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NON-LINEAR BI-HARMONIC CHOQUARD EQUATIONS

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Abstract. This note studies the fourth-order Choquard equation

$$i\dot{u} + \Delta^2 u \pm (I_\alpha * |u|^p)|u|^{p-2}u = 0.$$

In the mass super-critical and energy sub-critical regimes, a sharp threshold of global well-psedness and scattering versus finite time blow-up dichotomy is obtained.

1. **Introduction.** In this manuscript, we investigate the Cauchy problem for a bi-harmonic Choquard equation

$$\begin{cases} i\dot{u} + \Delta^2 u + \epsilon(I_\alpha * |u|^p)|u|^{p-2}u = 0; \\ u(0,.) = u_0, \end{cases} \tag{1.1}$$

where $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, for some $N \geq 1$, $\epsilon = \pm 1$, $0 < \alpha < N$ and the Riesz-potential is defined on $\mathbb{R}^N$ by

$$I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|\cdot|^{N-\alpha}}.$$

The classical Choquard equation is a model of quantum mechanics [17], non-relativistic quantum and Hartree-Fock theories [19, 9]. The particular case $p = 2$ with Laplacian operator (instead of bilaplacian) is called Hartree equation and models the dynamics of boson stars [6, 16].

Fourth-order Schrödinger equations, take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr non-linearity [12, 13].

If u is a solution to the Choquard problem (1.1), then the following scaled function solves the same problem

$$u_\lambda = \lambda^{\frac{4+\alpha}{2(p-1)}} u(\lambda^4., \lambda.), \quad \lambda > 0.$$

Using the next equality,

$$\|u_\lambda(t)\|_{\dot{H}^\mu} = \lambda^{\mu-\frac{N}{2}+\frac{4+\alpha}{2(p-1)}}\|u(\lambda^4 t)\|_{\dot{H}^\mu},$$

one obtains the unique invariant Sobolev norm under the previous scaling, called critical exponent

$$s_c := \frac{N}{2} - \frac{4+\alpha}{2(p-1)}.$$

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The exponent $s_c = 0$ is called mass-critical case and corresponds to $p_* := 1 + \frac{\alpha+4}{N}$. The energy-critical case $s_c = 2$ is equivalent to

$$p^* := \begin{cases} 1 + \dfrac{\alpha+4}{N-4}, & N > 4; \\ \infty, & 2 \leq N \leq 4. \end{cases}$$

The well-posedness issues for the mass-super-critical and energy sub-critical classical Choquard equation were investigated recently by many authors [7, 20, 23]. See also [8, 4, 22], for the fractional Choquard equation.

Recall the conservation laws for the Schrödinger problem (1.1),

$$\begin{aligned} &Mass := M(u(t)) := \int_{\mathbb{R}^N} |u(t,x)|^2 dx = M(u_0); \\ &Energy := E(u(t)) := \int_{\mathbb{R}^N} \Big(|\Delta u(t)|^2 + \frac{\epsilon}{p} (I_\alpha * |u(t)|^p)|u(t)|^p \Big) dx = E(u_0). \end{aligned}$$

The positive (respectively negative) sign of ϵ refers to the attractive or defocusing (respectively focusing) case, where a local solution in the energy space is claimed to be global and scatters (respectively blows-up in finite time).

It is the purpose of this manuscript to obtain a sharp dichotomy in the mass super-critical and energy sub-critical cases of global well-posedness and scattering versus finite time blow-up of solutions to the fourth-order Choquard problem (1.1), by use of a sharp Gagliardo-Nirenberg type inequality and the existence of ground states. In the scattering part, one uses the concentration-compactness-rigidity method, due to Kenig and Merle [14], which has a deep influence on asymptotic study of Schrödinger problems [5, 10].

To the author knowledge, this paper is the first one dealing with scattering of bi-harmonic Choquard equations.

The plan of this paper is as follows. Section two contains some classical estimates needed in the sequel. In the third section a sharp Gagliardo-Nirenberg type inequality is given. The existence of ground states is proved in section four. In section 5, local well-posedness in the energy space is given. A variance identity is established in section six. The existence of global/non global solutions to (1.1) are discussed in section 7. The goal of the last section is to investigate scattering of global solutions.

Here and hereafter C will denote a constant which may vary from line to line and if A and B are non-negative real numbers, $A \lesssim B$ means that $A \leq CB$.

Denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the standard norm $\|\cdot\|_r := \|\cdot\|_{L^r}$ and $\|\cdot\| := \|\cdot\|_2$. Take $H^2 := H^2(\mathbb{R}^N)$ the inhomogeneous Sobolev space endowed with the complete norm

$$\|\cdot\|_{H^2} := \big(\|\cdot\|^2 + \|\Delta\,.\,\|^2\big)^{\frac{1}{2}}.$$

If X is an abstract space $C_T(X) := C([0,T],X)$ stands for the set of continuous functions valued in X and X_{rd} is the set of radial elements in X, moreover for an eventual solution to (1.1), $T^* > 0$ denotes it's lifespan.

2. **Preliminary.**

This section contains some estimates needed in the sequel. Let us start with a Hardy-Littlewood-Sobolev inequality [18].

Lemma 2.1. *Let $0<\lambda<N\geq 1$ and $1<s,r<\infty$ be such that $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{N}=2$. Then,*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^\lambda}\,dx\,dy\leq C(N,s,\lambda)\|f\|_r\|g\|_s,\quad \forall f\in L^r(\mathbb{R}^N),\ \forall g\in L^s(\mathbb{R}^N).$$

The next consequence will be useful [22].

Corollary 2.2. *Let $0<\alpha<N\geq 1$ and $1<s,r,q<\infty$ be such that $\frac{1}{q}+\frac{1}{r}+\frac{1}{s}=1+\frac{\alpha}{N}$. Assume that $f\in L^s(\mathbb{R}^N)$ and $g\in L^q(\mathbb{R}^N)$. Then,*

$$\|(I_\alpha * f)g\|_{r'}\leq C(s,\alpha)\|f\|_s\|g\|_q.$$

Sobolev injections [2] give a meaning to several computations done in this note.

Lemma 2.3. *Let $N\geq 1$, then*

1. *$H^2\hookrightarrow L^q$ for any $q\in[2,\frac{2N}{N-4}]$ if $N\geq 5$ and any $2\leq q<\infty$ if $N\leq 4$;*
2. *the following injection $H^2_{rd}\hookrightarrow\hookrightarrow L^q$ is compact for any $q\in(2,\frac{2N}{N-4})$ if $N\geq 5$ and any $2<q<\infty$ if $2\leq N\leq 4$;*
3. *for all $\frac{1}{2}<\mu<\frac{N}{2}$,*

$$\sup_{x\neq 0}|x|^{\frac{N}{2}-\mu}|u(x)|\leq C(N,\mu)\|(-\Delta)^{\frac{\mu}{2}}u\|,\quad \forall u\in H^\mu_{rd}(\mathbb{R}^N). \tag{2.1}$$

Recall a Gagliardo-Nirenberg inequality [21].

Lemma 2.4. *Let $N\geq 1$, $1\leq p,q,r\leq\infty$ and $0\leq\frac{\mu}{m}\leq\theta\leq 1$ satisfying*

$$\frac{1}{p}=\frac{\mu}{N}+\theta(\frac{1}{r}-\frac{m}{N})+(1-\theta)\frac{1}{q}.$$

Then,

$$\|(-\Delta)^{\frac{\mu}{2}}\cdot\|_p\lesssim\|(-\Delta)^{\frac{m}{2}}\cdot\|_r^\theta\|\cdot\|_q^{1-\theta}. \tag{2.2}$$

Recall a fractional chain rule [3].

Lemma 2.5. *Let $s\in(0,1]$ and $1<p,p_i,q_i<\infty$ satisfying $\frac{1}{p}=\frac{1}{p_i}+\frac{1}{q_i}$. Thus,*

1. *if $G\in C^1(\mathbb{C})$, then $\||\nabla|^sG(u)\|_p\lesssim\|G'(u)\|_{p_1}\||\nabla|^su\|_{q_1}$;*
2. *$\||\nabla|^s(uv)\|_p\lesssim\||\nabla|^su\|_{p_1}\|v\|_{q_1}+\||\nabla|^sv\|_{p_2}\|u\|_{q_2}$.*

Definition 2.6. *A couple of real numbers (q,r) is said to be s admissible if*

$$\frac{2N}{N-2s}\leq r<\frac{2N}{N-4}\quad\text{and}\quad N(\frac{1}{2}-\frac{1}{r})=\frac{4}{q}-s.$$

Strichartz estimate [11, 24] is a classical tool to control solutions to (1.1).

Proposition 2.7. *Let $N\geq 2$, $0\leq s<2$, (q,r) be an admissible pair and $(\tilde{q},\tilde{r})$ be $-s$ admissible pair. Then, there exists $C:=C_{N,q,\tilde{q},s}$ such that if $u_0\in\dot{H}^s$,*

$$\|u\|_{L^q_t(L^r)}\leq C\Big(\|u_0\|_{\dot{H}^s}+\|i\dot{u}+\Delta^2u\|_{L^{\tilde{q}'}_t(L^{\tilde{r}'})}\Big).$$

Let us introduce [11] the linear profile decomposition for bounded radial sequences in H^2.

Proposition 2.8. *Let $N\geq 2$ and (u_n) be a bounded sequence in H^2_{rd}. Then for each integer M there exist a sub-sequence still denoted (u_n) and*

1. for every $1\leq j\leq M$, there exists a profile $\psi_j\in H^2$ and a sequence of time shifts t_n^j;

2. there exists a sequence of remainders $W_n^M \in H^2$, such that

$$u_n = \sum_{j=1}^{M} e^{-it_n^j\Delta^2}\psi^j + W_n^M.$$

The time sequences have the pairwise divergence property: For $1 \le i \neq j \le M$,

$$\lim_n |t_n^j - t_n^i| = \infty.$$

The remainder sequence has the following asymptotic smallness property

$$\lim_{M\to\infty}[\lim_{n\to\infty}\|e^{i.\Delta^2}W_n^M\|_{S(\mathbb{R})}] = 0.$$

For fixed M and any $0 \le \alpha \le 2$, the asymptotic Pythagorean expansions hold

$$\|u_n\|_{H^\alpha}^2 = \sum_{j=1}^{M}\|\psi^j\|_{H^\alpha}^2 + \|W_n^M\|_{H^\alpha}^2 + o_n(1);$$

$$E(u_n) = \sum_{j=1}^{M} E(e^{-it_n^j\Delta^2}\psi^j) + E(W_n^M) + o_n(1).$$

Proof. Taking account of [11], the last equality is the only point to prove. It is sufficient to prove that $Q(u) := \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx$ satisfies

$$Q(u_n) = \sum_{j=1}^{M} Q(e^{-it_n^j\Delta^2}\psi^j) + Q(W_n^M) + o_n(1).$$

Assume as a first case that there exists some j for which t_n^j converges to a finite number, which is supposed to be zero without loss of generality. From the proof of Lemma 5.3 in [11] and the compact embedding $H_{rd}^2 \hookrightarrow\hookrightarrow L^q$ for $2 < q < \frac{2N}{N-4}$, we get $W_n^{j-1} \to \psi^j$ in L^q for $2 < q < \frac{2N}{N-4}$. Write using Lemma 2.1, for $r := \frac{2N}{\alpha+N}$,

$$\begin{aligned}|Q(W_n^{j-1}) - Q(\psi^j)| &\le C\||W_n^{j-1}|^p - |\psi^j|^p\|_r(\|W_n^{j-1}\|_{rp}^p + \|W^j\|_{rp}^p)\\ &\le C\sum_{k=0}^{p-1}\||W_n^{j-1}| - |\psi^j|\|_{rp}\|W_n^{j-1}\|_{rp}^k\|\psi^j\|_{rp}^{p-k-1}.\end{aligned}$$

