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Topological dynamics on hyperspaces

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Abstract. In this paper we wish to relate the dynamics of the base map to the dynamics of the induced map. In the process, we obtain conditions on the endowed hyperspace topology under which the chaotic behaviour of the map on the base space is inherited by the induced map on the hyperspace. Several of the known results come up as corollaries to our results. We also discuss some metric related dynamical properties on the hyperspace that cannot be deduced for the base dynamics.

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1. Introduction

For past many years, use of topological methods to study the chaotic nature embroiled in dynamical systems has been of wide interest. Also, most of the dynamics observed seems to be collective phenomenon emerging out of many segregated components. This leads to the belief that most of these systems are collective (set valued) dynamics of many units of individual systems. Hence, arises the need of a topological treatment of such collective dynamics.

Some recent studies of dynamical systems, in branches of engineering and physical sciences, have revealed that the underlying dynamics is set valued or collective, instead of the normal individual kind which is usually studied (c.f. [10, 16, 18, 19]). With these illustrations of collective dynamics, some natural questions arise. What is the significance of the underlying topology for any kind of join of dynamics? If the individual dynamics of each unit as well as the topological details are known, what kind of collective dynamics will the combination display? Given a topological structure, how can the dynamical

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behaviour of a unit influence the collective behaviour? Similarly given a collective behaviour, under proper topological framework, what can be concluded about the individual dynamics of a specific unit?

So far, each of these questions is open. However, in this paper, we try to answer a part of the last two questions. We try to investigate the relation between individual dynamics and the induced dynamics on the "hyperspace" (i.e. set valued dynamics), considering all possible topological framework.

We derive relation between properties like dense periodicity, topological transitivity, weakly mixing and topologically mixing, existence of horseshoe, of the map on the base space with that of the induced map on the hyperspace. We also derive conditions on the topology of the hyperspace for these properties to be equivalent at both places. We also discuss the relation between some of the metric dependent properties like equicontinuity, sensitivity, expansivity, existence of scrambled sets etc. on the base space and the hyperspace.

Since our work is a convolution of 'topological dynamics' and 'hyperspace theory' we separately give some preliminaries on both these topics which we shall subsequently use, before a brief survey of the work done till now in this direction and thence our contribution.

1.1. **Dynamical systems.** By a dynamical system, we mean a pair (X, f) where X is a topological (metric) space and f is any continuous self map on X. We study the behavior of each point $x \in X$ under repeated actions of f.

A point $x \in X$ is called *periodic* if $f^n(x) = x$ for some positive integer n, where $f^n = f \circ f \circ f \circ \ldots \circ f$ (n times). The least such n is called the *period* of the point x. A map f is called *transitive* if for any pair of non-empty open sets U, V in X, there exist a positive integer n such that $f^n(U) \bigcap V \neq \phi$. A map f is called *weakly mixing* if for any pairs of non-empty open sets U_1, U_2 and V_1, V_2 in X, there exists $n \in \mathbb{N}$ such that $f^n(U_i) \bigcap V_i \neq \phi$ for $i = 1, 2$. It is known that for any continuous self map f, if f is weakly mixing and $U_1, U_2, \ldots U_n, V_1, V_2, \ldots V_n$ are non-empty open sets, then there exists a $k \geq 1$ such that $f^k(U_i) \bigcap V_i \neq \phi$ for $i = 1, 2, \ldots n$. A map f is called mixing or *topologically mixing* if for each pair of non-empty open sets U, V in X, there exists a positive integer k such that $f^n(U) \bigcap V \neq \phi$ for all $n \geq k$.

We now define the notion of *topological entropy.*

Let X be a compact space and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a finite subcover. Let $\mathcal{L}$ be the collection of all finite subcovers and let $\mathcal{U}^*$ be the subcover with minimum cardinality, say $N_{\mathcal{U}}$. Define $H(\mathcal{U}) = logN_{\mathcal{U}}$. Then $H(\mathcal{U})$ is defined as the *entropy* associated with the open cover $\mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are two open covers of X, define, $\mathcal{U} \vee \mathcal{V} = \{U \bigcap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. For a self map f on X, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is also an open cover of X. Define,

$$h_{f,\mathcal{U}} = \lim_{n\to\infty} \frac{H(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee f^{-2}(\mathcal{U}) \vee \ldots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

Then $\sup h_{f,\mathcal{U}}$, where $\mathcal{U}$ runs over all possible open covers of X is known as the *topological entropy of the map* f and is denoted by $h(f)$.

We say that X contains a *topological horseshoe* if there is a compact set Q in X such that $f(Q) = Q$ and $f|_Q$ factors over the shift on M symbols for some $M > 1$. In other words, we say that f has a *M-horseshoe* in X for $M > 1$; if we have a compact set Q with $Q = \bigcup_{i=1}^{M} Q_i$ such that $Q_i \bigcap Q_j = \phi$ and $\bigcup_{i=1}^{M} Q_i \subseteq \bigcap_{i=1}^{M} f(Q_i)$ with each Q_i compact.

A self map f on a metric space (X, d) is said to be *equicontinuous at a point* $x \in X$ if for each $\epsilon > 0$, there exists $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$, $y \in X$. The map f is called *equicontinuous* if it is equicontinuous at every point of X. A map f is said to be *uniformly equicontinuous* if for each $\epsilon > 0$, there exists $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$, $x, y \in X$. For $x \in X$, if there exists a $\delta > 0$ such that for each $\epsilon > 0$ there exists $y \in X$ and a positive integer n such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$, then f is said to be *sensitive* at x. If f is sensitive at each point $x \in X$, f has *sensitive dependence on initial conditions* or is simply called *sensitive.* A map f is called *δ-expansive* if for any pair of distinct elements $x, y \in X$, there exists $k \in \mathbb{Z}^+$ such that $d(f^k(x), f^k(y)) > \delta$. A set S is called *scrambled* for f if for any $x, y \in S$, $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ but $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. A system (X, f) is called *Li-Yorke chaotic* if there exists an uncountable scrambled set. See [4, 5, 6, 9, 12] for details.

While defining different dynamical notions, we observe that each such notion defined involves the topology of the space X. We shall consider only those topologies on X for which the self map f remains continuous. As the topology is made finer(or coarser), various dynamical properties behave differently. Some properties evolve as the topology is made finer(coarser), while some vanish. For example, if the periodic points are dense for the dynamical system (X, f) (when X is given topology τ), then they are dense when X is endowed with any coarser topology. However, denseness of periodic points may not be preserved when X is endowed with a finer topology. Similarly, if (X, f) is transitive, weakly mixing or topologically mixing, then the properties are preserved in any coarser topology. However, like previously, clouds of uncertainty prevail if the topology is made finer. One of the properties which may be preserved if we make the topology finer is the topological entropy. If we take a finer topology, we increase the number of open sets and hence the number of open covers. Thus, the topological entropy increases as the supremum is taken over a larger set.

However, for metrizable spaces, some dynamical properties like sensitive dependence on initial conditions, Li-Yorke sensitivity, existence of a Li-Yorke pair etc. depend only on the underlying metric, rather than the generated topology and hence are not preserved under some other equivalent metric.

