On Dirichlet problem for fractional p-Laplacian with singular nonlinearity

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Abstract

In this article, we study the following fractional p-Laplacian equation with critical growth singular nonlinearity

$$(-\Delta_p)^s u = \lambda u^{-q} + u^{\alpha}, u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > sp, s \in (0,1), \lambda > 0, 0 < q \leq 1$ and $\alpha \leq p_s^* - 1$. We use variational methods to show the existence and multiplicity of positive solutions of above problem with respect to parameter λ.

Key words: fractional p-Laplacian, Critical exponent, Singular nonlinearities

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1 Introduction

Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n > sp$. We consider the following problem with singular nonlinearity :

$$(P_\lambda): \qquad (-\Delta_p)^s u = \lambda u^{-q} + u^{\alpha}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

where $\lambda > 0, 0 < q \leq 1, \alpha \leq p_s^* - 1, p_s^* = \frac{np}{n-sp}$ and $(-\Delta_p)^s$ is the fractional p-Laplacian operator defined as

$$(-\Delta_p)^s u(x) = -2 \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy \text{ for all } x \in \mathbb{R}^n.$$

Recently a lot of attention is given to the study of fractional and non-local operators of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc.

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Semilinear Dirichlet problem for fractional Laplacian using variational methods is recently studied in [9, 42, 43]. The existence and multiplicity results for non-local operators like fractional Laplacian with combination of convex and concave type non linearity like $u^q + \lambda u^p, p, q > 0$ is studied in [4, 6, 36, 37, 44, 45]. Eigenvalue problem for the fractional p-Laplacian and properties like simplicity of smallest eigenvalue is studied in [34, 18]. The Brezis-Nirenberg type existence result is studied in [39]. Existence results with convex-concave type regular nonlinearities is studied in [27].

In the local setting ($s = 1$), the paper by Crandal, Rabinowitz and Tartar [13] is the starting point on semilinear problem with singular nonlinearity. A lot of work has been done related to existence and multiplicity results for Laplacian and p-Laplacian with singular non-linearity, see [1, 24, 25, 16, 12, 20, 21]. In [16, 12], the authors studied the singular problems of the type

$$-\Delta u = g(x, u) + h(x, \lambda u), \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, g(x, u) \in L^1(\Omega)$$

with $g(x, u) \sim u^{-\alpha}$. They studied the existence of solutions under suitable conditions on g and h. In [20] and [21], authors conside the singular problems of the type

$$-\Delta u + K(x)g(u) = \lambda f(x, u) + \mu h(x) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

where Ω is smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $\lambda > 0$. Here, $h, K \in C^{0,\gamma}(\Omega)$ for some $0 < \gamma < 1$ and $h > 0$ in Ω, $f : [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is positive on $\bar{\Omega} \times (0, \infty)$ that is sublinear at ∞ and of superlinear at 0. The function $g \in C^{0,\gamma}(0, \infty)$ for some $0 < \gamma < 1$ is non negative and non increasing such that $\lim_{s \to 0^+} g(s) = +\infty$. They proved several results related to existence and non existence of positive solutions of above problems taking into account both the sign of the potential K and the decay rate around the origin of the singular nonlinearity g. Several authors conside the problems of Lane-Emden-Fowler type with singular nonlinearity such as [17, 11, 22]. In addition, some bifurcation results has been proved in [22] for the problem

$$-\Delta u = g(u) + \lambda |\nabla u|^p + \mu f(x, u) \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

where $\lambda, \mu \geq 0$, $0 < p \leq 2$, f is non-decreasing with respect to the second variable and $g(u)$ behaves like $u^{-\alpha}$ around the origin. The asymptotic behaviour of the solutions is shown by constructing suitable sub- and supersolutions combined with the maximum principles. We also refer [26, 31] as a part of previous contributions to this field. For detailed study and recent results on singular problems we refer to [23].
In [24], authors studied the critical growth singular problem

$$-\Delta_p u = \lambda u^{-\delta} + u^q, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $0 < \delta < 1$ and $p - 1 < q \leq p^* - 1$ and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. Using the variational methods, they proved the existence of multiple solutions with restriction on p and q in the spirit of [19, 14]. Among the works dealing with elliptic equations with singular and critical growth terms, we cite also [1, 2, 29, 3, 10] and references there-in, with no attempt to provide a complete list.

Recently, the study of the fractional elliptic equations attracted lot of interest by researchers in nonlinear analysis. There are many works on existence of a solution for fractional elliptic equations with regular nonlinearities like $u^q + \lambda u^p,\ p,\ q > 0$. The sub critical growth problems are studied in [9, 42, 43] and critical exponent problems are studied in [6, 36, 37, 39]. Also, the multiplicity of solutions by the method of Nehari manifold and fibering maps has been investigated in [27, 45, 46]. For detailed study and recent results on this subject we refer to [38]. In [5] the authors the singular problem

$$(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + M u^p,\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < p < 2_s^* - 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. Here authors studied the existence of distributional solutions for small λ using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{n})^{-\gamma}$. In [40], the critical $(p = 2_s^* - 1)$ singular problem is studied where multiplicity results are obtained using the Nehari manifold approach.

There are many works on the study of p-fractional equations with polynomial type nonlinearities. In citess1 authors studied the subcritical problems using Nehari manifold and fibering maps. In [39], Brezis-Nirenberg type critical exponent problem is studied. We also [8, 28, 33] and references therein. To the best of our knowledge, there are no works on existence or multiplicity results with singular nonlinearities.

In this paper, we study the existence and multiplicity results with convex-concave type singular nonlinearity. Here we follow the approach as in [32]. We obtain our results by studying the existence of minimizers that arise out of structure of Nehari manifold. We would like to remark that the results proved here are new even for the case $q = 1$. Also the existence result is sharp in the sense that we show the existence of Λ such that $(0, \Lambda)$ is the maximal range for λ for which the solution exists. We show the existence of second solution in the sub-critical case for suitable range of λ where the fibering maps has two critical points. We also show some regularity results on weak solutions.

The paper is organized as follows: In section 2, we present some preliminaries on function spaces requi for variational settings. In section 3, we study the corresponding Nehari manifold and properties of minimizers. In section 4 and 5, we show the existence of minimizers and solutions and state the main results. In section 6, we show some regularity results and section 7 is devoted to the maximal range of λ for existence of solutions.

2 Preliminaries and Main Results

In [27], authors discussed the Dirichlet boundary value problem involving p-fractional Laplace operator using the variational techniques. Due to non-localness of the fractional Laplacian, they introduced the function space $(X_0, \|.\|_{X_0})$. The space X is defined as

$$X = \left\{ u | \ u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \frac{(u(x) - u(y))}{|x-y|^{\frac{n+sp}{p}}} \in L^p(Q) \right\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + [u]_X\,, \quad \text{where } [u]_X = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.$$

Then we define $X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$. Also, there exists a constant $C > 0$ such that $\|u\|_{L^p(\Omega)} \leq C[u]_X$ for all $u \in X_0$. Hence, $\|u\| = [u]_X$ is a norm on X_0 and X_0 is a Hilbert space. Note that the norm $\|.\|$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$. We denote $\|.\|_{L^p(\Omega)}$ as $|.|_p$ and $\|.\| = [.]_X$ for the norm in X_0. Now for each $\beta \geq 0$, we set

$$C_\beta = \sup \left\{ |u|_\beta^\beta \ : \ u \in X, \ \|u\| = 1 \right\}. \tag{2.1}$$

Then $C_0 = |\Omega| =$ Lebesgue measure of Ω and $\int_\Omega |u|^\beta dx \leq C_\beta \|u\|^\beta$, for all $u \in X_0$. From the embedding results in [27], we know that X_0 is continuously and compactly embedded in $L^r(\Omega)$ where $1 \leq r < p_s^*$ and the embedding is continuous but not compact if $r = p_s^*$. We define the best constant of the embedding S as

$$S = \inf\{\|u\|^p \ : \ u \in X_0, \ |u|_{p_s^*}^p = 1\}.$$

Definition 2.1 *We say $u \in X_0$ is a positive weak solution of (P_λ) if $u > 0$ in Ω and*

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}} \ dxdy - \int_\Omega \left(\lambda u^{-q} - u^\alpha\right) \psi \ dx = 0$$

for all $\psi \in C_c^\infty(\Omega)$.

We define the functional associated to (P_λ) as $I_\lambda : X_0 \to (-\infty, \infty]$ as

$$I_\lambda(u) = \frac{1}{p} \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy - \lambda \int_\Omega G_q(u) dx - \frac{1}{\alpha+1} \int_\Omega |u|^{\alpha+1} dx$$

where $G_q : \mathbb{R} \to [-\infty, \infty)$ is the function defined by

$$G_q(x) = \begin{cases} \frac{|x|^{1-q}}{1-q} & \text{if } 0 < q < 1 \\ \ln |x| & \text{if } q = 1 \end{cases}$$

for $x \in \mathbb{R}$. For each $0 < q \leq 1$, we set $X_+ = \{u \in X_0 : u \geq 0\}$ and

$$X_{+,q} = \{u \in X_+ : u \not\equiv 0, \ G_q(u) \in L^1(\Omega)\}.$$

Notice that $X_{+,q} = X_+ \setminus \{0\}$ if $0 < q < 1$ and $X_{+,1} \neq \emptyset$ if $\partial\Omega$ is, for example, of C^2. We will need the following important Lemma.

Lemma 2.2 *For each $w \in X_+$, there exists a sequence $\{w_k\}$ in X_0 such that,$w_k \to w$ strongly in X_0, where $0 \le w_1 \le w_2 \le \dots$ and w_k has compact support in Ω, for each k .*

Proof. Proof here is adopted from [32]. Let $w \in X_+$ and $\{\psi_k\}$ be sequence in $C_c^\infty(\Omega)$ such that ψ_k is non negative and converges strongly to w in X_0. Define $z_k = \min\{\psi_k, w\}$, then $z_k \to w$ converges strongly to w in X_0. Now, we set $w_1 = z_{r_1}$ where $r_1 > 0$ is such that $\|z_{r_1} - w\| \le 1$. Then $\max\{w_1, z_m\} \to w$ strongly as $m \to \infty$, thus we can find $r_2 > 0$ such that $\|\max\{w_1, z_{r_2}\} - w\| \le 1/2$. We set $w_2 = \max\{w_1, z_{r_2}\}$ and get $\max\{w_2, z_m\} \to w$ strongly as $m \to \infty$. Consequently, by induction we set, $w_{k+1} = \max\{w_k, z_{r_{k+1}}\}$ to obtain the desi sequence, since we can see that $w_k \in X_0$ has compact support, for each k and $\|\max\{w_k, z_{r_{k+1}}\} - w\| \le 1/(k+1)$ which says that $\{w_k\}$ converges strongly to w in X_0 as $k \to \infty$. □

Let $\phi_1 > 0$ be the eigenfunction of $(-\Delta_p)^s$ corresponding to the smallest eigenvalue λ_1. This is obtained as minimizer of the minimization problem

$$\lambda_1 = \min\{\|u\| \ : \ u \in X_0, \ \ \|u\|_{L^p(\Omega)} = 1\}.$$

In (see [39, 34]) it was shown that this minimizer is achieved by unique positive and bounded function ϕ_1. Moreover (λ_1, ϕ_1) is the solution of the eigenvalue problem

$$(-\Delta_p)^s u = \lambda_1 |u|^{p-2} u, \ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \Omega.$$

We assume $\|\phi_1\|_{L^\infty} = 1$. With these preliminaries, we state our main results.

