of (S). We claim that $\underline{u} \leq u_\lambda$ in Ω. If possible, let $x_0 \in \Omega$ be such that $x_0 := \arg \min\limits_{\overline{\Omega}} (u_\lambda - \underline{u})$ and $u_\lambda(x_0) \leq \underline{u}(x_0)$. Then using the fact that u_λ is a solution of (S) in the sense of distributions and $\underline{u}$ is a subsolution of (S), we get

$$\begin{aligned} &(u_\lambda - \underline{u})(x_0) + \lambda \int_\Omega \frac{(u_\lambda - \underline{u})(x_0) - (u_\lambda - \underline{u})(y)}{|x_0 - y|^{n+2s}} dy \\ &\quad \geq (u_\lambda - \underline{u})(x_0) + \lambda (-\Delta)^s (u_\lambda - \underline{u})(x_0) \geq \lambda \left(\frac{1}{u_\lambda^q(x_0)} - \frac{1}{\underline{u}^q(x_0)} \right). \end{aligned} \tag{3.12}$$

This gives a contradiction since left hand side of (3.12) is negative whereas right hand side of (3.12) is positive. Therefore we obtain

$$\underline{u} \leq u_\lambda \leq \overline{u}$$

which implies that $u_\lambda \in \mathcal{C}$, using (3.10) and (3.11). Now we show that $u_\lambda \in X_0(\Omega)$ and it is a weak solution of (S). Since $q(2s-1) < (2s+1)$, using the behavior of u_λ with respect to the δ function we get that $\int_\Omega u_\lambda^{1-q}\ dx < +\infty$. Also $\int_\Omega u_\lambda^{-q}\phi\ dx < +\infty$ for any $\phi \in X_0(\Omega)$ using Hardy's inequality. Therefore using $\overline{C_c^\infty}^{\|\cdot\|_{X_0(\Omega)}} = X_0(\Omega)$ and the Lebsegue dominated convergence theorem, we get that for any $\phi \in X_0(\Omega)$

$$\int_\Omega u_\lambda\phi + \lambda C_n^s \int_Q \frac{(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\ dxdy - \int_\Omega \left(\frac{\lambda}{u_\lambda^q} + g\right)\phi\ dx = 0.$$

That is $u_\lambda \in X_0(\Omega)\cap\mathcal{C}$ is a weak solution of (S). By Lemma 3.1, uniqueness of u_λ follows. Following the proof of Theorem 1.2 in [2], we get that $u \in C^\alpha(\mathbb{R}^n)$ where $\alpha = s$ if $q<1$, $\alpha = s-\epsilon$ if $q=1$, for any $\epsilon>0$ small enough and $\alpha = \frac{2s}{q+1}$ if $q>1$. This completes the proof. □

To prove the next result, we follow Lemma 3.6 and Theorem 3.7 of [8].

Proof of Theorem 2.5. Consider the following approximated problem

$$(P_{\lambda,k})\begin{cases} u_k + \lambda\left((-\Delta)^s u - \dfrac{1}{(u+\frac{1}{k})^q}\right) = g \text{ in } \Omega,\\ u_k = 0 \text{ in } \mathbb{R}^n\setminus\Omega.\end{cases}$$

By minimization argument, we know that the solution u_k to the problem $(P_{\lambda,k})$ belongs to $X_0(\Omega)$. By weak comparison principle, we get $u_k \le u_{k+1}$ for all k. From the proof of Theorem 2.4, we know that $m\phi_{1,s}$ and $\overline{u} = M_1 w$ forms subsolution and supersolution of $(P_{\lambda,k})$ respectively independent of k, where w solves (3.9) and m is a sufficiently small whereas M_1 is a sufficiently large positive constant. Therefore

$$0 \le m\phi_{1,s} \le u_k \le u_{k+1} \le \overline{u}, \text{ for all } k. \tag{3.13}$$

Since $g \in L^\infty(\Omega)$ so Proposition 1.1 of [20] gives that $u_k \in L^\infty(\Omega)\cap C^s(\mathbb{R}^n)$ for all k. Therefore if $\tilde{\Omega} \Subset \Omega$ then there exists a constant $c_{\tilde{\Omega}} > 0$ such that

$$u_k \ge c_{\tilde{\Omega}} > 0 \text{ in } \tilde{\Omega}. \tag{3.14}$$

Let $u_\lambda := \lim_{k\to\infty} u_k$. Then u_λ solves (S) in the sense of distributions. From the proof of Theorem 2.4, we also know that for sufficiently small $M_2>0$, $\underline{u} = M_2 w$ satisfies

$$\underline{u} + \lambda((-\Delta)^s\underline{u} - \underline{u}^{-q}) < g \text{ in } \Omega.$$

Then following the arguments in proof of Theorem 2.4 (refer (3.12)), we can show that $\underline{u} \le u_\lambda \le \overline{u}$ which implies that $u_\lambda \sim d^{\frac{2s}{q+1}}(x)$. Now for $b>1$ and $\beta \ge 1$, consider the function $\phi_\beta : [0,+\infty) \to [0,+\infty)$ defined as

$$\phi_\beta(r) = \begin{cases} r^\beta, \text{ if } 0 \le r < b,\\ \beta b^{\beta-1}r - (\beta-1)b^\beta, \text{ if } r \ge b > 1.\end{cases}$$

Then ϕ_β is a Lipschitz function with Lipschitz constant $\beta b^{\beta-1}$. We have $q>1$. So let

$$\beta > \max\left\{1, \left(1-\frac{1}{2s}\right)\left(\frac{q+1}{2}\right)\right\} \ge 1. \tag{3.15}$$

Then if $(2\beta - 1 - q) < 0$ then from $u_\lambda \sim d^{\frac{2s}{q+1}}(x)$ and (3.15) we get

$$\int_\Omega \frac{\phi'_\beta(u_\lambda)\phi_\beta(u_\lambda)}{u_\lambda^q}\, dx < +\infty. \tag{3.16}$$

Since $\phi'_\beta(u)\phi_\beta(u) \leq \beta u^{2\beta-1}$ so using (3.16), $u_k \uparrow u_\lambda$ as $k \to \infty$ and monotone convergence theorem we get that

$$\int_\Omega \frac{\phi'_\beta(u_k)\phi_\beta(u_k)}{u_k^q}\, dx < +\infty \text{ (independent of } k). \tag{3.17}$$

Also (3.17) holds true when $(2\beta - 1 - q) \geq 0$ which follows from the uniform bound on $\{u_k\}$ in $L^\infty(\Omega)$. Since it holds that

$$(-\Delta)^s \phi_\beta(u_k) \leq \phi'_\beta(u_k)(-\Delta)^s u_k,$$

therefore using (3.17) we get

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_\beta(u_k)(-\Delta)^s\phi_\beta(u_k) &\leq \frac{1}{\lambda}\int_\Omega (g - u_k)\phi'_\beta(u_k)\phi_\beta(u_k)\, dx + \int_\Omega \frac{\phi'_\beta(u_k)\phi_\beta(u_k)}{u_k^q}\, dx \\ &\leq \beta\left(\frac{\|g\|_\infty \|\overline{u}\|_{L^{2\beta-1}(\Omega)}}{\lambda} + C\right),\end{aligned}$$

where $C > 0$ is a constant independent of k. Passing on the limit as $b \to \infty$ we get $\{u_k^\beta\}$ is uniformly bounded in $X_0(\Omega)$. By weak lower semicontinuity of norms we have

$$\|u_\lambda^\beta\| \leq \liminf_{k\to\infty} \|u_k^\beta\| < +\infty$$

which implies $u_\lambda^\beta \in X_0(\Omega)$. Thus $u_\lambda^\beta \in L^{2_s^*}(\Omega)$ and since $\beta 2_s^* > 1$ we get $u_\lambda \in L^1(\Omega)$. Now let $\psi \in \Theta$ such that $\text{supp}(\psi) = \tilde{\Omega} \Subset \Omega$ then by Lebesgue dominated convergence theorem we get

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u_k(-\Delta)^s\psi\, dx = \int_{\mathbb{R}^n} u_\lambda(-\Delta)^s\psi\, dx < +\infty.$$

Using (3.14) we get

$$0 \leq \left|\left(\frac{g - u_k}{\lambda} + \frac{1}{(u_k + \frac{1}{k})^q}\right)\psi\right| \leq \left(\frac{|g| + |\overline{u}|}{\lambda} + \frac{1}{c_{\tilde{\Omega}}^q}\right)|\psi| \in L^1(\Omega).$$

Therefore using Lebesgue dominated convergence theorem again, we obtain

$$\int_{\mathbb{R}^n} u_\lambda(-\Delta)^s\psi = \lim_{k\to\infty}\int_\Omega \left(\frac{g - u_k}{\lambda} + \frac{1}{(u_k + \frac{1}{k})^q}\right)\psi\, dx = \int_\Omega \left(\frac{g - u_\lambda}{\lambda} + \frac{1}{u_\lambda^q}\right)\psi\, dx. \tag{3.18}$$

Now we claim that $u_\lambda \notin X_0(\Omega)$. On contrary, if $u_\lambda \in X_0(\Omega)$ then using Lemma 3.1 of [16] and monotone convergence theorem, we can easily show that (3.18) holds for any $\psi \in X_0(\Omega)$. Therefore $u_\lambda \in X_0(\Omega)$ solves (S) in the weak sense and we get

$$\frac{1}{u_\lambda^q} = \frac{1}{\lambda}(u_\lambda - g) + (-\Delta)^s u_\lambda \in (X_0(\Omega))^*.$$

This along with (3.13) implies that

$$\int_\Omega \overline{u}^{1-q}\, dx \leq \int_\Omega u_\lambda^{1-q}\, dx < +\infty$$

which contradicts the definition of $\overline{u}$. $\square$

4. **Existence of solution to (G_t^s) and its regularity.** We prove Theorem 2.7 and Proposition 1 in this section. We use the method of semi-discretization in time along with the implicit Euler method to prove Theorem 2.7.