Since, $p < p^*$, we get $2 < rp < \frac{2N}{N-4}$, which implies that $|Q(W_n^{j-1}) - Q(\psi^j)| \to 0$. Let $k \neq j$. Then, $|t_n^k| \to \infty$. Since $p > p_*$, from Lemma 2.1 and the L^p space-time decay estimates of the linear flow associated to (1.1),

$$|Q(e^{-it_n^k\Delta^2}\psi^k)|^{\frac{1}{2p}} \lesssim \|e^{-it_n^k\Delta^2}\psi^k\|_{\frac{2Np}{\alpha+N}} \lesssim \Big(\frac{1}{t_n^k}\Big)^{\frac{N}{4}(1-\frac{\alpha+N}{Np})}\|\psi^k\|_{(\frac{2Np}{\alpha+N})'} \to 0.$$

With the expansion of u_n,

$$u_n = \sum_{k=1}^{j-1} e^{-it_n^k\Delta^2}\psi^k + W_n^{j-1},$$

one gets $u_n \to \psi^j$ in L^q for $2 < q < \frac{2N}{N-4}$. As previously, it follows that $Q(u_n) \to Q(\psi^j)$. Finally, using the identity

$$W_n^M = W_n^{j-1} - \psi^j - \sum_{k=1+j}^{M} e^{-it_n^k\Delta^2}\psi^k,$$

one gets $W_n^M \to 0$ and $Q(W_n^M) \to 0$ for $M > j$. Similarly, we get the second case: for all j, $t_n^j \to \infty$. □

3. **Gagliardo-Nirenberg inequality.** Denote the real numbers

$$B := \frac{Np - N - \alpha}{2} \quad \text{and} \quad A := 2p - B.$$

The goal of this section is to prove a sharp Gagliardo-Nirenberg inequality related to the Choquard problem (1.1).

Theorem 3.1. *Let $0 < \alpha < N \geq 2$ and $1 + \frac{\alpha}{N} \leq p \leq p^*$. Then,*

1. there exists a positive constant $C(N,p,\alpha)$, such that for any $u \in H^2$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \leq C(N,p,\alpha)\|u\|^A \|\Delta u\|^B. \tag{3.1}$$

Moreover, if $1 + \frac{\alpha}{N} < p < p^*$, then

2. the minimization problem

$$\frac{1}{C(N,p,\alpha)} = \inf\left\{ J(u) := \frac{\|u\|^A \|\Delta u\|^B}{\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx}, \quad 0 \neq u \in H^2 \right\}$$

is attained in some $Q \in H^2$ satisfying $C(N,p,\alpha) = \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p \, dx$ and

$$B\Delta^2 Q + AQ - \frac{2p}{C(N,p,\alpha)}(I_\alpha * |Q|^p)|Q|^{p-2}Q = 0; \tag{3.2}$$

3. furthermore

$$C(N,p,\alpha) = \frac{2p}{A}\left(\frac{A}{B}\right)^{\frac{B}{2}} \|\phi\|^{-2(p-1)}, \tag{3.3}$$

where ϕ is a ground state solution to (4.1).

Proof. The proof contains three steps.

First, let us start by proving the interpolation inequality (3.1). Taking account of Lemma 2.4 and Corollary 2.2, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \leq & C_{N,p,\alpha}\|u\|^{2p}_{\frac{2Np}{\alpha+N}} \\ \leq & C_{N,p,\alpha}\|\Delta u\|^{2p\frac{N}{2}(\frac{1}{2} - \frac{\alpha+N}{2Np})}\|u\|^{2p[1-\frac{N}{2}(\frac{1}{2}-\frac{\alpha+N}{2Np})]} \\ \leq & C_{N,p,\alpha}\|\Delta u\|^B \|u\|^A.\end{aligned}$$

Second, one proves the equation (3.2). Denote $\beta := \frac{1}{C(N,p,\alpha)}$. Using (3.1), there exists a sequence (v_n) in H^2 such that $\beta = \lim_n J(v_n)$. Denoting for $a, b \in \mathbb{R}$, the scaling $u^{a,b} := au(b.)$, we compute

$$\|\Delta u^{a,b}\|^2 = a^2 b^{4-N}\|\Delta u\|^2; \quad \|u^{a,b}\|^2 = a^2 b^{-N}\|u\|^2;$$
$$\int_{\mathbb{R}^N} (I_\alpha * |u^{a,b}|^p)|u^{a,b}|^p \, dx = a^{2p}b^{-N-\alpha}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx.$$

It follows that

$$J(u^{a,b}) = J(u).$$

Now, we choose

$$\mu_n := \left(\frac{\|v_n\|}{\|\Delta v_n\|}\right)^{\frac{1}{2}} \quad \text{and} \quad \lambda_n := \frac{\|v_n\|^{\frac{N}{4}-1}}{\|\Delta v_n\|^{\frac{N}{4}}}.$$

Thus, $\psi_n := v_n^{\lambda_n,\mu_n}$ satisfies

$$\|\psi_n\| = \|\Delta\psi_n\| = 1 \quad \text{and} \quad \beta = \lim_n J(\psi_n).$$

Then, $\psi_n \rightharpoonup \psi$ in H^2 and using Sobolev injections, one gets for a sub-sequence denoted also (ψ_n),

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_n|^p)|\psi_n|^p\,dx \to \int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p\,dx.$$

In fact, thanks to Lemma 2.1 and Sobolev embedding,

$$\begin{aligned}
(I_n) &:= \int |(I_\alpha * |\psi_n|^p)|\psi_n|^p - (I_\alpha * |\psi|^p)|\psi|^p|\,dx \\
&\leq \int |(I_\alpha * [|\psi_n|^p - |\psi|^p])|\psi|^p\,dx - \int_{\mathbb{R}^N} (I_\alpha * |\psi_n|^p)[|\psi|^p - |\psi_n|^p]|\,dx \\
&\leq C\||\psi_n|^p - |\psi|^p\|_{\frac{2N}{N+\alpha}}[\|\psi_n\|^p_{\frac{2Np}{N+\alpha}} + \|\psi\|^p_{\frac{2Np}{N+\alpha}}] \\
&\leq C\||\psi_n|^p - |\psi|^p\|_{\frac{2N}{N+\alpha}}[\|\psi_n\|^p_{H^2} + \|\psi\|^p_{H^2}] \\
&\leq C\|\psi_n - \psi\|_{\frac{2Np}{N+\alpha}}[\|\psi_n\|^{2p-1}_{H^2} + \|\psi\|^{2p-1}_{H^2}] \to 0.
\end{aligned}$$

This implies that, when n goes to infinity

$$J(\psi_n) \to \frac{1}{\int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p\,dx}.$$

The semi continuity of $\|\cdot\|_{H^2}$ gives $\max\{\|\psi\|, \|\Delta\psi\|\} \leq 1$. Then,

$$\|\psi\| = \|\Delta\psi\| = 1,$$

because otherwise, one gets the absurdity $J(\psi) < \beta$. Thus,

$$\psi_n \to \psi \quad \text{in} \quad H^2 \quad \text{and} \quad \beta = J(\psi) = \frac{1}{\int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p\,dx}.$$

ψ satisfies (3.2) because the minimizer satisfies the Euler equation

$$\partial_\varepsilon J(\psi + \varepsilon\eta)_{|\varepsilon=0} = 0, \quad \forall \eta \in C_0^\infty \cap H^2.$$

Finally, let us establish the equation (3.3). Write $C(N,p,\alpha) = \frac{1}{\beta} = \int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p\,dx$, where ψ is given in (3.2). Define, the scaling $\psi = \phi^{a,b} := a\phi(b.)$, for $a, b \in \mathbb{R}$. Then, the equation

$$B\Delta^2\psi + A\psi - 2\beta p(I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0,$$

implies that

$$Aa\Big(\frac{B}{A}b^4\Delta^2\phi + \phi - 2\frac{\beta}{A}pa^{2(p-1)}b^{-\alpha}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi\Big) = 0.$$

Choosing

$$b = \Big(\frac{A}{B}\Big)^{\frac{1}{4}} \quad \text{and} \quad a = \Big(\Big(\frac{A}{B}\Big)^{\frac{\alpha}{4}}\frac{A}{2p\beta}\Big)^{\frac{1}{2(p-1)}},$$

it follows that

$$\Delta^2\phi + \phi - (I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0.$$

Now, since $\|\psi\| = 1 = ab^{-\frac{N}{2}}\|\phi\|$, we get

$$\beta = \frac{A}{2p}(\frac{A}{B})^{-\frac{B}{2}}\|\phi\|^{2(p-1)}.$$

The proof is closed. $\square$

4. **Existence of ground states.** For $u \in H^2$ and $a, b \in \mathbb{R}$, here and hereafter define the quantities

$$\underline{\mu} := \min\{2a + (N-4)b, 2a + Nb\},\ \bar{\mu} := \max\{2a + (N-4)b, 2a + Nb\};$$

$$\mathcal{A} := \Big\{(a,b) \in \mathbb{R}_+^* \times \mathbb{R} \quad \text{s.t} \quad \underline{\mu} \geq 0 \quad \text{and} \quad \bar{\mu} > 0\Big\};$$

$$u_{a,b}^\lambda := \lambda^a u(\lambda^{-b}.), \quad \mathcal{L}_{a,b}(u) := (\partial_\lambda u_{a,b}^\lambda)_{|\lambda=1};$$

$$K_{a,b}^Q(u) := (2a + (N-4)b)\|\Delta u\|^2 + (2a + Nb)\|u\|^2;$$

$$K_{a,b}^N(u) := -\frac{2ap + b(N+\alpha)}{p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\, dx;$$

$$S := M + E, \quad K_{a,b} := \mathcal{L}_{a,b} S = K_{a,b}^Q + K_{a,b}^N, \quad H_{a,b} := (1 - \frac{\mathcal{L}_{a,b}}{\bar{\mu}})S.$$

Definition 4.1. *We call ground state of* (1.1)*, any solution to*

$$\phi + \Delta^2\phi - (I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0, \quad 0 \neq \phi \in H^2, \tag{4.1}$$

which minimizes the problem

$$m_{a,b} := \inf_{0 \neq v \in H_{rd}^2} \Big\{S(v) \quad s.\ t \quad K_{a,b}(v) = 0\Big\}. \tag{4.2}$$

Remark 4.2. *The standing wave $e^{-it}\phi$ is a global solution to the Schrödinger problem* (1.1) *which gives the threshold between global well-posedness and finite time blow-up of solutions as proved in section 7.*

The following main result of this section follows with variational methods and ensures the existence of ground states.

Theorem 4.3. *Take $N \geq 2$, a couple of real numbers $(a,b) \in \mathcal{A}$ and $p_* < p < p^*$. Then,*

1. *$m := m_{a,b}$ is nonzero and independent of (a,b);*
2. *there is a ground state solution to* (4.1)-(4.2).

