Research Article

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Positive solutions of fractional elliptic equation with critical and singular nonlinearity

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Abstract: In this article, we study the following fractional elliptic equation with critical growth and singular nonlinearity:

$$(-\Delta)^s u = u^{-q} + \lambda u^{2_s^*-1}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\lambda > 0$, $q > 0$ and $2_s^* = \frac{2n}{n-2s}$. We use variational methods to show the existence and multiplicity of positive solutions with respect to the parameter λ.

Keywords: Nonlocal operator, fractional Laplacian, very singular nonlinearities, variational methods in non smooth analysis

MSC 2010: 35R11, 35R09, 35A15

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ (at least C^2), $n > 2s$ and $s \in (0,1)$. We consider the following problem with singular nonlinearity:

$$(-\Delta)^s u = u^{-q} + \lambda u^{2_s^*-1}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \tag{P_λ}$$

where $\lambda > 0$, $0 < q$, $2_s^* = \frac{2n}{n-2s}$ and $(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x) = 2C_s^n \left(\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\right),$$

where P.V. denotes the Cauchy principal value and $C_s^n = \pi^{-n/2} 2^{2s-1} s\Gamma(\frac{n+2s}{2})/\Gamma(1-s)$, with Γ being the Gamma function. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [3] for instance.

In the local setting ($s = 1$), the paper by Crandal, Rabinowitz and Tartar [10] is the starting point on semilinear problems with a singular nonlinearity. From this pioneering work, a lot of contributions have been made, related to existence, multiplicity, stability and regularity results on problems involving singular nonlinearities. We refer the survey papers [20, 29] for more details and references about the topic. Among the works dealing with elliptic equations with singular nonlinearities and critical growth terms, we

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cite [1, 17–19, 27, 28, 30, 31] and references therein, with no attempt to provide an exhaustive list. In [27], Haitao explored existence and multiplicity results for the maximal range of the parameter λ, when $0 < q < 1$, using monotone iterations and the mountain pass lemma in the spirit of [2]. The singular problem for the case $1 < q < 3$ is studied in [1, 12, 25], whereas, using the notion of very weak solutions introduced in [14, 15], Díaz, Hernández and Rakotoso in [13] proved the existence and regularity of weak solutions for any $q > 0$. In the quasilinear case with p-Laplacian, the multiplicity results are proved using Sobolev instead of Hölder minimizers when $0 < q < 1$. These results for $q > 1$ are still open in the non radial case. For related results, we refer to [11, 23, 24, 26, 28] and references therein. For the case $q > 3$, Hirano, Saccon and Shioji in [31] studied the existence of L^1_{loc} solutions u such that $(u-\epsilon)^+ \in H^1_0(\Omega)$ for all $\epsilon > 0$, using variational methods and the critical point theory of non-smooth analysis.

Recently, the study of fractional elliptic equations attracted lot of interests by researchers in nonlinear analysis. Subcritical growth problems (without singular nonlinearity) are studied in [8, 34–36, 42, 44] and Brezis–Nirenberg type critical exponent (and non singular) problems are studied in [6, 37, 38, 43, 45, 46]. We refer also to the survey about variational methods for non local equations [33]. In [5], Barrios et al. considered the problem

$$(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + Mu^p, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < p < 2^*_s - 1$ and $f \in L^m(\Omega)$, with $m \geq 1$, is a nonnegative function. Therein they studied the existence of distributional solutions using the uniform estimates of $\{u_n\}$, which are the unique solutions of regularized problems with the singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{n})^{-\gamma}$. They also discussed multiplicity results when $M > 0$ and for small λ in the subcritical case. The critical exponent problem with singular nonlinearity $\lambda u^{-q} + u^{2^*_s - 1}$, $0 < q < 1$, is recently studied in [39]. To the best of our knowledge, there are no works on existence results when $q > 1$.

In this paper we study the existence and multiplicity of positive solutions to a class of problems with a singular type nonlinearity $\lambda u^{-q} + u^{2^*_s - 1}$ for all $q > 0$ in the spirit of [31]. Besides, the functional

$$J(u) = \frac{C^n_s}{2} \|u\|^2_{H^s_0(\Omega)} - \frac{1}{1-q} \int_\Omega |u|^{1-q}\,dx - \frac{\lambda}{2^*_s} \int_\Omega |u|^{2^*_s}\,dx$$

(taking $q \neq 1$ for simplicity), associated to problem (P_λ), is not differentiable, even in the sense of Gâteaux. For the case $0 < q < 1$, the functional I is continuous on X_0, but when $q \geq 1$, the functional I is neither defined on the whole space nor it is continuous on $D(I) \equiv \{u \in H^s_0(\Omega) : I(u) < \infty\}$. With these difficulties and taking into account the non local feature of the operator, it is not easy to treat the problem with the usual variational approach. Another difficulty arises in showing that the weak solutions of (P_λ) are classical because the standard bootstrap arguments may not work. Overcoming these difficulties, we prove existence, multiplicity and regularity of solutions for (P_λ). For that we appeal to the critical point theory from non-smooth analysis. Precisely, we use a variant of the linking theorem (see Theorem 2.4) as in [31]. We also use a suitable positive subsolution combined with a weak comparison principle in the non local setting, in order to control the behavior of the singular nonlinearity in the variational setting of (P_λ).

The paper is organized as follows. In Section 2, we recall some results from non-smooth analysis and give the functional setting for the fractional Laplacian.

In Section 3, we prove the existence of the first solution by Perron's method for non-smooth functionals. Here, we adapt the variational approach in the work of Hirano, Saccon and Shioji [31] to the non local setting. We obtain our results using an approach based on non-smooth analysis, considering solutions of (P_λ) as critical points of I in some suitable non-smooth sense.

In Section 4, we prove the multiplicity result stated in Theorem 2.10. For that we show that the energy functional possesses a linking geometry and apply an appropriate version of the linking theorem. We point out that the multiplicity result obtained here is sharp in the sense that the problem has no solution outside the interval where multiplicity fails.

Finally, in Section 5, we extend the main results obtained in Section 3 and 4 to dimension one. In this case, the critical growth is given by the Orlicz space imbedding, stated in Theorem 5.1. Applying the harmonic extension introduced in [9], we study an equivalent local problem as in [8, 21, 22].

We use the following notations:
- For two real valued functions u and v, we define $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$.
- We say $u > v$ in Ω if $\operatorname{ess\,inf}_K u - v > 0$ for any compact subset K of Ω.
- We denote by $|\cdot|_p$ the standard norm in $L^p(\Omega)$, $1 \le p \le \infty$.
- For a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote the partial derivative $\frac{\partial f}{\partial u}(x, u)$ by $f'(x, u)$.
- We set $d(x) := \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$.

2 Preliminaries and main results

We recall some definitions for the critical point of a non-smooth function, definitions of function spaces and results that are required in later sections.

2.1 Some definitions and results from non smooth analysis

Definition 2.1. Let H be a Hilbert space and $I\colon H \to (-\infty, \infty]$ be a proper (i.e., $I \not\equiv \infty$) lower semicontinuous functional.

(i) Let $D(I) = \{u \in H : I(u) < \infty\}$ be the domain of I. For every $u \in D(I)$, we define the Fréchet sub-differential of I at u as the set
$$\partial^- I(u) = \Big\{\alpha \in H : \varliminf_{v \to u} \frac{I(v) - I(u) - \langle \alpha, v - u\rangle}{\|v - u\|_H} \ge 0\Big\}.$$

(ii) For every $u \in H$, we define
$$|||\partial^- I(u)||| = \begin{cases} \min\{\|\alpha\|_H : \alpha \in \partial^- I(u)\} & \text{if } \partial^- I(u) \neq \emptyset, \\ \infty & \text{if } \partial^- I(u) = \emptyset. \end{cases}$$

We know that $\partial^- I(u)$ is a closed convex set which may be empty. If $u \in D(I)$ is a local minimizer for I, then it can be seen that $0 \in \partial^- I(u)$.

Remark 2.2. We remark that if $I_0\colon H \to (-\infty, \infty]$ is a proper, lower semicontinuous, convex functional, $I_1\colon H \to \mathbb{R}$ is a C^1-functional and $I = I_1 + I_0$, then $\partial^- I(u) = \nabla I_1(u) + \partial I_0(u)$ for every $u \in D(I) = D(I_0)$, where ∂I_0 denotes the usual subdifferential of the convex functional I_0. Thus, u is said to be a critical point of I if $u \in D(I_0)$ and for every $v \in H$, we have
$$\langle \nabla I_1(u), v - u\rangle + I_0(v) - I_0(u) \ge 0.$$

Definition 2.3. For a proper, lower semicontinuous functional $I\colon H \to (-\infty, \infty]$, we say that I satisfies Cerami's variant of the Palais–Smale condition at level c (in short, I satisfies $(\mathrm{CPS})_c$), if any sequence $\{u_n\} \subset D(I)$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)|||\partial^- I(u_n)||| \to 0$ has a strongly convergent subsequence in H.

Analogous to the mountain pass theorem, we have the following linking theorem for non-smooth functionals.

Theorem 2.4 (see [31, Theorem 2]). *Let H be a Hilbert space. Assume $I = I_0 + I_1$, where $I_0\colon H \to (-\infty, \infty]$ is a proper, lower semicontinuous, convex functional and $I_1\colon H \to \mathbb{R}$ is a C^1-functional. Let D^n, S^{n-1} denote, respectively, the closed unit ball and its boundary in $\mathbb{R}^n$, and let $\psi\colon S^{n-1} \to D(I)$ be a continuous function such that*
$$\Phi := \{\varphi \in C(D^n, D(I)) : \varphi|_{S^{n-1}} = \psi\} \neq \emptyset.$$
Let A be a relatively closed subset of $D(I)$ such that
$$A \cap \psi(S^{n-1}) = \emptyset, \qquad A \cap \varphi(D^n) \neq \emptyset \quad \text{for all } \varphi \in \Phi \qquad \text{and} \qquad \inf I(A) \ge \sup I(\psi(S^{n-1})).$$
Define
$$c := \inf_{\varphi \in \Phi} \sup_{x \in D^n} I(\varphi(x)).$$
Assume that c is finite and that I satisfies $(\mathrm{CPS})_c$. Then there exists $u \in D(I)$ such that $I(u) = c$ and $0 \in \partial^- I(u)$. Furthermore, if $\inf I(A) = c$, then there exists $u \in A \cap D(I)$ such that $I(u) = c$ and $0 \in \partial^- I(u)$.

2.2 Functional setting and preliminaries

In [45], Servadei and Valdinoci discussed the Dirichlet boundary value problem for the fractional Laplacian using variational techniques. Due to the nonlocalness of the fractional Laplacian, they introduced the function space $(X_0, \|\cdot\|_{X_0})$. The space X is defined as

$$X = \left\{u \,\middle|\, u\colon \mathbb{R}^n \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x)-u(y))}{|x-y|^{n/2+s}} \in L^2(Q)\right\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + [u]_X\,,$$

where

$$[u]_X = \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} = \left(\frac{1}{C_s^n}\int_\Omega u(-\Delta)^s u\,dx\,dy\right)^{1/2}.$$

Then we define

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Also, there exists a constant $C > 0$ such that $\|u\|_{L^2(\Omega)} \le C[u]_X$ for all $u \in X_0$. Hence, $\|u\| = [u]_X$ is a norm on $(X_0, \|\cdot\|)$ and X_0 is a Hilbert space. Note that the norm $\|\cdot\|$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$. We denote $\|\cdot\| = [\,\cdot\,]_X$ the norm in X_0. From the embedding results, we know that X_0 is continuously, and compactly embedded in $L^r(\Omega)$ when $1 \le r < 2_s^*$ and the embedding is continuous but not compact if $r = 2_s^*$. We define

$$S_s = \inf_{u \in X_0\setminus\{0\}} \frac{\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\left(\int_\Omega |u|^{2_s^*}\,dx\right)^{2/2_s^*}}.$$

Consider the family of functions $\{U_\epsilon\}$ defined as

$$U_\epsilon(x) = \epsilon^{-(n-2s)/2} u^*\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^n,$$

where

$$u^*(x) = \bar{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \quad \bar{u}(x) = \frac{\tilde{u}(x)}{|u|_{2_s^*}} \quad \text{and} \quad \tilde{u}(x) = \alpha(\beta^2 + |x|^2)^{-(n-2s)/2},$$

with $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta > 0$ being fixed constants. Then, for each $\epsilon > 0$, U_ϵ satisfies

$$(-\Delta)^s u = |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^n,$$

and verifies the equality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} |U_\epsilon|^{2_s^*}\,dx = S_s^{n/(2s)}.$$

For a proof, we refer to [45].

Definition 2.5. A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is said to be a weak solution of (P_λ) if the following hold:
(i) $\inf_{x\in K} u(x) > 0$ for every compact subset $K \subset \Omega$,
(ii) u solves the PDE in (P_λ) in the sense of distributions,
(iii) $(u-\epsilon)^+ \in X_0$ for every $\epsilon > 0$.

In order to prove the existence results for (P_λ), we translate the problem by the solution of the purely singular problem:

$$(-\Delta)^s u = u^{-q}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \tag{P_0}$$

In [5], it is shown that the problem (P_0) has a minimal solution $\bar{u} \in L^\infty(\Omega)$ (by construction). Now we consider the following translated problem:

$$(-\Delta)^s u + \bar{u}^{-q} - (u+\bar{u})^{-q} = \lambda(u+\bar{u})^{2_s^*-1}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \tag{$\bar{P}_\lambda$}$$

Clearly, we can notice that $u+\bar{u}$ is a solution of (P_λ) if and only if $u \in X_0$ solves $(\bar{P}_\lambda)$ in the sense of distributions, and hence it is sufficient to show existence and multiplicity results for $(\bar{P}_\lambda)$. We define the function $g\colon \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$g(x,s) = \begin{cases} (\bar{u}(x))^{-q} - (s+\bar{u}(x))^{-q} & \text{if } s+\bar{u}(x) > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

We can easily see that g is nonnegative and non-decreasing in s. The required measurability of $g(\cdot\,,s)$ follows from [31, Lemmas 1 and 2]. We now define the notions of subsolution and supersolution for problem $(\bar{P}_\lambda)$.