For each $u \in X_{+,q}$ we define the fiber map $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ by $\phi_u(t) = I_\lambda(tu)$. Then we prove

Theorem 2.3 *Assume $0 < q \le 1$. In case $q = 1$, assume also $X_{+,1} \neq \emptyset$. Let Λ_1 be a constant defined by $\Lambda_1 = \sup\{\lambda > 0 : \text{ for each } u \in X_{+,q} \setminus \{0\}, \ \phi_u(t) \text{ has two critical points in } (0,\infty)\}$. Then $\Lambda_1 > 0$.*

Theorem 2.4 *For all $\lambda \in (0, \Lambda_1)$, (P_λ) has at least two distinct solutions in $X_{+,q}$ when $\alpha < p_s^* - 1$ and at least one solution in the critical case $\alpha = p_s^* - 1$.*

Definition 2.5 *We say $u \in X_0$ a weak sub solution of (P_λ) if $u > 0$ in Ω and*

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}}\, dxdy \le \int_\Omega \left(\lambda u^{-q} + u^\alpha\right)\psi \, dx = 0$$

for all $0 \le \psi \in C_c^\infty(\Omega)$. Similarly $u \in X_0$ is said to be a weak super solution to (P_λ) if in the above the reverse inequalities hold.

Next we study that the existence of solution with the parameter in maximal interval. For this we minimize the functional over the convex set $\{u \in X_{+,q} : \underline{u} \le u \le \overline{u}\}$ where $\underline{u}$ and $\overline{u}$ are sub and super solutions respectively. Using truncation techniques as in [30], we show that the minimizer is a solution.

Theorem 2.6 *Let $\alpha \le p_s^* - 1$ and $0 < q \le 1$. Then there exists $\Lambda > 0$ such that (P_λ) has a solution for all $\lambda \in (0, \Lambda)$ and no solution for $\lambda > \Lambda$.*

3 Nehari manifold and fibering maps

We denote $I_\lambda = I$ for simplicity now. One can easily verify that the energy functional I is not bounded below on the space X_0. We will show that it is bounded on the manifold associated to the functional I. In this section, we study the structure of this manifold. We define

$$\mathcal{N}_\lambda = \{u \in X_{+,q} | \left\langle I'(u), u \right\rangle = 0\}.$$

Theorem 3.1 *I is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. In case of $0 < q < 1$, since $u \in \mathcal{N}_\lambda$, using the embedding of X_0 in $L^{1-q}(\Omega)$, we get

$$\begin{aligned} I(u) &= \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)\|u\|^p - \lambda\left(\frac{1}{1-q} - \frac{1}{\alpha+1}\right)\int_\Omega |u|^{1-q}dx \\ &\geq c_1\|u\|^p - c_2\|u\|^{1-q} \end{aligned}$$

for some constants c_1 and c_2. This says that I is coercive and bounded below on $\mathcal{N}_\lambda$.
In case of $q = 1$, using the inequality $\ln|u| \leq |u|$ and embedding results for X_0, we can similarly get I as bounded below. □

From the definition of fiber map ϕ_u, we have

$$\phi_u(t) = \begin{cases} \dfrac{t^p}{p}\|u\|^p - \dfrac{t^{1-q}}{1-q}\displaystyle\int_\Omega |u|^{1-q}dx - \dfrac{t^{\alpha+1}}{\alpha+1}\displaystyle\int_\Omega |u|^{\alpha+1}dx & \text{if } 0 < q < 1 \\ \dfrac{t^p}{p}\|u\|^p - \lambda\displaystyle\int_\Omega \ln(t|u|)dx - \dfrac{t^{\alpha+1}}{\alpha+1}\displaystyle\int_\Omega |u|^{\alpha+1}dx & \text{if } q = 1. \end{cases}$$

which gives

$$\phi_u'(t) = t^{p-1}\|u\|^p - \lambda t^{-q}\int_\Omega |u|^{1-q}dx - t^\alpha\int_\Omega |u|^{\alpha+1;}dx$$

$$\phi_u''(t) = (p-1)t^{p-2}\|u\|^p + q\lambda t^{-q-1}\int_\Omega |u|^{1-q}dx - \alpha t^{\alpha-1}\int_\Omega |u|^{\alpha+1}dx.$$

It is easy to see that the points in $\mathcal{N}_\lambda$ are corresponding to critical points of ϕ_u at $t = 1$. So, it is natural to divide $\mathcal{N}_\lambda$ into three sets corresponding to local minima, local maxima and points of inflexion. Therefore, we define

$$\begin{aligned} \mathcal{N}_\lambda^+ =&\{u \in \mathcal{N}_\lambda |\ \phi_u'(1) = 0,\ \phi_u''(1) > 0\} = \{t_0u \in \mathcal{N}_\lambda |\ t_0 > 0,\ \phi_u'(t_0) = 0,\ \phi_u''(t_0) > 0\} \\ \mathcal{N}_\lambda^- =&\{u \in \mathcal{N}_\lambda |\ \phi_u'(1) = 0,\ \phi_u''(1) < 0\} = \{t_0u \in \mathcal{N}_\lambda |\ t_0 > 0,\ \phi_u'(t_0) = 0,\ \phi_u''(t_0) < 0\} \end{aligned}$$

and, $\mathcal{N}_\lambda^0 = \{u \in \mathcal{N}_\lambda | \phi_u'(1) = 0, \phi_u''(1) = 0\}$.

Lemma 3.2 *There exists $\lambda_* > 0$ such that for each $u \in X_{+,q}\backslash\{0\}$, there is unique $t_{\max}, t_1$ and t_2 with property that $t_< t_{max} < t_2$, $t_1 u \in \mathcal{N}_\lambda^+$ and $t_2 u \in \mathcal{N}_\lambda^-$ and for all $\lambda \in (0, \lambda_*)$.*

Proof. Define $A(u) = \int_\Omega |u|^{1-q}dx$ and $B(u) = \int_\Omega |u|^{\alpha+1}dx$. Let $u \in X_{+,q}$ then we have

$$\begin{aligned}\frac{d}{dt}I(tu) =& t^{p-1}\|u\|^p - t^{-q}\lambda A(u) - t^\alpha B(u)\\ =& t^{-q}\left(m_u(t) - \lambda A(u)\right)\end{aligned}$$

and we define $m_u(t) := t^{p-1+q}\|u\|^p - t^{\alpha+q}B(u)$. Since $\lim_{t\to\infty} m_u(t) = -\infty$, we can easily see that $m_u(t)$ attains its maximum at $t_{max} = \left[\frac{(p-1+q)\|u\|^p}{(\alpha+q)B(u)}\right]^{\frac{1}{\alpha+1-p}}$ and

$$m_u(t_{max}) = \left(\frac{\alpha+2-p}{p-1+q}\right)\left(\frac{p-1+q}{\alpha+q}\right)^{\frac{\alpha+q}{\alpha+1-p}} \frac{\|u\|^{\frac{p(\alpha+q)}{\alpha+1-p}}}{B(u)^{\frac{p-1+q}{\alpha+1-p}}}.$$

Now, $u \in \mathcal{N}_\lambda$ if and only if $m_u(t) = \lambda A(u)$ and we see that

$$\begin{aligned} m_u(t_{max}) - \lambda A(u) \geq & \ m_u(t_{max}) - \lambda|u|_{1-q}^{1-q}\\ \geq & \ \left(\frac{\alpha+2-p}{p-1+q}\right)\left(\frac{p-1+q}{\alpha+q}\right)^{\frac{\alpha+q}{\alpha+1-p}} \frac{\|u\|^{\frac{p(\alpha+q)}{\alpha+1-p}}}{B(u)^{\frac{p-1+q}{\alpha+1-p}}} - \lambda C_{1-q}\|u\|^{1-q} > 0\end{aligned}$$

if and only if $\lambda < \left(\frac{\alpha+2-p}{p-1+q}\right)\left(\frac{p-1+q}{\alpha+q}\right)^{\frac{\alpha+q}{\alpha+1-p}}(C_{\alpha+1})^{\frac{-p+1-q}{\alpha+1-p}}C_{1-q}^{-1}$(say), where C_β is defined as in (2.1).

Case(I) ($0 < q < 1$): We can also see that $m_u(t) = \lambda\int_\Omega |u|^{1-q}dx$ if and only if $\phi_u'(t) = 0$. So for $\lambda \in (0, \lambda_*)$, there exists exactly two points $0 < t_1 < t_{max} < t_2$ with $m_u'(t_1) > 0$ and $m_u'(t_2) < 0$ that is, $t_1u \in \mathcal{N}_\lambda^+$ and $t_2u \in \mathcal{N}_\lambda^-$. Thus, ϕ_u has local minimum at $t = t_1$ and local maximum at $t = t_2$, that is ϕ_u is decreasing in $(0, t_1)$ and increasing in (t_1, t_2).

Case(II)($q = 1$): Since $\lim_{t\to 0}\phi_u(t) = \infty$ and $\lim_{t\to\infty}\phi_u(t) = -\infty$ with similar reasoning as above we get t_1, t_2. That is, in both cases, ϕ_u has exactly two critical points t_1 and t_2 such that $0 < t_1 < t_{max} < t_2$, $\phi_u''(t_1) > 0$ and $\phi_u''(t_2) < 0$ that is $t_1u \in \mathcal{N}_\lambda^+$, $t_2u \in \mathcal{N}_\lambda^-$. □

Proof of Theorem 2.3: From Lemma 3.2, we see that Λ_1 is positive. If $I_\lambda(tu)$ has two critical points for some $\lambda = \lambda^*$, then $t \mapsto I_\lambda(tu)$ also has two critical points for all $\lambda < \lambda^*$. □

Corollary 3.3 $\mathcal{N}_\lambda^0 = \{0\}$ *for all* $\lambda \in (0, \Lambda_1)$.