Theorem 4.1. *If $h(t,x) \in L^\infty(\Lambda_T)$, $u_0 \in X_0(\Omega)\cap\mathcal{C}$, $q>0$ and $q(2s-1)<(2s+1)$, then there exists a unique weak solution $u \in \mathcal{A}(\Lambda_T)\cap\mathcal{C}$ of the problem (G_t^s).*

Proof. Let $\Delta_t = \frac{T}{n}$ and for $0 \le k \le n$, define $t_k := k\Delta_t$. Also define

$$h_k(x) := \frac{1}{\Delta_t}\int_{t_{k-1}}^{t_k} h(\tau,x)d\tau \text{ for } x \in \Omega.$$

Then since $h \in L^\infty(\Lambda_T)$, we get $h_k \in L^\infty(\Omega)$ and $\|h_k\|_\infty \le \|h\|_{L^\infty(\Lambda_T)}$. We define

$$h_{\Delta_t}(t,x) := h^k(x), \text{ when } t \in [t_{k-1},t_k),\ 1 \le k \le n$$

and get that $h_{\Delta_t} \in L^\infty(\Lambda_T)$. For $1<p<+\infty$,

$$\|h_{\Delta_t}\|_{L^p(\Lambda_T)} \le (|\Omega|T)^{\frac{1}{p}}\|h\|_{L^\infty(\Lambda_T)} \tag{4.1}$$

and $h_{\Delta_t} \to h$ in $L^p(\Lambda_T)$ as $\Delta_t \to 0$. Now taking $\lambda = \Delta_t$ and $g = \Delta_t h_k + u^{k-1} \in L^\infty(\Omega)$ in (S) and using Theorem 2.4 we define the sequence $\{u^k\} \subset X_0(\Omega)\cap\mathcal{C}$ as solution to the problem

$$\begin{cases} \dfrac{u^k-u^{k-1}}{\Delta_t} + (-\Delta)^s u^k - \dfrac{1}{(u^k)^q} = h_k \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\quad u^k = 0 \text{ in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{4.2}$$

where $u^0 = u_0 \in X_0(\Omega)\cap\mathcal{C}$. Now, for $1 \le k \le n$, we define

$$\forall t \in [t_{k-1},t_k), \begin{cases} u_{\Delta_t}(t,x) := u^k(x) \\ \tilde{u}_{\Delta_t}(t,x) := \dfrac{(u^k(x)-u^{k-1}(x))}{\Delta_t}(t-t_{k-1}) + u^{k-1}(x). \end{cases} \tag{4.3}$$

Then u_{Δ_t} and $\tilde{u}_{\Delta_t}$ satisfies

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} + (-\Delta)^s u_{\Delta_t} - \frac{1}{u_{\Delta_t}^q} = h_{\Delta_t} \in L^\infty(\Lambda_T). \tag{4.4}$$

At first, we establish some a priori estimates for u_{Δ_t} and $\tilde{u}_{\Delta_t}$ independent of Δ_t. Multiplying (4.2) by $\Delta_t u^k$, integrating it over $\mathbb{R}^n$ and summing it from $k=1$ to $n' \le n$, using Young's inequality and (4.1), we get for a constant $C>0$

$$\begin{aligned} &\sum_{k=1}^{n'}\int_\Omega (u^k-u^{k-1})u^k dx + \Delta_t\sum_{k=1}^{n'}\left(\|u^k\|^2_{X_0(\Omega)} - \int_\Omega (u^k)^{1-q}dx\right) \\ &= \Delta_t\sum_{k=1}^{n'}\int_\Omega h^k u^k dx \le \Delta_t\sum_{k=1}^{n'}\int_\Omega \frac{|h^k|^2}{2}dx + \Delta_t\sum_{k=1}^{n'}\int_\Omega\frac{|u^k|^2}{2}dx \\ &\le \frac{T}{2}|\Omega|\|h\|^2_{L^\infty(\Lambda_T)} + \frac{C\Delta_t}{2}\sum_{k=1}^{n'}\|u^k\|^2_{X_0(\Omega)}. \end{aligned} \tag{4.5}$$

As inequality (2.7) of Theorem 0.9 in [6], we can estimate the first term of (4.5) as follows

$$\begin{aligned}&\sum_{k=1}^{n'}\int_\Omega (u^k-u^{k-1})u^k dx\\ &=\frac{1}{2}\sum_{k=1}^{n'}\int_\Omega |u^k-u^{k-1}|^2dx+\frac{1}{2}\int_\Omega |u^{n'}|^2dx-\frac{1}{2}\int_\Omega |u_0|^2dx.\end{aligned}\tag{4.6}$$

Let v and w solves (3.9) and define

$$\underline{u}=mw \text{ and } \overline{u}=Mw$$

where $m>0$ is small enough and $M>0$ is large enough chosen in such a way that

$$\begin{cases}(-\Delta)^s\underline{u}-\dfrac{1}{\underline{u}^q}\le -\|h\|_{L^\infty(\Lambda_T)} \text{ in } \Omega,\\ (-\Delta)^s\overline{u}-\dfrac{1}{\overline{u}^q}\ge \|h\|_{L^\infty(\Lambda_T)} \text{ in } \Omega.\end{cases}$$

Since $u_0\in\mathcal{C}$, we can always choose $\underline{u}$ and $\overline{u}$ in such a way that it satisfies the above inequalities and $\underline{u}\le u_0\le\overline{u}$. Applying Lemma 3.1 iteratively we get $\underline{u}\le u^k\le\overline{u}$ for all k. This implies that for a.e. $(t,x)\in[0,T]\times\Omega$,

$$\underline{u}(x)\le u_{\Delta_t}(t,x),\tilde{u}_{\Delta_t}(t,x)\le\overline{u}(x)\tag{4.7}$$

i.e. $u_{\Delta_t},\tilde{u}_{\Delta_t}\in\mathcal{C}$ uniformly. Now, since $q(2s-1)<(2s+1)$ we can estimate the singular term in (4.5) as follows

$$\Delta_t\sum_{k=1}^{n'}\int_\Omega (u^k)^{1-q}dx\le\begin{cases}T\displaystyle\int_\Omega \overline{u}^{1-q}dx<+\infty \text{ if } q\le 1,\\ T\displaystyle\int_\Omega \underline{u}^{1-q}dx<+\infty \text{ if } q>1.\end{cases}\tag{4.8}$$

Since $u^k\in L^\infty(\Omega)$ for all k, by the definition of u_{Δ_t} and $\tilde{u}_{\Delta_t}$ we easily get that

$$u_{\Delta_t},\tilde{u}_{\Delta_t} \text{ is bounded in } L^\infty([0,T],L^\infty(\Omega)).\tag{4.9}$$

We see that for $t\in[t_{k-1},t_k)$,

$$\begin{aligned}\|\tilde{u}_{\Delta_t}(t,\cdot)\|_{X_0(\Omega)}&=\left\|\frac{(t-t_{k-1})}{\Delta_t}u^k+\frac{(\Delta_t-t+t_{k-1})}{\Delta_t}u^{k-1}\right\|_{X_0(\Omega)}\\ &\le\|u^k\|_{X_0(\Omega)}+\|u^{k-1}\|_{X_0(\Omega)}.\end{aligned}$$

Integrating both sides of (4.5) over (t_{k-1},t_k) and using (4.6), (4.8) we get that

$$u_{\Delta_t},\tilde{u}_{\Delta_t} \text{ is bounded in } L^2([0,T],X_0(\Omega)).\tag{4.10}$$

Now we try to obtain a second energy estimate. Multiplying (4.2) by u^k-u^{k-1}, integrating it over $\mathbb{R}^n$ and summing it from $k=1$ to $n'\le n$, using Young's inequality and (4.1) we get

$$\begin{aligned}&\Delta_t\sum_{k=1}^{n'}\int_\Omega\left(\frac{u^k-u^{k-1}}{\Delta_t}\right)^2dx+\sum_{k=1}^{n'}\int_{\mathbb{R}^n}((-\Delta)^su^k(x))(u^k-u^{k-1})(x)dx\\ &-\sum_{k=1}^{n'}\int_\Omega\frac{(u^k-u^{k-1})}{(u^k)^q}dx=\Delta_t\sum_{k=1}^{n'}\int_\Omega\frac{h^k(u^k-u^{k-1})}{\Delta_t}\,dx\end{aligned}$$

$$\leq \frac{\Delta_t}{2}\sum_{k=1}^{n'}\left(\int_\Omega |h^k|^2dx + \int_\Omega \left(\frac{u^k-u^{k-1}}{\Delta_t}\right)^2 dx\right) \quad (4.11)$$

which implies that

$$\begin{aligned}&\frac{\Delta_t}{2}\sum_{k=1}^{n'}\int_\Omega \left(\frac{u^k-u^{k-1}}{\Delta_t}\right)^2 dx + \sum_{k=1}^{n'}\int_{\mathbb{R}^n}((-\Delta)^s u^k(x))(u^k-u^{k-1})(x)dx\\ &\quad -\sum_{k=1}^{n'}\int_\Omega \frac{(u^k-u^{k-1})}{(u^k)^q}dx \leq \frac{|\Omega|T}{2}\|h\|^2_{L^\infty(\Lambda_T)}.\end{aligned} \quad (4.12)$$