Definition 2.6. $\phi \in X$ is called a subsolution (resp. a supersolution) of $(\bar{P}_\lambda)$ if the following hold:
(i) $\phi^+ \in X_0$ (resp. $\phi^- \in X_0$),
(ii) $g(\cdot\,,\phi) \in L^1_{\mathrm{loc}}(\Omega)$,
(iii) For all $w \in X_0$, $w \geq 0$, we have

$$C_s^n \int_Q \frac{(\phi(x)-\phi(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,\phi) - \lambda(\phi+\bar{u})^{2_s^*-1})w\,dx \leq 0 \quad (\text{resp.} \geq 0).$$

Definition 2.7. A function ϕ is a weak solution of $(\bar{P}_\lambda)$ if it is both a subsolution and a supersolution of $(\bar{P}_\lambda)$. That is, $\phi \in X_0$, $g(\cdot\,,\phi) \in L^1_{\mathrm{loc}}(\Omega)$ and for all $\psi \in C_0^\infty(\Omega)$,

$$C_s^n \int_Q \frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,\phi)\psi - \lambda(\phi+\bar{u})^{2_s^*-1}\psi)\,dx = 0.$$

Definition 2.8. A nonnegative function $u \in X_0$ is called positive weak solution to $(\bar{P}_\lambda)$ if u satisfies Definition 2.7 and $\operatorname{ess\,inf}_K u > 0$ for any compact set K of Ω.

Definition 2.9. We say ϕ is a strict subsolution (resp. strict supersolution) of $(\bar{P}_\lambda)$ if ϕ is a subsolution (resp. a supersolution) and

$$C_s^n \int_Q \frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,\phi)\psi - \lambda(\phi+\bar{u})^{2_s^*-1}\psi)\,dx < 0 \quad (\text{resp.} > 0)$$

for all $\psi \in X_0 \setminus \{0\}$ and $\psi \geq 0$.

With this introduction we state our main theorem.

Theorem 2.10. *There exist $\Lambda > 0$ and $\alpha \in (0,1)$ such that the following hold:*
(i) *(P_λ) admits at least two positive solutions in $C^\alpha_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ for every $\lambda \in (0,\Lambda)$.*
(ii) *(P_λ) admits no solution for $\lambda > \Lambda$.*
(iii) *(P_Λ) admits at least one positive solution $u_\Lambda \in C^\alpha_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$.*

3 Regularity of weak solutions of $(\bar{P}_\lambda)$

In this section, we shall prove some regularity properties of positive weak solutions of $(\bar{P}_\lambda)$. We will need the following important lemma.

Lemma 3.1. *For each $w \in X_0$, $w \geq 0$, there exists a sequence $\{w_k\}$ in X_0 such that $w_k \to w$ strongly in X_0, where $0 \leq w_1 \leq w_2 \leq \cdots$ and w_k has compact support in Ω for each k.*

Proof. Let $w \in X_0$, $w \geq 0$ and $\{\psi_k\}$ be sequence in $C_c^\infty(\Omega)$ such that ψ_k is nonnegative and converges strongly to w in X_0. Define $z_k = \min\{\psi_k, w\}$. Then $z_k \to w$ strongly to w in X_0. Now we set $w_1 = z_{r_1}$, where $r_1 > 0$ is such that $\|z_{r_1} - w\| \leq 1$. Then $\max\{w_1, z_m\} \to w$ strongly as $m \to \infty$, thus we can find $r_2 > 0$ such that $\|\max\{w_1, z_{r_2}\} - w\| \leq 1/2$. We set $w_2 = \max\{w_1, z_{r_2}\}$, and get that $\max\{w_2, z_m\} \to w$ strongly as $m \to \infty$.

Consequently, by induction, we set $w_{k+1} = \max\{w_k, z_{r_{k+1}}\}$ to obtain the desired sequence, since we can see that $w_k \in X_0$ has compact support for each k and $\|\max\{w_k, z_{r_{k+1}}\} - w\| \le 1/(k+1)$, which imply that $\{w_k\}$ converges strongly to w in X_0 as $k \to \infty$. □

Lemma 3.2. *Suppose that u is a nonnegative weak solution of $(\bar{P}_\lambda)$. Then, for each $w \in X_0$, $g(x,u)w \in L^1(\Omega)$ and*

$$C_s^n \int_Q \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,u) - \lambda(u+\bar{u})^{2_*^s-1})w\,dx = 0.$$

Proof. Let $w \in X_0$, $w \ge 0$. By Lemma 3.1, we obtain a sequence $\{w_k\} \in X_0$ such that $\{w_k\} \to w$ strongly in X_0, each w_k has compact support in Ω and $0 \le w_1 \le w_2 \le \cdots$. For each fixed k, we can find a sequence $\{\psi_n^k\} \subset C_c^\infty(\Omega)$ such that $\psi_n^k \ge 0$, $\bigcup_n \operatorname{supp} \psi_n^k$ is contained in a compact subset of Ω, $\{\|\psi_n^k\|_\infty\}$ is bounded and $\|\psi_n^k - w_k\| \to 0$ strongly as $n \to \infty$. Since u is a weak solution of $\bar{P}_\lambda$, we get

$$C_s^n \int_Q \frac{(u(x)-u(y))(\psi_n^k(x)-\psi_n^k(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega g(x,u)\psi_n^k\,dx + \lambda \int_\Omega (u+\bar{u})^{2_*^s-1}\psi_n^k\,dx.$$

By Lebesgue's dominated convergence theorem, as $n \to \infty$, we get

$$\int_\Omega g(x,u)w_k\,dx = -C_s^n \int_Q \frac{(u(x)-u(y))(w_k(x)-w_k(y))}{|x-y|^{n+2s}}\,dx\,dy + \lambda \int_\Omega (u+\bar{u})^{2_*^s-1}w_k\,dx.$$

Using the monotone convergence theorem and the nonnegativity of u, we obtain $g(x,u)w \in L^1(\Omega)$ and

$$\int_\Omega g(x,u)w\,dx = -C_s^n \int_Q \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy + \lambda \int_\Omega (u+\bar{u})^{2_*^s-1}w\,dx.$$

If $w \in X_0$, then $w = w^+ - w^-$ and $w^+, w^- \ge 0$. Since we proved the lemma for each $w \in X_0$, $w \ge 0$, we obtain the conclusion. □

Theorem 3.3. *Any nonnegative weak solution of $(\bar{P}_\lambda)$ belongs to $L^\infty(\Omega)$.*

Proof. We follow the bootstrap argument used in [4]. We use the following inequality for the fractional Laplacian:

$$(-\Delta)^s \varphi(u) \le \varphi'(u)(-\Delta)^s u, \tag{3.1}$$

where φ is a convex and differentiable function. We define

$$\varphi(t) = \varphi_{T,\beta}(t) \begin{cases} 0 & \text{if } t \le 0, \\ t^\beta & \text{if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta & \text{if } t \ge T, \end{cases}$$

where $\beta > 1$ and $T > 0$ is large. Then φ is Lipschitz with constant $M = \beta T^{\beta-1}$ and $\varphi(u) \in X_0$. Consequently,

$$\|\varphi(u)\| = \left(\int_Q \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} \le \left(\int_Q \frac{M^2|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} = M^2\|u\|.$$

Using $\|\varphi(u)\| = (1/C_s^n)^{1/2}\|(-\Delta)^{s/2}\varphi(u)\|_2$, we obtain

$$\frac{1}{C_s^n}\int_\Omega \varphi(u)(-\Delta)^s\varphi(u) = \|\varphi(u)\|^2 \ge S_s|\varphi(u)|_{2_s^*}^2, \tag{3.2}$$

where S_s is as defined in Section 1. Since φ is convex and $\varphi(u)\varphi'(u) \in X_0$, we obtain

$$\begin{aligned}\int_\Omega \varphi(u)(-\Delta)^s\varphi(u)\,dx &\le \int_\Omega \varphi(u)\varphi'(u)(-\Delta)^s u\,dx \\ &= \int_\Omega \varphi(u)\varphi'(u)(-g(x,u) + \lambda(u+\bar{u})^{2_s^*-1})\,dx. \end{aligned}\tag{3.3}$$

Therefore, using (3.2) and (3.3), we obtain

$$|\varphi(u)|_{2_s^*}^2 \le C \int_\Omega \varphi(u)\varphi'(u)(-g(x,u) + \lambda(u+\bar{u})^{2_s^*-1})\,dx$$

for some constant $C > 0$. We have $u\varphi'(u) \le \beta\varphi(u)$ and $\varphi'(u) \le \beta(1+\varphi(u))$, which gives

$$\begin{aligned}
C\int_\Omega \varphi(u)\varphi'(u)(-g(x,u) + \lambda(u+\bar{u})^{2_s^*-1})\,dx &\le C\lambda \int_\Omega \varphi(u)\varphi'(u)(u+\bar{u})^{2_s^*-1}\,dx \\
&\le 2^{2_s^*-2}C\lambda\beta\Big(\int_\Omega (\phi(u))^2 u^{2_s^*-2}\,dx + \int_\Omega (\varphi(u) + (\varphi(u))^2)\bar{u}^{2_s^*-1}\,dx\Big) \\
&\le 2^{2_s^*-2}C\lambda\beta\Big(\int_\Omega (\varphi(u))^2 u^{2_s^*-2}\,dx + \|\bar{u}\|_\infty^{2_s^*-1}\int_\Omega (\varphi(u) + (\varphi(u))^2)\,dx\Big) \\
&\le C_1\beta\Big(\int_\Omega (\varphi(u))^2 u^{2_s^*-2}\,dx + \int_\Omega (\varphi(u) + (\varphi(u))^2)\,dx\Big),
\end{aligned}$$

where $C_1 = 2^{2_s^*-2}\lambda C\max\{1, \|\bar{u}\|_\infty\}$. Thus, we have

$$|\varphi(u)|_{2_s^*}^2 \le C_1\beta\Big(\int_\Omega (\varphi(u))^2 u^{2_s^*-2}\,dx + \int_\Omega (\varphi(u) + (\varphi(u))^2)\,dx\Big). \tag{3.4}$$

Next we claim that $u \in L^{\beta_1 2_s^*}(\Omega)$, where $\beta_1 = 2_s^*/2$. Fixing some K whose appropriate value will be determined later, we can write

$$\begin{aligned}
\int_\Omega (\varphi(u))^2 u^{2_s^*-2}\,dx &= \int_{u\le K} (\varphi(u))^2 u^{2_s^*-2}\,dx + \int_{u>K} (\varphi(u))^2 u^{2_s^*-2}\,dx \\
&\le K^{2_s^*-2}\int_{u\le K} (\varphi(u))^2\,dx + \Big(\int_\Omega (\varphi(u))^{2_s^*}\,dx\Big)^{2/2_s^*}\Big(\int_{u>K} u^{2_s^*}\,dx\Big)^{(2_s^*-2)/2_s^*}.
\end{aligned}$$

Using the monotone convergence theorem, we choose K such that

$$\Big(\int_{u>K} u^{2_s^*}\,dx\Big)^{(2_s^*-2)/2_s^*} \le \frac{1}{2C_1\beta},$$

and this gives

$$\Big(\int_\Omega (\varphi(u))^{2_s^*}\,dx\Big)^{2/2_s^*} \le 2C_1\beta\Big(\int_\Omega (\varphi(u) + (\varphi(u))^2)\,dx + K^{2_s^*-2}\int_{u\le K} (\varphi(u))^2\,dx\Big). \tag{3.5}$$

Using $\varphi_{T,\beta_1}(u) \le u^{\beta_1}$ in the right-hand side of (3.5) and then letting $T \to \infty$ in the left-hand side, we obtain

$$\Big(\int_\Omega u^{2_s^*\beta_1}\,dx\Big)^{2/2_s^*} \le 2C_1\beta_1\Big(\int_\Omega (u^{2_s^*/2} + u^{2_s^*})\,dx + K^{2_s^*-2}\int_\Omega u^{2_s^*}\,dx\Big),$$

since $2\beta_1 = 2_s^*$. This proves the claim. Again, from (3.4), using $\varphi_{T,\beta}(u) \le u^\beta$ in the right-hand side and then letting $T \to \infty$ in the left-hand side, we obtain

$$\begin{aligned}
\Big(\int_\Omega u^{2_s^*\beta}\,dx\Big)^{2/2_s^*} &\le 2C_1\beta\Big(\int_\Omega (u^\beta + u^{2\beta})\,dx + \int_\Omega u^{2\beta+2_s^*-2}\,dx\Big) \\
&\le 2C_1\beta\Big(2|\Omega| + 2\int_{u\ge 1} u^{2\beta+2_s^*-2}\,dx + \int_\Omega u^{2\beta+2_s^*-2}\,dx\Big) \\
&\le 2C_2\beta\Big(1 + \int_\Omega u^{2\beta+2_s^*-2}\,dx\Big),
\end{aligned}$$

where $C_2 > 0$ is a constant (independent of β). With further simplifications, we get

$$\left(1+\int_\Omega u^{2_s^*\beta}\,dx\right)^{1/[2_s^*(\beta-1)]} \le C_\beta^{1/[2(\beta-1)]}\left(1+\int_\Omega u^{2\beta+2_s^*-2}\,dx\right)^{1/[2(\beta-1)]}, \tag{3.6}$$

where $C_\beta = 4C_2\beta(1+|\Omega|)$. For $m \ge 1$, let us define β_{m+1} inductively by

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m,$$

that is,

$$(\beta_{m+1}-1) = \frac{2_s^*}{2}(\beta_m - 1) = \left(\frac{2_s^*}{2}\right)^m(\beta_1 - 1).$$

Hence, from (3.6) it follows that

$$\left(1+\int_\Omega u^{2_s^*\beta_{m+1}}\,dx\right)^{1/[2_s^*(\beta_{m+1}-1)]} \le C_{\beta_{m+1}}^{1/[2(\beta_{m+1}-1)]}\left(1+\int_\Omega u^{2_s^*\beta_m}\,dx\right)^{1/[2_s^*(\beta_m-1)]},$$

where $C_{\beta_{m+1}} = 4C_2\beta_{m+1}(1+|\Omega|)$. Setting

$$D_{m+1} := \left(1+\int_\Omega u^{2_s^*\beta_m}\right)^{1/[2_s^*(\beta_m-1)]},$$

we obtain

$$D_{m+1} \le \{4C_2(1+|\Omega|)\}^{\sum_{i=2}^{m+1} 1/[2(\beta_i-1)]}\prod_{i=2}^{m+1}\left(1+\left(\frac{2_s^*}{2}\right)^{i-1}(\beta_1-1)\right)^{1/[2(2_s^*/2)^{i-1}(\beta_1-1)]} D_1.$$

It is not difficult to show that the following sequence is convergent:

$$\left(\{4C_2(1+|\Omega|)\}^{\sum_{i=2}^{m+1} 1/[2(\beta_i-1)]}\prod_{i=2}^{m+1}\left(1+\left(\frac{2_s^*}{2}\right)^{i-1}(\beta_1-1)\right)^{1/[2(2_s^*/2)^{i-1}(\beta_1-1)]}\right)_{m\in\mathbb{N}}$$

Therefore, there exists a constant $C_4 > 0$ such that $D_{m+1} \le C_4 D_1$, that is,

$$\left(1+\int_\Omega u^{2_s^*(\beta_{m+1})}\,dx\right)^{1/[2_s^*(\beta_{m+1}-1)]} \le C_4 D_1 \tag{3.7}$$

for all $m \ge 1$. Let us assume $\|u\|_\infty > C_4 D_1$. Then there exists $\eta > 0$ and a measurable subset $\Omega' \subset \Omega$ such that

$$u(x) > C_4 D_1 + \eta \quad \text{for all } x \in \Omega'.$$

It follows that

$$\liminf_{\beta_m\to\infty}\left(\int_{\Omega'}|u|^{2_s^*\beta_m}\,dx + 1\right)^{1/(2_s^*\beta_m-1)} \ge \liminf_{\beta_m\to\infty}(C_4D_1+\eta)^{\beta_m/(\beta_m-1)}(|\Omega'|)^{1/[2_s^*(\beta_m-1)]} = C_4D_1+\eta,$$

which contradicts (3.7). Hence, $\|u\|_\infty \le C_4 D_1$, that is, $u \in L^\infty(\Omega)$. □

Lemma 3.4. *Let $r > 0$ and let $v \in L^{(r+1)/r}(\Omega)$ be a positive function and $u \in X_0 \cap L^{r+1}(\Omega)$ a positive weak solution to*

$$(-\Delta)^s u + g(x,u) = v \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \tag{3.8}$$

Then $(u+\bar{u}-\epsilon_1)^+ \in X_0$ for every $\epsilon_1 > 0$. In particular, every positive weak solution u to $(\bar{P}_\lambda)$, belonging to $L^{r+1}(\Omega)$, satisfies $(u+\bar{u}-\epsilon_1)^+ \in X_0$ for every $\epsilon_1 > 0$.