Proof. Let $u \in \mathcal{N}_\lambda^0$ and $u \not\equiv 0$. Then $u \in \mathcal{N}_\lambda$. That is, $t = 1$ is a critical point of $\phi_u(t)$. By Lemma 3.2, ϕ_u has critical points corresponding to either local minima or local maxima. So, $t = 1$ is the critical point corresponding to either local minima or local maxima of ϕ_u. Thus, either $u \in \mathcal{N}_\lambda^+$ or $u \in \mathcal{N}_\lambda^-$, which is a contradiction. □

We can now show that I is bounded below on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ in following way:

Lemma 3.4 $\inf I(\mathcal{N}_\lambda^+) > -\infty$ *and* $\inf I(\mathcal{N}_\lambda^-) > -\infty$.

Proof. Let $u \in \mathcal{N}_\lambda^+$ and $v \in \mathcal{N}_\lambda^-$. Then we have

$$0 < \phi_u''(1) \le (p-1-\alpha)\|u\|^p + \lambda(\alpha+q)C_{1-q}\|a\|_\infty\|u\|^{1-q},$$
$$0 > \phi_v''(1) \ge (p-1+q)\|v\|^p - (\alpha+q)C_{\alpha+1}\|v\|^{\alpha+1}.$$

Thus we obtain

$$\|u\| \le \left(\frac{\lambda(\alpha+q)C_{1-q}}{\alpha+1-p}\right)^{\frac{1}{p+q-1}} \text{ and } \|v\| \ge \left(\frac{p-1+q}{(\alpha+q)C_{\alpha+1}}\right)^{\frac{1}{\alpha+1-p}}.$$

This implies that

$$\sup\{\|u\| : u \in \mathcal{N}_\lambda^+\} < \infty \;\text{ and }\; \inf\{\|v\| : v \in \mathcal{N}_\lambda^-\} > 0. \tag{3.1}$$

If $I(v) \le M$, using $\ln(|v|) \le |v|$ we get

$$\begin{aligned}
&\frac{\alpha+1-p}{p(\alpha+1)}\|v\|^p - \frac{\lambda(\alpha+q)C_{1-q}}{(\alpha+1)(1-q)}\|v\|^{1-q} \le M, \quad 0<q<1\\
&\text{and } \frac{\alpha+1-p}{p(\alpha+1)}\|v\|^p - \lambda C_1\|v\| + \frac{\lambda}{\alpha+1} \le M, \quad q=1.
\end{aligned} \tag{3.2}$$

which implies $\sup\{\|v\| : v \in \mathcal{N}_\lambda^-, I(v) \le M\} < \infty$ for each $M > 0$. Using (3.1) and (3.2), it is easy to show that $\inf I(\mathcal{N}_\lambda^+) > -\infty$ and $\inf I(\mathcal{N}_\lambda^-) > -\infty$. □

Lemma 3.5 *Suppose $u \in \mathcal{N}_\lambda^+$ and $v \in \mathcal{N}_\lambda^-$ be minimizers of I over $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then for each $w \in X_+$,*

1. *there exists $\epsilon_0 > 0$ such that $I(u+\epsilon w) \ge I(u)$ for each $\epsilon \in [0, \epsilon_0]$*
2. *$t_\epsilon \to 1$ as $\epsilon \to 0^+$, where t_ϵ is the unique positive real number satisfying $t_\epsilon(v+\epsilon w) \in \mathcal{N}_\lambda^-$.*

Proof.

1. Let $w \in X_+$ that is $w \in X_0$ and $w \ge 0$. We set

$$\rho(\epsilon) = (p-1)\|u+\epsilon w\|^p + \lambda q \int_\Omega |u+\epsilon w|^{1-q}dx - \alpha \int_\Omega |u+\epsilon w|^{\alpha+1}dx$$

for each $\epsilon \ge 0$. Then using continuity of ρ, $\rho(0) = \phi_u''(1) > 0$ and $u \in \mathcal{N}_\lambda^+$, there exist $\epsilon_0 > 0$ such that $\rho(\epsilon) > 0$ for $\epsilon \in [0, \epsilon_0]$. Since for each $\epsilon > 0$, there exists $t'_\epsilon > 0$ such that $t'_\epsilon(u+\epsilon w) \in \mathcal{N}_\lambda^+$. So, $t'_\epsilon \to 1$ as $\epsilon \to 0$ and for each $\epsilon \in [0, \epsilon_0]$ we have

$$I(u+\epsilon w) \ge I(t'_\epsilon(u+\epsilon w)) \ge \inf I(\mathcal{N}_\lambda^+) = I(u).$$

2. We define $h : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ by

$$h(t, l_1, l_2, l_3) = l_1 t^{p-1} - \lambda t^{-q} l_2 - t^\alpha l_3$$

for $(t, l_1, l_2, l_3) \in (0, \infty) \times \mathbb{R}^3$. Then h is C^∞ function. Then, we have

$$\frac{dh}{dt}(1, \|v\|^p, \int_\Omega |v|^{1-q}dx, \int_\Omega |v|^{\alpha+1}) = \phi_v''(1) < 0,$$

and for each $\epsilon \geq 0$, $h(t_\epsilon, \|v+\epsilon w\|^p, \int_\Omega |v+\epsilon w|^{1-q}dx, \int_\Omega |v|^{\alpha+1}) = \phi'_{v+\epsilon w}(t_\epsilon) = 0$. Also

$$h(1, \|v\|^p, \int_\Omega |v|^{1-q}dx, \int_\Omega |v|^{\alpha+1}) = \phi'_v(1) = 0.$$

Therefore, by implicit function theorem, there exists an open neighborhood $A \subset (0, \infty)$ and $B \subset \mathbb{R}^3$ containing 1 and $(\|v\|^p, \int_\Omega |v|^{1-q}dx, \int_\Omega |v|^{\alpha+1})$ respectively such that for all $y \in B$, $h(t, y) = 0$ has a unique solution $t = g(y) \in A$, where $g : B \to A$ is a continuous function. So, $(\|v+\epsilon w\|^p, \int_\Omega |v+\epsilon w|^{1-q}dx, \int_\Omega |v+\epsilon w|^{\alpha+1}) \in B$ and

$$g\left(\|v+\epsilon w)\|^p, \int_\Omega |v+\epsilon w|^{1-q}dx, \int_\Omega |v+\epsilon w|^{\alpha+1}\right) = t_\epsilon$$

since $h(t_\epsilon, \|v+\epsilon w)\|^p, \int_\Omega |v+\epsilon w|^{1-q}dx, \int_\Omega |v+\epsilon w|^{\alpha+1}) = 0$. Thus, by continuity of g, we get $t_\epsilon \to 1$ as $\epsilon \to 0^+$. □

Lemma 3.6 *Suppose $u \in \mathcal{N}_\lambda^+$ and $v \in N_\lambda^-$ are minimizers of I on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then for each $w \in X_+$, we have $u^{-q}w, v^{-q}w \in L^1(\Omega)$ and*

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}\,dxdy - \lambda\int_\Omega \left(u^{-q}+u^\alpha\right)wdx \geq 0, \tag{3.3}$$

$$\int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+sp}}\,dxdy - \lambda\int_\Omega \left(v^{-q}+v^\alpha\right)wdx \geq 0. \tag{3.4}$$

Proof. Let $w \in X_+$. For sufficiently small $\epsilon > 0$, by Lemma 3.5,

$$\begin{aligned} 0 \leq \frac{I(u+\epsilon w) - I(u)}{\epsilon} =& \frac{1}{p\epsilon}(\|u+\epsilon w\|^p - \|u\|^p) - \frac{\lambda}{\epsilon}\int_\Omega (G_q(u+\epsilon w) - G_q(u))dx \\ & - \frac{1}{\epsilon(\alpha+1)}\int_\Omega (|u+\epsilon w|^{\alpha+1} - |u|^{\alpha+1})dx \end{aligned} \tag{3.5}$$

We can easily verify that as $\epsilon \to 0^+$,

(i) $$\frac{(\|u+\epsilon w\|^p - \|u\|^p)}{\epsilon} \to p\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}\,dxdy$$

(ii) $$\int_\Omega \frac{(|u+\epsilon w|^{\alpha+1} - |u|^{\alpha+1})}{\epsilon}dx \to (\alpha+1)\int_\Omega |u|^{\alpha-1}uwdx.$$

which implies that $\frac{(G_q(u+\epsilon w) - G_q(u))}{\epsilon} \in L^1(\Omega)$. Also, for each $x \in \Omega$,

$$\frac{G_q(u(x)+\epsilon w(x)) - G_q(u(x))}{\epsilon} = \begin{cases} \frac{1}{\epsilon}\left(\frac{|u+\epsilon w|^{1-q}(x) - |u|^{1-q}(x)}{1-q}\right) & \text{if } 0 < q < 1 \\ \frac{1}{\epsilon}(\ln(|u+\epsilon w|) - \ln(|u|)) & \text{if } q = 1 \end{cases}$$

which increases monotonically as $\epsilon \downarrow 0$ and

$$\lim_{\epsilon\downarrow 0} \frac{G_q(u(x)+\epsilon w(x)) - G_q(u(x))}{\epsilon} = \begin{cases} 0 & \text{if } w(x)=0 \\ (u(x))^{-q}w(x) & \text{if } w(x)>0, u(x)>0 \\ \infty & \text{if } w(x)>0, u(x)=0. \end{cases}$$

So using monotone convergence theorem for $\{G_q\}$, we get $u^{-q}w \in L^1(\Omega)$. Letting $\epsilon \downarrow 0$ in both sides of (3.5), we get (3.3). Next, we will show these properties for v. For each $\epsilon > 0$, there exists $t_\epsilon > 0$ with $t_\epsilon(v+\epsilon w) \in \mathcal{N}_\lambda^-$. By Lemma 3.5(2), for sufficiently small $\epsilon > 0$, there holds

$$I(t_\epsilon(v+\epsilon w)) \geq I(v) \geq I(t_\epsilon v)$$

which implies $I(t_\epsilon(v+\epsilon w)) - I(v) \geq 0$ and thus, we have

$$\begin{aligned} \lambda \int_\Omega (G_q(t_\epsilon|v+\epsilon w|^{1-q}) - G_q(|v|^{1-q}))dx \leq & \frac{t_\epsilon^p}{p}(\|v+\epsilon w\|^p - \|v\|^p) \\ & - \frac{t_\epsilon^{\alpha+1}}{\alpha+1}\int_\Omega (|v+\epsilon w|^{\alpha+1} - |v|^{\alpha+1})dx. \end{aligned}$$

As $\epsilon \downarrow 0$, $t_\epsilon \to 1$. Thus, using similar arguments as above, we obtain $v^{-q}w \in L^1(\Omega)$ and (3.4) follows. □

