

Distinguishing environment-induced non-Markovianity from subsystem dynamics

Subhashish Banerjee,^{1,*} Javid Naikoo,^{1,†} R. Srikanth^{2,‡}

¹*Indian Institute of Technology Jodhpur, Jodhpur 342011, India*

²*Poornaprajna Institute of Scientific Research, Bangalore - 562164, India*

Quantum non-Markovianity modifies the environmental decoherence of a system. This situation is enriched in complex systems owing to interactions among subsystems. We consider the problem of distinguishing the multiple sources of non-Markovianity using a simple power spectrum technique, applied to a qubit interacting with another qubit via a Jaynes-Cummings type Hamiltonian and simultaneously subjected to some well known noise channels, such as, the random telegraph noise and non-Markovian amplitude damping, which exhibit both Markovian as well as non-Markovian dynamics under different parameter ranges.

I. INTRODUCTION

Any practical implementation of quantum information processing demands taking into account the effects of the ambient environment, resulting in the phenomena of decoherence and dissipation [1–6]. Until recently, noise was predominantly studied in the *Markovian* (memoryless) regime, where the environmental time scale is much smaller than the system time scale [7, 8], entailing the two weak conditions: (a) the monotonic fall of the distinguishability $D(\mathcal{E}[\rho_1], \mathcal{E}[\rho_2])$ of two states ρ_j , where \mathcal{E} is the noise superoperator; (b) the CP-divisibility of the noise dynamics, meaning that the intermediate map remains completely positive, essentially because the system and bath remain in an approximately product state during the evolution. It is known that (a) implies (b). Recently, [9] have argued that these two conditions are equivalent for bijective maps.

With a breakdown of condition (a), there is an increase in the distinguishability D , causing a recurrence or “back-flow” of information back from the environment into the system. With a breakdown of condition (b), the intermediate map is non-CP (NCP), essentially because the system-bath interaction generates system-bath entanglement. With the advancement of technology, one is now able to experimentally go beyond Markovian phenomena and enter into the non-Markovian regime [7, 10–16].

A complex dynamics, in general, will have a number of evolution patterns, in the form of different frequency components from different sources, superimposed to produce the resulting dynamics. This was seen in earlier works [17, 18] in the context of discrete time quantum walk (DTQW). There, the patterns superimposed on the position dynamics of the walker were the evolution due to its interaction degree of freedom traced out and noise due to an external source. Here, we elucidate the idea of disambiguating different sources of noise using a different model, viz., a qubit system evolved by a Jaynes Cummings type of evolution [19, 20] and a noise channel, such as the random telegraph noise (RTN) channel [21, 22], non-Markovian amplitude damping (NMAD) channel, and a concatenation of these channels. This could be very pertinent in complex as well as engineered systems where this scenario could be envisaged due to interaction among subsystems. Keeping this in mind, we consider a composite dynamics different from that studied in [17], in order to better illustrate the frequency-based noise disambiguation technique. The idea here is to associate the spectral power associated with different frequency components as a measure of their relative strengths of non-Markovianity (CP-indivisibility).

II. NON-MARKOVIAN NOISE

Random Telegraph Noise (RTN): We briefly describe the features of RTN noise, needed for our purpose. The random variable describing the noise fluctuation will be denoted by $\Omega(t)$ and by M the mean. The autocorrelation functions and the corresponding Kraus operators K_i are as follows:

$$M[\Omega(t), \Omega(s)] = a^2 e^{-|t-s|/\tau}, \quad K_1 = \sqrt{\frac{1 + \Lambda(t)}{2}} I, \quad K_2 = \sqrt{\frac{1 - \Lambda(t)}{2}} \sigma_3, \quad (1)$$