1.2. **Hyperspace topologies.** For a Hausdorff space (X, τ), a hyperspace (Ψ, Δ) comprises of a subfamily Ψ of all non-empty closed subsets of X endowed with the topology Δ, where the topology Δ is generated using the topology τ of X. The set Ψ may either comprise of all compact subsets of X, or all compact and connected subsets of X or all closed subsets of X. A hyperspace topology is called admissible if the map $x \to \{x\}$ is continuous. The topology Δ can be generated in several ways, however, we are interested in only those topologies Δ that are admissible. More generally, once Ψ and Δ are fixed, the space (Ψ, Δ) is called the hyperspace of the space (X, τ). Let,

$CL(X) = \{E \subseteq X :$ E is closed and non empty $\}$

$\mathcal{F}$(X) $=\{E \in CL(X) :$ E is finite $\}$

$\mathcal{F}_n$(X) $=\{E \in CL(X) : |E| = n\}$

$\mathcal{K}$(X) $= \{E \in CL(X) :$ E is compact in $X\}$

$\mathcal{K}_{\mathcal{C}}$(X) $= \{E \in CL(X) :$ E is compact and connected in $X\}$

$E^- = \{A \in CL(X) : A \bigcap E \neq \phi\}$

$E^+ = \{A \in CL(X) : A \subseteq E\}$

$E^{++} = \{A \in CL(X) : \exists\ \epsilon\ \geq 0$ and $S_\epsilon(A) \subseteq E\}$

where $S_\epsilon(A) = \bigcup\limits_{a \in A} S(a, \epsilon)$, where $S(a, \epsilon) = \{x \in X : d(a, x) < \epsilon\}$

Some of the standard hyperspace topologies are:

Let I be a finite index set and for all such I, let $\{U_i : i \in I\}$ be a collection of open subsets of X. Define for each such collection of open sets,

$\langle U_i \rangle_{i \in I} = \{E \in CL(X) : E \subseteq \bigcup_{i \in I} U_i$ and E $\bigcap U_i \neq \phi\ \forall i\}$

The topology generated by such collections is known as the *Vietoris topology.*

Let (X, d) be a metric space. For any two closed subsets A, B of X, define,

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq S_\epsilon(B) \text{ and } B \subseteq S_\epsilon(A)\}$$

It is easily seen that d_H defined above is a metric on $CL(X)$ and is called *Hausdorff metric* on $CL(X)$. This metric preserves the metric on X, i.e. $d_H(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$. The topology generated by this metric is known as the *Hausdorff metric topology* on $CL(X)$ with respect to the metric d on X.

It is known that the Hausdorff metric topology equals the Vietoris topology if and only if the space X is compact.

Let Φ be a subfamily of the collection of all non-empty closed subsets of X. The *Hit and Miss topology* determined by the collection Φ is the topology having subbasic open sets of the form U^- where U is open in X and $(E^c)^+$ with $E \in \Phi$. As a terminology, U is called the hit set and any member E of Φ is referred as the miss set. It has been proved that any topology on the hyperspace is of this type [14].

A typical member of the base for the *Lower Vietoris topology* on the hyperspace $CL(X)$ consists of the set, each of whose elements intersect or *hit* finitely many open sets U, i.e. a typical basic open set is the intersection of finitely many U^-. The Lower Vietoris topology is the smallest topology on the hyperspace containing all the sets U^- where U is open in X.

A typical basic open set for the *Upper Vietoris topology* on the hyperspace $CL(X)$ is of the form U^+ where U is open in X. Thus, given a closed set C, a typical member of the base in the Upper Vietoris topology is the set whose elements are the elements of the hyperspace disjoint from the closed set C.

The Vietoris topology equals the join of Upper Vietoris and Lower Vietoris topology, and is infact an example of a hit and miss topology.

Let (X, d) be a metric space.

For each element x in X we define a function d_x as:

$$d_x : CL(X) \longrightarrow \mathbb{R}$$
$$\text{such that } d_x(A) = d(x, A)$$

The *Wijsman topology* determined on $CL(X)$ is the weak topology determined by the family $\{d_x : x \in X\}$, i.e. the smallest topology on $CL(X)$ for which the family of above defined functions is continuous. It is the topology generated by the sets of the form $\{A \in CL(X) : d(x, A) < \alpha\}$ and $\{A \in CL(X) : d(x, A) > \alpha\}$, where x varies in X and $\alpha \geq 0$ varies in $\mathbb{R}$.

For a metric space (X, d) and a given collection Φ of closed subsets of X, the *Hit and Far Miss topology* determined by the collection Φ is the topology having subbasic open sets of the form U^- where U is open in X and $(E^c)^{++}$ with $E \in \Phi$.

Here the collection hits each open set U and far misses the complement of each member of Φ and hence forms a hit and far miss topology.

If we replace the family $\{U_k\}$ of hit sets in Lower Vietoris topology by pairwise disjoint family of open balls $S(x_k, \epsilon)$, then the topology thus obtained is known as the *Lower Discrete topology.*

It can be seen that Lower Discrete topology is coarser than the Hausdorff metric topology. Further, the topology determines the hit sets for the Hausdorff metric topology. Infact, the Hausdorff metric topology is the join of upper Vietoris topology and lower discrete topology.

The *ball proximal topology* on $CL(X)$ is defined by the collection of sets V^- and $(B^c)^{++}$ where V is an open subset of X and B is a closed ball.

In most cases, the Wijsman topology is same as the Ball Proximal topology.

In short, each topology Δ on the hyperspace is either hit and miss or hit and far miss type. And for some of the main hyperspace topologies, this can be briefly summarized as:

Topology	Hit Sets	Miss Sets	Far Miss Sets
Lower Vietoris	open sets	-	-
Upper Vietoris	-	closed sets	-
Vietoris	open sets	closed sets	-
Fell	open sets	compact sets	-
Wijsman	open sets	-	closed balls
Hausdorff Metric	discrete open balls	-	closed sets

It may be noted that if X is compact, each of the topologies Wijsman, Vietoris, Fell, Hausdorff and Ball Proximal coincide and hence are equal.

Again, as shown in [14], every hyperspace topology is of this type only, hence all topologies Δ on $\Psi \subseteq CL(X)$ can be obtained in this way. So for any $\Psi \subseteq CL(X)$ we can thus talk of the hyperspace (Ψ, Δ).

Let $f : X \to X$ be a continuous function. Then for such an f, there is a naturally induced map $\hat{f}$ on the hyperspace of all nonempty closed subsets of X defined as,

$$\hat{f} : CL(X) \to CL(X)$$
$$\text{such that } \hat{f}(K) = \overline{f(K)} = \overline{\{f(k) : k \in K\}}$$

where $\overline{A}$ denotes the closure of the set A.

If f is a closed map then this induced map can also be given as,

$$\overline{f} : CL(X) \to CL(X)$$
$$\text{such that } \overline{f}(K) = f(K) = \{f(k) : k \in K\}$$

However, in each of the above cases, the continuity of the induced maps $\overline{f}$ or $\hat{f}$ is not guaranteed by the continuity of the map f. It is very well known that if $CL(X)$ is assigned the Vietoris topology or the Hausdorff metric topology and f is a continuous self map on X, then $\hat{f} : CL(X) \to CL(X)$ is always continuous.