Proof. Let $\epsilon_1, \epsilon_2 > 0$ and set $\psi = \min\{u, \epsilon_1 - (\bar{u} - \epsilon_2)^+\} \in X_0$. Note that $u - \psi = (u + (\bar{u} - \epsilon_2)^+ - \epsilon_1)^+ \in X_0$. Since

$$0 \le \nu(u - \psi) \le \nu u + \nu \bar{u} \in L^1(\Omega),$$

using the arguments in the proof of Lemma 3.2, we can show that $g(\cdot\,, u)(u - \psi) \in L^1(\Omega)$ and

$$C_s^n \int_Q \frac{(u(x) - u(y))((u - \psi)(x) - (u - \psi)(y))}{|x - y|^{n+2s}}\, dx\, dy + \int_\Omega g(x, u)(u - \psi)\, dx - \int_\Omega \nu(u - \psi)\, dx = 0.$$

Let $0 \le \varphi \in C_c^\infty(\Omega)$. Then, using (3.1), we have

$$C_s^n \int_Q \frac{((\bar{u} - \epsilon_2)^+(x) - (\bar{u} - \epsilon_2)^+(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}}\, dx\, dy \le C_s^n \int_Q \frac{(\bar{u}(x) - \bar{u}(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}}\, dx\, dy = \int_\Omega \bar{u}^{-q}\varphi\, dx.$$

So, by arguing as in the proof of Lemma 3.2, we can show that

$$C_s^n \int_Q \frac{((\bar{u} - \epsilon_2)^+(x) - (\bar{u} - \epsilon_2)^+(y))((u - \psi)(x) - (u - \psi)(y))}{|x - y|^{n+2s}}\, dx\, dy \le \int_\Omega \bar{u}^{-q}(u - \psi)\, dx.$$

We have $u + \bar{u} \ge \epsilon_1$ when $u \ne \psi$, $(u + \bar{u})^{-q}(u - \psi) \in L^1(\Omega)$ and $\bar{u}(u - \psi) \in L^1(\Omega)$. Therefore, we have

$$\begin{aligned} C_s^n \int_Q & \frac{|(u + (\bar{u} - \epsilon_2)^+ - \epsilon_1)^+(x) - (u + (\bar{u} - \epsilon_2)^+ - \epsilon_1)^+(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \\ &\le \int_\Omega \bar{u}^{-q}(u - \psi)\, dx - \int_\Omega g(x, u)(u - \psi) dx + \int_\Omega \nu(u - \psi) dx \\ &= \int_\Omega (u + \bar{u})^{-q}(u - \psi)\, dx + \int_\Omega \nu(u - \psi)\, dx \\ &\le \epsilon_1^{-q} \int_\Omega (u - \psi)\, dx + \int_\Omega \nu(u - \psi)\, dx. \end{aligned}$$

Thus, for any $\epsilon > 0$, we have that $(u + (\bar{u} - \epsilon_2)^+ - \epsilon_1)^+$ is bounded in X_0 as $\epsilon_2 \to 0^+$. Hence, we conclude that $(u + \bar{u} - \epsilon_1)^+ \in X_0$ for every $\epsilon_1 > 0$. □

Lemma 3.5. *Let $F \in (X_0)^*$ (the dual of X_0) and let $z, v \in X$ be such that $z, v > 0$ a.e. in Ω, $z^{-q}, v^{-q} \in L^1_{\mathrm{loc}}(\Omega)$, $(z - \epsilon)^+ \in X_0$ for all $\epsilon > 0$ and*

$$C_s^n \int_Q \frac{(z(x) - z(y))(w(x) - w(y))}{|x - y|^{n+2s}}\, dx\, dy \le \int_\Omega z^{-q} w\, dx + \langle F, w\rangle,$$

$$C_s^n \int_Q \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{n+2s}}\, dx\, dy \ge \int_\Omega v^{-q} w\, dx + \langle F, w\rangle$$

for all compactly supported $w \in X_0 \cap L^\infty(\Omega)$ with $w \ge 0$. Then $z \le v$ a.e. in Ω.

Proof. Let us denote $\Phi_k : \mathbb{R} \to \mathbb{R}$ the primitive of the function

$$s \mapsto \begin{cases} \max\{-s^{-q}, -k\}, & s > 0, \\ -k, & s \le 0, \end{cases}$$

such that $\Phi_k(1) = 0$. We define a proper, lower semicontinuous, strictly convex functional $\hat{f}_{0,k} : L^2(\Omega) \to \mathbb{R}$ as follows:

$$\hat{f}_{0,k}(u) = \begin{cases} \frac{C_s^n}{2}\|u\|^2 + \int_\Omega \Phi_k(u)\, dx & \text{if } u \in X_0, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus X_0. \end{cases}$$

As we know, primitives are usually defined up to an additive constant. To prevent a possible unlikely choice we consider $f_{0,k} : L^2(\Omega) \to \mathbb{R}$ defined by

$$f_{0,k}(u) = \hat{f}_{0,k}(u) - \min \hat{f}_{0,k} = \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}),$$

where $u_{0,k} \in X_0$ is the minimum of $\hat{f}_{0,k}$. In general, for every $w \in (X_0)^*$, we define

$$\hat{f}_{w,k}(u) = \begin{cases} f_{0,k}(u) - \langle w, u - u_{0,k} \rangle & \text{if } u \in X_0, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus X_0. \end{cases}$$

Let $\epsilon > 0$ and $k > \epsilon^{-q}$, and let u be the minimum of the functional $f_{F,k}$ on the convex set

$$K = \{u \in X_0 : 0 \le u \le v \text{ a.e. in } \Omega\}.$$

Then, for all $\psi \in K$, we can get

$$C_s^n \int_Q \frac{(u(x) - u(y))((\psi - u)(x) - (\psi - u)(y))}{|x - y|^{n+2s}} \,dx\,dy \ge - \int_\Omega \Phi_k'(u)(\psi - u)\,dx + \langle F, \psi - u \rangle. \tag{3.9}$$

In particular, if $0 \le \psi \in C_c^\infty(\Omega)$ and $t > 0$, we can consider the above inequality with $\psi_t = \min\{u + t\psi, v\}$ as the test function. Since v is a supersolution of $(-\Delta)^s u = u^{-q} + F$, using the definition of Φ_k, we get v as a supersolution of $(-\Delta)^s u = -\Phi_k'(u) + F$. By definition, we have

$$u \le \psi_t \le v \quad \text{and} \quad \psi_t - u \le t\psi.$$

Now using these and (3.9), we get

$$\begin{aligned}
C_s^n & \int_Q \frac{((\psi_t - u)(x) - (\psi_t - u)(y))^2}{|x - y|^{n+2s}} \,dx\,dy - \int_\Omega (-\Phi_k'(\psi_t) + \Phi_k'(u))(\psi_t - u)\,dx \\
&= C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))((\psi_t - u)(x) - (\psi_t - u)(y))}{|x - y|^{n+2s}} \,dx\,dy \\
&\quad - C_s^n \int_Q \frac{(u(x) - u(y))((\psi_t - u)(x) - (\psi_t - u)(y))}{|x - y|^{n+2s}} \,dx\,dy \\
&\quad + \int_\Omega \Phi_k'(\psi_t)(\psi_t - u)\,dx - \int_\Omega \Phi_k'(u)(\psi_t - u)\,dx \\
&\le C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))((\psi_t - u)(x) - (\psi_t - u)(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(\psi_t)(\psi_t - u)\,dx - \langle F, \psi_t - u \rangle \\
&= C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))((\psi_t - u - t\psi)(x) - (\psi_t - u - t\psi)(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(\psi_t)(\psi_t - u - t\psi)\,dx \\
&\quad - \langle F, \psi_t - u - t\psi \rangle + t\bigg(C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(\psi_t)\psi\,dx - \langle F, \psi \rangle \bigg) \\
&\le C_s^n \int_Q \frac{(v(x) - v(y))((\psi_t - u - t\psi)(x) - (\psi_t - u - t\psi)(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(v)(\psi_t - u - t\psi)\,dx \\
&\quad - \langle F, \psi_t - u - t\psi \rangle + t\bigg(C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(\psi_t)\psi\,dx - \langle F, \psi \rangle \bigg) \\
&\le t\bigg(C_s^n \int_Q \frac{(\psi_t(x) - \psi_t(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} \,dx\,dy + \int_\Omega \Phi_k'(\psi_t)\psi\,dx - \langle F, \psi \rangle \bigg).
\end{aligned}$$

This gives

$$C_s^n \int_Q \frac{(\psi_t(x)-\psi_t(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \Phi_k'(\psi_t)\psi\,\mathrm{d}x - \langle F, \psi\rangle \ge -\int_\Omega |\Phi_k'(\psi_t)-\Phi_k'(u)|\frac{(\psi_t-u)}{t}\,\mathrm{d}x,$$

which implies

$$C_s^n \int_Q \frac{(\psi_t(x)-\psi_t(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \Phi_k'(\psi_t)\psi\,\mathrm{d}x - \langle F, \psi\rangle \ge -\int_\Omega |\Phi_k'(\psi_t)-\Phi_k'(u)|\psi\,\mathrm{d}x.$$

Since $\Phi_k'(\psi_t) \le -v^{-q}$, using Lebesgue's dominated convergence theorem and passing to the limit as $t \to 0^+$, we get

$$C_s^n \int_Q \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge -\int_\Omega \Phi_k'(u)\psi\,\mathrm{d}x - \langle F, \psi\rangle.$$

We can now easily show that the above equation holds for all $\psi \in X_0$ with $\psi \ge 0$ a.e. in Ω. In particular, since $u \ge 0$, we have $(z-u-\epsilon)^+ \in X_0$ and

$$C_s^n \int_Q \frac{(u(x)-u(y))\big((z-u-\epsilon)^+(x)-(z-u-\epsilon)^+(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge -\int_\Omega \Phi_k'(u)(z-u-\epsilon)^+\,\mathrm{d}x - \langle F, (z-u-\epsilon)^+\rangle. \tag{3.10}$$

Let us now consider $\sigma \in X_0$ such that $0 \le \sigma \le z$ a.e. in Ω. Let $\{\hat\sigma_m\}$ be a sequence in $C_c^\infty(\Omega)$ converging to σ in X_0 and set $\sigma_m = \min\{\hat\sigma_m, \sigma\}$. Then, since z is a subsolution of $(-\Delta)^s u = u^{-q} + F$, we have

$$-C_s^n \int_Q \frac{(z(x)-z(y))(\sigma_m(x)-\sigma_m(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge -\int_\Omega z^{-q}\sigma_m\,\mathrm{d}x - \langle F, \sigma_m\rangle.$$

If $z^{-q}\sigma \in L^1(\Omega)$, then passing to the limit as $m \to \infty$, we get

$$-C_s^n \int_Q \frac{(z(x)-z(y))(\sigma(x)-\sigma(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge -\int_\Omega z^{-q}\sigma\,\mathrm{d}x - \langle F, \sigma\rangle.$$

If $z^{-q}\sigma \notin L^1(\Omega)$, then the above inequality is obviously still true. In particular, we have

$$-C_s^n \int_Q \frac{(z(x)-z(y))\big((z-u-\epsilon)^+(x)-(z-u-\epsilon)^+(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge -\int_\Omega z^{-q}(z-u-\epsilon)^+\,\mathrm{d}x - \langle F, (z-u-\epsilon)^+\rangle. \tag{3.11}$$

Since $\epsilon^{-q} < k$, using (3.1), (3.10) and (3.11), we get

$$\begin{aligned}
C_s^n \int_Q & \frac{\big((z-u-\epsilon)^+(x)-(z-u-\epsilon)^+(y)\big)^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C_s^n \int_Q \frac{\big((z-u)(x)-(z-u)(y)\big)\big((z-u-\epsilon)^+(x)-(z-u-\epsilon)^+(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le \int_\Omega (z^{-q}+\Phi_k'(u))(z-u-\epsilon)^+\,\mathrm{d}x \\
&= \int_\Omega (-\Phi_k'(z)+\Phi_k'(u))(z-u-\epsilon)^+\,\mathrm{d}x \le 0.
\end{aligned}$$

Therefore, $z \le u+\epsilon \le v+\epsilon$ and the assertion follows from the arbitrariness of ϵ. □

Lemma 3.6. *Let $\lambda > 0$ and let $z \in X_0 \cap L^r(\Omega)$, $r > 1$, be a weak solution to (P_λ) as it is defined in Definition 2.5. Then $z - \bar u$ is a positive weak solution of $(\bar P_\lambda)$ belonging to $L^\infty(\Omega)$.*