*Electronic address: subhashish@iitj.ac.in

†Electronic address: naikoo.1@iitj.ac.in

‡Electronic address: srik@poornaprajna.org

where

$$\Lambda(t) = e^{-\gamma t} \left[\cos \left(\left[\sqrt{\left(\frac{2a}{\gamma}\right)^2 - 1} \right] \gamma t \right) + \frac{\sin \left(\left[\sqrt{\left(\frac{2a}{\gamma}\right)^2 - 1} \right] \gamma t \right)}{\sqrt{\left(\frac{2a}{\gamma}\right)^2 - 1}} \right], \quad (2)$$

represents the damped harmonic function, which encodes both the Markovian and non-Markovian behaviour of the noise. The reduced dynamics of a qubit subjected to RTN can be described by map $\mathcal{R} : \rho(t) = \mathcal{R}(t)[\rho(0)] = \sum_i K_i(t)\rho(0)K_i^\dagger(t)$, with the Kraus operators as given in Eq. (1).

In RTN, the function $\Lambda(t)$ has two regimes; the purely damping regime, where $a/\gamma < 0.5$, and damped oscillations for $a/\gamma > 0.5$. Corresponding to these regimes of $\Lambda(t)$, one observes Markovian and non-Markovian behavior, respectively.

Non-Markovian Amplitude Damping (NMAD): This model describes the dissipative interaction between a qubit and zero temperature Bosonic reservoir, which could be envisaged to be composed of a large number of harmonic oscillators. The channel \mathcal{A}_t acts on a general input state $\rho = (\rho_{11}, \rho_{12}; \rho_{21}, \rho_{22})$ as:

$$\mathcal{A}_t(\rho) = \begin{pmatrix} 1 - |G(t)|^2 \rho_{22} & G(t)\rho_{12} \\ G^*(t)\rho_{12} & |G(t)|^2 \rho_{22} \end{pmatrix}. \quad (3)$$

The dynamics can be described by the following Kraus operators:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & G(t) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{1 - |G(t)|^2} \\ 0 & 0 \end{pmatrix}. \quad (4)$$

The reservoir spectral density can be assumed to be a Lorentzian $J(\omega) = \frac{\gamma_M \lambda^2}{2\pi(\omega - \omega_c)^2 + \lambda^2}$, with width λ , peak frequency ω_c and γ_M an effective coupling constant. The function $G(t)$ becomes

$$G(t) = e^{-(\lambda - i\delta)t/2} \left[\cosh(\Omega t/2) + \frac{\lambda - i\delta}{\Omega} \sinh(\Omega t/2) \right]. \quad (5)$$

Here, $\Omega = \sqrt{\lambda^2 - 2i\delta\lambda - 4w^2}$, $w = \gamma_M \lambda/2$, $\delta = \omega_0 - \omega_c$, and ω_0 is the natural frequency corresponding to the input qubit state. The condition $\lambda/\gamma_M \gg 1$ corresponds to the flat and weakly coupled spectrum and hence pertains to the Markovian dynamics. In other words, the dynamics is non-Markovian for $\lambda/\gamma_M \ll 1$.

Under a Markovian evolution \mathcal{E} , given two distinct states ρ_1 and ρ_2 , distance measures \mathfrak{D} (such as relative entropy or trace distance) satisfy $\mathfrak{D}[\mathcal{E}(\rho), \mathcal{E}(\sigma)] \leq \mathfrak{D}[\rho, \sigma]$, while correlation measures \mathfrak{C} (such as fidelity or mutual information with a third system) satisfy $\mathfrak{C}[\mathcal{E}(\rho), \mathcal{E}(\sigma)] \geq \mathfrak{C}[\rho, \sigma]$. By contrast, non-Markovian dynamics can violate the above monotonicity property, resulting in information backflow from the environment into the system, manifesting as recurrence. In this work, we will use trace-distance (TD) applied to a simple model as the indicator of distinguishability of two states. Non-Markovian backflow has been also been studied using fidelity [23], relative entropy [24] and mutual information.