It can be seen that, for a closed map f,

$$\begin{aligned}
\overline{f}^{-1}(U^-) &= \{\overline{f}^{-1}(A) : A \in U^-\} \\
&= \{B \in CL(X) : f(B) = A, A \cap U \neq \phi\} \\
&= \{B \in CL(X) : f(B) \cap U \neq \phi\} \\
&= \{B \in CL(X) : B \cap f^{-1}(U) \neq \phi\} \\
&= (f^{-1}(U))^-
\end{aligned}$$

$$\begin{aligned}
\overline{f}^{-1}(U^+) &= \{\overline{f}^{-1}(A) : A \in U^+\} \\
&= \{B \in CL(X) : f(B) = A, A \subset U\} \\
&= \{B \in CL(X) : f(B) \subset U\} \\
&= \{B \in CL(X) : B \subset f^{-1}(U)\} \\
&= (f^{-1}(U))^+
\end{aligned}$$

This gives rise to the following theorem:

Theorem 1.1. *Let X and Y be topological spaces and let $CL(X)$ and $CL(Y)$) be the induced hyperspaces with collections of miss sets as Δ_X and Δ_Y respectively. A continuous and closed map $f : X \to Y$ induces a mapping $\overline{f} : CL(X) \to CL(Y)$. Then $\overline{f}$ is continuous if and only if $f^{-1}(A) \in \Delta_X$ for all $A \in \Delta_Y$.*

However, in general, $\overline{f}^{-1}(U^{++}) \neq (f^{-1}(U))^{++}$, as shown in the below example.

Example 1.2. Define f on each interval of the form $[2n-1, 2n]$ as,

$$f(x) = \begin{cases}
\frac{4}{n}x + n + \frac{3}{n} - 8, & 2n-1 \leq x \leq 2n - \frac{7}{8}; \\
n - \frac{1}{2n}, & 2n - \frac{7}{8} \leq x \leq 2n - \frac{3}{4}; \\
\frac{2}{n}x + n + \frac{1}{n} - 4, & 2n - \frac{3}{4} \leq x \leq 2n - \frac{1}{4}; \\
n + \frac{1}{2n}, & 2n - \frac{1}{4} \leq x \leq 2n - \frac{1}{8}; \\
\frac{4}{n}x + n + \frac{1}{n} - 8, & 2n - \frac{1}{8} \leq x \leq 2n;
\end{cases}$$

Define f in a suitable way on $[2n, 2n+1]$ so that the extended function remains continuous. Let g denote the extended function. Then $g : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $A = \bigcup_{n=1}^{\infty} [2n - \frac{3}{4}, 2n - \frac{1}{4}]$ and $U = \bigcup_{n=1}^{\infty} (n - \frac{1}{n}, n + \frac{1}{n})$

It is clear that images of A and $S_{\frac{1}{16}}(A)$ are same. Also, these images are contained in U. Thus, $A \in (g^{-1}(U))^{++}$.

However, $A \notin \overline{g}^{-1}(U^{++})$ as $\overline{g}(A) = \bigcup_{n=1}^{\infty} (n - \frac{1}{2n}, n + \frac{1}{2n})$ and any δ ball around $\overline{g}(A)$ cannot be contained in U.

On the other hand, a similar result does not hold when we consider the induced map $\hat{f}$.

It can be seen that,

$$\begin{aligned} \hat{f}^{-1}(U^-) &= \{\hat{f}^{-1}(A) : A \in U^-\} \\ &= \{B \in CL(X) : \overline{f(B)} = A, A \cap U \neq \phi\} \\ &= \{B \in CL(X) : \overline{f(B)} \cap U \neq \phi\} \\ &= \{B \in CL(X) : f(B) \cap U \neq \phi\} \\ &= \{B \in CL(X) : B \cap f^{-1}(U) \neq \phi\} \\ &= (f^{-1}(U))^- \end{aligned}$$

$$\begin{aligned} \hat{f}^{-1}(U^+) &= \{\hat{f}^{-1}(A) : A \in U^+\} \\ &= \{B \in \Psi : \overline{f(B)} = A, A \subset U\} \\ &= \{B \in \Psi : \overline{f(B)} \subset U\} \\ &\subset \{B \in \Psi : B \subset f^{-1}(U)\} \\ &= (f^{-1}(U))^+ \end{aligned}$$

Also, by giving an example similar to the previous one, it can be shown that, $\hat{f}^{-1}(U^{++}) \neq (f^{-1}(U))^{++}$ in general.

Thus, the relations, $\hat{f}^{-1}(U^+) = (f^{-1}(U))^+$ and $\hat{f}^{-1}(U^{++}) = (f^{-1}(U))^{++}$ may not hold in general. Conditions under which the induced map $\hat{f}$ can be continuous is still an open problem for investigation. However, some such conditions ensuring the continuity of $\hat{f}$ for various topologies on $CL(X)$ has been beautifully described in [7].

In this article, we shall consider only those hyperspace topologies under which the induced functions remain continuous.

See [2, 7, 8, 13, 14] for details.

1.3. **Recent results.** In recent years, some relations between the individual dynamics of a system and its collective dynamics have been studied. In [15], Heriberto Roman-Flores proved that for a metric space (X, d), for the hyperspace $\mathcal{K}(X)$, under Hausdorff metric topology, transitivity of the map $\overline{f}$ on the hyperspace implies transitivity of the map f on the base space. Further, he demonstrated by an illustration that transitivity of f need not imply transitivity of $\overline{f}$. In [1], Banks improved the result by proving that if the hyperspace $\mathcal{K}(X)$ is endowed with Vietoris topology, the transitivity of the map $\overline{f}$ is infact equivalent to weakly mixing of the map f. In [11], Dominik Kwietnaik and

Piotr Oprocha proved that positive topological entropy of the map $\overline{f}$ need not imply positive topological entropy for the map f. We briefly summarize their results.

Proposition 1.3 ([15]). *For continuous $f : X \to X$ and $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$, $\overline{f}$ transitive implies f transitive.*

Proposition 1.4 ([15]). *Let $f : X \to X$ and $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ be continuous maps. Then the following conditions are equivalent*

1. *f is transitive in (X, d).*
2. *$\overline{f}$ is transitive in the w^e topology.*

Remark 1.5. We note here that the w^e topology mentioned in [15] is actually the upper Vietoris topology.

Let $\mathcal{H} \subseteq CL(X)$ and let f be a continuous self map admissible with $\mathcal{H}$. Then f induces another self map on $\mathcal{H}$, denoted by $\overline{f}_{\mathcal{H}}$.

Let $\overline{f} = \overline{f}_{\mathcal{H}}$

Proposition 1.6 ([1]). *Suppose $\mathcal{H}$ is dense in $CL(X)$. Then the following are equivalent.*

1. *f is weakly mixing.*
2. *$\overline{f}$ is weakly mixing.*
3. *$\overline{f}$ is transitive.*

Proposition 1.7 ([1]). *Suppose $\mathcal{H}$ is dense in $CL(X)$. Then f is mixing if and only if $\overline{f}$ is mixing.*

Proposition 1.8 ([1]). *Suppose $\mathcal{F}(X) \subseteq \mathcal{H}$. If f has a dense set of periodic points, then so does $\overline{f}$.*

Example 1.9 ([11]). Let $\sum_2$ be the sequence space of all bi-infinite sequences of two symbols 0 and 1. For any two sequences $x = (x_i)$ and $y = (y_i)$, define

$$d(x, y) = \sum_{i=-\infty}^{\infty} \tfrac{|x_i - y_i|}{2^{|i|}}$$

It is easily seen that the metric d generates the product topology on $\sum_2$.

Let $S \subset \sum_2$ be the set of all bi-infinite sequences for which the symbol 1 occurs at most once.

$$\sigma : \textstyle\sum_2 \to \sum_2$$

$$\sigma(\ldots x_{-2}x_{-1}.x_0x_1\ldots) = \ldots x_{-2}x_{-1}x_0.x_1x_2x_3\ldots$$

The map σ is known as the shift map and is continuous with respect to the metric d defined.