Proof. Let us consider problem (3.8) with $v = \lambda z^{2_s^*-1}$. Then 0 is the strict subsolution of (3.8). Let

$$G(x,s) = \int_0^s g(x,\tau)\,\mathrm{d}\tau \quad \text{for } (x,s) \in \Omega \times \mathbb{R}.$$

We define the corresponding functional $\tilde{I}\colon X_0 \to (-\infty,\infty]$ by

$$\tilde{I}(u) = \begin{cases} \frac{C_s^n}{2}\|u\|^2 + \int_\Omega G(x,u)\,\mathrm{d}x - \lambda \int_\Omega z^{2_s^*-1} u\,\mathrm{d}x & \text{if } G(x,u) \in L^1(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

for every $u \in X_0$. Also, for every $u \in X_0$, we define the closed convex set $K_0 = \{u \in X_0 : u \geq 0 \text{ a.e.}\}$ and the functional $\tilde{I}_{K_0}$ as

$$\tilde{I}_{K_0}(u) = \begin{cases} \tilde{I}(u) & \text{if } u \in K_0 \text{ and } G(x,u) \in L^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Let $\{u_m\} \in K_0$ be the minimizing sequence of $\tilde{I}_{K_0}$ in K_0, i.e., $\tilde{I}_{K_0}(u_m) \to \inf_{K_0} \tilde{I}_{K_0}(u)$. It is easy to check that $\{u_m\}$ is bounded in X_0 and $\{G(\cdot,u_m)\}$ is bounded in $L^1(\Omega)$. Therefore, $u_m \rightharpoonup u$ (up to subsequence) weakly for some $u \in K_0$, and by Fatou's lemma,

$$\int_\Omega G(x,u)\,\mathrm{d}x \leq \liminf_{m\to\infty} \int_\Omega G(x,u_m)\,\mathrm{d}x < \infty.$$

Thus, $\tilde{I}_{K_0}(u) = \inf \tilde{I}_{K_0}(K_0)$. Hence, $0 \in \partial^- \tilde{I}_{K_0}(u)$, and by Proposition 4.2 we have that u is a nontrivial, nonnegative, weak solution of (3.8). Also, using Lemma 3.4, we have $(u + \bar{u} - \epsilon)^+ \in X_0$ for every $\epsilon > 0$. It can be shown that

$$C_s^n \int_Q \frac{((u+\bar{u})(x) - (u+\bar{u})(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega ((u+\bar{u})^{-q} - \lambda z^{2_s^*-1}) w\,\mathrm{d}x = 0$$

and

$$C_s^n \int_Q \frac{(z(x)-z(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega ((u+\bar{u})^{-q} - \lambda z^{2_s^*-1}) w\,\mathrm{d}x = 0$$

for $w \in X_0 \cap L^\infty(\Omega)$ with compact support in Ω. Then, using Lemma 3.5, we get $z = u + \bar{u}$, which implies that $u = z - \bar{u}$ is a positive weak solution of $(\bar{P}_\lambda)$. Thus, by Lemma 3.3, $u \in L^\infty(\Omega)$. □

4 Existence and multiplicity of positive solutions for (P_λ)

4.1 First solution

In this section, we prove the existence of a solution for problem (P_λ). We set the variational framework to problem $(\bar{P}_\lambda)$ in the space X_0. For this, recalling that $G(x,s) = \int_0^s g(x,\tau)\,\mathrm{d}\tau$ for $(x,s) \in \Omega \times \mathbb{R}$, we define the functional $I\colon X_0 \to (-\infty,\infty]$, corresponding to $(\bar{P}_\lambda)$, by

$$I(u) = \begin{cases} \frac{C_s^n}{2} \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega G(x,u)\,\mathrm{d}x - \frac{\lambda}{2_s^*} \int_\Omega |u+\bar{u}|^{2_s^*}\,\mathrm{d}x & \text{if } G(\cdot,u) \in L^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

For a convex subset $K \subset X_0$, we also define the restricted functional $I_K\colon X_0 \to (-\infty,\infty]$ by

$$I_K(u) = \begin{cases} I(u) & \text{if } u \in K \text{ and } G(\cdot,u) \in L^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

We note that $u \in D(I_K)$ if and only if $u \in K$ and $G(\cdot,u) \in L^1(\Omega)$. We now state a lemma which characterizes the set $\partial^- I_K(u)$.

Lemma 4.1. *Let K be a convex subset of X_0 and let $\alpha \in X_0$. Let also $u \in K$ with $G(\cdot, u) \in L^1(\Omega)$. Then the following two statements are equivalent:*

(i) $\alpha \in \partial^- I_K(u)$.

(ii) *For every $v \in K$ with $G(\cdot, v) \in L^1(\Omega)$, we have $g(\cdot, u)(v-u) \in L^1(\Omega)$ and*

$$C_s^n \int_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega g(x,u)(v-u)\,dx - \lambda \int_\Omega (u+\bar{u})^{2_s^*-1}(v-u)\,dx \geq \langle \alpha, v-u\rangle. \tag{4.1}$$

Moreover, as $G(\cdot, u)$ is convex, the last statement implies

$$C_s^n \int_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (G(x,v)-G(x,u))\,dx - \lambda \int_\Omega (u+\bar{u})^{2_s^*-1}(v-u)\,dx \geq \langle \alpha, v-u\rangle.$$

Proof. We follow the proof of [31, Lemma 3].

(i) $\Rightarrow$ (ii) Let $v \in K$ and $G(\cdot, v) \in L^1(\Omega)$, and set $w = v-u$. Then $g(\cdot, u)w$ is measurable and we have $G(\cdot, u) - G(\cdot, v) \in L^1(\Omega)$. Since $g(x,s)$ is non decreasing in s, we have $g(x,u)w \leq G(x,v) - G(x,u)$, which implies $(g(\cdot,u)w) \vee 0 \in L^1(\Omega)$. The function $t \mapsto (G(x, u+tw) - G(x,u))/t$, $(0,1] \to \mathbb{R}$, is increasing and

$$\frac{I_K(u+tw)-I_K(u)}{t} = C_s^n \int_Q \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy + \frac{tC_s^n}{2}\|w\|^2 + \int_\Omega \frac{(G(x,u+tw)-G(x,u))}{t}\,dx - \frac{1}{2_s^*}\int_\Omega \frac{(|u+\bar{u}+tw|^{2_s^*} - |u+\bar{u}|^{2_s^*})}{t}\,dx. \tag{4.2}$$

Letting $t \to 0$ on both sides of (4.2) and using the monotone convergence theorem, we get

$$\lim_{t\to 0} \frac{I_K(u+tw)-I_K(u)}{t} = C_s^n \int_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega g(x,u)(v-u)\,dx - \lambda \int_\Omega u^{2_s^*-1}(v-u)\,dx. \tag{4.3}$$

Also, $\alpha \in \partial^- I_K(u)$ implies

$$\lim_{t\to 0} \frac{I_K(u+tw)-I_K(u)}{t} \geq \langle \alpha, v-u\rangle.$$

Hence, we get (4.1) from (4.3). From (4.1), we have $(g(\cdot,u)w)\wedge 0 \in L^1(\Omega)$, and hence $(g(\cdot,u)w) \in L^1(\Omega)$.

(ii) $\Rightarrow$ (i) Let $v \in K$ and $G(\cdot, v) \in L^1(\Omega)$. Since $G(x,s)$ is convex in s, (ii) implies

$$\begin{aligned} I_K(v) - I_K(u) &= \frac{C_s^n}{2}\|(v-u)\|^2 + C_s^n \int_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \int_\Omega (G(x,v)-G(x,u))\,dx - \frac{\lambda}{2_s^*}\int_\Omega (|v+\bar{u}|^{2_s^*} - |u+\bar{u}|^{2_s^*})\,dx \\ &\geq \frac{C_s^n}{2}\|(v-u)\|^2 + \int_\Omega (G(x,v)-G(x,u)-g(x,u)(v-u))\,dx \\ &\quad - \lambda \int_\Omega \Big(\frac{|v+\bar{u}|^{2_s^*} - |u+\bar{u}|^{2_s^*}}{2_s^*} - |u+\bar{u}|^{2_s^*-1}(v-u)\Big)\,dx + \langle \alpha, v-u\rangle \\ &\geq \frac{C_s^n}{2}\|(v-u)\|^2 - \lambda \int_\Omega \Big(\frac{|v+\bar{u}|^{2_s^*} - |u+\bar{u}|^{2_s^*}}{2_s^*} - |u+\bar{u}|^{2_s^*-1}(v-u)\Big)\,dx + \langle \alpha, v-u\rangle, \end{aligned}$$

which implies $\alpha \in \partial^- I_K(u)$. □

For $\varphi, \psi \colon \Omega \to [-\infty, +\infty]$, we define

$$K_\varphi = \{u \in X_0 : \varphi \leq u \text{ a.e.}\}, \quad K^\psi = \{u \in X_0 : u \leq \psi \text{ a.e.}\} \quad \text{and} \quad K_\varphi^\psi = \{u \in X_0 : \varphi \leq u \leq \psi \text{ a.e.}\}.$$

We state the following proposition which can be thought of as Perron's method for non-smooth functionals.

Proposition 4.2. *Assume one of the following conditions:*
(i) φ_1 *is a subsolution of* $(\bar{P}_\lambda)$, $G(x, v(x)) \in L^1_{\mathrm{loc}}(\Omega)$ *for all* $v \in K_{\varphi_1}$, $u \in D(I_{K_{\varphi_1}})$ *and* $0 \in \partial^- I_{K_{\varphi_1}}(u)$.
(ii) φ_2 *is a supersolution of* $(\bar{P}_\lambda)$, $G(x, v(x)) \in L^1_{\mathrm{loc}}(\Omega)$ *for all* $v \in K^{\varphi_2}$, $u \in D(I_{K^{\varphi_2}})$ *and* $0 \in \partial^- I_{K^{\varphi_2}}(u)$.
(iii) φ_1 *and* φ_2 *are subsolution and supersolution of* $(\bar{P}_\lambda)$, $G(x, \varphi_1(x)), G(x, \varphi_2(x)) \in L^1_{\mathrm{loc}}(\Omega)$, $u \in D(I_{K^{\varphi_2}_{\varphi_1}})$ *and* $0 \in \partial^- I_{K^{\varphi_2}_{\varphi_1}}(u)$.
Then u is a weak solution of $(\bar{P}_\lambda)$.

Proof. We follow the proof of [31, Proposition 2]. We have that $G(\cdot, \varphi_1)$ and $g(\cdot, \varphi_1)$ are measurable and $G(x, \varphi_1(x)), G(x, u(x)) \in \mathbb{R}$ for a.e. $x \in \Omega$, since $G(\cdot, u), G(\cdot, \varphi_1) \in L^1_{\mathrm{loc}}(\Omega)$. So, $g(\cdot, u)$ is measurable by [31, Lemma 2 (ii)]. Since

$$g(x, \phi_1)\psi_0 \le g(x, u)\psi_0 \le G(x, u + \psi_0) - G(x, u)$$

for each $\psi_0 \in C^\infty_c(\Omega)$, we get $g(\cdot, u)\psi_0 \in L^1(\Omega)$. The arbitrariness of ψ_0 implies that $g(\cdot, u) \in L^1_{\mathrm{loc}}(\Omega)$. Let $\psi \in C^\infty_c(\Omega)$ and set $v_t = (u + t\psi) \vee \varphi_1$ for $0 < t \le 1$. Then $G(\cdot, v_t) \in L^1_{\mathrm{loc}}(\Omega)$ and $G(x, v_t) = G(x, u)$ on $\Omega \setminus \operatorname{supp}\psi$, which implies $v_t \in D(I_{K_{\varphi_1}})$. Setting $r_t = (\varphi_1 - (u + t\psi))^+$, we get $v_t - u = t\psi + r_t$. Clearly, r_t has a compact support and $|r_t(x)| \le t|\psi(x)|$ for each $x \in \Omega$. Using Lemma 4.1, we get $g(\cdot, u)(v_t - u) \in L^1(\Omega)$ and

$$\begin{aligned}
0 &\le C^n_s \int_Q \frac{(u(x) - u(y))\big((v_t - u)(x) - (v_t - u)(y)\big)}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2^*_s - 1})(v_t - u)\,dx \\
&\le tC^n_s \int_Q \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}}\,dx\,dy + t\int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2^*_s - 1})\psi\,dx \\
&\quad + C^n_s \int_Q \frac{(u(x) - u(y))(r_t(x) - r_t(y))}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2^*_s - 1})r_t\,dx.
\end{aligned} \tag{4.4}$$

Fix $t \in (0, 1]$ and let $\{w_k\}$ be a non-negative sequence of functions in $C^\infty_c(\Omega)$ such that $\bigcup_k \operatorname{supp} w_k$ is contained in a compact subset of Ω, $\{\|w_k\|_\infty\}$ is bounded and $\|w_k - r_t\| \to 0$ as $k \to \infty$.

Using the fact that φ_1 is a subsolution of $(\bar{P}_\lambda)$, for each k we get

$$C^n_s \int_Q \frac{(\varphi_1(x) - \varphi_1(y))(w_k(x) - w_k(y))}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, \varphi_1) - \lambda(\varphi_1 + \bar{u})^{2^*_s - 1})w_k\,dx \le 0.$$

Taking the limit as $k \to \infty$ and using Lebesgue's dominated convergence theorem, we obtain

$$C^n_s \int_Q \frac{(\varphi_1(x) - \varphi_1(y))(r_t(x) - r_t(y))}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, \varphi_1) - \lambda(\varphi_1 + \bar{u})^{2^*_s - 1})r_t\,dx \le 0. \tag{4.5}$$

From (4.4), (4.5) and since $-r_t - t\psi \le u - \phi_1$ in Ω, we get

$$\begin{aligned}
0 &\le t\bigg(C^n_s \int_Q \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2^*_s - 1})\psi\,dx\bigg) \\
&\quad - C^n_s\|r_t\|^2 - tC^n_s \int_Q \frac{(\psi(x) - \psi(y))(r_t(x) - r_t(y))}{|x - y|^{n+2s}}\,dx\,dy \\
&\quad + \int_\Omega \Big((g(x, u) - g(x, \varphi_1))r_t - \lambda((u + \bar{u})^{2^*_s - 1} - (\varphi_1 + \bar{u})^{2^*_s - 1})r_t\Big)\,dx,
\end{aligned}$$

which gives

$$\begin{aligned}
0 &\le C^n_s \int_Q \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2^*_s - 1})\psi\,dx \\
&\quad - C^n_s \int_Q \frac{(\psi(x) - \psi(y))(r_t(x) - r_t(y))}{|x - y|^{n+2s}}\,dx\,dy \\
&\quad + \int_\Omega \Big((g(x, u) - g(x, \varphi_1))\frac{r_t}{t} - \lambda((u + \bar{u})^{2^*_s - 1} - (\varphi_1 + \bar{u})^{2^*_s - 1})\frac{r_t}{t}\Big)\,dx.
\end{aligned}$$

Using the inequality $|r_t(x)| \le t|\psi(x)|$ for each $x \in \Omega$ and $0 < t \le 1$, the limits $\|r_t\| \to 0$ as $t \to 0^+$,

$$(g(x,u) - g(x,\varphi_1))\frac{r_t}{t} \to 0 \quad \text{and} \quad \lambda((u+\bar{u})^{2_s^*-1} - (\varphi_1+\bar{u})^{2_s^*-1})\frac{r_t}{t} \to 0 \quad \text{a.e. as } t \to 0^+,$$

and the fact that supp ψ is compact and $g(\cdot\,,u), g(\cdot\,,\varphi_1) \in L^1(\Omega)$, we get

$$0 \le C_s^n \int_Q \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,u) - \lambda(u+\bar{u})^{2_s^*-1})\psi\,dx.$$

Since $\psi \in C_c^\infty(\Omega)$ is arbitrary, u is a weak solution of $(\bar{P}_\lambda)$. The proofs of (ii) and (iii) are similar to those of [31, Proposition 2 (ii) and (iii)]. □

Let $\bar{\theta} \in X_0$ be the function which satisfies $(-\Delta)^s\bar{\theta} = 1/2$ in Ω in the sense of distributions. From [41, Proposition 1.1], $\bar{\theta} \in C^s(\mathbb{R}^n)$. For g and G, we have the following properties.