III. NON-MARKOVIAN WITNESSES: CP DIVISIBILITY VS. DISTINGUISHABILITY

As noted, non-Markovianity may be indicated by observing the backflow in the evolution [25, 26] or departure from CP-divisibility of the intermediate map connecting the density operators $\rho(t_2)$ and $\rho(t_1)$ at times t_2 and t_1 , with $t_2 > t_1$ [7, 10]. Both methods are discussed here and found to be equivalent for the noise models considered.

A. CP-divisibility criterion

Given a dynamical map $\mathcal{E}(t_2, t_0)$ linking a system's density operator at times t_0 and $t_2 > t_0$, the intermediate map $\mathcal{E}^{\text{IM}}(t_2, t_1)$ for some intermediate time t_1 such that $t_2 > t_1 > t_0$, is given by:

$$\mathcal{E}^{\text{IM}}(t_2, t_1) = \mathcal{E}(t_2, t_0)\mathcal{E}^{-1}(t_1, t_0). \quad (6)$$

provided that the inverse map $\mathcal{E}^{-1}(t_1, t_0)$ exists. The Choi matrix for the intermediate map can be obtained as:

$$M_{\text{Choi}} = (\mathcal{E}^{\text{IM}}(t_2, t_1) \otimes \mathbb{1}) |\Phi^+\rangle \langle \Phi^+|, \quad (7)$$

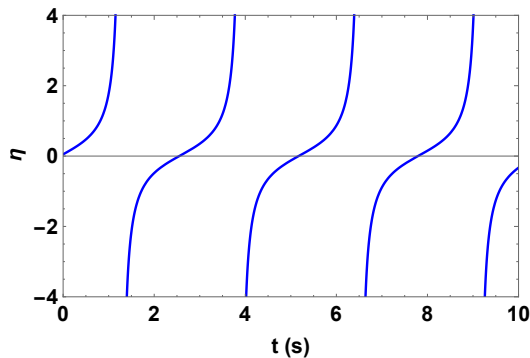


Figure 1: Plot of decoherence rate η for RTN with $\gamma = 0.05$ and $a = 0.6$ (non-Markovian regime). The decoherence rate takes both positive and negative values.

where $|\Phi^+\rangle \equiv |00\rangle + |11\rangle$. Following the method presented in [7], we briefly derive the intermediate dynamical maps of the RTN noisy channel, given in Eq. 1. Now, the dephasing master equation in its canonical form [12, 27] is

$$\dot{\rho} = \eta(-\rho + \sigma_3 \rho \sigma_3), \quad (8)$$

where η is the decoherence rate. It is known that a map is CP-divisible if and only if the map has only positive decoherence rate [28]. From the Kraus operator expressions in Eq. 1, we obtain $\rho \rightarrow \mathcal{E}(\rho)$. In particular, as the RTN noise is dephasing, we find:

$$\rho_0 \equiv \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \rightarrow \rho(t) = \begin{pmatrix} \rho_{00} & \lambda \rho_{01} \\ \lambda \rho_{10} & \rho_{11} \end{pmatrix}. \quad (9)$$

For this dephasing map, the decoherence rate is found by direct substitution in Eq. (8) to be:

$$\eta = -\frac{\dot{\lambda}}{2\lambda}. \quad (10)$$

For the channel under consideration $\lambda = \Lambda(t)$, which, in view of Eq. (2), yield the decoherence rate for this channel, which is plotted in Fig. (1). We note that RTN shows negative decoherence rates for certain intervals.

B. Distinguishability criterion

We will consider the composite dynamics comprising of the effect of RTN or NMAD on an otherwise unitary two-qubit dynamics governed by Jayne-Cummings (JC)-like Hamiltonian.

Trace Distance (TD) [29] is a measure of distinguishability between two states, defined as $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} \|\rho_1 - \rho_2\|$, where $\|A\|$ is the operator norm given by $\sqrt{A^\dagger A}$. It has been used as a measure of non-Markovianity to quantify the amount of backflow from the environment to the system. For the noise channel given in Eq. 1, and initial states $|\pm\rangle$, it is straightforward to compute the evolution of TD, which is $D(|+\rangle, |-\rangle) = \Lambda(t)$ for RTN. Thus, we find that the distinguishability criterion, like the divisibility criterion, is able to witness the non-Markovianity of RTN, because the process is P-indivisible. A similar conclusion can be made for the NMAD channel, discussed above. We now consider the case where RTN or NMAD is applied to a qubit subject to another non-Markovian noise. Our study will then focus on how to distinguish the non-Markovian signatures of these two composite processes through a simple frequency analysis.