Let $\eta > 0$ be such that $\phi = \eta\phi_1$ satisfies

$$\int_Q \frac{|\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}}dxdy \leq \lambda \int_Q \phi^{-q}\psi + \int_Q \phi^\alpha \psi \tag{3.6}$$

for all $\psi \in X_0$ (i.e ϕ is a sub-solution of (P_λ)) and $\phi^{\alpha+q}(x) \leq \lambda\left(\frac{q}{\alpha}\right)$, for each $x \in \Omega$. Then we have

Lemma 3.7 *Suppose $u \in \mathcal{N}_\lambda^+, v \in \mathcal{N}_\lambda^-$ are minimizers of I on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then $u \geq \phi$ and $v \geq \phi$ in Ω.*

Proof. By Lemma 2.2, let $\{w_k\}$ be a sequence in X_0 such that supp(w_k) is compact, $0 \leq w_k \leq (\phi-u)^+$ for each k and $\{w_k\}$ strongly converges to $(\phi-u)^+$ in X_0. Then

$$\frac{d}{dt}(\lambda t^{-q} + t^\alpha) = -q\lambda t^{-q-1} + \alpha t^{\alpha-1} \leq 0 \text{ if and only if } t^{\alpha+q} \leq \lambda\left(\frac{q}{\alpha}\right). \tag{3.7}$$

Using Lemma Lemma 3.6 and (3.6), we have

$$\int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(w_k(x)-w_k(y))\ dxdy - \int_\Omega (\lambda u^{-q} + u^\alpha)w_k dx + \int_\Omega (\lambda\phi^{-q} + \phi^\alpha)w_k dx \geq 0,$$

where $f(\xi) = |\xi(x)-\xi(y)|^{p-2}(\xi(x)-\xi(y))$. Since $\{w_k\}$ converges to $(\phi-u)^+$ strongly, we get a subsequence of $\{w_k\}$ such that $w_k(x) \to (\phi-u)^+(x)$ pointwise almost everywhere in Ω and we write $w_k(x) = (\phi-u)^+(x) + o(1)$ as $k \to \infty$. Then,

$$\begin{aligned} \int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(w_k(x)-w_k(y))\ dxdy = & \int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x) - (\phi-u)^+(y))\ dxdy \\ & + o(1)\int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}\ dxdy. \end{aligned}$$

Further we can see that

$$
\begin{aligned}
&\int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy \\
&\quad = \left(\int_{\Omega_1\times\Omega_1}+\int_{\Omega_1\times\Omega_2}+\int_{\Omega_2\times\Omega_1}+\int_{\Omega_2\times\Omega_2}\right)\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy
\end{aligned} \tag{3.8}
$$

where $\Omega_1=\{x:\phi(x)\geq u(x)\}$ and $\Omega_2=\{x:\phi(x)\leq u(x)\}$. Now, we separately estimate each integrals and to begin with, firstly we see that

$$
\int_{\Omega_2\times\Omega_2}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy=0. \tag{3.9}
$$

Next, we see that

$$
\begin{aligned}
&\int_{\Omega_1\times\Omega_1}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy \\
&=-\int_{\Omega_1\times\Omega_1}\frac{(f(\phi)-f(u))}{|x-y|^{n+sp}}((\phi-u)(x)-(\phi-u)(y))\ dxdy \\
&\leq -\frac{1}{2^{p-2}}\int_{\Omega_1\times\Omega_1}\frac{|(\phi-u)(x)-(\phi-u)(y)|^p}{|x-y|^{n+sp}}\ dxdy
\end{aligned} \tag{3.10}
$$

using $|a-b|^p\leq 2^{p-2}(|a|^{p-2}a-|b|^{p-2}b)(a-b),\ p\geq 2$ and $a,b\in\mathbb{R}$. Now, consider

$$
\begin{aligned}
&\int_{\Omega_1\times\Omega_2}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy \\
&=\int_{\Omega_1\times\Omega_2}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(\phi-u)(x)\ dxdy \\
&\leq -\frac{1}{2^{p-2}}\int_{\Omega_1\times\Omega_2}\frac{|(\phi-u)(x)-(\phi-u)(y)|^p}{|x-y|^{n+sp}}\ dxdy+ \\
&\quad +\int_{\Omega_1\times\Omega_2}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(\phi-u)(y)\ dxdy
\end{aligned} \tag{3.11}
$$

and similarly, we will get

$$
\begin{aligned}
&\int_{\Omega_2\times\Omega_1}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy \\
&\leq -\frac{1}{2^{p-2}}\int_{\Omega_2\times\Omega_1}\frac{|(\phi-u)(x)-(\phi-u)(y)|^p}{|x-y|^{n+sp}}\ dxdy-\int_{\Omega_2\times\Omega_1}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(\phi-u)(x)\ dxdy.
\end{aligned} \tag{3.12}
$$

Thus using (3.8)-(3.12), we get

$$
\begin{aligned}
&\int_Q \frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}((\phi-u)^+(x)-(\phi-u)^+(y))\ dxdy \\
&\qquad\leq -\frac{1}{2^{p-2}}\|(\phi-u)\|^p+\int_{\Omega_1\times\Omega_2}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(\phi-u)(y)\ dxdy \\
&\qquad\quad -\int_{\Omega_2\times\Omega_1}\frac{(f(u)-f(\phi))}{|x-y|^{n+sp}}(\phi-u)(x)\ dxdy \\
&\qquad = -\frac{1}{2^{p-2}}\|(\phi-u)\|^p.
\end{aligned}
$$

Since $\phi^{\alpha+q}(x) \le \lambda\left(\frac{q}{\alpha}\right)$, for each $x \in \Omega$, using (3.7) we get

$$\int_\Omega ((\lambda u^{-q} + u^\alpha) - (\lambda\phi^{-q} + \phi^\alpha))w_k dx$$
$$= \int_{\Omega\cap\{\phi\ge u\}} ((\lambda u^{-q} + u^\alpha) - (\lambda\phi^{-q} + \phi^\alpha))(\phi - u)^+(x)dx + o(1) \ge 0$$

which implies

$$\begin{aligned} 0 &\le -\frac{1}{2^{p-2}}\|(\phi-u)^+\|^2 - \int_\Omega (\lambda u^{-q} + u^\alpha)w_k dx + \int_\Omega (\lambda\phi^{-q} + \phi^\alpha)w_k dx + o(1) \\ &\le -\frac{1}{2^{p-2}}\|(\phi-u)^+\|^2 + o(1) \end{aligned}$$

and letting $k \to \infty$, we get $-\|(\phi-u)^+\|^2 \ge 0$. Thus, we showed $u \ge \phi$. Similarly, we can show $v \ge \phi$. □

4 Existence of minimizer on $\mathcal{N}_\lambda^+$

In this section, we will show that the minimum of I on $\mathcal{N}_\lambda^+$ is achieved in $\mathcal{N}_\lambda^+$. Also, we show that this minimizer is also solution of (P_λ).

Proposition 4.1 *For all $\lambda \in (0, \Lambda)$, there exist $u_\lambda \in \mathcal{N}_\lambda^+$ satisfying $I(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+} I(u)$.*

Proof. Assume $0 < q \le 1$ and $\lambda \in (0, \Lambda)$. We show that there exist $u_\lambda \in \mathcal{N}_\lambda^+$ such that $I(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+} I(u)$. Let $\{u_k\} \subset \mathcal{N}_\lambda^+$ be a sequence such that $I(u_k) \to \inf I(\mathcal{N}_\lambda^+)$ as $k \to \infty$. Now by (3.1) we can assume that there exists $u_\lambda \in X_0$ such that $u_k \rightharpoonup u_\lambda$ weakly in X_0 (up to subsequence). First we will show that $\inf I(\mathcal{N}_\lambda^+) < 0$. Let $u_0 \in \mathcal{N}_\lambda^+$, we have $\phi''_{u_0}(1) > 0$ which gives

$$\left(\frac{p-1+q}{\alpha+q}\right)\|u_0\|^p > \int_\Omega |u_0|^{\alpha+1}dx.$$

Therefore, using $\alpha > p-1$ we obtain

$$\begin{aligned} I(u_0) &= \left(\frac{1}{p} - \frac{1}{1-q}\right)\|u_0\|^p + \left(\frac{1}{1-q} - \frac{1}{\alpha+1}\right)\int_\Omega |u_0|^{\alpha+1}dx \\ &\le -\frac{(p+q-1)}{p(1-q)}\|u_0\|^p + \frac{(p+q-1)}{(\alpha+1)(1-q)}\|u_0\|^p = \left(\frac{1}{\alpha+1} - \frac{1}{p}\right)\left(\frac{p+q-1}{1-q}\right)\|u_0\|^p < 0 \end{aligned}$$

Case(I) ($\alpha < p_s^* - 1$) Firstly, we claim that $u_\lambda \in X_{+,q}$. When $0 < q < 1$, if $u_\lambda = 0$ then $0 = I(u_\lambda) \le \underline{\lim}\, I(u_k) < 0$, which is a contradiction. In the case $q = 1$, the sequence $\left\{\int_\Omega \ln(|u_k|)\right\}$ is bounded, since the sequence $\{I(u_k)\}$ and $\{\|u_k\|\}$ is bounded and using Fatou's Lemma and $\ln(|u_k|) \le u_k$, for each k, we get

$$-\infty < \overline{\lim_{k\to\infty}} \int_\Omega \ln(|u_k|)dx \le \int_\Omega \overline{\lim_{k\to\infty}} \ln(|u_k|)dx = \int_\Omega \ln(|u_\lambda|)dx.$$

which implies $u_\lambda \not\equiv 0$ and thus, in both cases we have shown $u_\lambda \in X_{+,q}$. We claim that $u_k \to u_\lambda$ strongly in X_0. Suppose not. Then, we may assume $\|u_k - u_\lambda\| \to c > 0$. Using Brezis-Lieb lemma and embedding results for X_0 in the subcritical case, we have

$$\lim_{k\to\infty} \phi'_{u_k}(1) = \phi'_{u_\lambda}(1) + c^p \tag{4.1}$$

which implies $\phi'_{u_\lambda}(1) + c^p = 0$, using $\phi'_{u_k}(1) = 0$ for each k. Since $\lambda \in (0, \Lambda)$, there exist $0 < t_1 < t_2$ (by fibering map analysis) such that $\phi'_{u_\lambda}(t_1) = \phi'_{u_\lambda}(t_2) = 0$ and $t_1 u_\lambda \in \mathcal{N}_\lambda^+$. By (4.1), we have $\phi'_{u_\lambda}(1) < 0$ which gives two cases : $1 < t_1$ or $t_2 < 1$. When $t_1 > 1$, we have

$$\inf I(\mathcal{N}_\lambda^+) = \lim I(u_k) = I(u_\lambda) + \frac{c^p}{p} = \phi_{u_\lambda}(1) + \frac{c^p}{p} > \phi_{u_\lambda}(1) > \phi_{u_\lambda}(t_1) \geq \inf I(\mathcal{N}_\lambda^+),$$