Proposition 4.3. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$, satisfying Definition 2.5 (i). Then $g(x,u(x)), G(x,u(x)) \in L^1_{\mathrm{loc}}(\Omega)$.*

Proof. Recall that $\inf_K \bar{u} > 0$ for any $K \Subset \Omega$. We have $0 \le g(x,u(x)) \le \bar{u}^{-q}$ and $0 \le G(\cdot\,,u) \le \bar{u}^{-q}u$ in Ω. Hence,

$$\int_K g(x,u(x))\,dx \le \int_K |\bar{u}(x)|^{-q}\,dx < \infty \quad \text{and} \quad \int_K |G(x,u(x))|\,dx \le \Big(\inf_K \bar{u}\Big)^{-\delta}\int_K |u|\,dx < \infty.$$ □

Lemma 4.4. *For each $x \in \Omega$, the following hold:*
(i) *$G(x,rt) \le t^2G(x,t)$ for each $r \ge 1$ and $t \ge 0$,*
(ii) *$G(x,r) - G(x,t) - (g(x,r)+g(x,t))(r-t)/2 \ge 0$ for each r, t with $r \ge t > -\bar{\theta}(x)$,*
(iii) *$G(x,r) - g(x,r)r/2 \ge 0$ for each $r \ge 0$.*

Proof. For a proof we refer to [31, Lemma 4]. □

We now proceed to prove some results to obtain the existence of a solution of $(\bar{P}_\lambda)$.

Lemma 4.5. *The following hold:*
(i) *0 is a strict subsolution of $(\bar{P}_\lambda)$ for all $\lambda > 0$.*
(ii) *$\bar{\theta}$ is a strict supersolution of $(\bar{P}_\lambda)$ for all sufficiently small $\lambda > 0$.*
(iii) *Any positive weak solution z of $(\bar{P}_\mu)$ is a strict super-solution of $(\bar{P}_\lambda)$ for $\mu > \lambda > 0$.*

Proof. (i) Let $\psi \in X_0 \setminus \{0\}$, $\psi \ge 0$. Since $g(x,0) = 0$, we get

$$C_s^n \int_Q \frac{(0(x)-0(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,0)\psi - \lambda(0+\bar{u})^{2_s^*-1}\psi)\,dx = -\lambda\int_\Omega |\bar{u}|^{2_s^*-1}\psi\,dx < 0.$$

(ii) We choose $\lambda > 0$ such that $1 - \lambda(\bar{\theta}+\bar{u}) > 0$ in Ω. We have $g(x,\bar{\theta}) \in L^1_{\mathrm{loc}}(\Omega)$ and g is nonnegative. So,

$$C_s^n \int_Q \frac{(\bar{\theta}(x)-\bar{\theta}(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,\bar{\theta})\psi - \lambda(\bar{\theta}+\bar{u})^{2_s^*-1}\psi)\,dx \ge \int_\Omega (1-\lambda|\bar{u}|^{2_s^*-1})\psi\,dx > 0.$$

(iii) Let $\lambda > 0$ and let z be a positive weak solution of $(\bar{P}_\mu)$ for some $\mu > \lambda$. We have $g(\cdot\,,z) \in L^1_{\mathrm{loc}}(\Omega)$ and g is nonnegative. So,

$$C_s^n \int_Q \frac{(z(x)-z(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (g(x,z)\psi - \lambda(z+\bar{u})^{2_s^*-1}\psi)\,dx = (\mu-\lambda)\int_\Omega |z+\bar{u}|^{2_s^*-1}\psi\,dx > 0,$$

which gives (iii). □

Let

$$\Lambda := \sup\{\lambda > 0 : (\bar{P}_\lambda) \text{ admits a weak solution}\}.$$

Remark 4.6. If $\Lambda > 0$, by Lemma 4.5, we can say that for any $\lambda \in (0,\Lambda)$, $(\bar{P}_\lambda)$ has a subsolution (the trivial function 0) and a positive strict supersolution (say z).

Theorem 4.7. *Let $\varphi_1, \varphi_2 \colon \Omega \to [-\infty, \infty]$ with $\varphi_1 \leq \varphi_2$ such that φ_1 is a strict supersolution of $(\bar{P}_\lambda)$. Let also $u \in D(I_{K_{\varphi_1}^{\varphi_2}})$ be a minimizer for $I_{K_{\varphi_1}^{\varphi_2}}$. Then u is a local minimizer for $I_{K_{\varphi_1}}$.*

Proof. For any $v \in K_{\varphi_1}$, define

$$\sigma(v) = \min\{v, \varphi_2\} = v - (v - \varphi_2)^+,$$

and for any $0 \leq w \in X_0$, define

$$H(w) = C_s^n \int_Q \frac{(\varphi_2(x) - \varphi_2(y))(w(x) - w(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega g(x, \varphi_2)w\,dx - \lambda \int_\Omega (\bar{u} + \varphi_2)^{2_s^*-1} w\,dx.$$

First we see that, there exists $0 \leq \theta \leq 1$ such that

$$\begin{aligned}
\frac{(\bar{u}+u)^{2_s^*-1} - (\bar{u}+v)^{2_s^*-1}}{(u-v)} &= (2_s^* - 1)\big((\bar{u}+u) + \theta(v-u)\big)^{2_s^*-2} \\
&\leq 2^{2_s^*-3}(2_s^*-1)\big[\bar{u}^{2_s^*-2} + \big((1-\theta)u + \theta v\big)^{2_s^*-2}\big] \\
&\leq c_1 \bar{u}^{2_s^*-2} + c_2 \max\{|u|, |v|\}^{2_s^*-2},
\end{aligned} \tag{4.6}$$

where c_1, c_2 are positive constants. For $x \in \Omega$, let us set

$$m_v(x) = \big(c_1 \bar{u}^{2_s^*-2} + c_2 \max\{|\varphi_2(x)|, |v(x)|\}^{2_s^*-2}\big)1_{\{v > \varphi_2\}}.$$

We know that $G(\cdot, \sigma(v)(\cdot))$ and $g(\cdot, \sigma(v)(\cdot))(v(\cdot) - \sigma(v)(\cdot))$ are measurable by [31, Lemma 2 (i) and (iii)]. Using the fact that $\sigma(v) \in K_{\varphi_1}^{\varphi_2}$, the inequality $\sigma(v) \leq v$, the convexity of $G(x, \cdot)$ and (4.6), we get

$$\begin{aligned}
I_{K_{\varphi_1}}(v) - I_{K_{\varphi_1}}(u) &\geq I_{K_{\varphi_1}}(v) - I_{K_{\varphi_1}}(\sigma(v)) \\
&= \frac{C_s^n}{2} \int_Q \frac{|(v-\sigma(v))(x) - (v-\sigma(v))(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + C_s^n \int_Q \frac{(\sigma(v)(x) - \sigma(v)(y))\big((v-\sigma(v))(x) - (v-\sigma(v))(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_\Omega (G(x,v) - G(x,\sigma(v)))\,dx - \frac{\lambda}{2_s^*} \int_\Omega ((\bar{u}+v)^{2_s^*} - (\bar{u}+\sigma(v))^{2_s^*})\,dx \\
&\geq \frac{C_s^n}{2}\|v - \sigma(v)\|^2 + C_s^n \int_Q \frac{(\sigma(v)(x) - \sigma(v)(y))\big((v-\sigma(v))(x) - (v-\sigma(v))(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_\Omega g(x, \sigma(v))(v - \sigma(v))\,dx - \lambda \int_\Omega (\bar{u} + \sigma(v))^{2_s^*-1}(v - \sigma(v))\,dx \\
&\quad - \frac{\lambda}{2_s^*} \int_\Omega ((\bar{u}+v)^{2_s^*} - (\bar{u}+\sigma(v))^{2_s^*} - 2_s^*(\bar{u}+\sigma(v))^{2_s^*-1}(v - \sigma(v)))\,dx \\
&= \frac{C_s^n}{2}\|v - \sigma(v)\|^2 + C_s^n \int_Q \frac{(\sigma(v)(x) - \sigma(v)(y))\big((v-\sigma(v))(x) - (v-\sigma(v))(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_\Omega g(x, \sigma(v))(v - \sigma(v))\,dx - \lambda \int_\Omega (\bar{u} + \sigma(v))^{2_s^*-1}(v - \sigma(v))\,dx \\
&\quad - \lambda \int_\Omega \int_{\sigma(v)}^{v} ((\bar{u}+t)^{2_s^*-1} - (\bar{u}+\sigma(v))^{2_s^*-1})\,dt\,dx \\
&\geq \frac{C_s^n}{2}\|v - \sigma(v)\|^2 + H(v - \sigma(v)) - \frac{1}{2}\int_\Omega m_v(x)(v - \sigma(v))^2\,dx.
\end{aligned} \tag{4.7}$$

This implies, for any $v \in D(I_{K_{\varphi_1}})$, that

$$I_{K_{\varphi_1}}(v) \geq I_{K_{\varphi_1}}(u) + \frac{C_s^n}{2}\|v - \sigma(v)\|^2 + H((v - \varphi_2)^+) - \frac{1}{2}|m_v|_{2_s^*/(2_s^*-2)}|(v - \varphi_2)^+|_{2_s^*}^2.$$

Suppose the conclusion of the above theorem does not hold under the considered assumptions. In this case, we can choose a sequence $\{v_k\} \subset X_0$ such that $v_k \in K_{\varphi_1}$ and

$$\|v_k - u\| \leq \frac{1}{2^k}, \quad I_{K_{\varphi_1}}(v_k) < I_{K_{\varphi_1}}(u) \quad \text{for all } k.$$

We set $l = u + \sum_{k=1}^{\infty} |v_k - u|$, which satisfies $|v_k| \leq l$ a.e. for all k. Also we set

$$\hat{m}_v(x) = (c_1 \bar{u}^{2_s^*-2} + c_2 \max\{|\varphi_2(x)|, |l(x)|\}^{2_s^*-2}) 1_{\{v>\varphi_2\}} \quad \text{for every } v \in D(I_{K_{\varphi_1}}).$$

Then we have

$$\begin{aligned}
0 &> I_{K_{\varphi_1}}(v_k) - I_{K_{\varphi_1}}(u)\\
&\geq I_{K_{\varphi_1}}(v_k) - I_{K_{\varphi_1}}(\sigma(v_k))\\
&\geq \frac{C_s^n}{2}\|(v_k - \varphi_2)^+\|^2 + H((v_k-\varphi_2)^+) - \frac{1}{2}\int_\Omega \hat{m}_{v_k}(x)((v-\varphi_2)^+)^2 \,\mathrm{d}x\\
&= \frac{C_s^n}{2}\|(v_k - \varphi_2)^+\|^2 + H((v_k-\varphi_2)^+) - \frac{1}{2}\int_{\{\hat{m}_{v_k} \leq R/C_s^n\}} \hat{m}_{v_k}(x)((v-\varphi_2)^+)^2 \,\mathrm{d}x\\
&\quad - \frac{1}{2}\int_{\{\hat{m}_{v_k} > R/C_s^n\}} \hat{m}_{v_k}(x)((v-\varphi_2)^+)^2 \,\mathrm{d}x\\
&\geq \frac{C_s^n}{2}\|(v_k - \varphi_2)^+\|^2 + H((v_k-\varphi_2)^+) - \frac{RC_s^n}{2}\int_\Omega |(v-\varphi_2)^+|^2 \,\mathrm{d}x\\
&\quad - \frac{1}{2S_s}\bigg(\int_{\{\hat{m}_{v_k} > R/C_s^n\}} |\hat{m}_{v_k}(x)|^{2_s^*/(2_s^*-2)} \,\mathrm{d}x\bigg)^{(2_s^*-2)/2_s^*} \|(v_k-\varphi_2)^+\|^2
\end{aligned}$$

for all $R > 0$ and k. As we can choose $R > 0$ such that

$$\frac{1}{2S_s}\bigg(\int_{\{\hat{m}_{v_k} > RC_s^n\}} |\hat{m}_{v_k}(x)|^{2_s^*/(2_s^*-2)} \,\mathrm{d}x\bigg)^{(2_s^*-2)/2_s^*} < \frac{C_s^n}{4} \quad \text{for all } k,$$

we get

$$0 > H((v_k - \varphi_2)^+) + \frac{C_s^n}{4}\|(v_k-\varphi_2)^+\|^2 - \frac{RC_s^n}{2}|(v_k-\varphi_2)^+|_2^2 \quad \text{for all } k. \tag{4.8}$$