By convexity of the term $\frac{-1}{1-q}\int_\Omega u^{1-q}dx$, we have

$$\frac{1}{1-q}\int_\Omega \left((u^{k-1})^{1-q}-(u^k)^{1-q}\right)dx \leq -\int_\Omega \frac{u^k-u^{k-1}}{(u^k)^q}\,dx. \quad (4.13)$$

Also

$$\frac{1}{2}\left(\|u^k\|^2_{X_0(\Omega)}-\|u^{k-1}\|^2_{X_0(\Omega)}\right) \leq \int_{\mathbb{R}^n}((-\Delta)^s u^k(x))(u^k-u^{k-1})(x)dx. \quad (4.14)$$

Therefore (4.12) gives

$$\begin{aligned}&\frac{\Delta_t}{2}\sum_{k=1}^{n'}\int_\Omega \left(\frac{u^k-u^{k-1}}{\Delta_t}\right)^2 dx + \frac{1}{2}\left(\|u^{n'}\|^2_{X_0(\Omega)}-\|u_0\|^2_{X_0(\Omega)}\right)\\ &\quad +\frac{1}{1-q}\int_\Omega \left((u_0)^{1-q}-(u^{n'})^{1-q}\right)dx \leq \frac{|\Omega|T}{2}\|h\|^2_{L^\infty(\Lambda_T)}.\end{aligned} \quad (4.15)$$

Integrating over (t_{k-1},t_k) on both sides of (4.15) and using (4.8), we get

$$\frac{\Delta_t}{2}\int_{\Lambda_T}\left|\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}\right|^2 dxdt < +\infty$$

which implies

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \text{ is bounded in } L^2(\Lambda_T) \text{ uniformly in } \Delta_t. \quad (4.16)$$

Using definition of u_{Δ_t} and $\tilde{u}_{\Delta_t}$, we have that

$$u_{\Delta_t} \text{ and } \tilde{u}_{\Delta_t} \text{ are bounded in } L^\infty([0,T];X_0(\Omega)) \text{ uniformly in } \Delta_t. \quad (4.17)$$

Moreover, there exists a constant $C>0$ (independent of Δ_t) such that

$$\|u_{\Delta_t}-\tilde{u}_{\Delta_t}\|_{L^\infty([0,T];L^2(\Omega))} \leq \max_{1\leq k\leq n}\|u^k-u^{k-1}\|_{L^2(\Omega)} \leq C(\Delta_t)^{\frac{1}{2}}. \quad (4.18)$$

Using (4.9) and (4.17), we get

$$u_{\Delta_t} \text{ and } \tilde{u}_{\Delta_t} \text{ are bounded in } L^\infty([0,T];X_0(\Omega)\cap L^\infty(\Omega)) \text{ uniformly in } \Delta_t.$$

Using weak* and weak compactness results, we say that as $\Delta_t \to 0^+$(i.e. $n\to\infty$), up to a subsequence

$$\begin{aligned}&\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u,\ \ u_{\Delta_t} \overset{*}{\rightharpoonup} v \text{ in } L^\infty([0,T];X_0(\Omega)\cap L^\infty(\Omega)) \text{ and}\\ &\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(\Lambda_T)\end{aligned} \quad (4.19)$$

where $u,v \in L^\infty([0,T];X_0(\Omega)\cap L^\infty(\Omega))$ such that $\frac{\partial u}{\partial t}\in L^2(\Lambda_T)$. From (4.18), we infer that $u\equiv v$. Also, from (4.7) we get that $\underline{u}\leq u\leq \overline{u}$. Thus, $u\in \mathcal{A}(\Lambda_T)\cap\mathcal{C}$.

Now we will prove that u is a weak solution of (G_t^s). At first we see that for a.e. $x\in\Omega$, $\tilde{u}_{\Delta_t}(\cdot,x)\in C([0,T])$. By (4.16), we get that $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ is bounded in $L^2(\Lambda_T)$

uniformly in Δ_t. Also, $\{\tilde{u}_{\Delta_t}\}$ is a bounded family in $X_0(\Omega)$ and the embedding of $X_0(\Omega)$ into $L^2(\Omega)$ is compact. If we define

$$W := \left\{ u \in C([0,T]; X_0(\Omega)) : \ \frac{\partial u}{\partial t} \in L^2(\Lambda_T) \right\},$$

then by Aubin-Lions-Simon Lemma, the embedding W into $C([0,T]; L^2(\Omega))$ is compact. Therefore, we get that $\{\tilde{u}_{\Delta_t}\}$ is compact in $C([0,T]; L^2(\Omega))$. Using $\underline{u} \leq \tilde{u}_{\Delta_t} \leq \overline{u}$ again, we get that $\{\tilde{u}_{\Delta_t}\}$ is compact in $C([0,T]; L^p(\Omega))$, $1 < p < \infty$ and therefore as $\Delta_t \to 0^+$, up to a subsequence

$$\tilde{u}_{\Delta_t} \to u \text{ in } C([0,T]; L^2(\Omega)). \tag{4.20}$$

This along with (4.18) gives that as $\Delta_t \to 0^+$,

$$u_{\Delta_t} \to u \text{ in } L^\infty([0,T]; L^2(\Omega)). \tag{4.21}$$

Using $(u_{\Delta_t} - u)$ as the test function in (4.4), we get

$$\int_0^T \int_{\mathbb{R}^n} \left(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} + (-\Delta)^s u_{\Delta_t} - u_{\Delta_t}^{-q} \right) (u_{\Delta_t} - u) dxdt = \int_{\Lambda_T} h_{\Delta_t} (u_{\Delta_t} - u) dxdt.$$

Also using (4.21), we know that $\int_{\Lambda_T} \frac{\partial u}{\partial t} (\tilde{u}_{\Delta_t} - u) dxdt \to 0$ as $\Delta_t \to 0^+$. Hence

$$\begin{aligned} \int_{\Lambda_T} \left(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t} \right) (\tilde{u}_{\Delta_t} - u) dxdt - \int_{\Lambda_T} u_{\Delta_t}^{-q} (u_{\Delta_t} - u) dxdt \\ + \int_0^T \langle (-\Delta)^s u_{\Delta_t}, (u_{\Delta_t} - u) \rangle dt = \int_{\Lambda_T} h_{\Delta_t} (u_{\Delta_t} - u) dxdt + o_{\Delta_t}(1). \end{aligned} \tag{4.22}$$

By (4.7), we have $u_{\Delta_t}^{-q} \leq \underline{u}^{-q}$. Also since $\underline{u} \leq u \leq \overline{u}$, we apply the Lebesgue Dominated convergence theorem with (4.21) to get

$$\int_0^T \int_\Omega u_{\Delta_t}^{-q} (u_{\Delta_t} - u) dxdt \leq \int_0^T \int_\Omega \underline{u}^{-q} (u_{\Delta_t} - u) dxdt = o_{\Delta_t}(1).$$

Similarly using (4.1) and (4.21) along with the Lebesgue Dominated convergence theorem, we get

$$\int_{\Lambda_T} h_{\Delta_t} (u_{\Delta_t} - u) dxdt = o_{\Delta_t}(1).$$

Using integration by parts and the fact that $\tilde{u}_{\Delta_t}(0,x) = u(0,x) = u_0$ in a.e. Ω, we get

$$2 \int_{\Lambda_T} \left(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t} \right) (\tilde{u}_{\Delta_t} - u) dxdt = \int_\Omega (\tilde{u}_{\Delta_t} - u)^2 (T) dt.$$

Therefore, (4.22) implies that

$$\frac{1}{2} \int_\Omega (\tilde{u}_{\Delta_t} - u)^2 (T) dt + \int_0^T \langle (-\Delta)^s u_{\Delta_t} - (-\Delta)^s u, u_{\Delta_t} - u \rangle dt = o_{\Delta_t}(1)$$

where we used the fact that $\int_0^T \langle (-\Delta)^s u, u_{\Delta_t} - u \rangle dt = o_{\Delta_t}(1)$ which follows from (4.21). Since $u \not\equiv 0$ identically in Λ_T, using (4.21) we get

$$\int_0^T \|(u_{\Delta_t} - u)(t, \cdot)\|_{X_0(\Omega)}^2 dt = o_{\Delta_t}(1).$$

Let $(X_0(\Omega))^*$ denotes the dual of $X_0(\Omega)$. Then the above equations suggest that as $\Delta_t \to 0$

$$(-\Delta)^s u_{\Delta_t} \to (-\Delta)^s u \text{ in } L^2([0,T]; (X_0(\Omega))^*). \tag{4.23}$$

From (4.7), for any $\phi \in X_0(\Omega)$, using Hardy's inequality and $q(2s-1) < (2s+1)$ we have

$$\int_\Omega |\phi (u_{\Delta_t})^{-q}| dx \leq \int_\Omega |\phi||\underline{u}^{-q}| dx$$
$$\leq \left(\int_\Omega \frac{1}{\delta^{2s(q-1)/(q+1)}(x)} dx \right)^{\frac{1}{2}} \left(\int_\Omega \frac{\phi^2}{\delta^{2s}(x)} dx \right)^{\frac{1}{2}} < +\infty.$$

Therefore using the Lebesgue Dominated convergence theorem we get

$$\frac{1}{(u_{\Delta_t})^q} \to \frac{1}{u^q} \text{ in } L^\infty([0,T];(X_0(\Omega))^*) \text{ as } \Delta_t \to 0^+. \tag{4.24}$$

Finally, we get $u \in \mathcal{A}(\Lambda_T)$ and for any $\phi \in \mathcal{A}(\Lambda_T)$ passing on the limit $\Delta_t \to 0^+$ in

$$\int_{\Lambda_T} \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \phi \ dxdt + \int_0^T \int_{\mathbb{R}^n} (-\Delta)^s u_{\Delta_t} \phi \ dxdt - \int_{\Lambda_T} \frac{1}{u_{\Delta_t}^q} \phi \ dxdt = \int_{\Lambda_T} h_{\Delta_t} \phi \ dxdt,$$

using (4.1), (4.19), (4.23) and (4.24), we get

$$\int_{\Lambda_T} \frac{\partial u}{\partial t} \phi \ dxdt + \int_0^T \int_{\mathbb{R}^n} (-\Delta)^s u \phi \ dxdt - \int_{\Lambda_T} \frac{1}{u^q} \phi \ dxdt = \int_{\Lambda_T} h \phi \ dxdt. \tag{4.25}$$

That is, u is a weak solution of (G_t^s).