IV. A SIMPLE MODEL

Consider a system of two qubits labeled as A and B , where A is the system and B serves as environment. At time $t = 0$, qubit B is prepared in the state $|\psi_B\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, whilst qubit A starts in state $|0\rangle$ or $|1\rangle$. The two qubits interact via the Jaynes-Cummings (JC) -like Hamiltonian [19]

$$H = \omega \left(|01\rangle\langle 10| + |10\rangle\langle 01| \right). \quad (11)$$

The reduced dynamics of qubit A can be described by map $\mathcal{J}(t) : \rho_A(t) = \mathcal{J}[\rho_A(0)] = \sum_i J_i(t)\rho_A(0)J_i^\dagger(t)$, with the following Kraus operators

$$J_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{i \sin(t\omega)}{\sqrt{2}} & \frac{\cos(t\omega)}{\sqrt{2}} \end{pmatrix}, \quad J_2 = \begin{pmatrix} \frac{\cos(t\omega)}{\sqrt{2}} & -\frac{i \sin(t\omega)}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (12)$$

The time evolved reduced state of A at time t , depending on whether it started in $|0\rangle$ or $|1\rangle$ at time $t = 0$, is respectively given by:

$$\rho_A^I(t) = \begin{pmatrix} \frac{1}{2} \cos^2(t\omega) + \frac{1}{2} & \frac{1}{2} i \sin(t\omega) \\ -\frac{1}{2} i \sin(t\omega) & \frac{1}{2} \sin^2(t\omega) \end{pmatrix}, \quad (13)$$

$$\rho_A^{II}(t) = \begin{pmatrix} \frac{1}{2} \sin^2(t\omega) & -\frac{1}{2} i \sin(t\omega) \\ \frac{1}{2} i \sin(t\omega) & \frac{1}{2} \cos^2(t\omega) + \frac{1}{2} \end{pmatrix}. \quad (14)$$

We find the trace distance $\mathcal{D}(\rho_A^I(t), \rho_A^{II}(t)) = \frac{1}{2} \sum_i |\lambda_i|$. Here λ_i are the eigenvalues of the matrix $\rho_A^I(t) - \rho_A^{II}(t)$. We obtain

$$\mathcal{D} = \frac{\sqrt{7 + \cos(4\omega t)}}{2\sqrt{2}} \quad (15)$$

The time evolution of qubit A during the interval $(0, t)$ is described by a composite map $\mathcal{R}(t) \circ \mathcal{J}(t)$, where $\mathcal{R}(t)$ denotes the RTN or NMAD channel, while $\mathcal{J}(t)$ indicates the dynamics generated by Eq. (12). We have

$$\sigma_A^\alpha(t) = (\mathcal{R}(t) \circ \mathcal{J}(t))\rho_A(0), \quad (16)$$

with $\alpha \in \{I, II\}$.

The trace distance between these (Eq. (16)) states, for RTN, turns out to be

$$\mathcal{D} = \left| \frac{\sqrt{4\Lambda^2 - 4(\Lambda^2 - 1)\cos(2t\omega) + \cos(4t\omega) + 3}}{2\sqrt{2}} \right|. \quad (17)$$

Note that the *noiseless* case pertains to $\Lambda = 1$, and the above expression reduces to Eq. (15).

The form of the channel parameter Λ is given by Eq. (2) such that $\frac{2a}{\gamma} > 1$ and $\frac{2a}{\gamma} < 1$ pertain to the non-Markovian and Markovian cases, respectively. Figure (2) depicts the variation of the trace distance in different scenarios.