which is a contradiction. Thus we have $t_2 < 1$. We set, for $t > 0$, $f(t) = \phi_{u_\lambda}(t) + \frac{c^p t^p}{2}, t > 0$. From (4.1), we get $f'(1) = 0$ and since $0 < t_2 < 1$, $f'(t_2) = t_2^{p-1} c^p > 0$. So, f is increasing on $[t_2, 1]$ and we obtain

$$\inf I(\mathcal{N}_\lambda^+) = I(u_\lambda) + \frac{c^p}{p} = \phi_{u_\lambda}(1) + \frac{c^p}{p} = f(1) > f(t_2) > \phi_{u_\lambda}(t_2) > \phi_{u_\lambda}(t_1) \geq \inf I(\mathcal{N}_\lambda^+),$$

which gives a contradiction. Hence, $c = 0$ and thus, $u_k \to u_\lambda$ strongly in X_0. Since $\lambda \in (0, \Lambda)$, we have $\phi''_{u_\lambda}(1) > 0$, so we obtain $u_\lambda \in \mathcal{N}_\lambda^+$ and $I(u_\lambda) = \inf I(\mathcal{N}_\lambda^+)$.
Case(II) ($\alpha = p_s^* - 1$ and $0 < q < 1$) We set $w_k := u_k - u_\lambda$ and claim that $u_k \to u_\lambda$ strongly in X_0. Suppose $\|w_k\|^p \to c^p \neq 0$ and $\int_\Omega |w_k|^{p_s^*} dx \to d^{p_s^*}$ as $k \to \infty$. Since $u_k \in \mathcal{N}_\lambda^+$, using Brezis-Lieb Lemma, we get

$$0 = \lim_{k\to\infty} \phi'_{u_k}(1) = \phi'_{u_\lambda}(1) + c^p - d^{p_s^*} \tag{4.2}$$

which implies

$$\|u_\lambda\|^p + c^p = \lambda \int_\Omega |u_\lambda|^{1-q} dx + \int_\Omega |u_k|^{p_s^*} dx + d^{p_s^*}.$$

We claim that $u_\lambda \in X_{+,q}$. Suppose $u_\lambda \equiv 0$. If $0 < q < 1$ and $c = 0$ then $0 > \inf I(\mathcal{N}_\lambda^+) = I(0) = 0$, which is a contradiction and if $c \neq 0$ then

$$\inf I(\mathcal{N}_\lambda^+) = I(0) + \frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*} = \frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*}. \tag{4.3}$$

But we have $\|u_k\|_{p_s^*}^p S \leq \|u_k\|^p$ which gives $c^p \geq S d^p$. Also from (4.2), we have $c^p = d^{p_s^*}$. Then (4.3) implies

$$0 > \inf I(\mathcal{N}_\lambda^+) = \left(\frac{1}{p} - \frac{1}{p_s^*}\right) c^p \geq \frac{s}{n} S^{\frac{n}{sp}},$$

which is again a contradiction. In the case $q = 1$, the sequence $\left\{\int_\Omega \ln(|u_k|)\right\}$ is bounded, since the sequence $\{I(u_k)\}$ and $\{\|u_k\|\}$ is bounded, using Fatou's Lemma and $\ln(|u_k|) \leq u_k$, for each k, we get

$$-\infty < \overline{\lim_{k\to\infty}} \int_\Omega \ln(|u_k|) dx \leq \int_\Omega \overline{\lim_{k\to\infty}} \ln(|u_k|) dx = \int_\Omega \ln(|u_\lambda|) dx.$$

which implies $u_\lambda \not\equiv 0$. Thus, in both cases we have shown that $u_\lambda \in X_{+,q}$. So, there exists $0 < t_1 < t_2$ such that $\phi'_{u_\lambda}(t_1) = \phi'_{u_\lambda}(t_2) = 0$ and $t_1 u_\lambda \in \mathcal{N}_\lambda^+$. Then, three cases arise:
(i) $t_2 < 1$,
(ii) $t_2 \geq 1$ and $\frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*} < 0$, and
(iii) $t_2 \geq 1$ and $\frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*} \geq 0$.
Case (i) Let $h(t) = \phi_{u_\lambda}(t) + \frac{c^p t^p}{p} - \frac{d^{p_s^*} t^{p_s^*}}{p_s^*}$ for $t > 0$. By (4.2) we get $h'(1) = \phi'_{u_\lambda}(1) + c^p - d^{p_s^*} = 0$ and

$$h'(t_2) = \phi'_{u_\lambda}(t_2) + t_2^p c^p - t_2^{p_s^*} d^{p_s^*} = t_2{}^p (c^p - t_2^{p_s^* - p} d^{p_s^*}) > t_2^p (c^p - d^{p_s^*}) > 0$$

which implies that h increases on $[t_2, 1]$. Then we get

$$\begin{aligned}
\inf I(\mathcal{N}_\lambda^+) &= \lim I(u_k) \geq \phi_u(1) + \frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*} = h(1) > h(t_2) \\
&= \phi_u(t_2) + \frac{c^p t_2^p}{p} - \frac{d^{p_s^*} t_2^{p_s^*}}{p_s^*} \geq \phi_u(t_2) + \frac{t_2^p}{p}(c^p - d^{p_s^*}) \\
&> \phi_u(t_2) > \phi_u(t_1) \geq \inf I(\mathcal{N}_\lambda^+),
\end{aligned}$$

which is a contradiction.
Case (ii) In this case, since $\lambda \in (0, \Lambda)$, we have $(c^p/p - d^{p_s^*}/p_s^*) < 0$ and $S d^p \leq c^p$. Also we see that, for each $u_0 \in \mathcal{N}_\lambda^+$

$$\begin{aligned}
0 < \phi''_{u_0}(1) &= (p-1)\|u_0\|^p + q\lambda \int_\Omega |u_0|^{1-q} dx - (p_s^* - 1) \int_\Omega |u_0|^{p_s^*} dx \\
&= (p - 1 + q)\|u_0\|^p + (-q - p_s^* + 1) \int_\Omega |u_0|^{p_s^*} dx
\end{aligned}$$

which implies $(p-1+q)\|u_0\|^p > (q + p_s^* - 1) \int_\Omega |u_0|^{p_s^*} dx = (q + p_s^* - 1)|u_0|_{p_s^*}^{p_s^*}$
or, $C_{p_s^*} \leq \left(\frac{p-1+q}{q+p_s^*-1}\right) \|u_0\|^{p-p_s^*}$ or, $\|u_0\|^p \leq \left(\frac{p-1+q}{q+p_s^*-1}\right)^{\frac{p}{p_s^*-p}} S^{\frac{p_s^*}{p_s^*-p}}$. Thus, we have

$$\sup\{\|u\|^p : u \in \mathcal{N}_\lambda^+\} \leq \left(\frac{p}{p_s^*}\right)^{\frac{p}{p_s^*-p}} S^{\frac{p_s^*}{p_s^*-p}} < c^p \leq \sup\{\|u\|^p : u \in \mathcal{N}_\lambda^+\},$$

which gives a contradiction. Consequently, in case (iii) we have

$$\inf I(\mathcal{N}_\lambda^+) = I(u_\lambda) + \frac{c^p}{p} - \frac{d^{p_s^*}}{p_s^*} \geq I(u_\lambda) = \phi_{u_\lambda}(1) \geq \phi_{u_\lambda}(t_1) \geq \inf I(\mathcal{N}_\lambda^+).$$

Clearly, this holds when $t_1 = 1$ and $(c^p/p - d^{p_s^*}/p_s^*) = 0$ which yields $c = 0$ and $u_\lambda \in \mathcal{N}_\lambda^+$. Thus, $u_k \to u_\lambda$ strongly in X_0 as $k \to \infty$ and $I(u_\lambda) = \inf I(\mathcal{N}_\lambda^+)$. □

Proposition 4.2 *u_λ is a positive weak solution of (P_λ).*

Proof. Let $\psi \in C_c^\infty(\Omega)$. By Lemma 3.7, since $\phi > 0$, we can find $\beta > 0$ such that $u_\lambda \geq \beta$ on support of ψ. Then $u_\lambda + \epsilon\psi \geq 0$, for small ϵ. With similar reasoning as in the proof of

Lemma 3.5, $I(u_\lambda + \epsilon\psi) \geq I(u_\lambda)$ for sufficiently small $\epsilon > 0$. Then we have

$$\begin{aligned}0 &\leq \lim_{\epsilon\to 0} \frac{I(u_\lambda + \epsilon\psi) - I(u_\lambda)}{\epsilon} \\ &= \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}\, dxdy - \lambda\int_\Omega u_\lambda^{-q}\psi dx - \int_\Omega u_\lambda^{\alpha}\psi\, dx.\end{aligned}$$

Since $\psi \in C_c^\infty(\Omega)$ is arbitrary, we conclude that u_λ is a positive weak solution of (P_λ). □

We recall the following comparison principle from [34].

Lemma 4.3 *Let $u, v \in X_0$ are such that $u \geq v$ in $\mathbb{R}^n\setminus\Omega$ and*

$$\int_Q \left(|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |v(x) - v(y)|^{p-2}(v(x) - v(y))\right) \frac{(\psi(x) - \psi(y))}{|x-y|^{n+ps}}\, dxdy \geq 0$$

for all non-negative $\psi \in X_0$. Then $u \geq v$ in Ω.

Proof. Proof follows by taking $\psi = (v-u)^+$ and using the equality

$$|b|^{p-2}b - |a|^{p-2}a = (p-1)(b-a)\int_0^1 |a + t(b-a)|^{p-2} dt.$$

□

As a consequence, we have

Lemma 4.4 $\Lambda_1 < \infty$.

Proof. Suppose $\Lambda_1 = \infty$. Then from Proposition 4.2, (P_λ) has a solution for all λ. Now choose λ large enough such that

$$\lambda t^{-q} + t^{p_s^*-1} > (\lambda_1 + \epsilon)t^{p-1}, \text{ for all } t \in (0,\infty).$$

Then $\overline{u} := u_\lambda$ is a super solution of the eigenvalue problem

$$(P_\epsilon) \quad u \in X_0; \text{ and } (-\Delta_p)^s u = (\lambda_1 + \epsilon)|u|^{p-2}u \text{ in } \Omega.$$

Also we can choose r small such that $\underline{u} := r\phi_1$ is a subsolution of (P_ϵ). Then by the boundedness of u_λ (see Theorem 6.4) and ϕ_1, we can choose r small such that $\underline{u} \leq \overline{u}$.
Now, we consider the monotone iterations

$$\begin{aligned} u_0 =& r\phi_1 \\ u_n \in X_0; \text{ and } (-\Delta_p)^s u_n =& (\lambda_1 + \epsilon)|u_{n-1}|^{p-2}u_{n-1} \text{ in } \Omega.\end{aligned}$$

Then by the weak comparison Lemma 4.3, we get

$$r\phi_1(x) \leq u_1(x) \leq u_2(x) \leq ... \leq u_{n-1}(x) \leq u_n(x) \leq .... \leq u_\lambda(x),\ \forall x \in \Omega$$

Therefore, the sequence $\{u_n\}$ is bounded in X_0 and hence has a weakly convergent subsequence $\{u_n\}$ that converges to u_0. Thus, u_0 is a solution of (P_ϵ). Since $\epsilon > 0$ is arbitrary, we get a contradiction to the simplicity and isolatedness of λ_1.