Now we show the uniqueness of u as a solution of (G_t^s) such that $u(t,\cdot) \in \mathcal{C}$, for all $t \in [0,T]$. On contrary, let v such that $v(t,\cdot) \in \mathcal{C}$, for all $t \in [0,T]$ be distinct from u and another weak solution of (G_t^s). Then for any $t \in [0,T]$, we have

$$\int_\Omega \frac{\partial (u-v)}{\partial t} (u-v)(t,x) \ dx + \int_{\mathbb{R}^n} ((-\Delta)^s (u-v))(u-v)(t,x) \ dx$$
$$- \int_\Omega \left(\frac{1}{u^q} - \frac{1}{v^q} \right) (u-v) dx = 0$$

which implies that

$$\frac{\partial}{\partial t} \left(\int_\Omega \frac{1}{2} (u-v)^2 (t,x) \ dx \right)$$
$$= -\|(u-v)(t,\cdot)\|^2_{X_0(\Omega)} + \int_\Omega \left(\frac{1}{u^q} - \frac{1}{v^q} \right) (u-v)(t,x) dx \leq 0.$$

So we see that the function $E : [0,T] \to \mathbb{R}$ defined by

$$E(t) := \frac{1}{2} \int_\Omega (u-v)^2 (t,x) \ dx$$

is a decreasing function. Then since u, v are distinct, we get $0 < E(t) \leq E(0) = 0$ which implies $E(t) = 0$, for all $t \in [0,T]$. Hence $u \equiv v$. □

Theorem 4.2. *The unique weak solution u of (G_t^s) (as obtained in Theorem 4.1) belongs to $C([0,T];X_0(\Omega))$ and $u(t) \in \mathcal{C}$ uniformly for each $t \in [0,T]$. Also, u satisfies* (2.1).

Proof. We first show that $u \in C([0,T];X_0(\Omega))$ and then establish (2.1) in order to complete the proof of this theorem. From (4.19), we already have $u \in C([0,T];L^2(\Omega))$ which implies that the map $\tilde{u} : [0,T] \to X_0(\Omega)$ defined as $[\tilde{u}(t)](x) := u(t,x)$ is weakly continuous. Also (4.20) gives $u \in L^\infty([0,T];X_0(\Omega))$, which implies that $\tilde{u}(t) \in X_0(\Omega)$ and $\|\tilde{u}(t)\|_{X_0(\Omega)} \leq \liminf_{t \to t_0} \|\tilde{u}(t)\|_{X_0(\Omega)}$ for all $t_0 \in [0,T]$.

Multiplying (4.2) by $u^k - u^{k-1}$, integrating it over $\mathbb{R}^n$ on both sides, summing it from $k = n''$ to n' (n' has been considered in (4.11)) and using (4.14), we get

$$\frac{\Delta_t}{2}\sum_{k=n''}^{n'}\int_\Omega\left(\frac{u^k-u^{k-1}}{\Delta_t}\right)^2dx+\frac{1}{2}\left(\|u^{n'}\|^2_{X_0(\Omega)}-\|u^{n''-1}\|^2_{X_0(\Omega)}\right)$$
$$+\frac{1}{1-q}\int_\Omega\left(\left(u^{n''-1}\right)^{1-q}-\left(u^{n'}\right)^{1-q}\right)dx\leq\sum_{k=n''}^{n'}\int_\Omega h_{\Delta_t}(u^k-u^{k-1})dx.$$

For any $t_1 \in [t_0, T]$, we take n'' and n' such that $n''\Delta_t \to t_1$ and $n'\Delta_t \to t_0$ as $\Delta_t \to 0^+$. Then using (4.1), (4.18), (4.21) and (4.24), from the above inequality we get

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dxdt+\frac{1}{2}\|u(t_1,\cdot)\|^2_{X_0(\Omega)}-\frac{1}{1-q}\int_\Omega u(t_1)^{1-q}dx\\ \leq\int_{t_0}^{t_1}\int_\Omega h\frac{\partial u}{\partial t}\,dxdt+\frac{1}{2}\|u(t_0,\cdot)\|^2_{X_0(\Omega)}-\frac{1}{1-q}\int_\Omega u(t_0)^{1-q}dx.\end{aligned}\tag{4.26}$$

Since $u \in L^\infty([0,T]; L^p(\Omega))$ for $1 < p < \infty$, passing on the limit $t_1 \to t_0^+$, we get

$$\limsup_{t_1\to t_0^+}\|u(t_1,\cdot)\|_{X_0(\Omega)}\leq\|u(t_0,\cdot)\|_{X_0(\Omega)}.$$

Therefore $\lim_{t\to t_0^+}\|u(t,\cdot)\|_{X_0(\Omega)} = \|u(t_0,\cdot)\|_{X_0(\Omega)}$ which implies that u is right continuous on $[0,T]$. Now let us prove the left continuity and assume $t_1 > t_0$. Let $0 < r \leq t_1 - t_0$ and define

$$\sigma_r(z):=\frac{u(z+r)-u(r)}{r}.$$

Then since u is a weak solution to (G_t^s), taking $\sigma_r(u)$ as the test function in (G_t^s), integrating over $(t_0, t_1) \times \mathbb{R}^n$ and using (4.13) we get

$$\int_{t_0}^{t_1}\int_\Omega\frac{\partial u}{\partial t}\sigma_r(u)\,dxdt+\frac{1}{2r}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}((-\Delta)^s u(t+r,x)-(-\Delta)^s u(t,x))dxdt$$
$$-\frac{1}{r(1-q)}\int_{t_0}^{t_1}\int_\Omega(u^{1-q}(t+r,x)-u^{1-q}(t,x))dxdt\geq\int_{t_0}^{t_1}\int_\Omega\sigma_r(u)dxdt.$$

Then it is an easy task to get

$$\begin{aligned}&\int_{t_0}^{t_1}\int_\Omega\frac{\partial u}{\partial t}\sigma_r(u)\,dxdt+\frac{1}{2r}\left(\int_{t_1}^{t_1+r}\int_{\mathbb{R}^n}(-\Delta)^s u(t,x)dxdt\right.\\ &\quad\left.-\int_{t_0}^{t_0+r}\int_{\mathbb{R}^n}(-\Delta)^s u(t,x)dxdt\right)-\frac{1}{r(1-q)}\left(\int_{t_1}^{t_1+r}\int_\Omega u^{1-q}(t,x)dxdt\right.\\ &\quad\left.-\int_{t_0}^{t_0+r}\int_\Omega u^{1-q}(t,x)dxdt\right)\\ &\geq\int_{t_0}^{t_1}\int_\Omega\sigma_r(u)dxdt.\end{aligned}\tag{4.27}$$

Since u is right continuous in $X_0(\Omega)$, using the Lebesgue Dominated Convergence theorem we get the following as $r \to 0^+$:

$$\frac{1}{r}\int_{t_1}^{t_1+r}\int_{\mathbb{R}^n}(-\Delta)^s u(t,x)dxdt\to\int_{\mathbb{R}^n}(-\Delta)^s u(t_1,x)dx,$$

$$\frac{1}{r}\int_{t_0}^{t_0+r}\int_{\mathbb{R}^n}(-\Delta)^s u(t,x)dxdt \to \int_{\mathbb{R}^n}(-\Delta)^s u(t_0,x)dx,$$
$$\frac{1}{r}\int_{t_1}^{t_1+r}\int_{\Omega}u^{1-q}(t,x)dxdt \to \int_{\Omega}u^{1-q}(t_1,x)dxdt,$$
$$\frac{1}{r}\int_{t_0}^{t_0+r}\int_{\Omega}u^{1-q}(t,x)dxdt \to \int_{\Omega}u^{1-q}(t_0,x)dxdt.$$

Using these estimates in (4.27), as $r \to 0^+$ we get

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2 dxdt + \frac{1}{2}\|u(t_1,\cdot)\|^2_{X_0(\Omega)} - \frac{1}{1-q}\int_{\Omega}u(t_1)^{1-q}dx\\ &\quad \geq \int_{t_0}^{t_1}\int_{\Omega}h\frac{\partial u}{\partial t}\,dxdt + \frac{1}{2}\|u(t_0,\cdot)\|^2_{X_0(\Omega)} - \frac{1}{1-q}\int_{\Omega}u(t_0)^{1-q}dx.\end{aligned} \tag{4.28}$$

The inequality (4.28) along with (4.26) gives the equality. Since the map $t \mapsto \int_\Omega u^{1-q}(t,x)dt$ is continuous, $u \in C([0,T];X_0(\Omega))$. Also, (2.1) is obtained by taking $t_1 = t \in [0,T]$ and $t_0 = 0$. □

Proof of Theorem 2.7. The proof follows from Theorem 4.1 and Theorem 4.2. □

Next, we present the proof of Proposition 1 and end this section. Through this Proposition, the solution obtained above for (G_t^s) can be proved to belong in $C([0,T];C_0(\overline{\Omega}))$ if the initial function $u_0 \in \overline{\mathcal{D}(L)}^{L^\infty}$. in Section 2.