Denote σ_S for $\sigma|_S$, the restriction of the shift map to S. Let $\overline{\sigma_S}$ denote the induced map on $\mathcal{K}(S)$. Then, topological entropy of σ_S, $h(\sigma_S) = 0$ but $h(\overline{\sigma}_S) = log2$.

2. Main results

Let (X, f) be a dynamical system. Let $\Psi \subseteq CL(X)$ be a collection admissible with the map f, i.e. $f(\Psi) \subseteq \Psi$. The topology Δ on Ψ is either a Hit and Miss or a Hit and Far Miss topology. We note that (X, d) is a metric space in case we consider a hit and far miss topology.

Further, we consider only those topologies Δ on Ψ with respect to which $\overline{f} : \Psi \to \Psi$ is continuous. Now the original dynamical system (X, f) induces another dynamical system $(\Psi, \overline{f})$.

We will be dealing with the question: Given a topological framework for a base space X and its associated hyperspace Ψ, how does the dynamics on one space effect the dynamics of the other. That is, under what conditions properties like dense set of periodic points transitivity, weakly mixing etc. in one space imply the same in the other. As the mentioned properties depend on the topology on Ψ, we shall derive suitable conditions on the topology Δ on Ψ for this to happen.

Proposition 2.1. *Let $\mathcal{F}(X) \subseteq \Psi$ and Ψ be endowed with any admissible hyperspace topology Δ. If (X, f) has dense set of periodic points, then so does $(\Psi, \overline{f})$.*

Proof. Let the topology Δ on the hyperspace Ψ be the hit and miss (or hit and far miss) topology determined by the collection $\mathcal{C}$. Let $\mathcal{U}$ be a non empty basic open set for the hyperspace (Ψ, Δ). Then $\mathcal{U}$ hits finitely many open sets, say $W_1, W_2, \ldots, W_n$ and misses(far misses) finitely many elements of $\mathcal{C}$ say, $T_1, T_2, \ldots, T_m$. Let $T = \bigcup T_j$. Thus each $V_i = W_i \bigcap T^c$ is non-empty, open in X. As periodic points of f are dense in X, each V_i contains a periodic point x_i of period k_i. As each x_i is periodic of period k_i, the set $\{x_1, x_2, ...x_n\}$ is periodic of period $r = lcm\{k_1, k_2, ...k_n\}$. Thus the point $\{x_1, x_2, ...x_n\} \in \mathcal{U}$ is a periodic point for $\overline{f}$. Hence the result holds. □

Remark 2.2. It can be noted that the above result holds for any hit and miss *or* hit and far miss topology. However, using denseness of periodic points, a periodic point generated in the hyperspace is a finite set and hence the condition $\mathcal{F}(X) \subseteq \Psi$ cannot be relaxed in the above proof. Again, as $\mathcal{F}(X) \nsubseteq \mathcal{K}_C(X)$, the condition $\mathcal{F}(X) \subseteq \Psi$ is not satisfied and hence the result is not true in this case, as noted in [1].

Also, the converse is not true. $(\Psi, \overline{f})$ may have a dense set of periodic points with (X, f) having no periodic point. An illustration for the same is given in [1].

Proposition 2.3. *If there exists a base β for the topology on X such that U^+ is non empty and $U^+ \in \Delta$ for every $U \in \beta$, then $\overline{f}$ transitive on Ψ implies that f is transitive on X.*

Proof. Let U and V be any two non-empty open sets in X. As β forms a base for topology on X, there exists $U_1, V_1 \in \beta$ such that $U_1 \subseteq U$ and $V_1 \subseteq V$. By given hypothesis, U_1^+ and V_1^+ are non empty open in the hyperspace Ψ. As $\overline{f}$ is transitive, there exists $n \in \mathbb{N}$ such that $\overline{f}^n(U_1^+) \bigcap V_1^+ \neq \phi$. As $U_1 \subseteq U$ and $V_1 \subseteq V$, f is transitive. □

Remark 2.4. For the above result to hold, it is firstly necessary that the set U^+ in (Ψ, Δ) is non-empty for any member U of the base. Since if $U^+ = \phi$ for some $U \in \beta$, the transitivity of $\overline{f}$ cannot be used to establish the transitivity of f. Secondly, it is sufficient to have sets of the form U^+ to be open in the hyperspace, for U open in X. This is basically the property of upper Vietoris topology. Thus, if the hyperspace Ψ contains all singletons and is endowed with any topology finer than the upper Vietoris topology, the transitivity of the induced map on the hyperspace ensures the transitivity of the base map on the space X. Since upper Vietoris topology is indeed coarser than topologies like Hausdorff metric topology and Vietoris topology, the result holds good in each of these topologies. However as known, when topologies like Wijsman topology, Ball proximal topology or Fell topology become finer than upper Vietoris topology, they actually coincide with the Vietoris topology. Hence the result may not hold for them in general.

Remark 2.5. Again, it can be seen that for the converse, the transitivity of f can guarantee transitivity of $\overline{f}$ when the hyperspace is endowed with upper Vietoris topology or any coarser hyperspace topology. Thus, if the hyperspace is endowed with upper Vietoris topology, the result holds in both directions and the transitivity of f is in fact equivalent to transitivity of $\overline{f}$ as also proved in [15].

Proposition 2.6. *Let $\mathcal{F}(X) \subseteq \Psi$. If f is weakly mixing, then so is $\overline{f}$. The converse holds if there exists a base β for topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.*

Proof. Let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$, $\mathcal{V}_2$ be non-empty open sets in the hyperspace such that $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$, $\mathcal{V}_2$ hits the open sets $W_{11}, W_{21}, \ldots W_{n_1 1}$; $W_{12}, W_{22}, \ldots W_{r_1 2}$; $R_{11}, R_{21}, \ldots R_{n_1 1}$ and $R_{12}, R_{22}, \ldots R_{r_1 2}$ and misses(far misses) the closed sets $T_{11}, T_{21}, \ldots T_{m_1 1}$; $T_{12}, T_{22}, \ldots T_{s_1 2}$; $S_{11}, S_{21}, \ldots S_{m_1 1}$ and $S_{12}, S_{22}, \ldots S_{s_1 2}$ respectively.

Let $T_i = \bigcup_j T_{ji}$. and $S_i = \bigcup_j S_{ji}$.

Let $M_j^i = W_{ji} \bigcap T_i^c$ and let $N_j^i = R_{ji} \bigcap S_i^c$.

Now, each of M_j^i, N_j^i are open sets and as f is weakly mixing, there exists $k \in \mathbb{N}$ such that $f^k(M_j^i) \bigcap N_j^i \neq \phi$, $\forall\ i, j$. Let $x_j^i \in M_j^i$ such that $f^k(x_j^i) \in N_j^i$. Then $X_i = \{x_j^i\}_j \in \mathcal{U}_i$ such that $\overline{f}^k(X_i) \in \mathcal{V}_i$. Hence $\overline{f}$ is weakly mixing.