Let us give some intermediate results.

Lemma 4.4. *Let $(a,b) \in \mathcal{A}$. Then,*

1. *$\min\big(\mathcal{L}_{a,b}H_{a,b}(u), H_{a,b}(u)\big) > 0$ for all $0 \neq u \in H^2$;*
2. *$\lambda \mapsto H_{a,b}(u^\lambda)$ is increasing.*

Proof. Compute

$$\begin{aligned}
H_{a,b}(u) :=& (1 - \frac{\mathcal{L}_{a,b}}{\bar{\mu}})S(u) = \frac{1}{\bar{\mu}}\big(\bar{\mu}S(u) - K_{a,b}(u)\big)\\
=& \frac{1}{\bar{\mu}}\Big[\Big(\bar{\mu} - (2a + (N-4)b)\Big)\|\Delta u\|^2 + \Big(\bar{\mu} - (2a + Nb)\Big)\|u\|^2\\
&+ \frac{1}{p}\Big(2ap + b(N+\alpha) - \bar{\mu}\Big)\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\, dx\Big].
\end{aligned}$$

Since $\underline{\mu} \geq 0$ and $p > p_*$, one obtains, if $b < 0$,

$$\begin{aligned}
2ap + b(\alpha + N) - \bar{\mu} =& 2a(p-1) + b(\alpha + 4)\\
&> 2a(p-1) - \frac{2a}{N}(4 + \alpha) > 2a(p - p_*) > 0.
\end{aligned} \tag{4.3}$$

If $b\geq 0$, then, $2ap+b(\alpha+N)-\bar\mu=2a(p-1)+b\alpha>0$. Hence, $H_{a,b}(u)>0$. Moreover,

$$\begin{aligned}\mathcal{L}_{a,b}H_{a,b}(u) =&\mathcal{L}_{a,b}\big(1-\frac{\mathcal{L}_{a,b}}{\bar\mu}\big)E(u)\\ =&\frac{-1}{\bar\mu}\big(\mathcal{L}_{a,b}-\bar\mu\big)\big(\mathcal{L}_{a,b}-\underline{\mu}\big)E(u)+\underline{\mu}\big(1-\frac{\mathcal{L}_{a,b}}{\bar\mu}\big)E(u)\\ =&\frac{-1}{\bar\mu}\big(\mathcal{L}_{a,b}-\bar\mu\big)\big(\mathcal{L}_{a,b}-\underline{\mu}\big)E(u)+\underline{\mu}H_{a,b}(u).\end{aligned}$$

Since $(\mathcal{L}_{a,b}-\bar\mu)(\mathcal{L}_{a,b}-\underline{\mu})\|u\|_{H^2}^2=0$, one gets

$$\begin{aligned}\mathcal{L}_{a,b}H_{a,b}(u) \geq&\frac{1}{\bar\mu}\big(\mathcal{L}_{a,b}-\bar\mu\big)\big(\mathcal{L}_{a,b}-\underline{\mu}\big)\Big(\frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\,dx\Big)\\ \geq&\frac{1}{p\bar\mu}\Big(2ap+b(N+\alpha)-\bar\mu\Big)\Big(2ap+b(N+\alpha)-\underline{\mu}\Big)\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p dx.\end{aligned}$$

Arguing as previously, it follows that $\mathcal{L}_{a,b}H_{a,b}(u)>0$.

The last point follows using the equality $\partial_\lambda H_{a,b}(u^\lambda)=\mathcal{L}_{a,b}H_{a,b}(u^\lambda)$. □

The next intermediate result is the following.

Lemma 4.5. *Let $(a,b)\in\mathcal{A}$ and $0\neq u_n$ be a bounded sequence of H^2 such that*

$$\lim_n(K_{a,b}^Q(u_n))=0.$$

Then, there exists $n_0\in\mathbb{N}$ such that $K_{a,b}(u_n)>0$ for all $n\geq n_0$.

Proof. We have

$$K_{a,b}(u_n)=K_{a,b}^Q(u_n)-\frac{2ap+b(N+\alpha)}{p}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^p)|u_n|^p\,dx.$$

If $b\leq 0$, then, $2a+(N-4)b=\bar\mu>0$ and if $b>0$, so, $\bar\mu=2a+Nb>0$, which implies that $b>-\frac{2a}{N}$. Then, $2a+(N-4)b>2a-\frac{2a}{N}(N-4)=\frac{4a}{N}>0$. Thus,

$$\|\Delta u_n\|^2\lesssim K_{a,b}^Q(u_n)\to 0.$$

Now, because $B>2$, using (3.1), for large n,

$$\int_{\mathbb{R}^N}(I_\alpha*|u_n|^p)|u_n|^p\,dx\leq C\|u_n\|^A\|\Delta u_n\|^B=o\Big(\|\Delta u_n\|^2\Big)=o(K_{a,b}^Q(u_n)).$$

Thus, when $n\to\infty$,

$$K_{a,b}(u_n)\simeq K_{a,b}^Q(u_n)>0.$$ □

One can express the minimizing problem (4.2), with negative constraint.

Lemma 4.6. *Let $(a,b)\in\mathcal{A}$. Then,*

$$m_{a,b}=\inf_{0\neq u\in H^2}\big\{H_{a,b}(u)\quad s.\ t\quad K_{a,b}(u)\leq 0\big\}.$$

Proof. Denoting by r the right hand side of the previous equality, it is sufficient to prove that $m_{a,b}\leq r$. Take $u\in H^2$ such that $K_{a,b}(u)<0$. Because $\lim_{\lambda\to 0}K_{a,b}^Q(u^\lambda)=0$, by the previous Lemma, there exists $\lambda\in(0,1)$ such that $K_{a,b}(u^\lambda)>0$. With a continuity argument there exists $\lambda_0\in(0,1)$ such that $K_{a,b}(u^{\lambda_0})=0$, then since $\lambda\mapsto H_{a,b}(u^\lambda)$ is increasing, we get

$$m_{a,b}\leq H_{a,b}(u^{\lambda_0})\leq H_{a,b}(u).$$

This closes the proof. □

Proof of Theorem 4.3. Let (ϕ_n) be a minimizing sequence, namely

$$0 \neq \phi_n \in H^2_{rd}, \quad K_{a,b}(\phi_n) = 0 \quad \text{and} \quad \lim_n H_{a,b}(\phi_n) = \lim_n S(\phi_n) = m_{a,b}. \tag{4.4}$$

• First step: (ϕ_n) is bounded in H^2.

First case $a > 0$ and $b > 0$. Denoting $\lambda := \frac{b}{2a}$, yields

$$\begin{aligned}&\|\phi_n\|^2_{H^2} - \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx\\ =&\lambda\Big(4\|\Delta\phi_n\|^2 - N\|\phi_n\|^2_{H^2} + \frac{\alpha+N}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx\Big)\end{aligned}$$

and

$$\|\phi_n\|^2_{H^2} - \frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx \to m_{a,b}.$$

So the following sequence is bounded

$$-4\lambda\|\Delta\phi_n\|^2 + \|\phi_n\|^2_{H^2} - (1 + \frac{\lambda\alpha}{p})\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx.$$

Thus, for any real number β, the following sequence is also bounded

$$4\lambda\|\Delta\phi_n\|^2 + (\beta - 1)\|\phi_n\|^2_{H^2} + (1 + \frac{\lambda\alpha - \beta}{p})\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx.$$

Choosing $\beta \in (1, p + \lambda\alpha)$, it follows that (ϕ_n) is bounded in H^2.

Second case $a > 0$ and $\frac{-2a}{N} < b \leq 0$. Compute

$$\begin{aligned}&(\bar{\mu} - \mathcal{L}_{a,b})S(\phi_n)\\ =&-4b\|\phi_n\|^2 + (2a(p-1) + (\alpha+4)b)\frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx\\ \geq&(2a(p-1) + (\alpha+4)b)\frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx.\end{aligned}$$

Moreover, if $b < 0$, $\bar{\mu} = 2a + (N-4)b$. Then, since $\underline{\mu} \geq 0$ and $p > p_*$, we obtain $2a(p-1) + (\alpha+4)b > 0$. Because $K_{a,b}(\phi_n) = 0$, this implies that

$$\begin{aligned}&\Big(\bar{\mu} + \big(2a(p-1) + (\alpha+4)b\big)\Big)S(\phi_n)\\ =&(\bar{\mu} - \mathcal{L}_{a,b})S(\phi_n) + (2a(p-1) + (\alpha+4)b)S(\phi_n) + \mathcal{L}_{a,b}S(\phi_n)\\ \geq&\big(2\alpha(p-1) + (\alpha+4)b\big)\|\phi_n\|^2_{H^2}.\end{aligned}$$

Hence, ϕ_n is bounded in H^2.

• Second step: the limit of (ϕ_n) is nonzero and $m > 0$.

Taking account of the compact injection in Lemma 2.3, take

$$\phi_n \rightharpoonup \phi \quad \text{in} \quad H^2$$

and for all $2 < p < \frac{2N}{N-4}$, where $\frac{2N}{N-4} = \infty$ if $N \leq 4$,

$$\phi_n \to \phi \quad \text{in} \quad L^p.$$

The equality $K_{a,b}(\phi_n) = 0$ implies that

$$(2a+(N-4)b)\|\Delta\phi_n\|^2 + (2a+Nb)\|\phi_n\|^2 = \frac{2ap+b(N+\alpha)}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p dx.$$

Assume that $\phi = 0$. Using Corollary 2.2, with the fact that $1 + \frac{\alpha}{N} < p < p^*$, write

$$\int_{\mathbb{R}^N}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx \lesssim \|\phi_n\|^{2p}_{\frac{2Np}{\alpha+N}} \to 0.$$

Now, by Lemma 4.5 yields $K_{a,b}(\phi_n)>0$ for large n. This contradiction implies that $\phi\neq 0$. Thanks to Lemma 2.1 and Sobolev embedding,

$$\begin{aligned}(J_n) :=& \int |(I_\alpha * |\phi_n|^p)|\phi_n|^p - (I_\alpha * |\phi|^p)|\phi|^p|\, dx\\ \leq& \int |(I_\alpha * [|\phi_n|^p - |\phi|^p])|\phi|^p\, dx - \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)[|\phi|^p - |\phi_n|^p]|\, dx\\ \leq& C\||\phi_n|^p - |\phi|^p\|_{\frac{2N}{N+\alpha}}[\|\phi_n\|^p_{\frac{2Np}{N+\alpha}} + \|\phi\|^p_{\frac{2Np}{N+\alpha}}]\\ \leq& C\||\phi_n|^p - |\phi|^p\|_{\frac{2N}{N+\alpha}}[\|\phi_n\|^p_{H^2} + \|\phi\|^p_{H^2}]\\ \leq& C\|\phi_n - \phi\|_{\frac{2Np}{N+\alpha}}[\|\phi_n\|^{2p-1}_{H^2} + \|\phi\|^{2p-1}_{H^2}] \to 0.\end{aligned}$$

So, with lower semi continuity of the H^2 norm, we have

$$\begin{aligned}0 =& \liminf_n K_{a,b}(\phi_n)\\ \geq& \frac{2a+(N-4)b}{2}\liminf_n \|\nabla\phi_n\|^2 + \frac{2a+Nb}{2}\liminf_n \|\phi_n\|^2\\ &- \frac{2ap+b(N+\alpha)}{p}\int_{\mathbb{R}^N}(I_\alpha * |\phi|^p)|\phi|^p\, dx\\ \geq& K_{a,b}(\phi).\end{aligned}$$

Similarly, we have $H_{a,b}(\phi)\leq m$. Moreover, thanks to Lemma 4.6, we assume that $K_{a,b}(\phi)=0$ and $S(\phi)=H_{a,b}(\phi)\leq m$. So, ϕ is a minimizer satisfying (4.4) and using previous computation

$$m = H_{a,b}(\phi) > 0.$$

- Third step: the limit ϕ is a solution to (4.1).