Proof of Proposition 1. Let $u_0 \in \overline{\mathcal{D}(L)}^{L^\infty}$, $\lambda > 0$ and $f_1, f_2 \in L^\infty(\Omega)$. Also, let $u, v \in X_0(\Omega) \cap \mathcal{C} \cap C_0(\overline{\Omega})$ be the unique solution to

$$\begin{cases} u + \lambda L(u) = f_1 \text{ in } \Omega,\\ v + \lambda L(v) = f_2 \text{ in } \Omega,\end{cases} \tag{4.29}$$

as obtained using Theorem 2.4. Then obviously, $u, v \in \mathcal{D}(L)$. Defining $w := (u - v - \|f_1 - f_2\|_\infty)^+$ and taking w as the test function, from (4.29) we get

$$\int_\Omega w^2 dx + \lambda\int_\Omega (L(u) - L(v))w\;dx \leq 0. \tag{4.30}$$

It is easy to compute that $\int_\Omega (L(u) - L(v))w\;dx \geq 0$. So if supp$(w)$ has nonzero measure, then

$$\int_\Omega w^2 dx + \lambda\int_\Omega (L(u) - L(v))w\;dx > 0$$

which contradicts (4.30). Therefore $(u-v) \leq \|f_1 - f_2\|_\infty$ and if we reverse the roles of u and v then we get $\|u - v\|_\infty \leq \|f_1 - f_2\|_\infty$. This proves that L is m-accretive in $L^\infty(\Omega)$. Let $\tilde{w} \in \mathcal{D}(L)$ and $a, b \in L^\infty(\Lambda_T)$. Then further proof of Proposition 1 can be obtained using Chapter 4, Theorem 4.2 and Theorem 4.4 of [7] or following the proof of Proposition 0.1 of [6]. □

5. **Existence of unique solution to** (Q^s).

We give the proof of Theorem 2.8 in this section. Before doing that, we prove a weak comparison principle which is needed to prove Theorem 2.8. We recall the following discrete Picone identity which will be required to prove the weak comparison principle.

Lemma 5.1. *(Lemma 6.2, [4]) Let $p \in (1, +\infty)$. For $u, v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $u \geq 0$, $v > 0$, we have*

$$M(u, v) \geq 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^n,$$

where $M(u,v) = |u(x)-u(y)|^p - |v(x)-v(y)|^{p-2}(v(x)-v(y))\left(\frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}}\right)$.

The equality holds in Ω if and only if $u = kv$ a.e. in Ω, for some constant k.

Theorem 5.2. *Let $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a Carathéodary function bounded from below such that the map $y \mapsto \frac{g(x,y)}{y}$ is decreasing in $\mathbb{R}^+$ for a.e. $x \in \Omega$. Let $u, v \in L^\infty(\Omega) \cap X_0(\Omega)$ be such that $u, v > 0$ in Ω,*

$$\int_\Omega u^{1-q}\, dx < +\infty, \quad \int_\Omega v^{1-q}\, dx < +\infty \tag{5.1}$$

and satisfies

$$(-\Delta)^s u \leq \frac{1}{u^q} + g(x,u) \text{ and } (-\Delta)^s v \geq \frac{1}{v^q} + g(x,v) \text{ weakly in } (X_0(\Omega))^*. \tag{5.2}$$

Moreover, if there exists $0 < w \in L^\infty(\Omega)$ such that $c_1 w \leq u, v \leq c_2 w$, for $c_1, c_2 > 0$ constants and

$$\int_\Omega |g(x, c_1 w)| w\, dx < +\infty, \quad \int_\Omega |g(x, c_2 w)| w\, dx < +\infty, \tag{5.3}$$

then $u \leq v$ in Ω.

Proof. For $k > 0$, let us define $u_k := u + \frac{1}{k}$ and $v_k := v + \frac{1}{k}$. Also let

$$\phi_k := \frac{u_k^2 - v_k^2}{u_k} \quad \text{and} \quad \psi_k := \frac{v_k^2 - u_k^2}{v_k}.$$

Then since $u, v \in L^\infty(\Omega)$, obviously $u_k, v_k \in L^\infty(\Omega)$ and thus $u_k, v_k \in L^2(\Omega)$. We assumed $u, v \in X_0(\Omega)$ which implies that $u, v \in H^s(\Omega)$. Since $\|u_k\|_{H^s(\Omega)} = \|u\|_{H^s(\Omega)}$ and $\|v_k\|_{H^s(\Omega)} = \|v\|_{H^s(\Omega)}$ we conclude that $u_k, v_k \in H^s(\Omega)$. Let

$$\eta_k := \frac{v_k^2}{u_k} \text{ and } \xi_k := \frac{u_k^2}{v_k}$$

then we claim that $\eta_k, \xi_k \in H^s(\Omega)$. Consider

$$\begin{aligned}
&|\eta_k(x) - \eta_k(y)| \\
&= \left| \frac{v_k^2(x) - v_k^2(y)}{u_k(x)} - \frac{v_k^2(y)(u_k(x) - u_k(y))}{u_k(x)u_k(y)} \right| \\
&\leq k|v_k(x) - v_k(y)||v_k(x) + v_k(y)| + \|v_k\|^2_{L^\infty(\Omega)} \frac{|u_k(x) - u_k(y)|}{u_k(x)u_k(y)} \\
&\leq 2k\|v_k\|_{L^\infty(\Omega)}|v_k(x) - v_k(y)| + k^2\|v_k\|^2_{L^\infty(\Omega)}|u_k(x) - u_k(y)| \\
&\leq C(k, \|v_k\|_{L^\infty(\Omega)})(|v_k(x) - v_k(y)| + |u_k(x) - u_k(y)|),
\end{aligned} \tag{5.4}$$

where $C(k, \|v_k\|_{L^\infty(\Omega)}) > 0$ is a constant. Since $u_k, v_k \in H^s(\Omega)$, we get $\eta_k \in H^s(\Omega)$. Similarly $\xi_k \in H^s(\Omega)$. Clearly, this implies that $\phi_k, \psi_k \in H^s(\Omega)$. We note that ϕ_k, ψ_k can also be written as

$$\phi_k = \frac{(u - v)(u_k + v_k)}{u_k} \text{ and } \psi_k = \frac{(v - u)(v_k + u_k)}{v_k}$$

which implies that $\phi_k, \psi_k = 0$ in $\mathbb{R}^n \setminus \Omega$ i.e. $\phi_k, \psi_k \in X_0(\Omega)$ since $\frac{u_k+v_k}{u_k}$ and $\frac{u_k+v_k}{v_k}$ in $L^\infty(\Omega)$. We set $\Omega^+ = \{x \in \Omega : u(x) > v(x)\}$ and $\Omega^- = \{x \in \Omega : u(x) \leq v(x)\}$.

Then $\phi_k \geq 0$ and $\psi_k \leq 0$ in Ω^+. Let $\tilde{\phi}_k = \chi_{\Omega^+}\phi_k$ and $\tilde{\psi}_k = \chi_{\Omega^+}\psi_k$. Since $\phi_k(x) \leq \phi_k(x) - \phi_k(y)$ for $(x,y) \in \Omega^+ \times \Omega^-$, we get

$$\begin{aligned}
&\int_Q \frac{|\tilde{\phi}_k(x) - \tilde{\phi}_k(y)|^2}{|x-y|^{n+2s}}\, dxdy \\
&= \int_{\Omega^+}\int_{\Omega^+} \frac{|\phi_k(x)-\phi_k(y)|^2}{|x-y|^{n+2s}}\, dxdy + 2\int_{\Omega^+}\int_{\Omega^-} \frac{|\phi_k(x)|^2}{|x-y|^{n+2s}}\, dxdy \\
&\quad + 2\int_{\Omega^+}\int_{\mathcal{C}\Omega} \frac{|\phi_k(x)|^2}{|x-y|^{n+2s}}\, dxdy \\
&\leq \int_{\Omega^+}\int_{\Omega^+} \frac{|\phi_k(x)-\phi_k(y)|^2}{|x-y|^{n+2s}}\, dxdy + 2\int_{\Omega^+}\int_{\Omega^-} \frac{|\phi_k(x)-\phi_k(y)|^2}{|x-y|^{n+2s}}\, dxdy \\
&\quad + 2\int_{\Omega}\int_{\mathcal{C}\Omega} \frac{|\phi_k(x)|^2}{|x-y|^{n+2s}}\, dxdy = \|\phi_k\|^2_{X_0(\Omega)} < +\infty.
\end{aligned}$$

This implies $\tilde{\phi}_k \in X_0(\Omega)$ since by definition $\tilde{\phi}_k = 0$ in $\mathbb{R}^n \backslash \Omega$. Similarly, $\tilde{\psi}_k \in X_0(\Omega)$. Using $\tilde{\phi}_k$ and $\tilde{\psi}_k$ as test functions in (5.2), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} ((-\Delta)^s u)\tilde{\phi}_k \, dx \leq \int_{\Omega^+} \left(\frac{1}{u^q} + g(x,u)\right)\phi_k \, dx, \\
&\int_{\mathbb{R}^n} ((-\Delta)^s v)\tilde{\psi}_k \, dx \leq \int_{\Omega^+} \left(\frac{1}{v^q} + g(x,v)\right)\psi_k \, dx.
\end{aligned} \tag{5.5}$$