5 Existence of minimizer on $\mathcal{N}_\lambda^-$

In this section we show the existence of second solution for (P_λ) in the subcritical case. We assume $\alpha < p_s^* - 1$.

Proposition 5.1 *For all $\lambda \in (0, \Lambda)$, there exist $v_\lambda \in \mathcal{N}_\lambda^-$ satisfying $I(v_\lambda) = \inf\limits_{v \in \mathcal{N}_\lambda^-} I(v)$.*

Proof. Assume $0 < q \leq 1$ and $\lambda \in (0, \Lambda)$. We will show that there exists $v_\lambda \in \mathcal{N}_\lambda^-$ with $I(v_\lambda) = \inf I(\mathcal{N}_\lambda^-)$. Let $\{v_k\} \subset \mathcal{N}_\lambda^-$ be a sequence such that $\lim\limits_{k \to \infty} I(v_k) = \inf I(\mathcal{N}_\lambda^-)$. Using Lemma 3.4, we can assume that $v_k \rightharpoonup v_\lambda$ weakly as $k \to \infty$ in X_0. We claim that $v_\lambda \in X_{+,q}$. When $0 < q < 1$, if $v_\lambda = 0$ then $\{v_k\}$ converges strongly to 0, which contradicts Lemma 3.4. If $q = 1$, we similarly have $-\infty < \int_\Omega \ln(|v_k|)dx$ as above. So, by both the cases, we get $v_\lambda \in X_{+,q}$. Next, we claim that $\{v_k\}$ converges strongly to v_λ in X_0. Suppose not. Then we may assume $\|v_k - v_\lambda\| \to d > 0$, and we have

1. $\inf I(\mathcal{N}_\lambda^-) = \lim I(v_k) \geq I(v_\lambda) + d^p/p$.
2. For each k, $\phi'_{v_k}(1) = 0$ and $\phi''_{v_k}(1) < 0 \implies \phi'_{v_\lambda}(1) + d^p = 0$ and $\phi''_{v_\lambda}(1) + d^p \leq 0$.

By (2), we have $\phi'_{v_\lambda}(1) < 0$ and $\phi''_{v_\lambda}(1) < 0$. So, there exists $t_2 \in (0,1)$ such that $\phi'_{v_\lambda}(t_2) = 0$ and $\phi''_{v_\lambda}(t_2) < 0$. Thus, $t_2 v_\lambda \in \mathcal{N}_\lambda^-$. Define $g : \mathbb{R}^+ \to \mathbb{R}$ as $g(t) = \phi_{v_\lambda}(t) + \frac{d^p t^p}{2}$, for $t > 0$. From (2), we get $g'(1) = 0$ and since $0 < t_2 < 1$, $g'(t_2) = d^p t_2^{p-1} > 0$. Then, g is increasing on $[t_2, 1]$. Now we obtain

$$\inf I(\mathcal{N}_\lambda^-) \geq I(v_\lambda) + \frac{d^p}{p} = \phi_{v_\lambda}(1) + \frac{d^p}{p} = g(1) \geq g(t_2) > \phi_{v_\lambda}(t_2) = I(t_2 v_\lambda) \geq \inf I(\mathcal{N}_\lambda^-),$$

which gives a contradiction. Hence, $d = 0$ and thus, $\{v_k\}$ converges strongly to v_λ in X_0. Since $\lambda \in (0, \Lambda)$, we have $\phi''_{v_\lambda}(1) < 0$. Therefore, we obtain $v_\lambda \in \mathcal{N}_\lambda^-$ and $I(v_\lambda) = \inf I(\mathcal{N}_\lambda^-)$. This completes the proof of this proposition in subcritical case. □

Proposition 5.2 *For $\lambda \in (0, \Lambda)$, v_λ is a positive weak solution of (P_λ).*

Proof. Let $\psi \in C_c^\infty(\Omega)$. Using Lemma 3.7, since $\phi > 0$ in Ω, we can find $\beta > 0$ such that $v_\lambda \geq \beta$ on *supp*(ψ). Also, $t_\epsilon \to 1$ as $\epsilon \to 0+$, where t_ϵ is the unique positive real number corresponding to $(v_\lambda + \epsilon\psi)$ such that $t_\epsilon(v_\lambda + \epsilon\psi) \in \mathcal{N}_\lambda^-$. Then, by Lemma 3.5 we have

$$\begin{aligned} 0 &\leq \lim_{\epsilon \to 0} \frac{I(t_\epsilon(v_\lambda + \epsilon\psi)) - I(v_\lambda)}{\epsilon} \leq \lim_{\epsilon \to 0} \frac{I(t_\epsilon(v_\lambda + \epsilon\psi)) - I(t_\epsilon v_\lambda)}{\epsilon} \\ &= \int_Q \frac{|v_\lambda(x) - v_\lambda(y)|^{p-2}(v_\lambda(x) - v_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+sp}}\, dxdy - \int_\Omega \left(\lambda v_\lambda^{-q} + v_\lambda^\alpha\right) \psi dx. \end{aligned}$$

Since $\psi \in C_c^\infty(\Omega)$ is arbitrary, we conclude that v_λ is positive weak solution of (P_λ). □

Proof of Theorem 2.4: Proof follows from Proposition 4.2 and Proposition 5.2. □

Remark 5.3 *To prove the existence of second positive solution in the critical case, one requires to know the classification of exact solutions of the problem*

$$(-\Delta_p)^s u = |u|^{p_s^*-2}u \text{ in } \mathbb{R}^n.$$

These are the minimizers of S, the best constant of the embedding X_0 into $L^{p_s^*}$. In [39, 8], authors obtained several estimates on these minimizers and conjectu that the solutions are dilations and translations of the radial function

$$U(x) = \frac{1}{(1+|x|^{p'})^{(N-sp)/p}},\ x \in \mathbb{R}^n$$

where $p' = \frac{p}{p-1}$. In case of $p = 2$, these classifications are proved in [41], where author proved that all solutions are classified by dilations and translations of $U(x)$. Using these classifications, in [40] it is shown that

$$\sup\{I(u_\lambda + tU_\epsilon) \ :\ t \geq 0\} < I(u_\lambda) + \frac{s}{n}S^{\frac{n}{2s}}.$$

where $U_\epsilon = \epsilon^{-(n-2s)/2}U(\frac{x}{\epsilon})$, $x \in \mathbb{R}^n$, $\epsilon > 0$ and u_λ is the minimizer on $\mathcal{N}_\lambda^+$. Then by carefully analysing the related fiber maps it is shown that $u_\lambda + tU_\epsilon \in \mathcal{N}_\lambda^-$, for large t. From this it follows

$$\inf I(\mathcal{N}_\lambda^-) < I(u_\lambda) + \frac{s}{n}S^{\frac{n}{2s}}$$

Then the existence of minimizer is shown using the analysis of fibering maps in Lemma 3.2.

6 Regularity of weak solutions

In this section, we shall prove some regularity properties of positive weak solutions of P_λ. We begin with the following lemma.

Lemma 6.1 *Suppose u is a weak solution of (P_λ), then for each $w \in X_0$, it satisfies $u^{-q}w \in L^1(\Omega)$ and*

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}\,dxdy - \int_\Omega \left(\lambda u^{-q} + u^\alpha\right) w dx = 0. \tag{6.1}$$

Proof. Let u be a weak solution of (P_λ) and $w \in X_+$. By Lemma 2.2, we get a sequence $\{w_k\} \in X_0$ such that $\{w_k\} \to w$ strongly in X_0, each w_k has compact support in Ω and $0 \leq w_1 \leq w_2 \leq \ldots$. Since each w_k has compact support in Ω and u is a positive weak solution of (P_λ), for each k we get

$$\lambda \int_\Omega u^{-q} w_k dx = \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w_k(x)-w_k(y))}{|x-y|^{n+sp}}\,dxdy - \int_\Omega u^\alpha w_k dx.$$

Using monotone convergence theorem, we get $u^{-q}w \in L^1(\Omega)$ and

$$\lambda \int_\Omega u^{-q} w dx = \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+sp}}\,dxdy - \int_\Omega u^\alpha w dx.$$

If $w \in X_0$ then $w = w^+ - w^-$ and $w^+, w^- \in X_+$. Since we proved the lemma for each $w \in X_+$, we obtain the conclusion. □

Before proving our next result, let us recall some estimates or inequalities from [7].

Lemma 6.2 *Let $1 < p < \infty$ and $f : \mathbb{R} \to \mathbb{R}$ be a C^1 convex function. If $\tau \geq 0$, t, a, $b \in \mathbb{R}$ and A, $B > 0$ then*

$$|f(a) - f(b)|^{p-2}(f(a) - f(b))(A - B) \leq |a - b|^{p-2}(a - b)(A|f'(a)|^{p-2}f'(a) - B|f'(b)|^{p-2}f'(b)).$$

Lemma 6.3 *Let $1 < p < \infty$ and $g : \mathbb{R} \to \mathbb{R}$ be an increasing function, then we have*

$$|G(a) - G(b)|^p \leq |a - b|^{p-2}(a - b)(g(a) - g(b))$$

where $G(t) = \int_0^t g'(\tau)^{\frac{1}{p}} d\tau$, for $t \in \mathbb{R}$.