Let

$$\nu = \inf\{H(w) : w \in A\},$$

where

$$A = \{w \in X_0 : w \geq 0,\ |w|_2 = 1,\ \|w\| \leq 2\sqrt{R}\}.$$

Clearly, A is weakly compact and using Fatou's lemma, we can show that H is weakly lower semicontinuous on A. So if $\{w_k\} \subset A$ be a minimizing sequence for ν such that $w_k \rightharpoonup w$ weakly as $k \to \infty$, then

$$H(w) \leq \liminf H(w_k).$$

Since φ_2 is a strict supersolution of $(\bar{P}_\lambda)$, $H(w) > 0$ for all $w \in A$. This implies $\nu > 0$. Since $v_k \to u$ in X_0, there exists k_0 such that $|(v_{k_0} - \varphi_2)^+|_2 \leq \nu/(RC_s^n)$. We consider two cases. If $\|(v_{k_0} - \varphi_2)^+\|^2 \geq 4R|(v_{k_0}-\varphi_2)^+|_2^2$, then from (4.8) we get

$$0 > \frac{C_s^n}{4}\|(v_{k_0}-\varphi_2)^+\|^2 - \frac{C_s^n}{8}\|(v_{k_0}-\varphi_2)^+\|^2 = \frac{C_s^n}{8}\|(v_{k_0}-\varphi_2)^+\|^2,$$

which is a contradiction. On the other hand, if $\|(v_{k_0} - \varphi_2)^+\|^2 \leq 4R|(v_{k_0}-\varphi_2)^+|_2^2$, then from (4.8) we get

$$0 > \Big(\nu - \frac{RC_s^n}{2}|(v_{k_0}-\varphi_2)^+|_2\Big)|(v_{k_0}-\varphi_2)^+|_2 \geq \frac{\nu}{4\sqrt{R}}\|(v_{k_0}-\varphi_2)^+\|,$$

which is again a contradiction. □

Theorem 4.8. *Suppose $\Lambda > 0$. Let $\lambda \in (0, \Lambda)$ and $0, z$ be the sub and super solutions of $(\bar{P}_\lambda)$, respectively, as in Remark 4.6. Let also $K = \{\phi \in H_0^1(\Omega) : 0 \le \phi \le z\}$. Then there exists a weak solution u_λ of $(\bar{P}_\lambda)$ with $u_\lambda \in K$ and $I_K(u_\lambda) = \inf_K I_K < 0$. Furthermore, u_λ is a local minimizer for I_{K_0}.*

Proof. We have $\inf_K I_K < 0$ since $I_K(0) < 0$. Let $\{u_m\}$ be a minimizing sequence for $\inf_K I_K$ in K. Then $0 \le u_m \le z$ for all m, that is, $\{u_m\}$ is bounded in X_0. So, there exist $u_\lambda \in K$ such that $u_m \rightharpoonup u_\lambda$ weakly in X_0 as $m \to \infty$. We have that the map $v \mapsto \int_\Omega G(x, v)\,dx$ is weakly sequentially lower semicontinuous and, by Lebesgue's dominated convergence theorem,

$$\lim_{m\to\infty} \int_\Omega |u_m + \bar{u}|^{2_s^*}\,dx = \int_\Omega |u_\lambda + \bar{u}|^{2_s^*}\,dx.$$

Thus, $I_K(u_\lambda) \le \liminf_{m\to\infty} I_K(u_m)$, which implies $I_K(u_\lambda) = \inf_K I_K$. Hence, $I_K(u_\lambda) < 0$ and $0 \in \partial^- I_K(u_\lambda)$. Thus, u_λ is a weak solution of $(\bar{P}_\lambda)$, by Proposition 4.2. Finally, using Theorem 4.7 with $\varphi_1 = 0$ and $\varphi_2 = z$, we conclude that u_λ is a local minimizer for I_{K_0}. □

Lemma 4.9. *We have $0 < \Lambda < \infty$.*

Proof. First, we prove that $\Lambda > 0$. From Lemma 4.5, we get 0 as a strict subsolution and $\bar{\theta}$ as a strict supersolution of $(\bar{P}_\lambda)$ for sufficiently small $\lambda > 0$. We define the convex set $K := \{\phi \in X_0(\Omega) : 0 \le \phi \le \bar{\theta}\}$. Then, arguing as in the proof of Theorem 4.8, we get that there exist $u \in D(I_K)$ such that $I_K(u) = \inf_K I_K$. In particular, $0 \in \partial^- I_K(u)$. Thus, u is a weak solution of $(\bar{P}_\lambda)$ for sufficiently small $\lambda > 0$, by Proposition 4.2. Thus, $\Lambda > 0$.

Next, we prove that $\Lambda < +\infty$. Suppose on the contrary that $\Lambda = +\infty$. So, there exists an increasing sequence $\{\lambda_m\} \in \mathbb{R}$ such that $\lambda_m \to +\infty$, and $(\bar{P}_{\lambda_m})$ admits a weak solution, say u_{λ_m} as given in Theorem 4.8. Consequently,

$$\frac{C_s^n}{2}\|u_{\lambda_m}\|^2 + \int_\Omega G(x, u_{\lambda_m})\,dx - \frac{\lambda_m}{2_s^*}\int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*}\,dx < 0. \tag{4.9}$$

Also, by the definition of a weak solution, we get

$$C_s^n\|u_{\lambda_m}\|^2 + \int_\Omega g(x, u_{\lambda_m})u_{\lambda_m}\,dx - \lambda_m\int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m}\,dx = 0. \tag{4.10}$$

From (4.9) and (4.10), we obtain

$$\int_\Omega \Big(G(x, u_{\lambda_m}) - \frac{1}{2}g(x, u_{\lambda_m})u_{\lambda_n}\Big)\,dx + \lambda_m\int_\Omega \Big(\frac{1}{2}|u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m} - \frac{|u_{\lambda_m} + \bar{u}|^{2_s^*}}{2_s^*}\Big)\,dx < 0.$$

By Lemma 4.4 (iii) we have $G(x, u_{\lambda_m}) - g(x, u_{\lambda_m})/2 \ge 0$, which implies

$$\int_\Omega \frac{1}{2}|u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m}\,dx < \int_\Omega \frac{|u_{\lambda_m} + \bar{u}|^{2_s^*}}{2_s^*}\,dx. \tag{4.11}$$

Next since $\bar{u} \in L^\infty(\Omega)$, we note that

$$\lim_{t\to\infty} \frac{|t + \bar{u}(x)|^{2_s^*}}{|t + \bar{u}(x)|^{2_s^*-1}t} = 1$$

uniformly with respect to $x \in \Omega$. Thus, for any $\epsilon > 0$ small enough, there exists $M = M_\epsilon > 0$ such that

$$\frac{1}{2_s^*}\int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*}\,dx < \frac{1}{2+\epsilon}\int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m}\,dx + M \quad \text{for all } m. \tag{4.12}$$

From (4.11) and (4.12), we get

$$\sup_m \int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m}\,dx < \infty.$$

Using (4.10), for each m, we obtain

$$C_s^n\|u_{\lambda_m}\|^2 \le \lambda_m \int_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1}u_{\lambda_m}\,dx,$$

which implies that the sequence $\{\lambda_m^{-1/2} u_{\lambda_m}\}$ is bounded in X_0. Set $v_{\lambda_m} := \lambda_m^{-1/2} u_{\lambda_m}$. Then, up to a subsequence, there exists $v \in X_0$ such that $v_{\lambda_m} \rightharpoonup v$ weakly in X_0 as $m \to \infty$. Let $\psi \in C_0^\infty(\Omega)$ be a non-trivial and non-negative function. Choose $m > 0$ such that $\bar{u} \geq m$ on the support of ψ. Then

$$
\begin{aligned}
C_s^n \int\limits_Q \frac{(v_{\lambda_m}(x) - v_{\lambda_m}(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\, dx\, dy &+ \int\limits_\Omega \frac{1}{m^\delta \sqrt{\lambda_n}} \psi\, dx \\
&\geq C_s^n \int\limits_Q \frac{(v_{\lambda_m}(x) - v_{\lambda_m}(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\, dx\, dy + \int\limits_\Omega \frac{g(x, u_{\lambda_m})}{\sqrt{\lambda_n}} \psi\, dx \\
&= \frac{\sqrt{\lambda_m}}{2_s^* - 1} \int\limits_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1} \psi\, dx \\
&\geq \sqrt{\lambda_m} \int\limits_\Omega |m + \bar{u}|^{2_s^*-1} \psi\, dx. \qquad (4.13)
\end{aligned}
$$

Since $\lambda_m \to \infty$ as $m \to \infty$, letting $m \to \infty$ in (4.13), we get

$$
C_s^n \int\limits_Q \frac{(v_{\lambda_m}(x) - v_{\lambda_m}(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\, dx\, dy = \infty,
$$

which is a contradiction. Hence, $\Lambda < \infty$. □

Theorem 4.10. *There exists a positive weak solution of $(\bar{P}_\Lambda)$.*

Proof. Let $\lambda_m \uparrow \Lambda$ and $\{u_{\lambda_m}\}$ be a sequence of positive weak solutions to $(\bar{P}_{\lambda_m})$ such that $u_{\lambda_m} \leq u_{\lambda_{m+1}}$ for all $m \in \mathbb{N}$. Then, as in the proof of Lemma 4.9, we have that $\{u_{\lambda_m}\}$ is uniformly bounded in X_0. Therefore, up to a subsequence, there exists $u_\Lambda \in X_0$ such that $u_{\lambda_m} \rightharpoonup u_\Lambda$ weakly in X_0 as $m \to \infty$. Now for any $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$, using the monotone convergence theorem, as $m \to \infty$, we have

$$
\int\limits_\Omega g(x, u_{\lambda_n})\phi\, dx \to \int\limits_\Omega g(x, u_\Lambda)\phi\, dx \quad \text{and} \quad \int\limits_\Omega |u_{\lambda_m} + \bar{u}|^{2_s^*-1} \phi\, dx \to \int\limits_\Omega |u_\Lambda + \bar{u}|^{2_s^*-1} \phi\, dx.
$$

Thus,

$$
C_s^n \int\limits_Q \frac{(u_\Lambda(x) - u_\Lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+2s}}\, dx\, dy + \int\limits_\Omega g(x, u_\Lambda)\phi\, dx - \Lambda \int\limits_\Omega |u_\Lambda + \bar{u}|^{2_s^*-1}\phi\, dx = 0.
$$

Now for any $\phi \in C_0^\infty(\Omega)$, taking $\phi = \phi^+ - \phi^-$ and arguing as above, it is easy to check that u_Λ is a positive weak solution of $(\bar{P}_\Lambda)$. □

4.2 Second solution

Now, we show the existence of at least two distinct positive weak solutions for $(\bar{P}_\lambda)$ with $\lambda \in (0, \Lambda)$. We fix $\lambda \in (0, \Lambda)$, and we denote by u the positive weak solution obtained in Theorem 4.8.

Proposition 4.11. *The functional I_{K_u} satisfies (CPS)$_c$ for each c satisfying*

$$
c < I_{K_u}(u) + \frac{s(C_s^n S_s)^{n/(2s)}}{n\lambda^{(n-2s)/(2s)}}.
$$

Proof. Let

$$
c < I_{K_u}(u) + \frac{S_s^{n/(2s)}}{n\lambda^{(n-2)/(2s)}}
$$

be fixed and choose a sequence $\{w_m\} \in D(I_{K_u})$ such that

$$
I_{K_u}(w_m) \to c \quad \text{and} \quad (1 + \|w_m\|)|||\partial^- I_{K_u}(w_m)||| \to 0 \quad \text{as } m \to \infty.
$$

There exists $\beta_m \in \partial^- I_{K_u}(w_m)$ such that $\|\beta_m\| = |||\partial^- I_{K_u}(w_m)|||$ for each $m \in \mathbb{N}$. Using Lemma 4.1, for every $m \in \mathbb{N}$ and $v \in D(I_{K_u})$, we have $g(\cdot, w_m)(v - w_m) \in L^1(\Omega)$ and

$$\begin{aligned}\langle \beta_m, v - w_m\rangle \leq C_s^n \int_Q \frac{(w_m(x) - w_m(y))((v - w_m)(x) - (v - w_m)(y))}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ + \int_\Omega g(x, w_m)(v - w_m)\,\mathrm{d}x - \lambda \int_\Omega (w_m + \bar{u})^{2_s^*-1}(v - w_m)\,\mathrm{d}x. \qquad (4.14)\end{aligned}$$

By Lemma 4.4 (ii) and since $G(\cdot, w_m) \in L^1(\Omega)$, we get $G(\cdot, 2w_m) \in L^1(\Omega)$, which implies $2w_m \in D(I_{K_u})$ for each m. Substituting $v = 2w_m$ in (4.14), we get

$$\langle \beta_m, w_m\rangle \leq C_s^n \|w_m\|^2 + \int_\Omega g(x, w_m)w_m\,\mathrm{d}x - \lambda \int_\Omega (w_m + \bar{u})^{2_s^*-1} w_m\,\mathrm{d}x.$$

Assuming $I_{K_u}(w_m) \leq c + 1$ for all m and using (4.12), we have

$$\begin{aligned}c + 1 &\geq \frac{C_s^n}{2}\|w_m\|^2 + \int_\Omega G(x, w_m)\,\mathrm{d}x - \frac{\lambda}{2_s^*}\int_\Omega (w_m + \bar{u})^{2_s^*}\,\mathrm{d}x \\ &\geq \frac{C_s^n}{2}\|w_m\|^2 + \int_\Omega G(x, w_m)\,\mathrm{d}x + \frac{1}{2+\epsilon}\bigg(\langle \beta_m, w_m\rangle - C_s^n\|w_m\|^2 - \int_\Omega g(x, w_m)w_m\,\mathrm{d}x\bigg) - \lambda M_\epsilon\end{aligned}$$

for $\epsilon > 0$ small enough. Using Lemma 4.4 (iii), it can be shown that $\{w_m\}$ is bounded in X_0. Thus, up to a subsequence, there exist $w \in X_0$ such that $w_m \rightharpoonup w$ weakly (and almost everywhere) in X_0 as $m \to \infty$. We assume, again up to a subsequence, that as $m \to \infty$,

$$\|w_m - w\|^2 \to a^2 \quad \text{and} \quad \int_\Omega |w_m - w|^{2_s^*}\,\mathrm{d}x \to b^{2_s^*}.$$

Also, we have

$$\begin{aligned}\int_\Omega G(x, w)\,\mathrm{d}x &\geq \int_\Omega G(x, w_m)\,\mathrm{d}x + \int_\Omega g(x, w_m)(w - w_m)\,\mathrm{d}x \\ &\geq \int_\Omega G(x, w_m)\,\mathrm{d}x - \lambda \int_\Omega (w_m + \bar{u})^{2_s^*-1}(w_m - w)\,\mathrm{d}x - \langle \beta_m, w_m - w\rangle \\ &\qquad + C_s^n \int_Q \frac{(w_m(x) - w_m(y))((w_m - w)(x) - (w_m - w)(y))}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which gives

$$\int_\Omega G(x, w)\,\mathrm{d}x \geq \int_\Omega G(x, w)\,\mathrm{d}x + C_s^n a^2 - \lambda b^{2_s^*}.$$