Similar analysis performed with the NMAD channel, Eqs. (3), (4), leads to following expression for trace distance

$$\begin{aligned} \mathcal{D} &= \frac{1}{2} \left| \frac{1}{4} (1 - G^2 - 2 \cos(2\omega t) [-1 + G^2]) \right. \\ &\quad \left. - \frac{1}{2} \sqrt{\cos^4(\omega t) [-1 + G^2]^2 + \frac{G}{2} [(7 + \cos(4\omega t))G + 8[-1 + G^2] \sin^2(\omega t)]} \right| \\ &\quad + \frac{1}{2} \left| \frac{1}{4} (1 - G^2 - 2 \cos(2\omega t) [-1 + G^2]) \right. \\ &\quad \left. + \frac{1}{2} \sqrt{\cos^4(\omega t) [-1 + G^2]^2 + \frac{G}{2} [(7 + \cos(4\omega t))G + 8[-1 + G^2] \sin^2(\omega t)]} \right|. \quad (18) \end{aligned}$$

For $G = 1$, the Kraus operators, given in Eq. (4), become $A_1 = I$, $A_2 = 0$, and we recover the noiseless Eq. (15).

V. DISAMBIGUATION OF MULTIPLE NON-MARKOVIAN EFFECTS

Here, we concern ourselves with the situation when non-Markovianity can have multiple sources of noise. In [17], one is an internal source (the coin degree of freedom) and the other an external source (external environment). For the system of two qubits driven by a JC-type of Hamiltonian, considered above, the oscillatory behavior of the trace distance with respect to time, as given in Eq. (15), reflects the non-Markovian character of the the dynamics. On

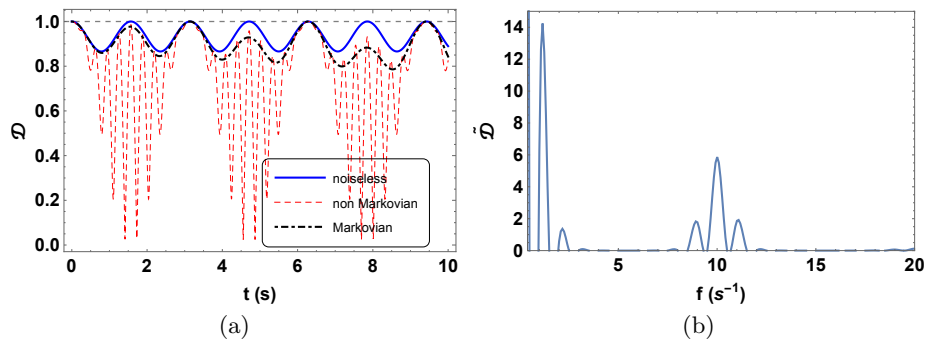


Figure 2: (a) Trace distance \mathcal{D} as defined in Eq. (17) for noiseless case $\Lambda = 1$ (blue curve), for non-Markovian RTN case with $a = 5$, $\gamma = 0.009$, and for Markovian case $a = 0.2$, $\gamma = 5$. In all the cases, the frequency ω which comes from the unitary evolution is set to one. (b) The Fourier transform $\tilde{\mathcal{D}}$ of trace distance function \mathcal{D} , defined by Eq. (19), in non-Markovian regime. The dominating contribution to the frequency comes from $f = 1$ from the noiseless dynamics and $f = 0.09$ and 10 from RTN.

top of this, we subject the system to the RTN or NMAD noise. Consequently, the trace distance, given by Eq. (17), now acquires a dependence on two frequency components, one from the subsystem dynamics and the other from the environmental noise, as depicted in Figs. 2 (a) and 3 (a), for the RTN and NMAD cases, respectively.