Conversely, let U_1, U_2, V_1, V_2 be open in X. As β is the base for the topology on X, $\exists\ U_{11}, U_{22}, V_{11}, V_{22} \in \beta$ such that $U_{ii} \subseteq U_i$ and $V_{ii} \subseteq V_i$ for $i = 1, 2$. By given hypothesis, U_{ii}^+ and V_{ii}^+ are non-empty open in the hyperspace Ψ. Hence, there exists $n \in \mathbb{N}$ such that $\overline{f}^n(U_{ii}^+) \bigcap V_{ii}^+ \neq \phi$. Thus, f is weakly mixing. $\square$

Proposition 2.7. *Let $\mathcal{F}(X) \subseteq \Psi$. If f is topologically mixing, then so is $\overline{f}$. The converse holds if there exists a base β for topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.*

Proof. Let $\mathcal{U}$, $\mathcal{V}$ be two non-empty open sets in the hyperspace (Ψ, Δ). Let $\mathcal{U}$ and $\mathcal{V}$ hit $W_1, W_2, \dots W_r$; $R_1, R_2, \dots R_r$ and miss $T_1, T_2, \dots T_m$; $S_1, S_2, \dots S_m$ respectively.

Let $T = \bigcup_j T_j$ and $U_i = W_i \bigcap T^c$.

Let $S = \bigcup_j S_j$ and $V_i = R_i \bigcap S^c$.

It may be noted that as $\mathcal{U}$, $\mathcal{V}$ are non-empty, each U_i, V_i are also non-empty. Now as f is topological mixing, for each pair of non empty open sets U_i, V_i, we obtain $n_i \in \mathbb{N}$ such that $f^k(U_i) \bigcap V_i \neq \phi$, $\forall\ k \geq n_i$. Let $n = \max \{n_i : i = 1, 2, \dots r\}$. Then $\overline{f}^k(\mathcal{U}) \bigcap \mathcal{V} \neq \phi \quad \forall k \geq n$. Thus, $\overline{f}$ is topological mixing.

Conversely, let $\overline{f}$ be topological mixing. Let U and V be non empty open subsets of X. As β is the base for the topology on X, there exists $U_1, V_1 \in \beta$ such that $U_1 \subseteq U$ and $V_1 \subseteq V$. By given hypothesis, U_1^+ and V_1^+ are open. As $\overline{f}$ is topological mixing, $\exists\ n \in \mathbb{N}$ such that $\overline{f}^k(U_1^+) \bigcap V_1^+ \neq \phi$, $\forall k \geq n$ which implies that f is topological mixing. □

Remark 2.8. For both the above results, we note that in the forward part, we need $\mathcal{F}(X) \subseteq \Psi$. Once again, for the converse part, we need the sets U^+ to be non-empty open for any member U of the base.

Remark 2.9. It is clear from the above proof that topological mixing of f is equivalent to the topological mixing of $\overline{f}$ when the hyperspace Ψ contains $\mathcal{F}(X)$ and is endowed with upper Vietoris topology or any of the finer hyperspace topologies.

Remark 2.10. In each the above proofs, to prove the existence of a dynamical property on the hyperspace, a finite set has been generated. As for any finite set A, if $\hat{f}$ is continuous, since $\overline{f}(A) = \hat{f}(A)$, therefore the above results hold good for $\hat{f}$ also.

Proposition 2.11. *Let(X, f) be a dynamical system and let $(\Psi, \overline{f})$ be the induced dynamical system on the hyperspace. Then, the system $(\Psi, \overline{f})$ has a positive topological entropy need not imply the same for the system (X, f).*

Proof. We demonstrate the proof by giving a counterexample. We prove the result on the same lines as done in [11] for Vietoris topology.

Let $\sum_2$ be the sequence space of all bi-infinite sequences of two symbols 0 and 1. Let $S \subset \sum_2$ be the set of all bi-infinite sequences for which the symbol 1 occurs atmost once. Denote σ_S for $\sigma|_S$, the restriction of the shift map to S. Let $\overline{\sigma_S}$ denote the induced map on $\mathcal{K}(S)$. Let $\Psi = \mathcal{K}(S)$ be endowed with any hyperspace topology such that $\overline{\sigma_S}$ is continuous. Then, $h(\sigma_S) = 0$, but, $h(\overline{\sigma}_S) = log2$.

As S is countable, $h(\sigma_S) = 0$. Further, define,

$$\phi : \textstyle\sum_2 \to \mathcal{K}(S)$$
$$\phi(x = \dots x_{-2}x_{-1}.x_0x_1 \dots) = \overline{\{a_n : n \in A_x\}}$$

where $A_x = \{n \in \mathbb{Z} : x_n = 1\}$.

Then, ϕ is a continuous, onto function. Further, any $K \in \mathcal{K}(S)$ has atmost two preimages. Thus, ϕ is a uniformly finite-to-one semiconjugacy between the systems $(\sum_2, \sigma)$ and $(\mathcal{K}(S), \overline{\sigma}_S)$. Thus, $h(\overline{\sigma}_S) = h(\sigma) = log2$ (by [11]). □

Remark 2.12. In the above example, we correct the uniformly finite-to-one semiconjugacy given in [11]. Further, we establish the result for a general hit and miss(hit and far-miss) topology on the hyperspace and hence generalize the result given in [11].

Proposition 2.13. *Let $\mathcal{F}_1(X) \subseteq \Psi$. If (X, f) has a M-horseshoe, then so does $(\Psi, \overline{f})$.*

Proof. Let $S_1, S_2, \ldots S_M$ constitute a M-horseshoe for the system (X, f). Then, $\mathcal{K}(S_1) \bigcap \Psi, \mathcal{K}(S_2) \bigcap \Psi, \ldots \mathcal{K}(S_M) \bigcap \Psi$ constitutes a M-horseshoe for the system $(\Psi, \overline{f})$. □

We now give an example to show that the converse is not true, i.e. existence of a horseshoe for the induced system need not guarantee the same for the original system.

Example 2.14. Let $\sum_2$ be the sequence space of one sided infinite sequences of two symbols 0 and 1. For any two sequences $x = (x_i)$ and $y = (y_i)$, define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

The metric d generates the product topology on $\sum_2$. Let $S = \{a_n : n \in \mathbb{N}\} \bigcup \{a_0\}$, where $a_n \in \sum_2$ such that a_n is sequence with all 0 and a 1 only at the n-th place and a_0 is the sequence of all 0.

As any two infinite subsets of S intersect, there cannot exist a horseshoe for any self map f on S. We now define a self map f on S such that $(\mathcal{K}(S), \overline{f})$ has a 2-horseshoe. Define,

$$f(a_k) = \begin{cases} a_{2n-1}, & k = 4n-1; \\ a_{2n}, & k = 4n+1; \\ a_{n-1}, & k = 2n, n \neq 0; \\ a_0, & k = 0; \end{cases}$$

We claim that the induced system $(\mathcal{K}(S), \overline{f})$ has a 2-horseshoe.

Let $S_1 = \{a_1, a_3, a_5, \ldots a_{2n-1}, \ldots\}$ and $S_2 = \{a_2, a_4, a_6, \ldots a_{2n}, \ldots\}$.

Let $\mathcal{J}_1$ be a subset of the hyperspace with elements of the form $\{a_{n_k} : k \in \mathbb{N}\} \bigcup \{a_0\}$ where $n_1 = 1$ and each $a_{n_k} \in S_1$.

Similarly, let $\mathcal{J}_2$ be a subset of the hyperspace with elements of the form $\{a_{n_k} : k \in \mathbb{N}\} \bigcup \{a_0\}$ where $n_1 = 2$ and each $a_{n_k} \in S_2$.