This in turn yields $\lambda b^{2_s^*} \geq C_s^n a^2$. Since u is a positive weak solution, we have

$$C_s^n \int_Q \frac{(u(x) - u(y))((w_m - u)(x) - (w_m - u)(y))}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega (g(x, u) - \lambda(u + \bar{u})^{2_s^*-1})(w_m - u)\,\mathrm{d}x = 0. \qquad (4.15)$$

Since $G(\cdot, w_m), G(\cdot, 2w_m) \in L^1(\Omega)$ and $u \leq 2w_m - u \leq 2w_m$, we have that $2w_m - u \in D(I_{K_u})$. Substituting $v = 2w_m - u$ in (4.14), we get

$$\begin{aligned}\langle \beta_m, w_m - u\rangle \leq C_s^n \int_Q \frac{(w_m(x) - w_m(y))((w_m - u)(x) - (w_m - u)(y))}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ + \int_\Omega (g(x, w_m) - \lambda(w_m + \bar{u})^{2_s^*-1})(w_m - u)\,\mathrm{d}x. \qquad (4.16)\end{aligned}$$

By (4.15), (4.16) and Lemma 4.4 (ii), we get

$$
\begin{aligned}
I_{K_u}(w_m) - I_{K_u}(u) &= \frac{C_s^n}{2}\|w_m\|^2 + \int_\Omega G(x, w_m)\,\mathrm{d}x - \frac{\lambda}{2_s^*}\int_\Omega |w_m + \bar{u}|^{2_s^*}\,\mathrm{d}x \\
&\quad - \bigg(\frac{C_s^n}{2}\|u\|^2 + \int_\Omega G(x, u)\,\mathrm{d}x - \frac{\lambda}{2_s^*}\int_\Omega |u + \bar{u}|^{2_s^*}\,\mathrm{d}x\bigg) \\
&\geq \int_\Omega \Big(G(x, w_m) - G(x, u) - \frac{1}{2}g(x, w_m)(w_m - u) - \frac{1}{2}g(x, u)(w_m - u)\Big)\,\mathrm{d}x \\
&\quad + \lambda \int_\Omega \bigg(\frac{1}{2}|w_m + \bar{u}|^{2_s^*-1}(w_m - u) - \frac{1}{2_s^*}|w_m + \bar{u}|^{2_s^*} \\
&\quad + \frac{1}{2}|u + \bar{u}|^{2_s^*-1}(w_m - u) + \frac{1}{2_s^*}|u + \bar{u}|^{2_s^*}\bigg)\,\mathrm{d}x + \frac{1}{2}\langle \beta_m, w_m - u\rangle \\
&\geq \lambda \int_\Omega \bigg(\frac{1}{2}|w_m + \bar{u}|^{2_s^*-1}(w_m - u) - \frac{1}{2_s^*}|w_m + \bar{u}|^{2_s^*} \\
&\quad + \frac{1}{2}|u + \bar{u}|^{2_s^*-1}(w_m - u) + \frac{1}{2_s^*}|u + \bar{u}|^{2_s^*}\bigg)\,\mathrm{d}x + \frac{1}{2}\langle \beta_m, w_m - u\rangle.
\end{aligned}
$$

Since the map $t \mapsto |t + \bar{u}|^{2_s^*-1}$ is convex, using the Brezis–Lieb lemma (see [7]) and letting $m \to \infty$ on both sides, we get

$$
\begin{aligned}
c - I_{K_u}(u) &\geq \frac{\lambda s b^{2_s^*}}{n} + \lambda \int_\Omega \bigg(\frac{1}{2}|w + \bar{u}|^{2_s^*-1}(w - u) - \frac{1}{2_s^*}|w + \bar{u}|^{2_s^*}\,\frac{1}{2}|u + \bar{u}|^{2_s^*-1}(w - u) + \frac{1}{2_s^*}|u + \bar{u}|^{2_s^*}\bigg)\,\mathrm{d}x \\
&\geq \frac{\lambda s b^{2_s^*}}{n} + \lambda \int_\Omega \bigg(\frac{1}{2}|w + \bar{u}|^{2_s^*-1}(w - u) + \frac{1}{2}|u + \bar{u}|^{2_s^*-1}(w - u)\int_u^w |t + \bar{u}|^{2_s^*-1}\,\mathrm{d}t\bigg)\,\mathrm{d}x \\
&\geq \frac{\lambda s b^{2_s^*}}{n}.
\end{aligned}
$$

Suppose $a > 0$. Then $\lambda b^{2_s^*} \geq C_s^n a^2$ and $a^2 \geq S_s b^2$ together imply

$$
\frac{\lambda s b^{2_s^*}}{n} \geq \frac{s(C_s^n S_s)^{n/(2s)}}{n\lambda^{(n-2s)/(2s)}},
$$

which contradicts our hypothesis. Thus, a must be 0, and hence $\|w_m\|$ strongly converges to w in X_0. Therefore, I_{K_u} satisfies $(\mathrm{CPS})_c$. □

For the sake of simplicity, we assume $0 \in \Omega$. In order to extend U_ϵ (defined in Section 1) by zero outside Ω, we fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$ and let $\zeta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \zeta \leq 1$ in $\mathbb{R}^n$, $\zeta \equiv 0$ in $\mathbb{R}^n \setminus B_{2\delta}$ and $\zeta \equiv 1$ in B_δ. For each $\epsilon > 0$ and $x \in \mathbb{R}^n$, we define

$$
\Phi_\epsilon(x) := \zeta(x)U_\epsilon(x).
$$

Moreover, since u is positive and bounded (see Lemma 3.2), we can find $m, M > 0$ such that for each $x \in B_{2\delta}$, $m \leq u(x) \leq M$.

Lemma 4.12. *For any sufficiently small $\epsilon > 0$,*

$$
\sup\{I_{K_u}(u + t\Phi_\epsilon) : t \geq 0\} < I_{K_u}(u) + \frac{s(C_s^n S_s)^{n/(2s)}}{n\lambda^{(n-2s)/(2s)}}.
$$

Proof. We assume $\epsilon > 0$ to be sufficiently small. Using [45, Proposition 21], we have

$$
\int_Q \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \leq S_s^{n/(2s)} + o(\epsilon^{n-2s}),
$$

which implies that we can find $r_1 > 0$ such that

$$\int_Q \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le S_s^{n/(2s)} + r_1\epsilon^{n-2s}.$$

Now, we have

$$\begin{aligned}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx &= \int_{\mathbb{R}^n} |U_\epsilon|^{2_s^*}\,dx + \int_{\mathbb{R}^n} (\zeta(x)^{2_s^*} - 1)|U_\epsilon(x)|^{2_s^*}\,dx \\ &= S_s^{n/(2s)} + \int_{\mathbb{R}^n\setminus B_\delta} (\zeta(x)^{2_s^*} - 1)|U_\epsilon(x)|^{2_s^*}\,dx \\ &= S_s^{n/(2s)} + \epsilon^{-n}\int_{\mathbb{R}^n\setminus B_\delta} (\zeta(x)^{2_s^*} - 1)\Big|u^*\Big(\frac{x}{\epsilon}\Big)\Big|^{2_s^*}\,dx \\ &\ge S_s^{n/(2s)} - \epsilon^n \int_{\mathbb{R}^n\setminus B_\delta} |x|^{-2n}\,dx \\ &\ge S_s^{n/(2s)} - r_2\epsilon^n\end{aligned}$$

for some constant $r_2 > 0$. We now fix $1 < \rho < \min\{2, \frac{n}{n-2s}\}$ and have

$$\int_\Omega |\Phi_\epsilon|^\rho\,dx = \epsilon^{-(n-2s)\rho/2}\int_{B_{2\delta}} \Big|\zeta(x)u^*\Big(\frac{x}{\epsilon}\Big)\Big|^\rho\,dx = O(\epsilon^{(n-2s)\rho/2}) \le r_3\epsilon^{(n-2s)\rho/2}$$

for a constant $r_3 > 0$. Now we see that

$$\int_{B_\epsilon} |\Phi_\epsilon|^{2_s^*-1}\,dx = \alpha^{2_s^*-1}\beta^{-(n+2s)}\epsilon^{(n-2s)/2}\int_{|y|<1/(\beta S_s^{1/(2s)})} (1+|y|^2)^{-(n+2s)/2}\,dy \ge r_4\epsilon^{(n-2s)/2}$$

for some constant $r_4 > 0$. We also have

$$G(x, r+t) - G(x,r) - g(x,r)t = \int_r^{r+t} (g(x,\tau) - g(x,r))\,d\tau = \int_r^{r+t} ((r+\bar{u}(x))^{-q} - (\tau+\bar{u})^{-q})\,d\tau \le \int_r^{r+t} (r^{-q} - \tau^{-q})\,d\tau.$$

Thus, we can find $\gamma > 0$ such that

$$G(x, r+t) - G(x,r) - g(x,r)t \le \gamma t^\rho \quad \text{for each } x \in \Omega,\ r \ge m \text{ and } t \ge 0.$$

We can find an appropriate constant $\rho_s > 0$ such that the following inequalities hold:

$$\begin{aligned}&\frac{(c+d)^{2_s^*}}{2_s^*} - \frac{c^{2_s^*}}{2_s^*} - c^{2_s^*-1}d \ge \frac{d^{2_s^*}}{2_s^*} && \text{for all } c, d \ge 0,\\ &\frac{(c+d)^{2_s^*}}{2_s^*} - \frac{c^{2_s^*}}{2_s^*} - c^{2_s^*-1}d \ge \frac{d^{2_s^*}}{2_s^*} + \frac{\rho_s c d^{2_s^*-1}}{r_4 m(2_s^*-1)} && \text{for all } 0 \le c \le M,\ d \ge 1.\end{aligned}$$

Since u is a positive weak solution of (P_λ), using the above inequalities, we obtain

$$\begin{aligned}I_{K_u}(u+t\Phi_\epsilon) - I_{K_u}(u) &= I_{K_u}(u+t\Phi_\epsilon) - I_{K_u}(u) - t\bigg(C_s^n\int_Q \frac{(u(x)-u(y))(\Phi_\epsilon(x)-\Phi_\epsilon(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \int_\Omega (g(x,u)\Phi_\epsilon - \lambda(u+\bar{u})^{2_s^*-1}\Phi_\epsilon)\,dx\bigg) \\ &= \frac{t^2C_s^n}{2}\int_Q \frac{|\Phi_\epsilon(x)-\Phi_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega \frac{1}{2_s^*}(|u+t\Phi_\epsilon+\bar{u}|^{2_s^*} - |u+\bar{u}|^{2_s^*})\,dx \\ &\quad + \lambda t\int_\Omega (u+\bar{u})^{2_s^*-1}\Phi_\epsilon\,dx + \int_\Omega (G(x,u+t\Phi_\epsilon) - G(x,u) - g(x,u)(t\Phi_\epsilon))\,dx\end{aligned}$$

$$\leq \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx + \gamma\int_\Omega |t\Phi_\epsilon|^\rho\,dx$$
$$\leq \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}(S_s^{n/(2s)} - r_2\epsilon^n) + \gamma r_3 t^\rho \epsilon^{(n-2s)\rho/2}$$

for $0 \leq t < \lambda^{-(n-2s)/(4s)}/2$. Since we can assume $t\Phi_\epsilon \geq 1$ for each $t \geq \lambda^{-(n-2s)/(4s)}/2$ and $|x| \leq \epsilon$, we have

$$\begin{aligned} I_{K_u}(u + t\Phi_\epsilon) - I_{K_u}(u) &\leq \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx \\ &\quad - \frac{\lambda\rho_s t^{2_s^*-1}}{r_4 m(2_s^*-1)}\int_{|x|\leq\epsilon}(u+\bar{u})|\Phi_\epsilon|^{2_s^*-1}\,dx + \gamma\int_\Omega |t\Phi_\epsilon|^\rho\,dx \\ &\leq \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}(S_s^{n/(2s)} - r_2\epsilon^n) - \frac{\lambda\rho_s t^{2_s^*-1}}{(2_s^*-1)}\epsilon^{(n-2s)/2} + \gamma r_3 t^\rho \epsilon^{(n-2s)\rho/2}. \end{aligned}$$

Now, we define a function $h_\epsilon : [0,\infty) \to \mathbb{R}$ by

$$h_\epsilon(t) = \begin{cases} \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}(S_s^{n/(2s)} - r_2\epsilon^n) + \gamma r_3 t^\rho\epsilon^{(n-2s)\rho/2}, & t \in [0, \frac{\lambda^{(n-2s)/4s}}{2}), \\ \frac{t^2 C_s^n}{2}(S_s^{n/(2s)} + r_1\epsilon^{n-2s}) - \frac{\lambda t^{2_s^*}}{2_s^*}(S_s^{n/(2s)} - r_2\epsilon^n) - \frac{\lambda\rho_s t^{2_s^*-1}}{(2_s^*-1)}\epsilon^{(n-2s)/2} + \gamma r_3 t^\rho\epsilon^{(n-2s)\rho/2}, & t \in [\frac{\lambda^{(n-2s)/4s}}{2}, \infty). \end{cases}$$

With some computations, it can be checked that h_ϵ attains its maximum at

$$t = \left(\frac{C_s^n}{\lambda}\right)^{(n-2s)/(4s)} - \frac{\rho_s\epsilon^{(n-2s)/2}}{(2_s^*-2)S_s^{n/(2s)}} + o(\epsilon^{(n-2s)/2}),$$

so we get

$$\begin{aligned} \sup\{I(u + t\Phi_\epsilon) - I(u) : t \geq 0\} &\leq (C_s^n)^{n/(2s)}\frac{sS_s^{n/(2s)}}{n\lambda^{(n-2s)/(2s)}} - \frac{\rho_s(C_s^n)^{(n+2s)/4s}\epsilon^{(n-2s)/2}}{(2_s^*-1)\lambda^{(n-2s)/(4s)}} + o(\epsilon^{(n-2s)/2}) \\ &< \frac{s(C_s^n S_s)^{n/(2s)}}{n\lambda^{(n-2s)/(2s)}}. \end{aligned}$$

This completes the proof. □

Proposition 4.13. *For each $\lambda \in (0,\Lambda)$, there exist a second positive weak solution of $(\bar{P}_\lambda)$.*

Proof. Let $\Phi = \Phi_\epsilon$ for some sufficiently small $\epsilon > 0$, as obtained in the previous lemma. From Theorem 4.8, u is a local minimizer of I_{K_u}. So we can choose $\alpha > 0$ small enough such that $I_{K_u}(v) \geq I_{K_u}(u)$ for every $v \in K_u$ with $\|v - u\| \leq \alpha$. We know that $I_{K_u}(u + tw) \to -\infty$ as $t \to \infty$, which implies that we can choose $t > \alpha/\|w\|$ such that $I_{K_u}(u + tw) \leq I_{K_u}(u)$. Let us set

$$\Phi = \{\phi \in C([0,1], D(I_{K_u})) : \phi(0) = u,\ \phi(1) = u + tw\},$$
$$A = \{v \in D(I_{K_u}) : \|v - u\| = \alpha\} \quad\text{and}\quad c = \inf_{\phi\in\Phi}\sup_{0\leq r\leq 1} I_{K_u}(\phi(r)).$$

The functional I_{K_u} satisfies $(\mathrm{CPS})_c$, by Proposition 4.11 and Lemma 4.12. If $c = I_{K_u}(u)$, then $u \notin A$, $u + tw \notin A$, $\inf I_{K_u}(A) \geq I_{K_u}(u) \geq I_{K_u}(u + tw)$, and for each $\phi \in \Phi$, there exist $r \in [0,1]$ such that $\|\phi(r) - u\| = \alpha$. Hence, by Theorem 2.4, we have $v \in D(I_{K_u})$ such that $v \neq u$, $I_{K_u}(v) = c$ and $0 \in \partial^- I_{K_u}(v)$. Using Proposition 4.2 (i), we have that v is a positive weak solution of $(\bar{P}_\lambda)$. □

Proof of Theorem 2.10. The proof of Theorem 2.10 follows from Theorems 4.8, 4.10 and 4.13, and Lemmas 3.3–3.6. □

5 Fractional problem in the critical dimension $n = 1$

In the critical dimension $n = 1$, the critical growth nonlinearities for the fractional Laplacian is explored in [22]. The analogous critical problem in this case is

$$(-\Delta)^{1/2}u = u^{-q} + \lambda u^{p+1}\exp(u^2), \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.1}$$

where p, q, λ are positive parameters. Fractional problems with exponential growth nonlinearities are motivated by the following version of the Moser–Trudinger inequality [32].