There is a Lagrange multiplier $\eta\in\mathbb{R}$ such that $S'(\phi)=\eta K'_{a,b}(\phi)$. Thus,

$$\begin{aligned}&0 = K_{a,b}(\phi) = \mathcal{L}_{a,b}S(\phi) = \langle S'(\phi), \quad \mathcal{L}_{a,b}(\phi)\rangle = \eta\langle K'_{a,b}(\phi),\\ &\mathcal{L}_{a,b}(\phi)\rangle = \eta\mathcal{L}_{a,b}K_{a,b}(\phi) = \eta\mathcal{L}^2_{a,b}S(\phi).\end{aligned}$$

Taking account of previous computations,

$$-\mathcal{L}^2_{a,b}S(\phi) - \bar{\mu}\underline{\mu}S(\phi) = -(\mathcal{L}_{a,b}-\bar{\mu})(\mathcal{L}_{a,b}-\underline{\mu})S(\phi) > 0.$$

Therefore, $\mathcal{L}^2_{a,b}S(\phi)<0$. Thus, $\eta=0$ and $S'(\phi)=0$. So, ϕ is a ground state and m is independent of (a,b). □

Let us end this section with the so-called generalized Pohozaev identity [15].

Lemma 4.7. *$\phi\in H^2$ is solution to (4.1) if and only if $S'(\phi)=0$. Moreover, in a such case*

$$K_{a,b}(\phi)=0, \quad \textit{for any} \quad (a,b)\in\mathbb{R}^2.$$

5. **Well-posedness in the energy space.** Using a classical fixed point argument and taking account of Strichartz estimates and Sobolev injections, one can obtain the following result.

Proposition 5.1. *Let $N\geq 2$, $0<\alpha<N$ such that $\alpha\geq N-8$, $2\leq p\leq p^*$ and $u_0\in H^2$. Then, there exists $T>0$ such that (1.1) admits a unique local solution*

$$u\in C_T(H^2).$$

Moreover,

1. *the solution satisfies the mass and energy conservation laws;*

2. *u is global if*

$$(a)\ \epsilon = 1 \text{ and } p < p^*; \quad (b)\ p < p_*; \quad (c)\ p = p_* \text{ and } M(u_0) < \Big(\frac{p}{C(N,p,\alpha)}\Big)^{\frac{2}{A}}.$$

Remark 5.2. *1. Thanks to the inequality* (3.1)*, the energy is well-defined for* $1+\frac{\alpha}{N} \le p \le p^*$*. So, the condition* $p \ge 2$ *which gives a restriction on the space dimension, seems to be technical;*

2. the proof is omitted because it follows as in [22].

6. **Virial type identity.** This section is devoted to prove a Virial type identity, which will be useful in order to obtain finite time blow-up of some solutions to the Choquard problem (1.1). Here and hereafter, denote $\psi_R := R^2\psi(\frac{\cdot}{R})$, $R>0$, where $\psi \in C_0^\infty(\mathbb{R}^n)$ is a radial function satisfying $\psi'' \le 1$ and

$$\psi(x) = \begin{cases} \frac{1}{2}|x|^2, & |x| \le 1; \\ 0, & |x| \ge 2. \end{cases}$$

A direct computation gives

$$\psi_R'' \le 1, \quad \psi_R'(r) \le r \quad \text{and} \quad \Delta\psi_R \le N.$$

Denote the localized Virial

$$M_\psi[u(t)] := 2\Im \int_{\mathbb{R}^N} \bar{u}(t)\nabla\psi\nabla u(t)\,dx.$$

Define the self-adjoint differential operator $\Gamma_\psi := -i(\nabla.\nabla\psi + \nabla\psi.\nabla)$, which acts on functions

$$\Gamma_\psi f = -i\Big[\nabla.((\nabla\psi)f) + (\nabla\psi).(\nabla f)\Big].$$

Then,

$$M_\psi[u(t)] = < u(t), \Gamma_\psi u(t) > .$$

The main result of this section reads as follows.

Theorem 6.1. *Let* $N \ge 2$*,* $0 < \alpha < N$ *such that* $\alpha \ge N-8$*,* $2 \le p \le p^*$ *and* $u \in C_T(H^2_{rd})$ *be a solution of* (1.1)*. Then, on* $[0,T)$*, for any* $R>0$ *and* $\frac{1}{2} < \mu < 2$*,*

$$\begin{aligned}\frac{d}{dt}M_{\psi_R}[u] \le& 4BE[u] - 2N(p-p_*)\|\Delta u\|^2 + CR^{-4} \\ &+ CR^{-2}\|\nabla u\|^2 + \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}}\|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})}.\end{aligned}$$

Proof. Taking account of the equation (1.1), one gets

$$\frac{d}{dt}M_\psi[u(t)] = < u(t), [\Delta^2, i\Gamma_\psi]u(t) > + < u(t), [-(I_\alpha * |u|^p)|u|^{p-2}, i\Gamma_\psi]u(t) >,$$

where $[X,Y] := XY - YX$ denotes the commutator of X and Y. According to computation done in [1], one has

$$< u(t), [\Delta^2, i\Gamma_{\psi_R}]u(t) > \le 8\|\Delta u(t)\|^2 + O(R^{-4} + R^{-2}\|\nabla u(t)\|^2).$$

Using computations in [22], it follows that

$$\begin{aligned}(N) :=& < u(t), [-(I_\alpha * |u|^p)|u|^{p-2}, i\Gamma_{\psi_R}]u(t) > \\ =& -\frac{4B}{p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u(x)|^p\,dx + O\Big(\int_{\{|x|>R\}}(I_\alpha * |u|^p)|u|^p\,dx\Big).\end{aligned}$$

Thanks to (2.1), one has

$$\begin{aligned}(I) &:= \int_{\{|x|>R\}} (I_\alpha * |u|^p)|u|^p\,dx\\ &\lesssim \|u\|^p_{\frac{2Np}{\alpha+N}} \Big(\int_{\{|x|>R\}} |u|^{\frac{2Np}{\alpha+N}}\,dx\Big)^{\frac{\alpha+N}{2N}} \lesssim \|\Delta u\|^p \|u\|^{p-1-\frac{\alpha}{N}}_{L^\infty(|x|>R)} \|u\|^{\frac{\alpha+N}{N}}.\end{aligned}$$

Take $\frac{1}{2} < \mu < \min\{2, \frac{N}{2}\}$. Taking account of (2.1) and (2.2), write

$$\begin{aligned}(I) &\lesssim \|\Delta u\|^p \|u\|^{p-1-\frac{\alpha}{N}}_{L^\infty(|x|>R)} \lesssim \|\Delta u\|^p \Big(R^{-\frac{N}{2}+\mu} \|(-\Delta)^{\frac{\mu}{2}} u\|\Big)^{p-1-\frac{\alpha}{N}}\\ &\lesssim \|\Delta u\|^p \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}} \|(-\Delta)^{\frac{\mu}{2}} u\|^{p-1-\frac{\alpha}{N}}\\ &\lesssim \|\Delta u\|^p \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}} \Big(\|u\|^{1-\frac{\mu}{2}} \|\Delta u\|^{\frac{\mu}{2}}\Big)^{p-1-\frac{\alpha}{N}}\\ &\lesssim \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}} \|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})},\end{aligned}$$

Finally

$$\begin{aligned}\frac{d}{dt} M_{\psi_R}[u] &= < u, [\Delta^2, i\Gamma_{\psi_R}]u > + < u, [-(I_\alpha * |u|^p)|u|^{p-2}, i\Gamma_{\psi_R}]u >\\ &\leq 8\|\Delta u\|^2 + CR^{-4} + CR^{-2}\|\nabla u\|^2\\ &\quad - \frac{4B}{p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u(x)|^p\,dx + O\Big(\int_{\{|x|>R\}} (I_\alpha * |u|^p)|u|^p\,dx\Big)\\ &\leq 4BE - 2N(p-p_*)\|\Delta u\|^2 + CR^{-4}\\ &\quad + CR^{-2}\|\nabla u\|^2 + \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}} \|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})}.\end{aligned}$$

□

7. **Global/non global existence of solutions.** In this section, we prove a sharp criteria of finite time blow-up/global existence of solutions to the Choquard problem (1.1) in the focusing regime. In this section one takes $\epsilon = -1$. Here and hereafter, denote, for $u \in H^2$, the scale invariant quantities

$$\mathcal{ME}[u] := \frac{E[u]^{s_c} M[u]^{2-s_c}}{E[\phi]^{s_c} M[\phi]^{2-s_c}}; \quad \mathcal{G}[u] := \frac{\|\Delta u\|^{s_c} \|u\|^{2-s_c}}{\|\Delta\phi\|^{s_c} \|\phi\|^{2-s_c}}.$$

The main result of this section reads.

Theorem 7.1. *Let $N \geq 2$, $0 < \alpha < N$ such that $\alpha > N-8$, $0 < s_c < 2$, ϕ be a ground state solution to (4.1) and a maximal solution $u \in C_{T^*}(H^2_{rd})$ of (1.1). Suppose that*

$$\mathcal{ME}[u] < 1. \tag{7.1}$$

1. *Assume that $p < 3$ and*

$$\mathcal{G}[u] > 1. \tag{7.2}$$

Then, u blows-up in finite time, i.e, $0 < T^* < \infty$ and

$$\limsup_{t \to T^*} \|\Delta u(t)\| = +\infty;$$

2. *Assume that $E(u_0) \geq 0$ and*

$$\mathcal{G}[u] < 1. \tag{7.3}$$

Then, $T^* = \infty$ and u scatters. Precisely, there exists $\psi \in H^2$ such that

$$\limsup_{t\to\infty} \|u(t) - e^{it\Delta^2}\psi\|_{H^2} = 0.$$

Remark 7.2. *1. The unnatural condition $p < 3$ which seems to be technical is due to a lack of a Virial identity similar to the NLS case;*

2. the radial condition is required for the Virial identity in the first case and is assumed for simplicity in the second case;

3. scattering is proved in the next section;

4. the proof of next auxiliary result is omitted because it follows like in [22].

Lemma 7.3. *The next conditions are invariant under the flow of the problem (1.1), 1. (7.1) and (7.2); 2. (7.1) and (7.3).*

Remark 7.4. *The global well-posedness part of Theorem 7.1 is a consequence of the second point in Lemma 7.3.*

In order to prepare the finite time blow-up part of Theorem 7.1, let us give an intermediate result about the localized variance.

Lemma 7.5. *Assume that $E(u_0) \neq 0$ and there exist $t_0 > 0$ and $\delta > 0$ such that*

$$M_{\psi_R}[u(t)] \leq -\delta \int_{t_0}^{t} \|\Delta u(\tau)\|^2 \, d\tau, \quad \forall t \geq t_0.$$

Then, $T^* < \infty$.