Let $\mathcal{K}_1 = \overline{\mathcal{J}_1}$ and let $\mathcal{K}_2 = \overline{\mathcal{J}_2}$.

Then we claim that $\mathcal{K}_1, \mathcal{K}_2$ constitute a 2-horseshoe for the system $(\mathcal{K}(S), \overline{f})$.

Any element $\{a_{2n_k-1} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_1$ is image of $\{a_1\} \bigcup \{a_{4n_k-1} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_1$ and $\{a_2\} \bigcup \{a_{4n_k} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_2$.

Again, any element $\{a_{2n_k} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_2$ is image of $\{a_1\} \bigcup \{a_{4n_k+1} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_1$ and $\{a_2\} \bigcup \{a_{4n_k+2} : k \in \mathbb{N}\} \bigcup \{a_0\} \in \mathcal{K}_2$.

Also any other element of $\mathcal{K}_1$ is of the form $\{a_0, a_1, a_{2r_1-1}, a_{2r_2-1}, \dots a_{2r_k-1}\}$ which is image of $\{a_0, a_1, a_{4r_1-1}, a_{4r_2-1}, \dots a_{4r_k-1}\} \in \mathcal{K}_1$ and $\{a_0, a_2, a_{4r_1}, a_{4r_2}, \dots a_{4r_k}\} \in \mathcal{K}_2$.

Lastly, any other element of $\mathcal{K}_2$ is of the form $\{a_0, a_2, a_{2r_1}, a_{2r_2}, \dots a_{2r_k}\}$ which is image of $\{a_0, a_1, a_{4r_1+1}, a_{4r_2+1}, \dots a_{4r_k+1}\} \in \mathcal{K}_1$ and $\{a_0, a_2, a_{4r_1+2}, a_{4r_2+2}, \dots a_{4r_k+2}\} \in \mathcal{K}_2$.

Hence $\mathcal{K}_1 \bigcup \mathcal{K}_2 \subseteq \overline{f}(\mathcal{K}_1) \bigcap \overline{f}(\mathcal{K}_2)$.

Hence, $\mathcal{K}_1, \mathcal{K}_2$ is a 2-horseshoe for the system $(\mathcal{K}(S), \overline{f})$.

Hence, existence of a horseshoe for the induced system does not imply the same for the original system.

For a hyperspace $\Psi \subset CL(X)$, and a topology Δ finer than the upper Vietoris topology, we have the following lemma.

Lemma 2.15. *Let (X, τ) be a topological space and let (Ψ, Δ) be the induced hyperspace such that $\Psi \subseteq \mathcal{K}(X)$. Let Δ be finer than the upper Vietoris topology. Then,*

$$\mathcal{B} \in \mathcal{K}(\Psi, \Delta) \Rightarrow (\bigcup_{E \in \mathcal{B}} E) \in \mathcal{K}(X)$$

Proof. Let $A = \bigcup_{E \in \mathcal{B}} E$. Let $\bigcup_{i \in I} U_i$ be an open cover of A. Let $E \in \mathcal{B}$. Then, as each $E \in \mathcal{B}$ is also contained in A, $\bigcup_{i \in I} U_i$ is also an open cover of E. Thus there exists U_1^E, U_2^E, $\dots U_{n_E}^E$ such that $E \subseteq \bigcup_{i=1}^{n_E} U_{n_E}^E$ and thus $E \in < \bigcup_{i=1}^{n_E} U_{n_E}^E >$. Thus $\bigcup_{E \in \mathcal{B}} < \bigcup_{i=1}^{n_E} U_{n_E}^E >$ is an open cover of $\mathcal{B}$. As $\mathcal{B}$ is compact, there exists $E_1, E_2, \dots E_k$ such that $\mathcal{B} \subseteq \bigcup_{i=1}^{k} < \bigcup_{j=1}^{n_{E_i}} U_j^{E_i} >$. Hence $A \subseteq \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_{E_i}} U_j^{E_i}$ and thus has a finite subcover.

Hence A is compact.

□

Proposition 2.16. *Let $\mathcal{F}_1(X) \subseteq \Psi \subseteq \mathcal{K}(X)$ and let Δ be finer than the upper Vietoris topology. Then the following are equivalent:*

(1) *There exists compact sets $S_1, S_2, \dots S_M$ constituting a horseshoe for (X, f).*
(2) *There exists compact sets of the hyperspace $\mathcal{J}_1$, $\mathcal{J}_2$, $\dots \mathcal{J}_M$ constituting a M-horseshoe on the hyperspace such that the sets $Q_i = \bigcup_{K \in \mathcal{J}_i} K$ are pairwise disjoint.*

Proof. The proof of (1) $\implies$ (2) follows from Result 2.13.

For (2) $\implies$ (1) let $\mathcal{J}_1$, $\mathcal{J}_2$, $\dots \mathcal{J}_M$ constitute a M-horseshoe on the hyperspace. Then, the sets $Q_i = \bigcup_{K \in \mathcal{J}_i} K$ are compact by lemma 2.15 and thus form a horseshoe for the system (X, f). □

We now deal with metric related properties. As observed in the previous section, since these properties are highly metric dependent, we can only think of discussing these properties on the hyperspace Ψ with the Hausdorff metric, d_H. We recall here that the metric d_H on Ψ preserves the metric d on X.

Proposition 2.17. *Let $\mathcal{F}_1(X) \subseteq \Psi$. Then, $(\Psi, \overline{f})$ is equicontinuous $\Rightarrow$ (X, f) is equicontinuous.*

Proof. If $\overline{f}$ is equicontinuous, corresponding to $\epsilon > 0$ and $\{x\} \in \Psi$, there exists $\eta > 0$ such that $d_H(\{x\}, K) < \eta$ implies $d_H(\overline{f}^n(\{x\}), \overline{f}^n(K)) < \epsilon$ for all $n \in \mathbb{N}$, $K \in \Psi$. Thus, whenever $d(x, y) = d_H(\{x\}, \{y\}) < \eta$, $d_H(\overline{f}^n(\{x\}), \overline{f}^n(\{y\})) = d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$ and hence the result holds. □

Proposition 2.18. *Let $\mathcal{F}(X) \subseteq \Psi \subseteq \mathcal{K}(X)$. (X, f) is uniformly equicontinuous if and only if $(\Psi, \overline{f})$ is uniformly equicontinuous.*

Proof. Let $\epsilon > 0$ be given. As f is uniformly equicontinuous, corresponding to $\frac{\epsilon}{2} > 0$, there exists $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^n(x), f^n(y)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$, $x, y \in X$. Let $K \in \Psi$ be a non-empty compact subset of X. As K is compact, there exists a finite subset $\{x_1, x_2, \ldots x_r\}$ such that $K \subseteq \bigcup\limits_{i=1}^{r} S_{\frac{\eta}{4}}(x_i)$. Now, if $d_H(\{x_1, x_2, \ldots x_r\}, A) < \eta$, then it is clear that $d_H(f^n(\{x_1, x_2, \ldots x_r\}), f^n(A)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ and $A \in \Psi$. Thus, by triangle inequality, for any compact set K^* such that $d_H(K, K^*) < \frac{\eta}{4}$, $d_H(\{x_1, x_2, \ldots x_r\}, K^*) < \frac{\eta}{2} < \eta$ and hence $d_H(f^n(\{x_1, x_2, \ldots x_r\}), f^n(K^*)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. Thus, $d_H(f^n(K), f^n(K^*)) < \epsilon$ for all $n \in \mathbb{N}$.