Theorem 6.4 *Let u be a positive solution of (P_λ). Then $u \in L^\infty(\Omega)$.*

Proof. Proof here is adopted from Brasco and Parini [7]. Let $\epsilon > 0$ be very small and define

$$f_\epsilon(t) = (\epsilon^2 + t^2)^{\frac{1}{2}}$$

which is smooth, convex and Lipschitz. Let $0 < \psi \in C_c^\infty(\Omega)$ and we take $\varphi = \psi|f_{\epsilon'}(u)|^{p-2}f'_\epsilon(u)$ as the test function in (6.1). By taking the choices

$$a = u(x), \quad b = u(y), \quad A = \psi(x), \quad B = \psi(y)$$

in Lemma 6.2, we get

$$\int_Q \frac{|f_\epsilon(u(x)) - f_\epsilon(u(y))|^{p-2}(f_\epsilon(u(x)) - f_\epsilon(u(y)))(\psi(x) - \psi(y))}{|x - y|^{n+sp}} dxdy \leq \int_\Omega \left(|\lambda u^{-q} + u^\alpha|\right) |f'_\epsilon(u)|^{p-1}\psi dx \tag{6.2}$$

As $t \to 0$, $f_\epsilon(t) \to |t|$ and we have $|f'_\epsilon(t)| \leq 1$. So using Fatou's Lemma, we let $\epsilon \to 0$ in above inequality and get

$$\int_Q \frac{|\,|u(x)| - |u(y)|\,|^{p-2}\,(|u(x)| - |u(y)|)(\psi(x) - \psi(y))}{|x - y|^{n+sp}} dxdy \leq \int_\Omega \left(|\lambda u^{-q} + u^\alpha|\right) \psi\, dx, \tag{6.3}$$

for every $0 < \psi \in C_c^\infty(\Omega)$. The above inequality still holds for $0 \leq \psi \in X_0$ (similar proof as of Lemma 6.1). Now, let us define $u_K = \min\{(u-1)^+, K\} \in X_0$, for $K > 0$. For $\beta > 0$ and $\rho > 0$, we take $\psi = (u_K + \rho)^\beta - \rho^\beta$ as test function in (6.3) and get

$$\begin{aligned} &\int_Q \frac{|\,|u(x)| - |u(y)|\,|^{p-2}\,(|u(x)| - |u(y)|)((u_K(x) + \rho)^\beta - (u_K(y) + \rho)^\beta)}{|x - y|^{n+sp}} dxdy \\ &\quad \leq \int_\Omega \left(|\lambda u^{-q} + u^\alpha|\right) \left((u_K + \rho)^\beta - \rho^\beta\right) dx. \end{aligned} \tag{6.4}$$

Then, by using Lemma 6.3 with the function

$$g(u) = (u_K + \rho)^\beta,$$

we get

$$\begin{aligned}
&\int_Q \frac{|(u_K(x)+\rho)^{\frac{\beta+p-1}{p}} - (u_K(y)+\rho)^{\frac{\beta+p-1}{p}}|}{x-y}^{n+sp} dxdy \\
\leq\ & \frac{(\beta+p-1)^p}{\beta p^p}\int_Q \frac{|\,|u(x)|-|u(y)|\,|^{p-2}\,(|u(x)|-|u(y)|)((u_K(x)+\rho)^\beta-(u_K(y)+\rho)^\beta)}{|x-y|^{n+sp}}\,dxdy \\
\leq\ & \frac{(\beta+p-1)^p}{\beta p^p}\int_\Omega \lambda|u^{-q}|((u_K+\rho)^\beta-\rho^\beta)\ dx + \int_\Omega |u^\alpha|((u_K+\rho)^\beta-\rho^\beta)\ dx.
\end{aligned} \tag{6.5}$$

Now, from the support of u_K we have

$$\begin{aligned}
&\int_\Omega \lambda|u^{-q}|((u_K+\rho)^\beta-\rho^\beta)\ dx + \int_\Omega |u^\alpha|((u_K+\rho)^\beta-\rho^\beta)\ dx \\
&= \int_{\{u\geq 1\}} \lambda|u^{-q}|((u_K+\rho)^\beta-\rho^\beta)\ dx + \int_{\{u\geq 1\}} |u^\alpha|((u_K+\rho)^\beta-\rho^\beta)\ dx \\
&\leq C_1\int_{\{u\geq 1\}} (1+|u|^\alpha)((u_K+\rho)^\beta-\rho^\beta)\ dx \\
&\leq 2C_1\int_{\{u\geq 1\}} |u|^\alpha((u_K+\rho)^\beta-\rho^\beta)\ dx \\
&\leq 2C_1|u|^\alpha_{p_s^*}\ |(u_K+\rho)^\beta|_r
\end{aligned} \tag{6.6}$$

where $C_1=\max\{\lambda,1\}$ and $r=\frac{p_s^*}{p_s^*-\alpha}$. By using Sobolev inequality given in Theorem 1 of [35], we get

$$\begin{aligned}
\int_Q \frac{|(u_K(x)+\rho)^{\frac{\beta+p-1}{p}} - (u_K(y)+\rho)^{\frac{\beta+p-1}{p}}|}{x-y^{n+sp}}\ dxdy &\geq \frac{1}{T_{p,s}}|(u_K+\rho)^{\frac{\beta+p-1}{p}}-\rho^{\frac{\beta+p-1}{p}}|^p_{p_s^*} \\
&\geq \frac{1}{T_{p,s}}\left(\left(\frac{\rho}{2}\right)^{p-1}|(u_K+\rho)^{\beta_p}|^p_{p_s^*}-\rho^{\beta+p-1}|\Omega|^{\frac{p}{p_s^*}}\right),
\end{aligned}$$

where $T_{p,s}$ is a nonnegative constant and the last inequality follows from triangle inequality and $(u_K+\rho)^{\beta+p-1}\geq \rho^{p-1}(u_K+\rho)^\beta$. Using all these estimates, we now have

$$|(u_K+\rho)^{\frac{\beta}{p}}|^p_{p_s^*} \leq C\left(T_{p,s}\left(\frac{2}{\rho}\right)^{p-1}\left(\frac{(\beta+p-1)^p}{\beta p^p}\right)|u|^\alpha_{p_s^*}\ |(u_K+\rho)^\beta|_r+\rho^\beta|\Omega|^{\frac{p}{p_s^*}}\right),$$

where $C=C(p)>0$ is a constant. By convexity of the map $t\mapsto t^p$, we can show that

$$\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p \geq 1.$$

Using this we can also check that

$$\rho^\beta|\Omega|^{\frac{p}{p_s^*}} \leq \frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}|(u_K+\rho)^\beta|_r.$$

Hence we have

$$|(u_K+\rho)^{\frac{\beta}{p}}|_{p_s^*}^p \leq C\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p |(u_K+\rho)^\beta|_r \left(\frac{T_{p,s}|u|_{p_s^*}^\alpha}{\rho^{p-1}}+|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right), \tag{6.7}$$

for $C=C(p)>0$ is constant. We now suitably choose

$$\rho=\left(T_{p,s}|u|_{p_s^*}^\alpha\right)^{\frac{1}{p-1}}|\Omega|^{\frac{-1}{p-1}\left(1-\frac{1}{r}-\frac{sp}{n}\right)}$$

and let $\beta\geq 1$ be such that

$$\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p\leq \beta^{p-1}.$$

In addition, if we let $\tau=\beta r$ and $\nu=\frac{p_s^*}{pr}>1$, then the above inequality uces to

$$|(u_K+\rho)|_{\nu\tau}\leq \left(C|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right)^{\frac{r}{\tau}}\left(\frac{\tau}{r}\right)^{\frac{(p-1)r}{\tau}}|(u_K+\rho)|_\tau \tag{6.8}$$

At this stage itself, if we take $K\to\infty$, we can say that $(u-1)^+\in L^m(\Omega)$, for all m. This will imply that $u\in L^m(\Omega)$, for all m. Now, we iterate (6.8) using $\tau_0=r$ and

$$\tau_{m+1}=\nu\tau_m=\nu^{m+1}r$$

which gives

$$|(u_K+\rho)|_{\tau_{m+1}}\leq \left(C|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right)^{\sum_{i=0}^m\frac{r}{\tau_i}}\left(\prod_{i=0}^m\left(\frac{\tau_i}{r}\right)^{\frac{r}{\tau_i}}\right)^{p-1}|(u_K+\rho)|_r. \tag{6.9}$$

Since $\nu>1$,

$$\sum_{i=0}^\infty\frac{r}{\tau_i}=\sum_{i=0}^m\frac{1}{\nu^i}=\frac{\nu}{\nu-1}$$

and

$$\prod_{i=0}^\infty\left(\left(\frac{\tau_i}{r}\right)^{\frac{r}{\tau_i}}\right)^{p-1}=\nu^{\frac{\nu}{(\nu-1)^2}}.$$

Taking limit as $n\to 0$ in (6.9), we finally get

$$|u_K|_\infty\leq\left(C\nu^{\frac{\nu}{(\nu-1)^2}}\right)^{p-1}\left(|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right)^{\frac{\nu}{\nu-1}}|(u_K+\rho)|_r.$$

Since $u_K\leq(u-1)^+$, using triangle inequality in above inequality we get,

$$|u_K|_\infty\leq C\left(\nu^{\frac{\nu}{(\nu-1)^2}}\right)^{p-1}\left(|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right)^{\frac{\nu}{\nu-1}}\left(|(u-1)^+|_r+\rho|\Omega|^{\frac{1}{r}}\right)$$

for some constant $C=C(p)>0$. If we now let $K\to\infty$, we get

$$|(u-1)^+|_\infty\leq C\left(\nu^{\frac{\nu}{(\nu-1)^2}}\right)^{p-1}\left(|\Omega|^{1-\frac{1}{r}-\frac{sp}{n}}\right)^{\frac{\nu}{\nu-1}}\left(|(u-1)^+|_r+\rho|\Omega|^{\frac{1}{r}}\right).$$

Hence in particular, we say that $u\in L^\infty(\Omega)$. □

Theorem 6.5 *Let u be a positive solution of P_λ. Then there exist $\gamma \in (0, s]$ such that $u \in C^{\gamma}_{loc}(\Omega')$, for all $\Omega' \subset\subset \Omega$.*

Proof. Let $\Omega' \subset\subset \Omega$. Then using lemma 3.7 and above regularity result, for any $\psi \in C_c^\infty(\Omega)$ we get

$$\lambda \int_{\Omega'} u^{-q}\psi dx + \int_{\Omega'} u^{\alpha}\psi dx \leq \lambda \int_{\Omega'} \phi_1^{-q}\psi dx + \|u\|_\infty^\alpha \int_{\Omega'} \psi dx \leq C \int_{\Omega'} \psi dx$$

for some constant $C > 0$, since we can find $k > 0$ such that $\phi_1 > k$ on Ω'. Thus we have $|(-\Delta_p)^s u| \leq C$ weakly on Ω'. So, using theorem 4.4 of [33] and applying a covering argument on inequality in corollary 5.5 of [33], we can prove that there exist $\gamma \in (0, s]$ such that $u \in C^{\gamma}_{loc}(\Omega')$, for all $\Omega' \Subset \Omega$. □

7 Global existence of solution

Let us define $\Lambda = \sup\{\lambda > 0 : (P_\lambda) \text{ has a solution}\}$.

Lemma 7.1 $\Lambda < +\infty$.

Proof. The proof follows similarly as the proof of Lemma 4.4. □

In the following lemmas, we will show the existence of solution of (P_λ).