Proof. Using the properties of ψ, write

$$|M_{\psi_R}[u(t)]| \leq 2\|\nabla\psi_R\|_\infty \|u(t)\|\|\nabla u(t)\| \leq CR\|u_0\|\|\nabla u(t)\| \leq CR\|u_0\|^{\frac{3}{2}}\|\Delta u(t)\|^{\frac{1}{2}}.$$

Thus,

$$M_{\psi_R}[u(t)] \leq -C_R \int_{t_0}^{t} |M_{\psi_R}[u(\tau)]|^4 \, d\tau.$$

Take $z(t) := \int_{t_0}^{t} |M_{\psi_R}[u(\tau)]|^4 \, d\tau$. Then, $z' \geq C_R^4 z^4 > 0$ for $t > t_0$. Integrating the previous inequality, one obtains for some $t_* > 0$,

$$\lim_{t_*} M_{\psi_R}[u(t)] \leq -C_R \lim_{t_*} z(t) = -\infty.$$

Then, u cannot be global. Hence $T^* < \infty$. □

We are ready to prove Theorem 7.1. Assume that (7.1)-(7.2) are satisfied and take $\eta > 0$ satisfying

$$E(u_0)^{s_c} M(u_0)^{2-s_c} < [(1-\eta)E(\phi)]^{s_c} M(\phi)^{2-s_c}$$

Then, thanks to (7.2), one gets

$$(1-\eta)(B-2)\|\Delta u(t)\|^2 > BE(u_0).$$

With Theorem 6.1, for $O_R(1) \to 0$ uniformly in time, and using Young inequality via the fact that $p < 3$, one gets

$$\begin{aligned}\frac{d}{dt} M_{\psi_R}[u(t)] \leq& 4BE(u_0) - 2N(p-p_*)\|\Delta u\|^2 + CR^{-4} \\ &+ CR^{-2}\|\nabla u\|^2 + \frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}}\|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})} \\ \leq& 2\Big(2(1-\eta)(B-2) - N(p-p_*) + O_R(1)\Big)\|\Delta u\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}}\|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})}+O_R(1)\\
\leq&(-4\eta(B-2)+O_R(1))\|\Delta u(t)\|^2+O_R(1)\\
&+\frac{1}{R^{(\frac{N}{2}-\mu)(p-1-\frac{\alpha}{N})}}\|\Delta u\|^{p+\frac{\mu}{2}(p-1-\frac{\alpha}{N})}\\
\leq&[-4\eta(B-2)+O_R(1)]\|\Delta u(t)\|^2+O_R(1)\leq -2\eta(B-2)\|\Delta u(t)\|^2.
\end{aligned}
$$

The proof is a consequence of Lemma 7.5.

8. **Scattering.** This section is devoted to prove scattering of global solutions to (1.1), precisely the second part of Theorem 7.1 is proved. For a slab $I\subset\mathbb{R}$ and $p>p_*$, define the spaces

$$
S(I):=L^{2p}(I,L^{\frac{2Np(p-1)}{4+\alpha p}})\quad\text{and}\quad W(I):=L^{2p}(I,L^{\frac{2Np}{Np-4}}).
$$

Remark 8.1. *Thanks to Sobolev injection, one has*

$$
\|\cdot\|_{S(I)}\leq C\|\,|\nabla|^{s_c}.\|_{W(I)}.
$$

Proposition 8.2 (Small data)**.** *Let $u_0\in H^2$. Then, there exists $\delta>0$ such that if $\|e^{i.\Delta^2}u_0\|_{S(I)}\leq\delta$, then there exists $u\in C(I,H^2)$ solving (1.1) satisfying*

$$
\|u\|_{S(I)}\leq 2\delta\quad\textit{and}\quad\|(1+\Delta)u\|_{W(I)\cap L^\infty(I,L^2)}<cA.
$$

Proof. First, let us use a fixed point argument. For $T>0$ and $I:=(0,T)$, take the set

$$
X_{\delta,M}:=\{v\in S(I),\quad\|v\|_{S(I)}\leq 2\delta\quad\text{and}\quad\|(1+\Delta)v\|_{W(I)\cap L^\infty(I,L^2)}\leq M\}
$$

equipped with the complete distance

$$
d(u,v):=\|u-v\|_{W(I)}.
$$

Set the function

$$
\tilde v:=\phi_{u_0}(v):=e^{i.\Delta^2}u_0+i\int_0^{\cdot}e^{i(.-s)\Delta^2}(I_\alpha*|v|^p)|v|^{p-2}v(s)ds.
$$

By the Strichartz estimate Hölder and Hardy-Littlewood-Sobolev inequalities, one gets for $(q,r):=(2p,\frac{2Np}{Np-4})$ and $w:=u-v$,

$$
\begin{aligned}
d(\tilde u,\tilde v)\leq&C\|(I_\alpha*|u|^p)|u|^{p-2}u-(I_\alpha*|v|^p)|v|^{p-2}v\|_{L^{q'}(I,L^{r'})}\\
\lesssim&\|(I_\alpha*|u|^p)[|u|^{p-2}u-|v|^{p-2}v]\|_{L^{q'}(I,L^{r'})}\\
&+\|(I_\alpha*[|u|^p-|v|^p])|v|^{p-2}v\|_{L^{q'}(I,L^{r'})}\\
\lesssim&\|(I_\alpha*|u|^p)[|u|^{p-2}+|v|^{p-2}]w\|_{L^{q'}(I,L^{r'})}\\
&+\|(I_\alpha*[|u|^{p-1}+|v|^{p-1}]w)|v|^{p-2}v\|_{L^{q'}(I,L^{r'})}\\
\lesssim&(\|u\|_{S(I)}^{2(p-1)}+\|v\|_{S(I)}^{2(p-1)})\|w\|_{L^q(I,L^r)}\leq C\delta^{2(p-1)}d(u,v).
\end{aligned}
$$

Now, by the Strichartz estimate, Hardy-Littlewood-Sobolev inequality and fractional chain rule, one gets for $cA:=\frac{M}{2}$,

$$
\begin{aligned}
(I):=&\|(1+\Delta)\tilde v\|_{W(I)\cap L^\infty(I,L^2)}\\
\leq&c\|u_0\|_{\dot H^2}+C\|(1+\Delta)[(I_\alpha*|v|^p)|v|^{p-2}v]\|_{L^{q'}(I,L^{r'})}\\
\leq&\frac{M}{2}+C\|v\|_{S(I)}^{2(p-1)}\|(1+\Delta)v\|_{W(I)}\leq\frac{M}{2}+C\delta^{2(p-1)}M.
\end{aligned}
$$

Thanks to the Sobolev injection in the previous remark, yields

$$\|\tilde v\|_{S(I)} \leq \delta + C\|(1+\Delta)(\tilde v - e^{i.\Delta^2}u_0)\|_{W(I)} \leq \delta + C\delta^{2(p-1)}M.$$

Taking $\delta > 0$ small enough, it follows that ϕ_{u_0} is a contraction of $X_{\delta,M}$. Then, the fixed point principle gives the result. □

Proposition 8.3 (Long time perturbation theory). *Let $0\in I\subset\mathbb{R}$, a time slab. Take $u \in C(I,H^2)$ a solution of (1.1). Let $\tilde u \in L^\infty(I,H^2)$ satisfying $\|\tilde u\|_{L^\infty(I,H^2)\cap S(I)} \leq A$, for some constant $A > 0$. Assume that*

$$i\dot{\tilde u} + \Delta\tilde u + (I_\alpha * |\tilde u|^p)|\tilde u|^{p-2}\tilde u = e$$

and that for $(q,r) := (2p, \frac{2Np}{Np-4})$, $\epsilon > 0$,

$$\|(1+\Delta)e\|_{L^{q'}(I,L^{r'})} \leq \epsilon, \quad \|e^{i.\Delta^2}[u_0 - \tilde u_0]\|_{S(I)} \leq \epsilon.$$

Then, there exists $\epsilon_0 := \epsilon_0(A)$ such that for $0 < \epsilon < \epsilon_0$,

$$\|u\|_{S(I)} \leq C(A).$$

Proof. For $\delta = \delta(A) > 0$ small enough, split $I \subset \cup_j I_j$ such that $\|\tilde u\|_{S(I_j)} \leq \delta$. Using Duhamel formula and arguing as previously, one gets for $1 - C\delta^{2(p-1)} > 0$,

$$\|(1+\Delta)\tilde u\|_{W(I_j)} \leq CA + C\|\tilde u\|^{2(p-1)}_{S(I_j)}\|(1+\Delta)\tilde u\|_{W(I_j)} + C\|e\|_{L^{q'}(I,W^{2,r'})} \leq C(A+\epsilon).$$

Letting $I_j := [t_{-1+j}, t_j]$, one gets

$$\begin{aligned} w(t) :=& u(t) - \tilde u(t) \\ =& \int_{t_j}^t e^{i(t-t')\Delta^2}[(I_\alpha * |\tilde u + w|^p)|\tilde u + w|^{p-2}(\tilde u + w) - (I_\alpha * |\tilde u|^p)|\tilde u|^{p-2}\tilde u]\,dt' \\ &+ e^{i(t-t_j)\Delta^2}w(t_j) - \int_{t_j}^t e^{i(t-t')\Delta^2}e(t')\,ds. \end{aligned}$$

With a Picard fixed point argument and arguing as in Proposition 8.2, one solves the previous integral equation in $I_1 = [t_0, t_1] := [0, t_1]$, precisely

$$\|w\|_{S(I_1)} \leq 2\epsilon, \quad \|(1+\Delta)w\|_{W(I_1)} \leq C(\epsilon, A).$$

Now, by taking $t = t_1$ in the previous integral equality and applying $e^{i(t-t_1)\Delta^2}$, yields

$$\begin{aligned} e^{i(t-t_1)\Delta^2}w(t_1) =& \int_{t_0}^{t_1} e^{i(t-t')\Delta^2}[(I_\alpha * |\tilde u + w|^p)|\tilde u + w|^{p-2}(\tilde u + w) - (I_\alpha * |\tilde u|^p)|\tilde u|^{p-2}\tilde u]dt' \\ &+ e^{i(t-t_0)\Delta^2}w(t_0) + \int_{t_0}^{t_1} e^{i(t-t')\Delta^2}e(t')\,ds. \end{aligned}$$

Then, with similar to previous computation, one obtains

$$\|e^{i(.-t_1)\Delta^2}w(t_1)\|_{S(I)} \leq 2(\|e^{i(.-t_0)\Delta^2}w(t_0)\|_{S(I)} + C\epsilon).$$

Now, iterate the beginning with $j = 0$, and we obtain

$$\|e^{i(.-t_j)\Delta^2}w(t_j)\|_{S(I)} \leq 2^j\|e^{i(.-t_0)\Delta^2}w(t_0)\|_{S(I)} + C2^j\epsilon \leq C2^{1+j}\epsilon.$$

This finishes the proof. □

Proposition 8.4 (Scattering)**.** *Let $u \in C(\mathbb{R}, H^2)$ be a global solution to (1.1) with Strichartz norm*

$$\|u\|_{S(\mathbb{R})} < \infty \quad \textit{and} \quad \|u\|_{L^\infty(\mathbb{R},H^2)} < \infty,$$

then $u(t)$ scatters in H^2 as $t \to \infty$. Precisely, there exists $\phi \in H^2$ such that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta^2}\phi\|_{H^2} = 0.$$