Conversely, when $\overline{f}$ is uniformly equicontinuous, corresponding to $\epsilon > 0$, there exists $\eta > 0$, such that $d_H(\{x\}, \{y\}) = d(x, y) < \eta$ implies $d_H(\overline{f}^n(\{x\}), \overline{f}^n(\{y\})) = d(f^n(x), f^n(y)) < \epsilon$ and hence the result holds. □

Proposition 2.19. *(X, f) is uniformly equicontinuous if and only if $(CL(X), \overline{f})$ is uniformly equicontinuous.*

Proof. Let $\epsilon > 0$ be given. Then, as f is uniformly equicontinuous, corresponding to $\frac{\epsilon}{2} > 0$, there exists $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^n(x), f^n(y)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$, $x, y \in X$. Let K be a non-empty closed subset of X. For $\alpha > 0$, an α-discrete subset of X is the set such that for any two distinct elements x, y in the set, $d(x, y) \geq \alpha$. Let A be a $\frac{\eta}{4}$ discrete subset such that $d_H(K, A) < \frac{\eta}{4}$. Now, for any non-empty closed set B with $d_H(A, B) < \eta$, $d_H(f^n(A), f^n(B)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. Thus, for any closed set K^* such that $d_H(K, K^*) < \frac{\eta}{4}$, $d_H(A, K^*) < \frac{\eta}{2} < \eta$ and hence $d_H(f^n(A), f^n(K^*)) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. Hence, $d_H(f^n(K), f^n(K^*)) < \epsilon$ for all $n \in \mathbb{N}$.

Conversely, if $\overline{f}$ is uniformly equicontinuous, corresponding to $\epsilon > 0$, there exists $\eta > 0$, such that $d_H(\{x\}, \{y\}) = d(x, y) < \eta$ implies $d_H(\overline{f}^n(\{x\}), \overline{f}^n(\{y\})) = d(f^n(x), f^n(y)) < \epsilon$ and hence the result holds. □

Remark 2.20. The equivalence of the dynamical property of uniform equicontinuity on the base space and the hyperspace holds for any $\Psi \subseteq \mathcal{K}(X)$ and for $CL(X)$. However, the result may not hold for any $\Psi \subset CL(X)$ as the existence of an appropriate α-discrete set in the desired neighborhood cannot be guaranteed.

Proposition 2.21. *Let $\mathcal{F}(X) \subseteq \Psi \subseteq \mathcal{K}(X)$. (X, f) is equicontinuous $\Rightarrow$ $(\Psi, \overline{f})$ is almost equicontinuous.*

Proof. Let (X, f) be equicontinuous and let $\epsilon > 0$ be given. To establish our claim, we prove that every finite set is a point of equicontinuity for the map $\overline{f}$.

Let $A = \{x_1, x_2, \ldots, x_r\}$ be a finite set in the hyperspace. Thus, corresponding to $\epsilon > 0$, by equicontinuity of f at x_i, there exists $\eta_i > 0$ such that $d(x_i, y) < \eta_i$ implies $d(f^n(x_i), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$, $y \in X$. Let $\eta =$min $\{\eta_i : i = 1, 2, \ldots, r\}$. Then, for $K \in \Psi$ with $d_H(A, K) < \eta$, $d_H(f^n(A), f^n(K^*)) < \epsilon$ for all $n \in \mathbb{N}$. Hence the system $(\Psi, \overline{f})$ is almost equicontinuous. □

Remark 2.22. From the above result, we can infer that, if (X, f) has no points of sensitivity, then, Ψ(as above) has a dense set on which $\overline{f}$ is not sensitive.

However, the case of sensitivity is a bit involved. It is observed that sensitivity of f on (X, d) implies sensitivity of $\overline{f}$ for some (Ψ, d_H) but not for any Ψ in general. The converse also holds true for some restricted cases. Such results are discussed in [17].

Stronger than sensitivity is the property of expansivity.

Proposition 2.23. *Let $\mathcal{F}_1(X) \subseteq \Psi \subseteq CL(X)$. Then, $(\Psi, \overline{f})$ is δ-expansive implies (X, f) is δ-expansive.*

Proof. Let $(\Psi, \overline{f})$ be δ-expansive and let $x, y \in X$. Then, as $\overline{f}$ is expansive, for $\{x\}, \{y\} \in \Psi$, there exists $k \in \mathbb{Z}^+$ such that

$d(f^k(x), f^k(y)) = d_H(\overline{f}^k(\{x\}), \overline{f}^k(\{y\})) \geq \delta.$

Thus, (X, f) is also δ-expansive. □

The converse does not hold true in general. We provide an example to show that the converse is not true.

Example 2.24. Let $\sum_2$ be the sequence space of two symbols 0 and 1 and let $\mathcal{K}(\sum_2)$ be the hyperspace of all non empty compact subsets of $\sum_2$. It can be easily observed that $(\sum_2, \sigma)$ is expansive with expansivity constant $\frac{1}{2}$. However, we prove that the system $(\mathcal{K}(\sum_2), \overline{\sigma})$ is not expansive.

Let if possible, $(\mathcal{K}(\sum_2), \overline{\sigma})$ be expansive with expansivity constant δ. Let $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \delta$. Let $S_1 = \{0^k 1^r 0^\infty : k \geq 0, r \leq n\}$ and let $S_2 = \{0^k 1^r 0^\infty : k \geq 0, r \leq n+1\}$. Then, $d_H(S_1, S_2) = \frac{1}{2^{n+1}}$,

Also, $\overline{\sigma}(S_i) = S_i$, $i = 1, 2$.

Thus, for any $k \in \mathbb{N}$, $d_H((\overline{\sigma}^k(S_1), \overline{\sigma}^k(S_2) = d_H(S_1, S_2) = \frac{1}{2^{n+1}} < \delta$ which is a contradiction.

Thus, the system $(\mathcal{K}(\sum_2), \overline{\sigma})$ is not expansive.

Proposition 2.25. *Let Ψ contain the set of all singletons. If (X, f) is Li-Yorke chaotic, so is $(\Psi, \overline{f})$.*

Proof. Let $\{a_\lambda : \lambda \in \Lambda\}$ be an uncountable scrambled set in X. Then, $\{\{a_\lambda\} : \lambda \in \Lambda\}$ is the desired scrambled set in the hyperspace. □

As observed in [11], there is a dynamical system (X, f) with zero topological entropy but $(\mathcal{K}(X), \overline{f})$ has positive topological entropy. As proved in [3], if any system (X, f) has positive topological entropy, then it is Li-Yorke chaotic. Thus, it can be conduced that there can exist a system (X, f) which is not Li-Yorke chaotic but its induced counterpart on some hyperspace is Li-Yorke chaotic. Taking an example similar to that of [11], we now show that the existence of a scrambled set for the map $\overline{f}$ on the hyperspace need not guarantee the same for the map f on the base space X.

Example 2.26. Let $\sum_2$ be the sequence space of one sided infinite sequences of two symbols 0 and 1.

Let $S = \{a_n : n \in \mathbb{N}\} \bigcup \{a\}$, where $a_n \in \sum_2$ such that a_n is a sequence with 1 only at the n-th place and 0 at all other places and let a be the sequence of all 0's. It can be seen that the set S is compact in $\sum_2$ and is invariant under σ. Hence we can talk of $\sigma : S \to S$.