Consider

$$\begin{aligned}
&\int_{\Omega^+}\int_{\Omega^+} \frac{(u(x)-u(y))(\phi_k(x)-\phi_k(y))}{|x-y|^{n+2s}}\, dxdy \\
&\quad + \int_{\Omega^+}\int_{\Omega^+} \frac{(v(x)-v(y))(\psi_k(x)-\psi_k(y))}{|x-y|^{n+2s}}\, dxdy \\
&= \int_{\Omega^+}\int_{\Omega^+} \frac{(u_k(x)-u_k(y))^2 + (v_k(x)-v_k(y))^2}{|x-y|^{n+2s}}\, dxdy \\
&\quad - \int_{\Omega^+}\int_{\Omega^+} \frac{(v_k(x)-v_k(y))\left(\frac{u_k^2(x)}{v_k(x)} - \frac{u_k^2(y)}{v_k(y)}\right)}{|x-y|^{n+2s}}\, dxdy \\
&\quad \int_{\Omega^+}\int_{\Omega^+} \frac{(u_k(x)-u_k(y))\left(\frac{v_k^2(x)}{u_k(x)} - \frac{v_k^2(y)}{u_k(y)}\right)}{|x-y|^{n+2s}}\, dxdy \\
&= \int_{\Omega^+}\int_{\Omega^+} \frac{M(u_k,v_k) + M(v_k,u_k)}{|x-y|^{n+2s}}\, dxdy \geq 0,
\end{aligned} \tag{5.6}$$

using Lemma 5.1 with $p = 2$. We have

$$\int_{\Omega^+} \left(\frac{\phi_k}{u^q} + \frac{\psi_k}{v^q}\right) dx \leq 0.$$

Using this, we get

$$\begin{aligned}
&\int_{\Omega^+} \left(\frac{1}{u^q} + g(x,u)\right)\phi_k \, dx + \int_{\Omega^+} \left(\frac{1}{v^q} + g(x,v)\right)\psi_k \, dx \\
&\leq \int_{\Omega^+} (g(x,u)\phi_k + g(x,v)\psi_k)\, dx \\
&= \int_{\Omega^+} \left(\frac{g(x,u)}{u}\left(\frac{u}{u_k}\right) - \frac{g(x,v)}{v}\left(\frac{v}{v_k}\right)\right)(u_k^2 - v_k^2)\, dx.
\end{aligned} \tag{5.7}$$

Since $\frac{u}{u_k} \to 1$ and $\frac{v}{v_k} \to 1$ a.e. in Ω as $k \to +\infty$, using (5.3) and the Lebesgue Dominated convergence theorem with (5.7) we get $\lim_{k\to+\infty} \int_{\Omega^+}(g(x,u)\phi_k + g(x,v)\psi_k)\, dx = 0$. Therefore (5.7) implies that

$$\lim_{k\to+\infty} \left(\int_{\Omega^+} \left(\frac{1}{u^q} + g(x,u) \right) \phi_k \, dx + \int_{\Omega^+} \left(\frac{1}{v^q} + g(x,v) \right) \psi_k \, dx \right) \le 0. \tag{5.8}$$

From (5.5), we have that

$$\begin{aligned} &\int_{\Omega^+} (((-\Delta)^s u)\phi_k + ((-\Delta)^s v)\psi_k) \, dx \\ &\le \int_{\Omega^+} \left(\left(\frac{1}{u^q} + g(x,u) \right) \phi_k + \left(\frac{1}{v^q} + g(x,v) \right) \psi_k \right) \, dx, \end{aligned} \tag{5.9}$$

We claim that

$$\begin{aligned} &\int_Q \frac{(u(x)-u(y))(\tilde{\phi}_k(x)-\tilde{\phi}_k(y))}{|x-y|^{n+2s}} \, dxdy + \int_Q \frac{(v(x)-v(y))(\tilde{\psi}_k(x)-\tilde{\psi}_k(y))}{|x-y|^{n+2s}} \, dxdy \\ &\ge \int_{\Omega^+}\int_{\Omega^+} \frac{(u(x)-u(y))(\phi_k(x)-\phi_k(y)) + (v(x)-v(y))(\psi_k(x)-\psi_k(y))}{|x-y|^{n+2s}} \, dxdy \end{aligned} \tag{5.10}$$

To prove this, we consider

$$\begin{aligned} &\int_Q \frac{(u(x)-u(y))(\tilde{\phi}_k(x)-\tilde{\phi}_k(y))}{|x-y|^{n+2s}} \, dxdy + \int_Q \frac{(v(x)-v(y))(\tilde{\psi}_k(x)-\tilde{\psi}_k(y))}{|x-y|^{n+2s}} \, dxdy \\ &= \int_{\Omega^+}\int_{\Omega^+} \frac{(u(x)-u(y))(\phi_k(x)-\phi_k(y)) + (v(x)-v(y))(\psi_k(x)-\psi_k(y))}{|x-y|^{n+2s}} \, dxdy \\ &\quad + 2\int_{\Omega^+}\int_{\Omega^-} \frac{(u(x)-u(y))\phi_k(x) + (v(x)-v(y))\psi_k(x)}{|x-y|^{n+2s}} \, dxdy \\ &\quad\quad + 2\int_{\Omega^+}\int_{\mathcal{C}\Omega} \frac{(u(x)-u(y))\phi_k(x) + (v(x)-v(y))\psi_k(x)}{|x-y|^{n+2s}} \, dxdy. \end{aligned}$$

Since $\phi_k u_k + \psi_k v_k = 0$ by definition and $\phi_k + \psi_k \le 0$ in Ω^+ and Ω^- both, we get

$$\begin{aligned} &\int_{\Omega^+}\int_{\Omega^-} \frac{(u(x)-u(y))\phi_k(x) + (v(x)-v(y))\psi_k(x)}{|x-y|^{n+2s}} \, dxdy \\ &= \int_{\Omega^+}\int_{\Omega^-} \frac{(u_k(x)-u_k(y))\phi_k(x) + (v_k(x)-v_k(y))\psi_k(x)}{|x-y|^{n+2s}} \, dxdy \\ &= -\int_{\Omega^+}\int_{\Omega^-} \frac{u_k(y)\phi_k(x) + v_k(y)\psi_k(x)}{|x-y|^{n+2s}} \, dxdy \\ &\ge -\int_{\Omega^+}\int_{\Omega^-} \frac{v_k(y)(\phi_k(x)+\psi_k(x))}{|x-y|^{n+2s}} \, dxdy \ge 0. \end{aligned}$$

Similarly

$$\begin{aligned} &\int_{\Omega^+}\int_{\mathcal{C}\Omega} \frac{(u(x)-u(y))\phi_k(x) + (v(x)-v(y))\psi_k(x)}{|x-y|^{n+2s}} \, dxdy \\ &\quad = \int_{\Omega^+}\int_{\mathcal{C}\Omega} \frac{-(\phi_k(x)+\psi_k(x))}{k|x-y|^{n+2s}} \, dxdy \ge 0. \end{aligned}$$

This establishes our claim. Therefore using (5.6), (5.8), (5.9), (5.10) and Fatou's Lemma, we get

$$\begin{aligned}0 &\le \int_{\Omega^+}\int_{\Omega^+} \frac{M(u,v)+M(v,u)}{|x-y|^{n+2s}}\ dxdy\\ &\le \lim_{k\to+\infty}\left(\int_{\mathbb{R}^n}((-\Delta)^s u)\tilde{\phi}_k\ dx + \int_{\mathbb{R}^n}((-\Delta)^s v)\tilde{\psi}_k\ dx\right)\\ &\le \lim_{k\to+\infty}\int_{\Omega^+}\left(\left(\frac{1}{u^q}+g(x,u)\right)\phi_k + \left(\frac{1}{v^q}+g(x,v)\right)\psi_k\right)\ dx \le 0.\end{aligned}$$

This implies that

$$\int_{\Omega^+}\int_{\Omega^+} \frac{M(u,v)+M(v,u)}{|x-y|^{n+2s}}\ dxdy = 0.$$

Therefore $M(u,v)=0=M(v,u)$ a.e. in Ω^+. So using Lemma 5.1, we have $u=kv$ a.e. in Ω^+, for some constant $k>0$. By definition of Ω^+, we have $k>1$. Consider

$$\begin{aligned}&\int_{\Omega^+}(((-\Delta)^s u)u - ((-\Delta)^s kv)kv)\ dx\\ &= \int_{\Omega^+}((-\Delta)^s u - (-\Delta)^s kv)kv)\ dx = \int_{\Omega^+}((-\Delta)^s (u-kv))kv\ dx\\ &= 2C_n^s\int_{\Omega^+}\left(P.V.\int_{\mathbb{R}^n}\frac{(u-kv)(x)-(u-kv)(y)}{|x-y|^{n+2s}}dy\right)kv(x)dx\\ &= 2C_n^s\int_{\Omega^+}P.V.\int_{\Omega^-}\frac{(kv-u)(y)}{|x-y|^{n+2s}}kv(x)dxdy\\ &\ge 2C_n^s k^2\int_{\Omega^+}P.V.\int_{\Omega^-}\frac{(v-u)(y)}{|x-y|^{n+2s}}v(x)dxdy \ge 0.\end{aligned} \tag{5.11}$$

From (5.1) and (5.2) we get

$$\begin{aligned}\int_{\Omega^+}((-\Delta)^s u)u\ dx &\le \int_{\Omega^+}\left(\frac{g(x,kv)}{kv}(kv)^2 + k^{1-q}v^{1-q}\right)\ dx \text{ and}\\ k^2\int_{\Omega^+}((-\Delta)^s v)v\ dx &\ge \int_{\Omega^+}\left(\frac{g(x,v)}{v}(kv)^2 + k^2v^{1-q}\right)dx\end{aligned} \tag{5.12}$$

which implies that $k\le 1$ by (5.11). This gives a contradiction which implies $u\le v$ in Ω. □