Proof. Write with the integral formula

$$\begin{aligned} u &= e^{i.\Delta^2}u_0 + i\int_0^{.} e^{i(t-s)\Delta^2}[(I_\alpha * |u|^p)|u|^{p-2}u]\,ds;\\ \phi &= u_0 + i\int_0^{\infty} e^{-is\Delta^2}[(I_\alpha * |u|^p)|u|^{p-2}u]\,ds;\\ u - e^{i.\Delta^2}\phi &= -i\int_{.}^{\infty} e^{i(t-s)\Delta^2}[(I_\alpha * |u|^p)|u|^{p-2}u]\,ds. \end{aligned}$$

Using Corollary 2.2, write

$$\begin{aligned} \|\Delta\Big(u - e^{i.\Delta^2}u_0\Big)\|_{L_T^{q'}(L^{r'})} \lesssim& \|(I_\alpha * \Delta(|u|^p))|u|^{p-2}u + (I_\alpha * |u|^p)\Delta(|u|^{p-2}u)\\ &+ (I_\alpha * \nabla(|u|^p))\nabla(|u|^{p-2}u)\|_{L_T^{q'}(L^{r'})}\\ :=& (A) + (B) + (C). \end{aligned}$$

Thus, using the identity $|\Delta(|u|^p)| \leq C_p(|\Delta u||u|^{p-1} + |\nabla u|^2|u|^{p-2})$, denoting $S(I) := L^{2p}(I, L^a)$, $\frac{1}{b} = \frac{1}{2}(\frac{1}{r} + \frac{1}{a})$ and taking account of the inequality $\|\nabla\cdot\|_b^2 \leq C\|\Delta\cdot\|_r\|\cdot\|_a$ via Hardy-Littlewood-Sobolev inequality, one gets

$$(A) \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)}\|\Delta u\|_{W(\mathbb{R})} + \|\|\nabla u\|_b^2\|u\|_a^{2p-3}\|_{q'} \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)}\|\Delta u\|_{W(\mathbb{R})}.$$

With the same way, for $p \geq 3$,

$$(A) + (B) + (C) \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)}\|\Delta u\|_{W(\mathbb{R})}.$$

Now, by previous computation

$$\|\Delta u\|_{W(t,\infty)} \leq C\|u\|_{L^\infty(\mathbb{R},H^2)} + C\|u\|_{S(t,\infty)}^{2(p-1)}\|\Delta u\|_{W(t,\infty)}.$$

Taking $t > 0$ large enough such that $\|u\|_{S((t,\infty))} << 1$, then a partition of $[0,t) \subset \cup I_j$ with $\sup_j \|u\|_{S(I_j)} << 1$, this implies that

$$\|\Delta u\|_{W(\mathbb{R})} < \infty.$$

Thus, when $t \to \infty$,

$$\|\Delta(u - e^{i.\Delta^2}\phi)\|_{W(t,\infty)\cap L^\infty((t,\infty),L^2)} \leq C\|u\|_{S(t,\infty)}^{2(p-1)}\|\Delta u\|_{W(t,\infty)} \to 0.$$

With the same way, we prove that when $t \to \infty$,

$$\|u - e^{i.\Delta^2}\phi\|_{L^\infty((t,\infty),L^2)} \to 0.$$

Finally when $t \to \infty$,

$$\|u - e^{i.\Delta^2}\phi\|_{L^\infty((t,\infty),H^1)} \to 0.$$

□

8.1. **Critical solution and compactness.** In this section, we prepare the proof of the scattering part of Theorem 7.1. Let u be the solution of (1.1) such that the assumptions of the second part of Theorem 7.1 hold. Then, we know that u is global. Thus, combined with Proposition 8.4, the goal is to show that

$$u \in S(\mathbb{R}).$$

Let us prove the claim: there exists $\delta > 0$ such that if

$$E[u_0]M[u_0]^{\frac{2}{s_c}-1} < \delta \quad \text{and} \quad \|u_0\|^{\frac{2}{s_c}-1}\|\Delta u_0\| < \|\phi\|^{\frac{2}{s_c}-1}\|\Delta\phi\|,$$

then $u \in S(\mathbb{R})$. Indeed, write

$$\begin{aligned} E(u) =& \|\Delta u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx \\ \geq& \|\Delta u\|^2\Big(1 - \frac{C_{N,p,\alpha}}{p}\|u\|^A\|\Delta u\|^{B-2}\Big) \\ \geq& \|\Delta u\|^2\Big(1 - \frac{2}{A}(\frac{A}{B})^{\frac{B}{2}}\frac{\|u\|^A\|\Delta u\|^{B-2}}{\|\phi\|^{2(p-1)}}\Big). \end{aligned}$$

Taking account of Pohozaev identity, one gets $\|\Delta\phi\|^2 = \frac{B}{A}\|\phi\|^2$. Then,

$$\begin{aligned} E(u) \geq& \|\Delta u\|^2\Big(1 - \frac{2}{B}(\frac{\|\phi\|}{\|\Delta\phi\|})^{B-2}\frac{\|u\|^A\|\Delta u\|^{B-2}}{\|\phi\|^{2(p-1)}}\Big) \\ \geq& \|\Delta u\|^2\Big(1 - \frac{2}{B}\frac{\|u\|^A\|\Delta u\|^{B-2}}{\|\phi\|^A\|\Delta\phi\|^{B-2}}\Big) \\ \geq& \|\Delta u\|^2\Big(1 - \frac{2}{B}\Big[\frac{\|u\|^{\frac{A}{B-2}}\|\Delta u\|}{\|\phi\|^{\frac{A}{B-2}}\|\Delta\phi\|}\Big]^{B-2}\Big) \\ \geq& \|\Delta u\|^2\Big(1 - \frac{2}{B}\Big[\frac{\|u\|^{\frac{2}{s_c}-1}\|\Delta u\|}{\|\phi\|^{\frac{2}{s_c}-1}\|\Delta\phi\|}\Big]^{B-2}\Big) \geq \|\Delta u\|^2\Big(1 - \frac{2}{B}\Big). \end{aligned} \tag{8.1}$$

Since $p > p_*$, $B > 2$, $E(u)$ is conserved implies that $\|\Delta u(t)\|$ is bounded. The claim follows by Proposition 8.2.

Now, for each $\delta > 0$, define the set

$$S_\delta := \{u_0 \in H^2,\ E[u_0]M[u_0]^{\frac{2}{s_c}-1} < \delta \ \text{ and } \ \|u_0\|^{\frac{2}{s_c}-1}\|\Delta u_0\| < \|\phi\|^{\frac{2}{s_c}-1}\|\Delta\phi\|\}.$$

Define also $(ME)_c := \sup\{\delta > 0 \text{ s. t } u_0 \in S_\delta \Rightarrow\ u \in S(\mathbb{R})\}$. The goal is to prove that $(ME)_c = M[\phi]^{\frac{2}{s_c}-1}E[\phi]$. By contradiction, assume that

$$(ME)_c < M[\phi]^{\frac{2}{s_c}-1}E[\phi]. \tag{8.2}$$

Proposition 8.5 (Existence of wave operator)**.** *Let ϕ be a ground state solution to (4.1) and $\psi \in H^2$ satisfying*

$$\|\psi\|^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi\|^2 < \|\phi\|^{\frac{2(2-s_c)}{s_c}}E(\phi).$$

Then, there exists $v \in C(\mathbb{R}, H^2)$ a solution to (1.1) which satisfies

$$\|v_0\|^{\frac{2-s_c}{s_c}}\|\Delta v(t)\| < \|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|, \quad M(v) = \|\psi\|^2, \quad E(v) = \|\Delta\psi\|^2$$

and

$$\lim_{t\to\infty}\|v(t) - e^{it\Delta^2}\psi\|_{H^2} = 0.$$

Proof. Arguing as in the proof of Proposition 8.2, one can solve for large $t>0$, the integral equation

$$v(t):=e^{it\Delta^2}\psi - i\int_t^\infty e^{i(t-s)\Delta^2}[(I_\alpha*|v|^p)|v|^{p-2}v]\,ds.$$

Indeed, taking $t>0$ such that $\|e^{i.\Delta^2}\psi\|_{S(t,\infty)}<\delta$, where δ is given in Proposition 8.2, there exist $v\in C((t,\infty),H^2)$ a solution to (1.1) such that $\|v\|_{S(t,\infty)}\le 2\delta$ and $\|(1+\Delta)v\|_{W(t,\infty)\cap L^\infty((t,\infty),L^2)}<cA$. Write as $t\to\infty$,

$$\|v-e^{i.\Delta^2}\psi\|_{L^\infty((t,\infty),H^2)}\le C\|v\|_{S(t,\infty)}^{2(p-1)}(\|v\|_{W(t,\infty)}+\|\Delta v\|_{W(t,\infty)})\to 0.$$

This implies that $M(v)=\|\psi\|^2$. Since $p>p_*$, from Lemma 2.1 and the L^p space-time decay estimates of the linear flow associated to (1.1), one gets

$$Q(e^{it\Delta^2}\psi)\to 0 \quad \text{as} \quad t\to\infty.$$

Then, $E(v)=\lim_{t\to\infty}E(v(t))=\|\Delta\psi\|^2$. This implies that

$$M(v)^{\frac{2-s_c}{s_c}}E(v)<M(\phi)^{\frac{2-s_c}{s_c}}E(\phi).$$

Moreover,

$$\begin{aligned}\lim_{t\to\infty}\|v(t)\|^{\frac{2(2-s_c)}{s_c}}\|\Delta v(t)\|^2 =&\|\psi\|^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi\|^2\\ \le& M(\phi)^{\frac{2-s_c}{s_c}}E(\phi)=\frac{B-2}{B}\|\Delta\phi\|^2\|\phi\|^{\frac{2(2-s_c)}{s_c}}.\end{aligned}$$

Then, by Lemma 7.3, v is global, which concludes the proof. □

Proposition 8.6 (Existence of a critical solution)**.** *Assume that* $(ME)_c<M[\phi]^{\frac{2-s_c}{s_c}}E[\phi]$. *Then, there exists a global solution* u_c *to* (1.1) *with data* $u_{c,0}$ *such that* $\|u_{c,0}\|=1$,

$$\|\Delta u_{c,0}\|<\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|,\quad E[u_c]=(ME)_c\quad \textit{and}\quad \|u_c\|_{S(\mathbb{R})}=\infty.$$

Proof. There exists a sequence of solutions u_n to (1.1) with H^2 data $u_{n,0}$ (rescaled to satisfy $\|u_n\|=1$) such that $\|\Delta u_{n,0}\|<\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|, E[u_{n,0}]\to(ME)_c$ and for any n, $\|u_n\|_{S(\mathbb{R})}=\infty$. Using the profile decomposition, one gets

$$u_{n,0}=\sum_{j=1}^M e^{-it_n^j\Delta^2}\psi^j+W_n^M; \tag{8.3}$$

$$E(u_n)=\sum_{j=1}^M E(e^{-it_n^j\Delta^2}\psi^j)+E(W_n^M)+o_n(1).$$

Then,

$$(ME)_c=\sum_{j=1}^M\lim_n E(e^{-it_n^j\Delta^2}\psi^j)+\lim_n E(W_n^M).$$

With the profile decomposition,

$$\|\Delta u_{n,0}\|^2=\sum_{j=1}^M\|\Delta\psi^j\|^2+\|\Delta W_n^M\|^2+o_n(1);$$

$$1=\sum_{j=1}^M\|\psi^j\|^2+\|W_n^M\|^2+o_n(1).$$

Then, $\sum_{j=1}^M \|\Delta\psi^j\|^2 \leq \limsup_n \|\Delta u_{n.0}\|^2$ and $\sum_{j=1}^M \|\psi^j\|^2 \leq 1$. So, $\|\Delta\psi_j\| < \|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|$ and with the same way $\lim_n \|\Delta W_n^M\| < \|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|$. Thus, by (8.1), $E(e^{-it_n^j\Delta^2}\psi^j) \geq 0$, $\lim_n E(W_n^M) \geq 0$ and so

$$\lim_n E(e^{-it_n^j\Delta^2}\psi^j) \leq (ME)_c.$$

Claim : only one $\psi_j \neq 0$.