It can be seen that under iterative application of the map σ, any point a_n reaches a in finitely many steps. For any $n_1, n_2 \in \mathbb{N}$, if $n = \max\{n_1, n_2\}$, then $f^n(a_{n_1}) = f^n(a_{n_2}) = a$. Thus, there exists no scrambled set for the map σ on S.

We, however show the existence of an uncountable scrambled set for $(\mathcal{K}(S), \overline{\sigma})$.

Let $(A_n) = (2, 3, 5, 9, 17, \ldots) = (2^{n-1}+1)_{n\in\mathbb{N}}$ and $(B_n) = (3, 4, 6, 10, 18, \ldots) = (2^{n-1} + 2)_{n\in\mathbb{N}}$ be two fixed sequences of natural numbers.

Consider the collection $\mathcal{P}$ of all subsets of $\sum$ such that any two distinct sequences in any set in $\mathcal{P}$ differs at infinitely many places. Then, the collection $\mathcal{P}$ of all such sets is a poset under the usual set inclusion. Let $\mathcal{A}$ be its maximal element.

We show that $\mathcal{A}$ is uncountable. Any sequence $(x_n) \in \mathcal{A}^c$ eventually coincides with some sequence in $\mathcal{A}$. If not so, then the sequence (x_n) differs from every sequence in $\mathcal{A}$ at infinitely many places. Thus, $\mathcal{A} \bigcup \{(x_n)\}$ violates the maximality of $\mathcal{A}$ and thus any sequence in $\mathcal{A}^c$ eventually coincides with some sequence in $\mathcal{A}$. For any sequence (y_n), the number of sequences eventually coinciding with (y_n) are countable. Thus, if $\mathcal{A}$ were countable, its complement will also be countable. This would imply that $\sum_2$ is countable, which is a contradiction. Thus, $\mathcal{A}$ is an uncountable set whose each element is itself a sequence.

For any sequence $z = (z_n) \in \mathcal{A}$, define a sequence (b_n^z) of natural numbers as,

$$b_n^z = \begin{cases} A_n, & \text{if } z_n = 0 \\ B_n, & \text{if } z_n = 1 \end{cases}$$

Let $\mathcal{B}$ be the set of all sequences thus generated. Then, $\mathcal{B}$ is a collection of sequences of natural numbers. As any two sequences in $\mathcal{A}$ differ at infinitely many places, and so any two sequences in $\mathcal{B}$ will also differ at infinitely many places. Also, as $\mathcal{A}$ is uncountable, $\mathcal{B}$ is uncountable.

For any $s = (s_n) \in \mathcal{B}$, define $K_s = \{a_{s_n} : n \in \mathbb{N}\} \bigcup \{a\}$. Then K_s is an element of $\mathcal{K}(S)$. As $\mathcal{B}$ is uncountable, we now have an uncountable subset of $\mathcal{K}(S)$ which we claim to be a scrambled set for $\overline{\sigma}$. Let this set be denoted by $\mathcal{D}$.

Let $K_r, K_s \in \mathcal{D}$. Then, $K_s = \{a_{s_n} : n \in \mathbb{N}\} \bigcup \{a\}$, $K_r = \{a_{r_n} : n \in \mathbb{N}\} \bigcup \{a\}$.
For $s_k \neq r_k$,

$$d_H(\overline{\sigma}^{2^{k-2}+2}(K_s), \overline{\sigma}^{2^{k-2}+2}(K_r)) = \frac{1}{2^{2^{k-2}-1}}$$

and

$$d_H(\overline{\sigma}^{2^{k-1}}(K_s), \overline{\sigma}^{2^{k-1}}(K_r)) = \frac{1}{2}.$$

As s_k and r_k differ for infinitely many k, the above relation also hold for infinitely many k.

So $\liminf\limits_{n\to\infty} d_H(\overline{\sigma}^n(K_r), \overline{\sigma}^n(K_s)) = 0$ but $\limsup\limits_{n\to\infty} d_H(\overline{\sigma}^n(K_r), \overline{\sigma}^n(K_s)) \geq \frac{1}{2}$. Thus, $\mathcal{D}$ is an uncountable scrambled set in the hyperspace $\mathcal{K}(S)$.

Thus, existence of a scrambled set in the hyperspace does not guarantee the same in the base space.

3. Conclusion

In section 2, we have studied relations between the dynamical properties of the system (X, f) and the induced system $(\Psi, \overline{f})$, where $\Psi \subseteq CL(X)$ is any hyperspace endowed with some topology Δ. Each of these topologies Δ is of the form *hit and miss* or *hit and far miss*.

We have seen that whenever Ψ contains all finite subsets of X, the property of dense periodicity is preserved in $(\Psi, \overline{f})$ for any topology Δ on Ψ. However, for any $\Psi \subseteq CL(X)$ with any topology Δ on Ψ, such property in $(\Psi, \overline{f})$ need not conduce the same on (X, f).

However, the property of transitivity is equivalent for both (X, f) and $(\Psi, \overline{f})$ whenever Ψ is large enough and is endowed with the upper Vietoris topology. Also it is guaranteed that transitivity will be preserved in $(\Psi, \overline{f})$ when Ψ is endowed with a topology coarser than the upper Vietoris topology. Again, transitivity in $(\Psi, \overline{f})$ ensures the transitivity in (X, f) when Ψ is large enough and is endowed with the upper Vietoris topology or any finer topology.

If Ψ contains all finite subsets of X, then given any topology Δ, the properties of weakly mixing and mixing are preserved in $(\Psi, \overline{f})$. However, such properties on $(\Psi, \overline{f})$ conduce the same on (X, f) only when the topology Δ is endowed with the upper Vietoris topology or any finer topology. Also such properties on $(\Psi, \overline{f})$ are preserved if Ψ is endowed with any topology coarser than the upper Vietoris topology.

Thus, we can conclude that if the collection Ψ contains all singletons and the topology Δ on Ψ is atleast upper Vietoris, then these properties are equivalent on both (X, f) and $(\Psi, \overline{f})$. This generalizes the results in [1, 15] where Ψ is either $\mathcal{K}(X)$ or $CL(X)$ and Δ is either the Hausdorff metric topology or the Vietoris topology.

However, if topologies like Wijsman topology, Ball Proximal topology or Fell topology become finer than the upper Vietoris topology, then they actually coincide with the Vietoris topology. Since such topologies in general are not finer than the upper Vietoris topology, hence most of our observations cannot be established when Ψ is endowed with any of these topologies.

Again when Ψ is large enough to contain $\mathcal{F}_1(X)$ then the existence of the horseshoe on the base space implies the same on the hyperspace. The converse need not be true. But when $\mathcal{F}_1(X) \subseteq \Psi \subseteq \mathcal{K}(X)$ then for any topology finer than the upper Vietoris topology the property 'existence of horseshoe' is equivalent for both the individual dynamics and the induced dynamics under some conditions.

For the metric dependent properties, the comparison in the dynamic behaviour of the base map and the induced map is valid only when the hyperspace is endowed with the Hausdorff metric topology. Since the Hausdorff metric preserves the metric on the base space.

Here again 'uniform equicontinuity' is preserved whenever the hyperspace $\Psi \subset \mathcal{K}(X)$. However, if Ψ is big enough to contain $\mathcal{K}(X)$, then an exact equivalence does not hold true.

And finally, for Li-Yorke chaos the implication holds only in one direction.

All these observations conduce that the dynamics become more complex when studied in a set valued form. This may lead to an uncertainty in any prediction or observation made in a set valued form.

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