Lemma 7.2 *If $\underline{u} \in X_0$ is a weak sub-solution and $\overline{u} \in X_0$ is a weak super-solution of (P_λ) such that $\underline{u} \leq \overline{u}$ a.e. in Ω, then there exists a weak solution $u \in X_0$ satisfying $\underline{u} \leq u \leq \overline{u}$.*

Proof. We follow [30]. We know that the functional I is non- differentiable in X_0. Let $M := \{u \in X_0 : \underline{u} \leq u \leq \overline{u}\}$, then M is closed, convex and I is weakly lower semicontinuous on M. We can see that if $\{u_k\} \subset M$ and $u_k \rightharpoonup u$ in X_0 as $k \to \infty$, we may assume $u_k \to u$ pointwise a.e. in Ω (along a subsequence). Since $u \in M$, $\int_\Omega |\overline{u}|^{\alpha+1} dx < +\infty$ and $\int_\Omega |\overline{u}|^{1-q} dx < +\infty$, then by Lebesgue Dominated Convergence theorem,

$$\int_\Omega |u_k|^{\alpha+1} dx \to \int_\Omega |u|^{\alpha+1} dx \quad \text{and} \quad \int_\Omega |u_k|^{1-q} dx \to \int_\Omega |u|^{1-q} dx.$$

So, $\underline{\lim}_{k\to\infty} I(u_k) \geq I(u)$. Thus, there exist $u \in M$ such that $I(u) = \inf_{u_0 \in M} I(u_0)$. We claim that u is a weak solution of (P_λ). For $\epsilon > 0$ and $\varphi \in X_0$, define $v_\epsilon = u + \epsilon\varphi - \varphi^\epsilon + \varphi_\epsilon \in M$ where $\varphi^\epsilon = (u + \epsilon\varphi - \overline{u})^+ \geq 0$ and $\varphi_\epsilon = (u + \epsilon\varphi - \underline{u})^- \geq 0$. For $t \in (0, 1)$, $u + t(v_\epsilon - u) \in M$ and we have

$$\begin{aligned}
0 &\leq \frac{I(u + t(v_\epsilon - u)) - I(u)}{t} \\
&= \lim_{t\to 0} \left(\frac{1}{pt}(\|u + t(v_\epsilon - u)\|^p - \|u\|^p) + \lambda \int_\Omega \frac{(G_q(u + t(v_\epsilon - u)) - G_q(u))}{t} dx \right. \\
&\qquad \left. - \frac{1}{\alpha+1} \int_\Omega \frac{|u + t(v_\epsilon - u)|^{\alpha+1} - |u|^{\alpha+1}}{t} dx \right) \\
&= \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))((v_\epsilon - u)(x) - (v_\epsilon - u)(y))}{|x - y|^{n+sp}}\, dxdy - \lambda \int_\Omega u^{-q}(v_\epsilon - u) dx \\
&\qquad - \int_\Omega u^\alpha (v_\epsilon - u) dx
\end{aligned}$$

which gives

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}dxdy - \int_\Omega (\lambda u^{-q}+u^\alpha)\varphi dx \geq \frac{1}{\epsilon}(H^\epsilon - H_\epsilon) \quad (7.1)$$

where

$$H^\epsilon = \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi^\epsilon(x)-\varphi^\epsilon(y))}{|x-y|^{n+sp}}\ dxdy - \int_\Omega (\lambda u^{-q}+u^\alpha)\varphi^\epsilon dx$$
$$H_\epsilon = \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi_\epsilon(x)-\varphi_\epsilon(y))}{|x-y|^{n+sp}}\ dxdy - \int_\Omega (\lambda u^{-q}+u^\alpha)\varphi_\epsilon dx.$$

Now we consider

$$\frac{1}{\epsilon}H^\epsilon = \frac{1}{\epsilon}\left(\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi^\epsilon(x)-\varphi^\epsilon(y))}{|x-y|^{n+sp}}\ dxdy - \int_\Omega (\lambda u^{-q}+u^\alpha)\varphi^\epsilon dx\right)$$

Let $\Omega_1 = \{u+\epsilon\varphi \geq \overline{u} > u\}$ and $\Omega_2 = \{u+\epsilon\varphi < \underline{u}\}$, then using the technique of Lemma 3.7, we get

$$\begin{aligned}
&\frac{1}{\epsilon}\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi^\epsilon(x)-\varphi^\epsilon(y))}{|x-y|^{n+sp}}\ dxdy \\
&\quad= \frac{1}{\epsilon}\left(\int_{\Omega_1\times\Omega_1}+\int_{\Omega_1\times\Omega_2}+\int_{\Omega_2\times\Omega_1}\right)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi^\epsilon(x)-\varphi^\epsilon(y))}{|x-y|^{n+sp}}dxdy \\
&\quad= \frac{1}{\epsilon}\int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))((u-\overline{u})(x)-(u-\overline{u})(y))}{|x-y|^{n+sp}}\ dxdy \\
&\qquad+ \int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dxdy \\
&\qquad+ \frac{1}{\epsilon}\int_{\Omega_1\times\Omega_2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}(u-\overline{u})(x)\ dxdy \\
&\qquad+ \int_{\Omega_1\times\Omega_2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\varphi(x)\ dxdy \\
&\qquad- \frac{1}{\epsilon}\int_{\Omega_2\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}(u-\overline{u})(y)\ dxdy \\
&\qquad- \int_{\Omega_2\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\varphi(y)\ dxdy \\
&\quad\geq \frac{3}{\epsilon 2^{p-2}}\int_{\Omega_1\times\Omega_1} \frac{|(u-\overline{u})(x)-(u-\overline{u})(y)|^p}{|x-y|^{n+sp}}\ dxdy \\
&\qquad+ \int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dxdy \\
&\quad\geq \int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dxdy
\end{aligned}$$

where we used the inequality $|a-b|^p \le 2^{p-2}(|a|^{p-2}a - |b|^{p-2}b)(a-b)$, for $p \ge 2$ and $a, b \in \mathbb{R}$. Thus,

$$\begin{aligned}\frac{1}{\epsilon}H^\epsilon &\ge \int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dxdy - \int_{\Omega_1}(\lambda u^{-q}+u^\alpha)\varphi^\epsilon dx\\ &\ge \int_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dxdy - \int_{\Omega_1}|\lambda\overline{u}^{-q}-u^{-q}||\varphi|dx\\ &= o(1)\end{aligned}$$

as $\epsilon \to 0$, since meas$(\Omega_1) \to 0$ as $\epsilon \to 0$. Similarly, as $\epsilon \to 0$ we can show that $\frac{1}{\epsilon}H_\epsilon \le o(1)$. Therefore, from (7.1) taking $\epsilon \to 0$, we get

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dxdy - \int_\Omega \left(\lambda u^{-q}+u^\alpha\right)\varphi dx \ge o(1).$$

Since $\varphi \in X_0$ is arbitrary, for all $\varphi \in X_0$ we get

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dxdy - \int_\Omega (\lambda u^{-q}+u^\alpha)\varphi dx = 0. \qquad \square$$

Proposition 7.3 *For $\lambda \in (0,\Lambda)$, (P_λ) has a weak solution $u_\lambda \in X_0$.*

Proof. We fix $\lambda \in (0,\Lambda)$. By definition of Λ, there exists $\lambda_0 \in (\lambda,\Lambda)$ such that (P_{λ_0}) has a solution u_{λ_0} (say). Then $\overline{u} = u_{\lambda_0}$ becomes a super-solution of (P_λ). Now consider the function ϕ_1 as the eigenfunction of $(-\Delta_p)^s$ corresponding to the smallest eigenvalue λ_1. Then $\phi_1 \in L^\infty(\Omega)$ and

$$(-\Delta_p)^s\phi_1 = \lambda_1|\phi_1|^{p-2}\phi_1,\ \phi_1 > 0 \text{ in } \Omega, \quad \phi_1 = 0 \text{ on } \mathbb{R}^n\setminus\Omega.$$

Let us choose $t > 0$ such that $t\phi_1 \le \overline{u}$ and $t^{p+q-1}\phi_1^{p+q-1} \le \lambda/\lambda_1$. If we define $\underline{u} = t\phi_1$, then

$$\begin{aligned}(-\Delta_p)^s\underline{u} &= \lambda_1 t^{p-1}\phi_1^{p-1} \le \lambda t^{-q}\phi_1^{-q}\\ &\le \lambda t^{-q}\phi_1^{-q} + t^\alpha\phi_1^\alpha = \lambda\underline{u}^{-q} + \underline{u}^\alpha.\end{aligned}$$

that is, $\underline{u}$ is a sub-solution of (P_{λ_0}) and $\underline{u} \le \overline{u}$. Applying Lemma 7.2 shows that (P_λ) has a solution for all $\lambda \in (0,\Lambda)$. This completes the proof. $\square$

Proof of Theorem 2.6: Proof follows from Proposition 7.3 and Lemma 7.1. $\square$

Remark 7.4 *We remark that the method in Lemma 7.2 we can show the existence of solution for pure singular problem:*

$$(-\Delta_p)^s u = \lambda u^{-q} \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n\setminus\Omega. \tag{7.2}$$

where $0 < q < 1$. We define u to be a positive weak solution of (7.2) if $u > 0$ in Ω, $u \in X_0$ and

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}}\,dxdy - \lambda\int_\Omega u^{-q}dx = 0 \text{ for all } \psi \in X_0.$$

Also, we say $u \in X_0$ to be a positive weak sub-solution of (7.2) *if $u > 0$ and*

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}}\, dxdy \leq \lambda \int_\Omega u^{-q}dx \text{ for all } \psi \in X_0.$$

We define the functional $J_\lambda : X_0 \to (-\infty, \infty]$ by

$$J_\lambda(u) = \frac{1}{p}\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy - \lambda \int_\Omega G_q(u)dx$$

where G_q is as defined in section 2. One can easily see that J_λ is coercive, bounded below and weakly lower semicontinuous in X_0. Thus there exist a $u_0 \in X_0$ such that $\inf_{u\in X_0} I(u) = I(u_0)$. We claim that u_0 is a positive weak solution of (7.2). We choose $t > 0$ such that $t\phi_1 \leq u_0$ in Ω and $t\phi_1$ is a sub-solution of (7.2) (ϕ_1 is defined in proposition 7.3). Let us define $M := \{u \in X_0 : \underline{u} \leq u\}$, where $\underline{u}$ is a weak sub-solution of (7.2). Then $u_0 \in M$ and following the proof of lemma 7.2 with $v_\epsilon = u_0 + \epsilon\varphi + \varphi_\epsilon$ where $\epsilon > 0, \varphi_\epsilon = (u_0 + \epsilon\varphi - \underline{u})^-$ and $\varphi \in X_0$, we can show that u_0 is a positive weak solution of (7.2)*.*

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