Assume the contrary of the claim. Then, $M[\psi_j] < 1$ for any j and so for large n,

$$M(e^{-it_n^j\Delta^2}\psi^j)^{\frac{2-s_c}{s_c}} E(e^{-it_n^j\Delta^2}\psi^j) < (ME)_c.$$

If $|t_n^j| \to +\infty$, assume that up to a sub-sequence, $t_n^j \to \pm\infty$. In this case, by the decay of the linear flow,

$$\lim_n \; Q(e^{-it_n^j\Delta^2}\psi^k) = 0, \quad \forall k.$$

Then,

$$\|\psi^j\|^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi^j\|^2 = \|e^{-it_n^j\Delta^2}\psi^j\|^{\frac{2(2-s_c)}{s_c}}\|\Delta[e^{-it_n^j\Delta^2}\psi^j]\|^2 < (ME)_c.$$

Then, from the existence of wave operators (Proposition 8.5) there exists $\tilde{\psi}^j$ such that $\tilde{v}$ the solution of (1.1) with data $\tilde{\psi}_j$ satisfies

$$\lim_n \|\tilde{v}(-t_n^j) - e^{-it_n^j\Delta^2}\psi^j\|_{H^2} = 0,$$

$$\|\tilde{\psi}_j\|^{\frac{2-s_c}{s_c}}\|\Delta\tilde{v}(t)\| < \|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|, \quad M(\tilde{\psi}^j) = M(\psi), \quad E(\tilde{v}) = \|\Delta\psi^j\|^2.$$

Then,

$$M(\tilde{\psi}^j)^{\frac{2-s_c}{s_c}} E(\tilde{\psi}^j) < (ME)_c, \quad \tilde{v} \in S(\mathbb{R}).$$

If, $t_n^j \to t'$ finite, then by the continuity of the linear flow in H^2, we have

$$\lim_n \|e^{-it_n^j\Delta^2}\psi^j - e^{-it'\Delta^2}\psi^j\|_{H^2} = 0.$$

Let $\tilde{\psi}^j = BNLS(t')[e^{-it'\Delta^2}\psi^j]$ so that $BNLS(-t')[\tilde{\psi}^j] = e^{-it'\Delta^2}\psi^j$.

In both cases, there is a new profile $\tilde{\psi}^j$ associated to each original profile ψ^j such that

$$\lim_n \|BNLS(-t_n^j)[\tilde{\psi}^j] - e^{-it_n^j\Delta^2}\psi^j\|_{H^2} = 0.$$

So, one can replace $e^{-it_n^j\Delta^2}\psi_j$ by $BNLS(-t_j^n)\tilde{\psi}_j$ in (8.3) to obtain

$$u_{n,0} = \sum_{j=1}^M BNLS(-t_j^n)\tilde{\psi}_j + \tilde{W}_n^M,$$

where

$$\lim_{M\to\infty}[\lim_{n\to\infty}\|e^{i.\Delta^2}\tilde{W}_n^M\|_{S(\mathbb{R})}] = 0.$$

Denote $v^j = BNLS(.)\tilde{\psi}_j$, $u_n = BNLS(.)u_{n,0}$, and $\tilde{u}_n = \sum_{j=1}^M v^j(. - t_n^j)$. Then,

$$i\dot{\tilde{u}}_n + \Delta^2\tilde{u}_n - (I_\alpha * |\tilde{u}_n|^p)|\tilde{u}_n|^{p-2}\tilde{u}_n = e_n,$$

where

$$-e_n = (I_\alpha * |\tilde{u}_n|^p)|\tilde{u}_n|^{p-2}\tilde{u}_n - \sum_{j=1}^M (I_\alpha * |v^j(. - t_n^j)|^p)|v^j(. - t_n^j)|^{p-2}v_j(. - t_n^j).$$

Using the profile decomposition, write

$$\|e^{-i.\Delta^2}(\tilde{u}_n - u_n)(0)\|_{S(\mathbb{R})}$$

$$\leq \sum_{j=1}^{M}\|e^{-i.\Delta^2}(v^j(-t_n^j)-e^{it_n^j\Delta^2}\psi^j)\|_{S(\mathbb{R})}+\|e^{-i.\Delta^2}W_n^M\|_{S(\mathbb{R})}$$
$$\leq \sum_{j=1}^{M}\|v^j(-t_n^j)-e^{-it_n^j\Delta^2}\psi^j\|_{\dot H^{s_c}}+\|e^{-i.\Delta^2}W_n^M\|_{S(\mathbb{R})}.$$

Then,

$$\lim_M \limsup_n \|e^{-i.\Delta^2}(\tilde u_n-u_n)(0)\|_{S(\mathbb{R})}=0.$$

Let us prove two claims.

Claim 1: There exists a large constant A such that for any M, there exists $n_0:=n_0(M)$ such that for $n>n_0$, $\|\tilde u_n\|_{S(\mathbb{R})}<A$.

Claim 2: For each M and $\epsilon>0$, there exist $n_1=n_1(M,\epsilon)$ such that for $n>n_1$, $\|(1+\Delta)e_n\|_{W'(\mathbb{R})}<\epsilon$.

Let M_0 be sufficiently large such that $\|e^{i.\Delta^2}\tilde W_n^{M_0}\|_{S(\mathbb{R})}<\frac{\delta}{2}$ (defined in Proposition 8.2). Thus, from the definition of $\tilde W_n^{M_0}$ that for any $j>M_0$, $\|e^{i.\Delta^2}v^j(-t_n^j)\|_{S(\mathbb{R})}<\delta$. By Proposition 8.2, one obtains

$$\|v^j(.-t_n^j)\|_{S(\mathbb{R})}<2\|e^{i.\Delta^2}v^j(-t_n^j)\|_{S(\mathbb{R})}<2\delta;$$
$$\|(1+\Delta)v^j(.-t_n^j)\|_{W(\mathbb{R})}<c\|v^j(-t_n^j)\|_{H^2}.$$

Using the identity $\lim_n\|v^j(-t_n^j)-e^{-it_n^j\Delta^2}\psi^j\|_{\dot H^2}=0$, one gets

$$\|(1+\Delta)v^j(.-t_n^j)\|_{W(\mathbb{R})}<c\|e^{-it_n^j\Delta^2}\psi^j\|_{H^2}<c\|\psi^j\|_{H^2}.$$

Thus, by elementary calculation,

$$\|(1+\Delta)\tilde u_n\|_{W(\mathbb{R})}\leq \sum_{j=1}^{M_0}\|(1+\Delta)v^j\|_{W(\mathbb{R})}+\sum_{j=1+M_0}^{M}\|(1+\Delta)v^j\|_{W(\mathbb{R})}$$
$$\leq \sum_{j=1}^{M_0}\|(1+\Delta)v^j\|_{W(\mathbb{R})}+c\sum_{j=1+M_0}^{M}\|\psi^j\|_{\dot H^2}.$$

On the other hand, by the profile decomposition,

$$\|\Delta u_{n,0}\|^2=\sum_{j=1}^{M_0}\|\Delta\psi^j\|^2+\sum_{j=1+M_0}^{M}\|\Delta\psi^j\|^2+\|\Delta W_n^M\|^2+o_n(1).$$

Then, $\sum_{j=1+M_0}^{M}\|\psi^j\|_{\dot H^2}^2$ is bounded independently of M and so $\|(1+\Delta)\tilde u_n\|_{W(\mathbb{R})}$ is bounded independently of M, for large n. By Sobolev injection $\|\tilde u_n\|_{S(\mathbb{R})}$ is bounded.Then, Claim 1 holds.

Write the expansion of e_n,

$$-e_n=(I_\alpha*|\tilde u_n|^p)|\tilde u_n|^{p-2}\tilde u_n-\sum_{j=1}^{M}(I_\alpha*|v_n^j|^p)|v_n^j|^{p-2}v_j^n$$
$$=(I_\alpha*|\sum_{j=1}^{M}v_n^j|^p)|\sum_{j=1}^{M}v_n^j|^{p-2}\sum_{j=1}^{M}v_n^j-\sum_{j=1}^{M}(I_\alpha*|v_n^j|^p)|v_n^j|^{p-2}v_n^j.$$

Then,

$$\begin{aligned}-e_n =&(I_\alpha * [|\sum_{j=1}^M v_n^j|^p - \sum_{j=1}^M |v_n^j|^p])|\sum_{j=1}^M v_n^j|^{p-2}\sum_{j=1}^M v_n^j \\ &+\sum_{j=1}^M (I_\alpha * |v_n^j|^p)|\sum_{j=1}^M v_n^j|^{p-2}\sum_{j=1}^M v_n^j - \sum_{j=1}^M (I_\alpha * |v_n^j|^p)|v_n^j|^{p-2}v_n^j \\ =&(I_\alpha * [|\sum_{j=1}^M v_n^j|^p - \sum_{j=1}^M |v_n^j|^p])|\sum_{j=1}^M v_n^j|^{p-2}\sum_{j=1}^M v_n^j \\ &+\sum_{j=1}^M (I_\alpha * |v_n^j|^p)|\sum_{j=1}^M v_n^j|^{p-2}\sum_{j\neq k=1}^M v_n^k.\end{aligned}$$

Then, taking a cross term and arguing as previously and using the inequality

$$|(\sum_{j=1}^M a_j)^r - \sum_{j=1}^M a_j^r| \le C_M \sum_{1\le j\neq k\le M} a_j a_k^{p-1}, \quad a_j \ge 0,$$

one gets as previously

$$\begin{aligned}(A) :=&\|(1+\Delta)\Big[(I_\alpha * |v_n^l|^{p-1}|v_n^m|)|v_n^j|^{p-2}v_n^k\Big]\|_{W'(\mathbb{R})} \\ =&\Big\|(1+\Delta)\Big[(I_\alpha * |v^l(.-(t_n^l-t_n^j))|^{p-1}|v^m(.-(t_n^m-t_n^j))|) \\ &|v^j(t)|^{p-2}v^k(.-(t_n^k-t_n^j))\Big]\Big\|_{W'(\mathbb{R})} \\ \lesssim&\|v^l\|_{S(\mathbb{R})}^{p-1}\|v^m\|_{S(\mathbb{R})}\|v^j\|_{S(\mathbb{R})}^{p-2}\|(1+\Delta)v^k(.-(t_n^k-t_n^j))\|_{W(\mathbb{R})}.\end{aligned}$$

By the fact that $|t_n^j - t_n^k| \to \infty$, for $1 \le k \neq j \le M$, the cross terms go to zero as $n \to \infty$ and Claim 2 is proved.

