Intuitionistic Fuzzy Logit Model of Discrete Choice

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***Abstract*—The criteria evaluations are often vague (or not crisp) in the real world multi-criteria decision making (MCDM). The existing choice models are difficult to apply in such situations. In this paper, we introduce an intuitionistic fuzzy variant of multinomial logit model, so as to give a decision-maker's likely choices with vague evaluations. The applicability of the proposed model is shown through a real multi-criteria decision-making application.**

***Index Terms*—Decision analysis; choice behaviour; discrete choice probability; intuitionistic fuzzy**

I. Introduction

The discrete choice models provide a useful tool to reperesent *probabilistic* uncertainty for precisely defined random events. However, these models are rendered unusable for imprecisely defined notions with *possibilistic* uncertainty, like: *high income*, *low temperature*, etc. that are described by fuzzy set [37] and intuitionistic fuzzy set (IFS) [4] theories. Very often in the real world situations, both probabilistic and possibilistic uncertainties co-exist. For example, *it is highly likely that it will be a warm day.* This conveys probabilistic information about fuzzy events. The notions of an event and its probability is extended to the fuzzy domain with the concept of *probability of a fuzzy event* in [38, 35]. In a similar vein, the notion of *probability of an intuitionistic fuzzy event* is conceived in [27].

These existing defintions are derived from extending the basic probability concept to the fuzzy and intuitionistic fuzzy domains. The underlying set-theoretic premise for considering the probability and its calculus is an experiment $\mathcal{E}$ that is to be performed. Let its generic uncertain outcome be denoted as X, x denote a generic outcome of $\mathcal{E}$ after it actually taking place, and let Ω denote the set of all conceived outcomes of $\mathcal{E}$. Thus $x \in \Omega$. Let $\mathcal{F}$ denote a family of subsets of Ω, which are referred to as *events*. In classical probability theory, it is presumed that events are precisely defined in the sense that there is no ambiguity in declaring whether an outcome x belongs to an event A of Ω, or not. In contrast, with fuzzy sets, there is an ambiguity in determining the degree of belonging (membership) of x in A, because A is not sharply defined. In the case of intuitionistic fuzzy domain, there is also an associated element of hesitancy associated with the graded membership of x in A.

Since $\mathcal{E}$ is yet to be performed, the occurrence of any x and thus A is uncertain. We denote this uncertainty by a number $P(A)$ in the range $[0, 1]$, giving the probability of event A. For a given "probability measure space" $(\Omega, \mathcal{F}, \mathcal{P})$, if $A \in \mathcal{F}$ is a crisp set with characteristic function $I_A(x)$ such that $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$, otherwise. Then

$$\mathcal{P}(A) = \sum_x I_A(x)\mathcal{P}(x), \;\; x \in \Omega, \tag{1}$$

where $\mathcal{P}(x)$ is the probability that the outcome of $\mathcal{E}$ is x. In the case of fuzzy subset $\acute{A}$ of Ω, which is defined as *fuzzy event* in [38], the probability measure of $\acute{A}$ is given as

$$\Pi(\acute{A}) = \int_\Omega \mu_{\acute{A}}(x) d\mathcal{P}(x) = \mathbf{E}[\mu_{\acute{A}}(x)] \tag{2}$$

where $\mu_{\acute{A}}(x)$ is the membership function of $\acute{A}$ and $\mathbf{E}$ denotes expectation. This definition is extended in [27] for intuitionistic fuzzy event $\tilde{A}$ by replacing $\mu_{\acute{A}}$ with intuitionistic fuzzy membership function $\tilde{\mu}_{\tilde{A}}$

The measure in (2) is essentially the same as that in (1), as in both the cases, the probabilistic uncertainty pertains to the uncertain outcome $X = x$ of $\mathcal{E}$. The reason for having (2) is to consider another facet of uncertainty, termed as possibilistic uncertainty, regarding the membership of x in $\acute{A}$ in the fuzzy domain (or membership of x in $\tilde{A}$ in intuitionistic fuzzy domain), which is not required in the case of crisp set. Hence, in order to apply (2) in practice, we require to have a complete knowledge of $\mathcal{P}(x)$.