Proof of Theorem 2.8. Under the hypothesis on f, we let $l,\mu>0$ be such that $-l\le f(x,y)\le \mu y + l$. Let μ be such that $0<\mu<\lambda_{1,s}(\Omega)$. Suppose w is a solution of (3.9). For $\eta>0$, we define

$$\underline{u} = \eta w. \tag{5.13}$$

Since $w\in \mathcal{C}\cap C_0(\overline{\Omega})$ (see (3.10)-(3.11)), we can choose $\eta>0$ small enough such that

$$(-\Delta)^s\underline{u} - \frac{1}{\underline{u}^q} \le -l \le f(x,\underline{u}) \text{ in } \Omega,\ \ \underline{u}=0 \text{ in } \mathbb{R}^n\setminus\Omega. \tag{5.14}$$

Let $0<M, M'$ and

$$\overline{u} = Mw + M'\phi_{1,s} \tag{5.15}$$

Also, let $\epsilon>0$ and define $\Omega_\epsilon := \{x\in\Omega:\ \text{dist}(x,\partial\Omega)<\epsilon\}$. Then since we know that $w=0$ in $\mathbb{R}^n\setminus\Omega$, we can choose $\epsilon>0$ small enough such that $0\le w\le c$ in Ω_ϵ

where $c > 0$ is such that

$$\left(M - \frac{1}{M^q}\right)\frac{1}{c^q} \geq \mu M c + l$$

which is possible for $c > 0$ sufficiently small. Therefore in Ω_ϵ, we get

$$\begin{aligned}(-\Delta)^s \overline{u} - \frac{1}{\overline{u}^q} &= \left(M - \frac{1}{M^q}\right)\frac{1}{w^q} + M'\lambda_{1,s}\phi_{1,s} \\ &\geq \left(M - \frac{1}{M^q}\right)\frac{1}{c^q} + M'\mu\phi_{1,s} \\ &\geq \mu M c + l + M'\mu\phi_{1,s} \geq \mu M w + l + M'\mu\phi_{1,s} = \mu\overline{u} + l.\end{aligned} \tag{5.16}$$

Now consider the set $\Omega \setminus \Omega_\epsilon = \{x \in \Omega : \ d(x, \partial\Omega) \geq \epsilon\}$. Then there exists a constant $c_1 > 0$ (depending on ϵ) such that $0 < c_1 \leq \phi_{1,s}$ in $\Omega \setminus \Omega_\epsilon$. Since $\mu < \lambda_{1,s}$ and M is fixed now, we choose

$$M' \geq \frac{\mu M \|w\|_\infty + l}{c_1(\lambda_{1,s} - \mu)}.$$

Then in $\Omega \setminus \Omega_\epsilon$, we get

$$\begin{aligned}(-\Delta)^s \overline{u} - \frac{1}{\overline{u}^q} &= \left(M - \frac{1}{M^q}\right)\frac{1}{w^q} + M'\lambda_{1,s}\phi_{1,s} \\ &\geq M'\lambda_{1,s}\phi_{1,s} \geq \mu M w + l + M'\mu\phi_{1,s} = \mu\overline{u} + l.\end{aligned} \tag{5.17}$$

Therefore (5.16) and (5.17) implies that $\overline{u}$ satisfies

$$(-\Delta)^s \overline{u} - \frac{1}{\overline{u}^q} \geq \mu\overline{u} + l \geq f(x, \overline{u}) \text{ in } \Omega, \ \ \overline{u} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{5.18}$$

By construction, $\underline{u}, \overline{u} \in \mathcal{C}$. Since f is uniformly locally Lipschitz with respect to the second variable, we can find appropriate constant $K_0 > 0$ such that the map $t \mapsto K_0 t + f(x,t)$ is non-decreasing in $[0, \|\overline{u}\|_{X_0(\Omega)}]$, for a.e. $x \in \Omega$. We define an iterative scheme to obtain a sequence $\{u_k\} \subset X_0(\Omega) \cap \mathcal{C} \cap C_0(\overline{\Omega})$ (using Theorem 2.7) as solution of the problem

$$\left\{ (-\Delta)^s u_k - \frac{1}{u_k^q} + K_0 u_k = f(x, u_{k-1}) + K_0 u_{k-1} \text{ in } \Omega, \ \ u_k = 0, \text{ in } \mathbb{R}^n \setminus \Omega, \right. \tag{5.19}$$

where $u_0 := \underline{u}$. This scheme is well defined because by the choice of K_0 and using weak comparison principle (Lemma 3.1), we get that

$$\underline{u} \leq u_k \leq \overline{u}, \tag{5.20}$$

for all k. This implies for each k, right hand side of (5.19) is in $L^\infty(\Lambda_T)$ and hence Theorem 2.7 is applicable for (5.19). Again, using Lemma 3.1 and monotonicity of the map $t \mapsto K_0 t + f(x,t)$, we have that the sequence $\{u_k\}$ is a monotone increasing sequence. From (5.19) we have $(-\Delta)^s u_k = g_k \in L^\infty(\Omega')$, where $g_k := u_k^{-q} - K_0 u_k + f(x, u_{k-1}) + K_0 u_{k-1} \leq \underline{u}^{-q} - K_0\underline{u} + f(x, \overline{u}) + K_0\overline{u}$ and Ω' is a compact subset of Ω. Following the proof of Theorem 1.2 of [2], we get that $u_k \in C^{s-\epsilon}(\mathbb{R}^n)$ for each $\epsilon > 0$ small enough when $q = 1$ and $u_k \in C^{\frac{2s}{q+1}}(\mathbb{R}^n)$ when $q > 1$. Also since (5.20) holds, we get that $\{u_k\}$ is a uniformly bounded sequence in $C_0(\overline{\Omega}) \cap \mathcal{C}$. Therefore by Arzela Ascoli theorem, we know that there exists a $\tilde{u} \in C_0(\overline{\Omega}) \cap \mathcal{C}$ such that $u_k \uparrow \tilde{u}$ in $C_0(\overline{\Omega}) \cap \mathcal{C}$ as $k \to \infty$. Therefore it must be Cauchy in $C_0(\overline{\Omega}) \cap \mathcal{C}$ and this alongwith (5.19) gives that $\{u_k\}$ is Cauchy in $X_0(\Omega)$ which converges to $\tilde{u}$ in $X_0(\Omega)$. Now passing on to the limits as $k \to \infty$ and using the Lebesgue Dominated convergence theorem (since $u_k \leq \overline{u}$, for all k) in (5.19), we obtain $\tilde{u}$ to be a solution of (Q^s). Lastly, uniqueness of $\tilde{u}$ follows from Theorem 5.2. □

6. **Existence of solution to** (P_t^s) **and its regularity.** We devote this section to study the problem (P_t^s) which is our concern for this article. Precisely, we will prove Theorem 2.9 and Proposition 2.

Proof of Theorem 2.9. We will closely make use of arguments in the proof of Theorem 2.7 while proving this theorem. Since $T > 0$, we define $\Delta_t := \frac{T}{n}$, where $n \in \mathbb{N}^*$. Taking $u^0 = u_0$, we obtain a sequence $\{u^k\} \subset \mathcal{C} \cap X_0(\Omega) \subset L^\infty(\Omega)$ as solutions to following iterative scheme

$$u^k - \Delta_t \left((-\Delta)^s u^k + \frac{1}{(u^k)^q} \right) = \Delta_t f(x, u^{k-1}) + u^{k-1} \text{ in } \Omega. \tag{6.1}$$

Since $u^0 \in \mathcal{C} \cap X_0(\Omega)$ and $\Delta_t f(x, u^{k-1}) + u^{k-1} \in L^\infty(\Lambda_T)$ for each k, we can apply Theorem 2.7 to obtain the sequence $\{u^k\} \subset \mathcal{C} \cap X_0(\Omega) \subset L^\infty(\Omega)$. In (5.13) and (5.15), we can choose $\eta, M, M' > 0$ appropriately such that $\underline{u} \leq u_0 \leq \overline{u}$ (since $u_0 \in \mathcal{C}$). Using $-l \leq f(x,y) \leq \mu y + l$ and applying Lemma 3.1 iteratively, we can get $\underline{u} \leq u^k \leq \overline{u}$, for all k. We remark that it is clear from the definition in (5.13) that $\underline{u}$ and $\overline{u}$ are independent of Δ_t. Let u_{Δ_t} and $\tilde{u}_{\Delta_t}$ be as defined in (4.3) alongwith the assumption that $u_{\Delta_t}(t) = u_0$, when $t < 0$. Then it is easy to see that (4.4) is satisfied with $h_{\Delta_t}(t,x) := f(x, u_{\Delta_t}(t - \Delta_t, x))$, for $t \in [0,T]$ and $x \in \Omega$. Using (4.7), we have $\underline{u} \leq u_{\Delta_t} \leq \overline{u}$. Therefore,

$$h_{\Delta_t}(t,x) \leq \mu u_{\Delta_t}(t - \Delta_t, x) + l \in L^\infty(\Lambda_T)$$

independent of Δ_t. Hence we can use similar techniques as in the proof of Theorem 2.7 to get

$$\begin{aligned} &u_{\Delta_t}, \tilde{u}_{\Delta_t} \in L^\infty([0,T]; X_0(\Omega) \cap \mathcal{C}),\ u_{\Delta_t}, \tilde{u}_{\Delta_t} \in L^\infty(\Lambda_T), \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \in L^2(\Lambda_T), \\ &\|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{L^2(\Omega)} \leq C(\Delta_t)^{\frac{1}{2}} \text{ and } \frac{1}{(u_{\Delta_t})^q} \in L^\infty([0,T]; (X_0(\Omega))^*) \end{aligned} \tag{6.2}$$

uniformly in Δ_t. So we can use the Banach Alaoglu theorem and (6.2) to get $u \in L^\infty([0,T]; X_0(\Omega))$ and $u \in L^\infty(\Lambda_T)$ such that, upto a subsequence,