The different source contributions to non-Markovian dynamics can be demarcated by using simple Discrete Fourier Transform (DFT). The trace distance, Eq. (17) and (18), apart from the frequency ω , which comes from dynamics governed by JC-like two qubit Hamiltonian, also depends on the memory kernels Λ of RTN and G of the NMAD noise, respectively. The form of Λ , as given in Eq. (2), reveals the dependence on two frequencies γ and $\gamma\sqrt{(2a/\gamma)^2 - 1}$. Figures 2 (b), 3 (b), depict the DFT of the trace distance defined in Eq. (17) and (18), respectively, which brings out a reasonably clear separation of the different contributing factors. The corresponding power spectrum is given by $|\tilde{\mathcal{D}}(f)|^2$, where $\tilde{\mathcal{D}}$ is the Fourier transform of the $\mathcal{D}(t)$, given by

$$\tilde{\mathcal{D}}(f) = \int \mathcal{D}(t)e^{-2\pi ift} dt, \quad (19)$$

implemented numerically. As the two noise contributions (JC-like dynamics and RTN or NMAD) occur at different frequencies, the TD spectral power at those frequencies can be directly read off as representing the relative strength of their contribution to the non-Markovianity. The power spectral plot in Figure 2 (b) shows that the dominant contribution is seen to be at the frequency of the RTN with $\gamma = 0.009$, $\gamma\sqrt{(2a/\gamma)^2 - 1} = 10$, with a lower contribution due to JC-like dynamics with $\omega = 1$ (again, identified by the associated frequency). Similarly, when the system is subjected to NMAD channel, one finds two dominating contributions to the frequency spectrum, at $(\lambda - i\delta)/2 = 0.3$ and $\Omega/2 = \frac{1}{2}\sqrt{\lambda^2 - 2i\delta\lambda - 4w^2} = 30$ coming from NMAD dynamics, and around $\omega = 1$ due to noiseless dynamics. Further, the frequency $f = 15$ is also seen to contribute actively, which may be a consequence of the complicated structure of the memory kernel $G(t)$, Eq. (5).

A more complicated scenario: We now consider a more complex situation wherein the qubit state ρ is acted upon by the composition of RTN and NMAD channels as

$$\rho' = \sum_{\mu,\nu=1,2} K_\mu A_\nu \rho A_\nu^\dagger K_\mu^\dagger. \quad (20)$$

Here, K_μ and A_ν are defined in Eqs. (1), and (4), respectively. Trace distance is a useful witness for non-Markovian dynamics, as seen above in case of RTN and NMAD. Here, we analyze the effect of memory due to the composite map on the coherence of a qubit state. We use the l_1 norm definition of coherence given by the sum of absolute values of the off-diagonal elements of density matrix, i.e., $C = \sum_{i \neq j} |\rho_{ij}|$ [14, 30, 31]. With initial state $\rho = |+\rangle \langle +|$ subjected to the composite map, Eq. (20), the coherence of output state turns out to be

$$C = \Lambda(t)G(t), \quad (21)$$

where $\Lambda(t)$ and $G(t)$ are defined in Eqs. (2), and (5), respectively. Thus, coherence is given by the product of two memory kernels, capturing the non-Markovian features of both the channels. In Fig. (4) is depicted the coherence and its Fourier transform. The frequencies $f = 0.1$, and 10 come from RTN and $f = 0.3$ and 30 from NMAD dynamics. A clear disambiguation of the effect of two channels is observed. Further, with fundamental frequency of $f_0 = 15$, one observes higher harmonics at nf_0 , ($n = 2, 3, \dots$). This invites further investigation in more complicated scenarios.