Claim 1 and Claim 2 give a contradiction with Proposition 8.3. This implies that the profile expansion is reduced to the case $\psi^1 \neq 0$ and $\psi_j = 0$ for all $j > 1$.

Let us show the existence of a critical solution. By the profile decomposition, $M(\psi^1) \le 1$ and with previously, $\lim_n E(e^{it_n^1\Delta}\psi^1) \le (ME)_c$. If $\lim_n t_n^1 = 0$, take $\tilde\psi^1 = \psi^1$ so that

$$\lim_n \|BNLS(-t_n^1)\tilde\psi^1 - e^{-it_n^1\Delta^2}\psi^1\|_{H^2} = 0.$$

If $t_n^1 \to \infty$, by the decay of the linear flow associated to (1.1), $Q(e^{-it_n^1\Delta^2}\psi^1) \to 0$. So

$$\|\Delta\psi^1\|^2 = \lim_n E(e^{-it_n^1\Delta^2}\psi^1) \le (ME)_c.$$

Therefore, by Proposition 8.5, there exist $\tilde\psi^1$ such that

$$M(\tilde\psi^1) = M(\psi^1) \le 1, \quad E(\tilde\psi^1) = \|\Delta\psi^1\|^2 \le (ME)_c$$

and

$$\lim_{n\to\infty} \|BNLS(-t_n^1)\tilde\psi^1 - e^{-it_n^1\Delta^2}\psi^1\|_{H^2} = 0.$$

Take $\tilde W_n^M = W_n^M - (BNLS(-t_n^1)\tilde\psi^1 - e^{-it_n^1\Delta^2}\psi^1)$, by Strichartz and Sobolev estimates

$$\|e^{-i.\Delta^2}\tilde W_n^M\|_{S(\mathbb{R})} \le \|e^{-i.\Delta^2}W_n^M\|_{S(\mathbb{R})} + c\|BNLS(-t_n^1)\tilde\psi^1 - e^{-it_n^1\Delta^2}\psi^1\|_{H^2}.$$

So

$$\lim_n \|e^{-i.\Delta^2}\tilde{W}_n^M\|_{S(\mathbb{R})} = \lim_n \|e^{-i.\Delta^2}W_n^M\|_{S(\mathbb{R})}.$$

Write

$$u_{n,0} = BNLS(-t_n^1)\tilde{\psi}^1 + \tilde{W}_n^M,$$

$M(\tilde{\psi}) \leq 1$, $E(\tilde{\psi}^1) \leq (ME)_c$ and $\lim_M[\lim_n \|e^{i.\Delta^2}\tilde{W}_n^M\|_{S(\mathbb{R})}] = 0$.

Let u_c be the solution to (1.1) with data $u_{c,0} := \tilde{\psi}^1$. Suppose that

$$\|BNLS(.-t_n^1)\tilde{\psi}^1\|_{S(\mathbb{R})} = \|BNLS(.)\tilde{\psi}^1\|_{S(\mathbb{R})} = \|u_c\|_{S(\mathbb{R})} < \infty.$$

Taking large M, n such that $\|e^{i.\Delta^2}\tilde{W}_n^M\|_{S(\mathbb{R})}$ is small enough, then applying the long-time perturbation theory Proposition 8.3, one obtains $\|u_n\|_{S(\mathbb{R})} < \infty$. This contradiction gives $\|u_c\|_{S(\mathbb{R})} = \infty$, which implies that $M[u_c] = 1$ and $E[u_c] = (ME)_c$. This finishes the proof. □

Proposition 8.7 (pre-compactness of the flow of the critical solution)**.** *Let u_c be as in the previous Proposition, then, the following set is pre-compact in H^2,*

$$\{u_c(t,.), \quad t \geq 0\}.$$

Proof. Denote $u := u_c$. By contradiction, suppose that $\exists \eta > 0$ and a sequence $t_n \to \infty$ such that for all $n \neq m$,

$$\|u(t_n) - u(t_m)\|_{H^2} > \eta.$$

Take the profile decomposition, $\phi_n := u(t_n) = \sum_{j=1}^M e^{-it_n^j\Delta^2}\psi^j + W_n^M$. With the energy Pythagorean expansion, one gets

$$(ME)_c = E(\phi_n) = \sum_{j=1}^M \lim_n E(e^{-it_n^j\Delta^2}\psi^j) + \lim_n E(W_n^M).$$

Since as previously, by (8.1) each energy is positive, for any j,

$$(ME)_c \geq \lim_n E(e^{-it_n^j\Delta^2}\psi^j).$$

By the profile decomposition expansion properties

$$1 = M(\phi_n) = \sum_{j=1}^M \lim_n M(\psi^j) + \lim_n M(W_n^M).$$

Following the proof of the previous Proposition, we have $\psi^1 \neq 0 = \psi^j$, for any $j \neq 1$. Thus,

$$\phi_n = e^{-it_n^1\Delta^2}\psi^1 + W_n^M.$$

Arguing as in the proof of the previous Proposition, one gets

$$1 = M(\psi^1), \quad \lim_n E(e^{-it_n^1\Delta^2}\psi^1) = (ME)_c, \quad \lim_n E(W_n^M) = 0.$$

Suppose that $t_n^1 \to \infty$ and write

$$\|e^{i.\Delta^2}u(t_n)\|_{S(\mathbb{R})} \leq \|e^{-i(t_n^1-.)\Delta^2}\psi^1\|_{S(\mathbb{R})} + \|e^{i.\Delta^2}W_n^M\|_{S(\mathbb{R})}.$$

Since for large n, $\|e^{i.\Delta^2}W_n^M\|_{S(\mathbb{R})} \leq \delta$ and $\lim_n \|e^{-i(t_n^1-.)\Delta^2}\psi^1\|_{S(\mathbb{R})} = 0$, one gets a contradiction with the small data scattering. Then, $t_n^1 \to t^1$ up to a sub-sequence. In such a case, because $e^{it_n^1\Delta^2}\psi^1 \to e^{it^1\Delta^2}\psi^1$ in H^2, this implies that ϕ_n converges in H^2, which contradicts the beginning and concludes the proof. □

Proposition 8.8. *Let u be a solution to* (1.1) *such that $\{u(t)|, \quad t>0\}$ is pre-compact in H^2. Then, for each $\epsilon>0$, there exists $R>0$ such that*

$$\int_{|x|>R}\Big(|\Delta u|^2+|u|^2+\frac{1}{p}(I_\alpha*|u|^p)|u|^p\Big)\,dx<\epsilon.$$

Proof. Otherwise, there exist $\epsilon>0$ and a real numbers sequence t_n such that for any $R>0$,

$$\int_{|x|>R}\Big(|\Delta u(t_n)|^2+|u(t_n)|^2+\frac{1}{p}(I_\alpha*|u(t_n)|^p)|u(t_n)|^p\Big)\,dx>\epsilon.$$

Since $\{u(t)|, \quad t>0\}$ is pre-compact, for a sub-sequence $u(t_n)\to\phi$ in H^2. Then, for any $R>0$,

$$\int_{|x|>R}\Big(|\Delta\phi|^2+|\phi|^2+\frac{1}{p}(I_\alpha*|\phi|^p)|\phi|^p\Big)\,dx\geq\epsilon.$$

This contradiction ends the proof. □

8.2. **Rigidity Theorem.** In this section, let us prove a Liouville-type theorem.

Proposition 8.9. *Let $N\geq2$, $0<\alpha<N$ such that $\alpha>N-8$, $0<s_c<2$, ϕ be a ground state solution to* (4.1) *satisfying* (7.1) *and* (7.3)*. Let $u\in C(\mathbb{R},H^2)$ be a global solution of* (1.1)*. If $\{u(t),\, t>0\}$ is pre-compact, then $u_0=0$.*

Proof. With the previous computation via Proposition 8.8 and the previous proposition

$$\begin{aligned}\frac{d}{dt}M_\psi[u(t)]=&8\|\Delta u(t)\|^2-\frac{4B}{p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u(x)|^p\,dx\\&+O(R^{-4}+R^{-2}\|\nabla u(t)\|^2)+O\Big(\int_{\{|x|>R\}}(I_\alpha*|u|^p)|u|^p\,dx\Big)\\\leq&8\|\Delta u(t)\|^2-\frac{4B}{p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u(x)|^p\,dx+O_R(1).\end{aligned}$$

Claim: there exists $\delta>0$ such that for large $R>0$,

$$4\|\Delta u\|^2-\frac{2B}{p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u(x)|^p\,dx+o_R(1)>\delta\|\Delta u_0\|^2.$$

This implies that

$$|M_{\psi_R}(t)-M_{\psi_R}(0)|\geq\delta t\|\Delta u_0\|^2.$$

On the other hand

$$|M_{\psi_R}(t)-M_{\psi_R}(0)|\leq C_R\|\psi\|^2_{H^2}.$$

Then, $u_0=0$.

It remains to prove the claim. Indeed, since u_0 satisfies (7.1) and (7.3), there exists $\delta>0$ such that

$$E(u)^{s_c}M(u)^{2-s_c}<(1-\delta)E(\phi)^{s_c}M(\phi)^{2-s_c};\quad \|\Delta u_0\|^2<(1-\delta)x_1,$$

where we take the notations of the proof of Lemma 7.3. Now, $f((1-\delta)x_1)=(1-\frac{2}{B}[(1-\delta)x_1]^{\frac{B}{2}-1})(1-\delta)x_1>(1-\delta)f(x_1)$. Then,

$$f(X(t))\leq E(u)<(1-\delta)f(x_1)<f((1-\delta)x_1);\quad X(0)<(1-\delta)x_1.$$

A continuity argument gives

$$\|\Delta u(t)\|^2<(1-\delta)x_1,\quad \text{on}\quad \mathbb{R}.$$

Take the function $F(x) := x^2 - x^B$ and compute using Theorem 3.1,

$$\begin{aligned}
&F\Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}\|\Delta u\|}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)\\
=&\Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}\|\Delta u\|}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^2 - \Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}\|\Delta u\|}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^B\\
\leq&\Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}\|\Delta u\|}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^2 - \Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^B\Big(\frac{\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx}{C_{N,p,\alpha}\|u\|^A}\Big)\\
\leq&\Big(\frac{\|u\|^{\frac{2-s_c}{s_c}}\|\Delta u\|}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^2 - \frac{B}{2p}\Big(\frac{1}{\|\phi\|^{\frac{2-s_c}{s_c}}\|\Delta\phi\|}\Big)^2 M(u_0)^{\frac{2-s_c}{s_c}}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx.
\end{aligned}$$

Now, since $B > 2$, there exists $C_\delta > 0$ such that $F(x) > C_\delta x^2$ for $0 < x < 1-\delta$. Then, on $\mathbb{R}$,

$$\|\Delta u\|^2 - \frac{B}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx > C_\delta\|\Delta u\|^2.$$

The claim follows by the previous inequality via (8.1). □

8.3. **Proof of scattering.** Thanks to Proposition 8.7, the critical solution u_c constructed in Proposition 8.6 satisfies the hypotheses in Proposition 8.9. Therefore, to complete the proof of Theorem 7.1, we apply Proposition 8.9 to u_c and find that $u_{c,0} = 0$, which contradicts the fact that $\|u_c\|_{S(\mathbb{R})} = \infty$. This contradiction shows that (8.2) is false. Thus, by Proposition 8.4, H^2 scattering holds.

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