Theorem 5.1. *For $u \in H^{1/2}((-1,1))$, $\exp(\alpha u^2) \in L^1((-1,1))$ for any $\alpha > 0$. Moreover, there exists a constant $C > 0$ such that*

$$\sup_{\|(-\Delta)^{1/4}u\|_{L^2(-1,1)}\leq 1} \left(\int_{-1}^{1} \exp(\alpha u^2)\,\mathrm{d}x \right) \leq C \quad \text{for all } \alpha \leq \pi.$$

Problem (5.1) can be transformed into a local problem by Dirichlet–Neumann maps introduced by Cafarelli and Silvestre [9]. For any $v \in H^{1/2}(\mathbb{R})$, the unique function $w(x,y)$ that minimizes the weighted integral

$$\mathcal{E}_{1/2}(w) = \int_0^\infty \int_{\mathbb{R}} |\nabla w(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y,$$

over the set

$$\{w(x,y) : \mathcal{E}_{1/2}(w) < \infty,\ w|_{y=0} = v\},$$

satisfies

$$\int_{\mathbb{R}} |(-\Delta)^{1/2} v|^2 = \mathcal{E}_{1/2}(w).$$

Moreover, $w(x,y)$ solves the boundary value problem

$$-\operatorname{div}(\nabla w) = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \qquad w|_{y=0} = v, \qquad \frac{\partial w}{\partial \nu} = (-\Delta)^{1/2} v(x),$$

where $\frac{\partial w}{\partial \nu} = -\lim_{y\to 0^+} \frac{\partial w}{\partial y}(x,y)$. So the extension problem corresponding to (5.1) is

$$\begin{cases} -\operatorname{div}(\nabla w) = 0, \quad w > 0 & \text{in } \mathcal{C} := (-1,1)\times(0,\infty), \\ \dfrac{\partial w}{\partial \nu} = w^{-q} + \lambda w^{p+1}\exp(w^2) & \text{on } \Omega \times \{0\}. \end{cases} \tag{5.2}$$

To solve this, we closely follow the arguments used in [40]. The natural space to look for the solution of this extension problems is the Sobolev space

$$H^1_{0,L}(\mathcal{C}) = \{v \in H^1(\mathcal{C}) : v = 0 \text{ a.e. in } (-1,1)\times(0,\infty)\},$$

equipped with the norm $\|w\| = (\int_{\mathcal{C}} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}y)^{1/2}$. Now using the relation between the space $H^{1/2}((-1,1))$ and the square root Laplacian operator (see [16]), we get

$$\|(-\Delta)^{1/4} u\|_{L^2((-1,1))} = \frac{1}{\sqrt{2\pi}} [u]_{H^{1/2}(\mathbb{R})} = \|w\|,$$

where

$$[u]_{H^{1/2}(\mathbb{R})} = \left(\iint_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

If w solves the extension problem (5.2), then the trace$(w) = w(x,0)$ solves the given nonlocal problem and vice-versa.

Definition 5.2. A function $w \in L^1_{\mathrm{loc}}(\mathcal{C})$ is said to be a weak solution of (5.2) if the following hold:
(i) $\inf_{(x,y)\in K} w(x,y) > 0$ for every compact subset $K \subset \Omega \times [0,\infty)$,
(ii) w solves the PDE in (5.2) in the sense of distributions,
(iii) $(w-\epsilon)^+ \in H^1_{0,L}(\mathcal{C})$ for every $\epsilon > 0$.

Let w_0 be the minimal weak solution (in the sense of Definition 5.2) of

$$-\Delta w = 0 \quad \text{in } (-1,1)\times(0,\infty), \qquad \frac{\partial w}{\partial y} = w^{-q} \quad \text{in } (-1,1)\times\{0\}.$$

The existence of w_0 can be obtained by solving the corresponding equivalent problem (P_0) with $\Omega = (-1, 1)$ and by following the approach used in [5] (see Section 3). Precisely, regularizing the singular nonlinearity in (P_0), we introduce for $n \in \mathbb{N}^*$ the following approximated problem:

$$(-\Delta)^s u = \Big(u + \frac{1}{n}\Big)^{-q}, \qquad u > 0 \quad \text{in } (-1, 1), \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus (-1, 1). \tag{P_n}$$

This problem admits a unique solution w_n in $\tilde{H}^{1/2}(-1, 1)$, the Lions–Magenes space defined by

$$\begin{aligned}\tilde{H}^{1/2}(-1, 1) &:= \Big\{u \in H^1(-1, 1) : \int_{-1}^{1} \frac{u^2}{d(x)} \mathrm{d}x < \infty\Big\}\\ &= \{u \in H^{1/2}(\mathbb{R}) : u \equiv 0 \text{ in } \mathbb{R} \setminus (-1, 1)\}\\ &= [H_0^1(-1, 1), L^2(-1, 1)]_{1/2}.\end{aligned}$$

Then, passing w_n to the limit as $n \to \infty$ in the sense of distributions, we obtain $w_0(x, 0)$. Using a similar proof to that of Theorem 3.3, we can show that $w_0 \in L^\infty(\mathcal{C})$. We can translate the problem, as in Section 3, by w_0 as follows:

$$\begin{cases} -\Delta w = 0, \quad w > 0 & \text{in } (-1, 1) \times (0, \infty),\\ \dfrac{\partial w}{\partial y} + w_0^{-q} - (w + w_0)^{-q} = \lambda (w + w_0)^{p+1} \exp((w + w_0)^2) & \text{in } (-1, 1) \times \{0\}. \end{cases} \tag{$\bar{P}'_\lambda$}$$

Note that $w + w_0$ is a solution of (5.2) if $w \in H^1_{0,L}(\mathcal{C})$ is a nonnegative distributional solution of $(\bar{P}'_\lambda)$. Hence, it is enough to show existence and multiplicity results for $(\bar{P}'_\lambda)$. It is possible to give a variational framework for problem $(\bar{P}'_\lambda)$ in the space $H^1_{0,L}(\mathcal{C})$. Following the arguments used in [40], we define the functions $g, f\colon (-1, 1) \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, s) = \begin{cases} (s + w_0(x, 0))^{p+1} \exp((s + w_0(x, 0))^\beta) & \text{if } s + w_0(x, 0) > 0,\\ 0 & \text{otherwise,} \end{cases}$$

$$g(x, s) = \begin{cases} (w_0(x, 0))^{-q} - (s + w_0(x, 0))^{-q} & \text{if } s + w_0(x) > 0,\\ -\infty & \text{otherwise.} \end{cases}$$

It is easy to see that both g and f are nonnegative and nondecreasing in s. The required measurability of $g(\cdot\,, s)$ and $f(\cdot\,, s)$ follows from [31, Lemmas 1 and 2]. We define the primitives $F\colon (-1, 1) \times \mathbb{R} \to \mathbb{R}$ and $G\colon (-1, 1) \times \mathbb{R} \to (-\infty, \infty]$, respectively, by

$$F(x, s) = \int_0^s f(x, \tau)\, \mathrm{d}\tau \quad \text{and} \quad G(x, s) = \int_0^s g(x, \tau)\, \mathrm{d}\tau \quad \text{for } (x, s) \in (-1, 1) \times \mathbb{R}.$$

Then we note that there exist $M > 0$, $\theta > 2$ such that for all $s > 0$, $x \in (-1, 1)$,

$$F(x, s) \le M(f(x, s) + 1) \quad \text{and} \quad \theta F(x, s) \le f(x, s)s.$$

Define a functional $I\colon H^1_{0,L}(\mathcal{C}) \to (-\infty, \infty]$ corresponding to (5.2) by

$$I(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{-1}^1 G(x, w(x, 0))\, \mathrm{d}x - \lambda \int_{-1}^1 F(x, w(x, 0))\, \mathrm{d}x & \text{if } G(\cdot\,, u) \in L^1(\Omega),\\ \infty & \text{otherwise.} \end{cases}$$

Now we can define the weak sub and super solutions and, by following the arguments used in Section 3, we can show the existence of the first solution w_λ. Moreover, for $I_K := I|_K$, the following theorem follows from [40, Theorem 3.19].

Theorem 5.3. *Take $\lambda \in (0, \Lambda)$. Let z be a strict super-solution of $(\bar{P}'_\lambda)$. Let also $w_\lambda \in D(I_K)$ be a minimizer for I_K, where $K = \{u \in H^1_{0,L}(\mathcal{C}) : 0 \le u \le z\}$. Then w_λ is a local minimizer for I_{H^+}, where $H^+ = \{v \in H^1_{0,L}(\mathcal{C}), v \ge 0\}$.*

To prove the existence of another solution to problem (5.2), as in [40], we translate this problem about the first solution w_λ as follows:

$$\begin{cases} -\Delta w = 0, \quad w > 0 & \text{in } \mathcal{C}, \\ \dfrac{\partial w}{\partial y} + g(x, w + w_\lambda) - g(x, w_\lambda) = \lambda(f(x, w + w_\lambda) - f(x, w_\lambda)), \quad w > 0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{TP_λ}$$

Clearly, w is a solution of (TP_λ) if and only if $(w + w_\lambda)$ solves (5.2). Define

$$\tilde{g}(x, s) = \begin{cases} g(x, s + w_\lambda) - g(x, w_\lambda) & s > 0, \\ 0 & s \le 0, \end{cases} \qquad \tilde{f}(x, s) = \begin{cases} f(x, s + w_\lambda) - f(x, w_\lambda) & s > 0, \\ 0 & s \le 0. \end{cases}$$

Define the respective primitives:

$$\tilde{G}(x, u) = \int_0^u \tilde{g}(x, s)\,ds, \quad \tilde{F}(x, u) = \int_0^u \tilde{f}(x, s)\,ds.$$

Thanks to the nondecreasing nature of g and hence $\tilde{g}$, we obtain the following inequality:

$$\tilde{G}(x, s) \le \tilde{g}(x, s)s \quad \text{for all } s \ge 0.$$

Let us define the energy functional $E\colon H^1_{0,L}(\mathcal{C}) \to (-\infty, +\infty]$, associated with (TP_λ), as follows:

$$E(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \int_{-1}^1 \tilde{G}(x, w)\,dx - \lambda \int_{-1}^1 \tilde{F}(x, w)\,dx & \text{if } \tilde{G}(\cdot, u) \in L^1(-1, 1), \\ \infty & \text{otherwise.} \end{cases}$$

Recalling the definition of I, we note that

$$E(u) = I(w^+ + w_\lambda) - I(w_\lambda) + \frac{1}{2}\|w^-\|^2 \quad \text{for all } w \in H^1_{0,L}(\mathcal{C}).$$

It follows that

$$D(E) \cap H^+ = D(I) \cap H^+.$$

Since w_λ is a local minimum of I_{H^+}, it follows that 0 is a local minimum of $E(u)$ in H^+. Thus, there exists $\rho_0 > 0$ such that $E(u) \ge E(0) = 0$ for all $u \in H^+$ with $\|u\| \le \rho_0$.

We recall the following version of the Lions compactness lemma (see [21, Lemma 2.3]).

Theorem 5.4. *Let $\{w_k : \|w_k\| = 1\}$ be a sequence of $H^1_{0,L}(\mathcal{C})$ functions converging weakly to a non zero function u. Then, for all $p < (1 - \|w\|)^{-1}$,*

$$\sup_k \left(\int_{-1}^1 \exp(\pi p |w_k|^2)\,dx \right) < \infty.$$

To show the existence of mountain-pass solution, we need the following sequence of Moser functions concentrating on the boundary, see [22].

Lemma 5.5. *There exists a sequence $\{\phi_k\} \subset H^1_{0,L}(\mathcal{C})$ satisfying the following:*
(i) $\phi_k \ge 0$, $\operatorname{supp}(\phi_k) \subset B(0, 1) \cap \mathbb{R}^2_+$ *and* $\|\phi_k\| = 1$,
(ii) ϕ_k *is constant on* $x \in B(0, \frac{1}{k}) \cap \mathbb{R}^2_+$ *and* $\phi_k^2 = \frac{1}{\pi}\log k + O(1)$ *for* $x \in B(0, \frac{1}{k}) \cap \mathbb{R}^2_+$.

Now we have the following estimate on the level. The proof follows as in [40, Lemma 4.4].

Lemma 5.6. *We have*

$$\sup_{t>0} E(t\phi_n) < \frac{\pi}{2} \quad \textit{for all large } n.$$

Now the proof of the existence of the second solution follows from theorem 5.4 and by closely following the proofs of [40, Lemma 4.9 and Proposition 4.10].

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