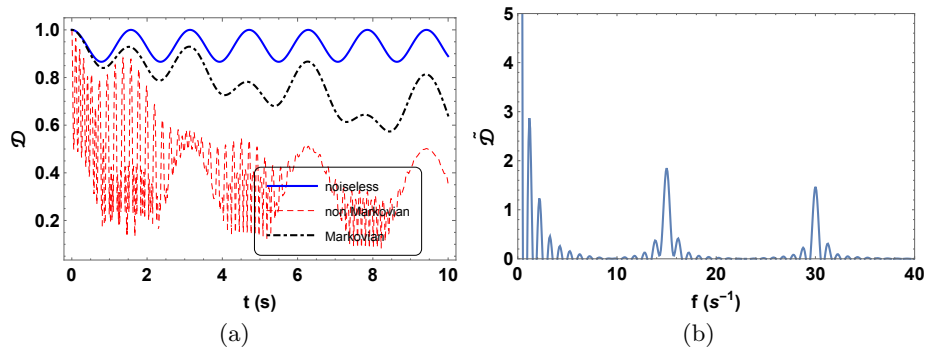


Figure 3: (a) Trace distance \mathcal{D} as defined in Eq. (18) for noiseless case $G(t) = 1$ (blue curve), for Markovian AD case with $\lambda = 10, \gamma_M = 0.1$, and for non-Markovian case $\lambda = 0.6, \gamma_M = 100$. In all the cases, the frequency ω which comes from the unitary evolution is set to one. (b) The Fourier transform $\tilde{\mathcal{D}}$ of trace distance function \mathcal{D} defined by Eq. (18). The frequencies $f = 0.3$ and 30 correspond to NMAD, and $f = 1$ to the noiseless dynamics.

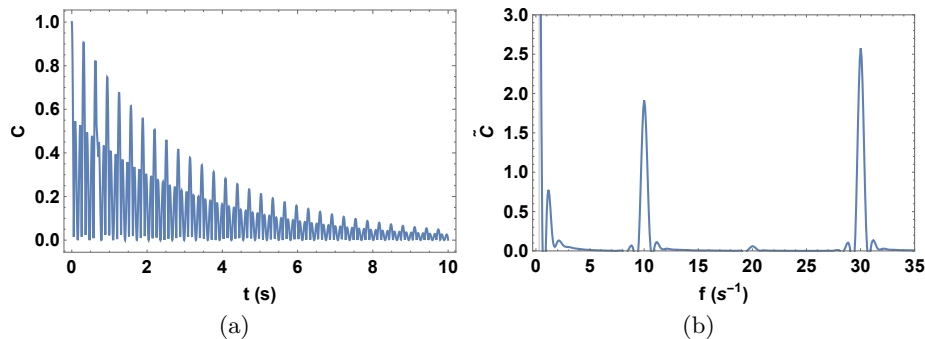


Figure 4: Depicting coherence C as defined in Eq. (21) (a) and its Fourier transform \tilde{C} (b), when a qubit is subjected to dynamics generated by composition of RTN and NMAD channels both operating in non-Markovian regime. The various parameters used are the same as in Figs. (2) and (3). The dominating contribution to the frequency due to RTN is at $f = 0.01$, and 10 , and from NMAD at $f = 0.3$ and 30 .

VI. CONCLUSION

Quantum non-Markovianity is a fundamental memory effect in open system dynamics. A simple manifestation thereof is a departure from the monotonic fall of distinguishability between states, or of the CP-divisibility of the system's evolution. In complex systems, subsystem dynamics can compound this effect by having noisy contributions due to different factors. Here, we studied the non-Markovian effects on the evolution of a qubit subjected to two types of noise: one due to a JC-type interaction with a second qubit, and another due to RTN (NMAD) channel.

Both noisy channels are P-indivisible, and thus the distinguishability criterion is sufficient to study their effect. We derived the trace distance dependence of the system's evolution on characteristic frequencies for the JC-type dynamics induced noise and RTN (NMAD) channel. Their corresponding frequencies can be demarcated using simple discrete Fourier transform, distinguishing non-Markovian contributions to the dynamics coming from the JC-like dynamics source and the external (environment) source. The technique is also seen to work in the more complicated case of a qubit subjected to a concatenation of RTN and NMAD noise channels, both realistic noises arising via interactions with proper baths. In the later case, a fundamental frequency and its harmonics are found to occur in the power spectrum which invites for further investigation in more complicated scenarios. They may present challenges that would make the unraveling of the time-scales harder. It would be interesting to extend the present work to incorporate such cases.

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