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} L^\infty([0,T]; X_0(\Omega)) \text{ and in } L^\infty(\Lambda_T),\ \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(\Lambda_T) \tag{6.3}$$

as $\Delta_t \to 0^+$. Also similar to the proof of Theorem 2.7, we get

$$u_{\Delta_t},\ \tilde{u}_{\Delta_t} \to u \text{ in } L^\infty([0,T]; L^2(\Omega)) \text{ and } u \in C([0,T]; L^2(\Omega)). \tag{6.4}$$

In addition, if $M > 0$ denotes the Lipschitz constant for f then for $t \in [0,T]$

$$\begin{aligned} \|h_{\Delta_t}(t,\cdot) - f(\cdot, u(t,\cdot))\|_{L^2(\Omega)} &= \|f(\cdot, u_{\Delta_t}(t - \Delta_t, \cdot)) - f(\cdot, u(t,\cdot))\|_{L^2(\Omega)} \\ &\leq M \|u_{\Delta_t}(t - \Delta_t, \cdot) - u(t,\cdot)\|_{L^2(\Omega)}. \end{aligned} \tag{6.5}$$

From (6.4) and (6.5), we deduce that $h_{\Delta_t}(t,x) \to f(x, u(x))$ in $L^\infty([0,T]; L^2(\Omega))$. Finally, following exactly the last part of the proof of Theorem 2.7, we can show that $u \in \mathcal{A}(\Lambda_T)$ and u is a weak solution of (P_t^s).

It remains to prove the uniqueness. For that, let $v \in \mathcal{A}(\Lambda_T)$ be another weak solution of (P_t^s). For fix $t_0 \in [0,T]$, we have

$$\int_0^{t_0} \int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)\ dxdt + \int_0^{t_0} \int_{\mathbb{R}^n} ((-\Delta)^s (u-v))(u-v)\ dxdt$$

$$\begin{aligned} &- \int_0^{t_0} \int_\Omega \left(\frac{1}{u^q} - \frac{1}{v^q} \right) (u - v)\ dxdt \\ &= \int_0^{t_0} \int_\Omega (f(x, u(x) - f(x, v(x))))(u - v)\ dxdt. \end{aligned} \tag{6.6}$$

From (6.6), $u(0, x) = v(0, x) = u_0(x)$ in Ω and f being locally Lipschitz uniformly in Ω, we get

$$\begin{aligned} &\frac{1}{2} \|(u - v)(t_0)\|_{L^2(\Omega)} + \int_0^{t_0} \int_{\mathbb{R}^n} ((-\Delta)^s (u - v))(u - v)\ dxdt \\ &- \int_0^{t_0} \int_\Omega \left(\frac{1}{u^q} - \frac{1}{v^q} \right) (u - v)\ dxdt \leq M \int_0^{t_0} \int_\Omega |u - v|^2\ dxdt, \end{aligned} \tag{6.7}$$

where M is Lipschitz constant for f. From Lemma 3.1, we know that the operator A is strictly monotone which gives

$$\begin{aligned} 0 < &\int_0^{t_0} \int_\Omega |(u - v)|^2\ dxdt + \int_0^{t_0} \int_{\mathbb{R}^n} ((-\Delta)^s (u - v))(u - v)\ dxdt \\ &- \int_0^{t_0} \int_\Omega \left(\frac{1}{u^q} - \frac{1}{v^q} \right) (u - v)\ dxdt. \end{aligned}$$

Using this with (6.7), we get

$$\frac{1}{2} \|(u - v)(t_0)\|_{L^2(\Omega)} \leq M_0 \int_0^{t_0} \int_\Omega |u - v|^2\ dxdt,$$

where $M_0 > 0$ is a constant. By Gronwall's inequality, we get $\|(u-v)(t_0, \cdot)\|_{L^2(\Omega)} \leq \|(u-v)(0, \cdot)\|_{L^2(\Omega)} \exp(M_0 t_0)$. Since $u(0, \cdot) = v(0, \cdot)$ and this holds for all $t_0 \in [0, T]$, we get $u \equiv v$. This completes the proof. □

Now we give the proof of Proposition 2.

Proof of Proposition 2. Using Proposition 1 above and following the proof of Proposition 0.2 of [6], the result can be similarly obtained. □

7. **Asymptotic behavior.**

In this section, we present the proof of Theorem 2.10.

Proof of Theorem 2.10. Let $\underline{u}, \overline{u} \in \mathcal{C} \cap X_0(\Omega) \cap C_0(\overline{\Omega})$ be the sub and supersolution respectively of

$$\begin{cases} (-\Delta)^s u - \dfrac{1}{u^q} = f(x, u) \text{ in } \Omega, \\ \qquad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{7.1}$$

where $\underline{u}, \overline{u}$ is defined in (5.13). We can choose $\eta > 0$ small enough and $M > 0$ large enough so that $\underline{u} \leq u_0 \leq \overline{u}$ which is possible because we took $u_0 \in \mathcal{C} \cap X_0(\Omega)$. Let u be the solution of (P_t^s) and v_1 and v_2 be unique solutions of (P_t^s) with initial datum $\underline{u}$ and $\overline{u}$. The existence of v_1 and v_2 are justified through Theorem 2.9. We claim that $\underline{u}, \overline{u} \in \overline{\mathcal{D}(L)}^{L^\infty(\Omega)}$. Let $g, h \in (X_0(\Omega))^*$ be functions such that $L(\underline{u}) = g$ and $L(\overline{u}) = h$. Using (5.14), we have $g \leq 0$ and $h \geq 0$. Now, let $\{g_k\} = \max\{g, -k\}$, $\{h_k\} = \min\{h, k\}$ and $\{u_k\}, \{w_k\}$ be two sequences in $\mathcal{D}(L)$ defined by $L(u_k) = g_k$, $L(w_k) = h_k$. Since L is a monotone operator, as Lemma 3.1 we can show a similar kind of weak comparison principle concerning L. Using that, we can get $\{u_k\}$ is non increasing while $\{w_k\}$ is non decreasing. By definition of g_k, h_k, we can show that $g_k \to g$ and $h_k \to h$ in $(X_0(\Omega))^*$ as $k \to \infty$. This implies that $u_k \to \underline{u}$ and $w_k \to \overline{u}$ in $X_0(\Omega)$ as $k \to \infty$. Therefore, upto a subsequence, $u_k \to \underline{u}$ and $w_k \to \overline{u}$

pointwise a.e. in Ω as $k \to \infty$. Using Dini's theorem, we get $u_k \to \underline{u}$ and $w_k \to \overline{u}$ in $L^\infty(\Omega)$ as $k \to \infty$. This proves our claim.

Now we can use Theorem 2.9 and Proposition 2 to obtain $v_1, v_2 \in C([0,T]; C_0(\overline{\Omega}))$. Taking $\underline{u}^0 = \underline{u}$(respectively $\overline{u}^0 = \overline{u}$), we consider the sequence $\{\underline{u}^k\}$(respectively $\{\overline{u}^k\}$) which is non decreasing(respectively non increasing) as solutions to the iterative scheme given by (6.1), for $0 < \Delta_t < 1/M$ where M denotes the Lipschitz constant of f on $[\underline{u}, \overline{u}]$. If the sequence $\{u^k\}$ denotes the one that is obtained in (6.1), then by the choice of Δ_t we can show that

$$\underline{u}^k \leq u^k \leq \overline{u}^k. \tag{7.2}$$

Let u denotes the weak solution of (P_t^s) as obtained in the proof of Theorem 2.9. We follow the proof of Theorem 2.9 and use (7.2) to obtain

$$v_1(t) \leq u(t) \leq v_2(t). \tag{7.3}$$

Consider the maps $t \mapsto v_1(t,x)$ and $t \mapsto v_2(t,x)$ which are non decreasing and non increasing respectively. Assume $v_1(t) \to \tilde{v}_1$ and $v_2(t) \to \tilde{v}_2$ as $t \to \infty$. Now let $S(t)$ denotes the semigroup on $L^\infty(\Omega)$ generated by the given evolution equation $u_t + \lambda L(u) = f(x,u)$. Then we know

$$\tilde{v}_1 = \lim_{t' \to +\infty} S(t'+t)(\underline{u}) = S(t) \lim_{t' \to +\infty} S(t')(\underline{u}) = S(t) \lim_{t' \to +\infty} v_1(t') = S(t)\tilde{v}_1$$

and analogously, we obtain

$$\tilde{v}_2 = S(t)\tilde{v}_1.$$

Then $\tilde{v}_1$ and $\tilde{v}_2$ are stationary solutions of (P_t^s) i.e. solves (Q^s). But by uniqueness of solution to (Q^s) as shown in Theorem 2.8, we get $\tilde{v}_1 = \tilde{v}_2 = \hat{u} \in C(\overline{\Omega})$. Therefore, by Dini's theorem we get

$$v_1(t) \to \hat{u} \text{ and } v_2(t) \to \hat{u} \text{ in } L^\infty(\Omega) \text{ as } t \to \infty.$$

Using (7.3), we conclude that $u(t) \to \hat{u}$ in $L^\infty(\Omega)$ as $t \to \infty$. □

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