In the context of multi criteria decision making (MCDM), x indicates an alternative (option) that is chosen by a decision maker (DM) among several others. An alternative is described by multiple criteria evaluations (utilities), and the DM chooses the alternative yielding maximum utility. There is often an unobservable utility associated with the DM's choice, hence it is difficult to predict with certainty the DM's choice. To this end, probabilistic models of discrete choice are commonly used to give a DM's choice probabilities for various alternatives. Multinomial logit model (MNL) [23] is perhaps the most popular model due to its easy interpretability. The popularity of the discrete choice models can be gauged through their applications in diverse domains in the recent times. They are applied in severity analysis [9, 36, 22],

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price optimization [33], revenue optimization [13], location planning [16], choice analysis problems [25, 24, 20, 15, 19], risk analysis [6, 32, 39, 7, 18, 2], demand analysis [31, 11], data analytics [14, 8, 5], regression analysis [21, 10, 26], causal inference in medicine [28], and forecasting [17], to name a few.

However, it can only be used when the criteria evaluations are in terms of crisp values. In the real world decision making, very often, the decision makers have only a partial knowledge to concretely evaluate the alternatives against multiple criteria. The fuzzy and IFS theories are quite useful to imprecisely evaluate the alternatives, in MCDM under uncertainty. Imprecision also arises in the real world decision making wherein the goals, the constraints, and the consequences of actions cannot be precisely specified. Our objective in this note is to show how the notion of crisp event (of a DM's discrete choice) with imprecise criteria evaluations can be described. To this end, we extend the discrete choice models to fuzzy and intuitionistic fuzzy domains.

The proposed class of logit models would be able to address the situations where different DMs, with the same criteria values and utility coefficients, may still have the different choices, as per their individual degrees of satisfaction derived from the criteria values. MNL model always predicts the same choice probability in such situations. We present an intuitionistic fuzzy variant of MNL model. Tbe main contributions are summarized herewith:

- We give the background of the study in Section II.
- The intuitionistic fuzzy variant of MNL model is introduced in Section III, along with the motivation.
- In Section IV, we give a real application of the proposed work.
- Section V concludes the paper with an outlook on the future work.

II. Background

A. Review of MNL Model

A decision-maker (DM), faces a choice among n alternatives. The DM derives a certain level of utility (or enjoyment) from each of the criteria associated with an alternative. The net utility that decision-maker D obtains from alternative $\boldsymbol{a}_i$ is $U_i^D, j = 1, \ldots, K$. The DM chooses the alternative that provides the highest utility. The behavioral model is therefore: choose alternative $\boldsymbol{a}_i$ (between alternatives $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$) if and only if $U_i^D > U_j^D$. We ease the notations by obviating the superscript D from the notations.

An alternative $\boldsymbol{a}_i$ can be represented in terms of it's criteria values as following:

$$\boldsymbol{a}_i = \left(a_i^{(1)}, \ldots, a_i^{(M)}\right), \tag{3}$$

where, M is the number of criteria associated with $\boldsymbol{a}_i$, and $a_i^{(m)}$ is the value $\boldsymbol{a}_i$ takes for m^{th} criterion c_m. The goal in any econometric model is to determine the utility value U_i that depends on a 'representative' utility that is a function of the observable criteria $a_i^{(m)}, m = 1, \ldots, M$, and an unobservable component ϵ_i corresponding to $\boldsymbol{a}_i$. That is:

$$U_i = V_i + \epsilon_i \tag{4}$$

where, ϵ_i represents the additive random component of the utility, due to the unobservable factors. We denote the representative utility that alternative $\boldsymbol{a}_i$ holds for the DM, by virtue of it's observable criteria, as:

$$V_i = V(\boldsymbol{\beta}, \boldsymbol{a}_i), \tag{5}$$

where $\boldsymbol{\beta}$ is the vector of the coefficients that the DM attaches to the given criteria. More explicitly, it is represented as

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_M), \tag{6}$$

where $\beta^{(m)}$ is the DM's utility coefficient for $a^{(m)}$. The vector $\boldsymbol{\beta}$ is specific to a DM. In [23], the function $V_i = V(\boldsymbol{\beta}, \boldsymbol{a}_i)$ is taken as

$$V_i = \sum_{m=1}^{M} \beta_m a_i^{(m)} \tag{7}$$

Since, the ϵ_i component of the utility value U_i cannot be determined, it is not possible to predict with certainty about the best choice of alternative, relying only upon an utility maximization model. In this regard, it is simpler to predict the probability with which an alternative would be chosen by a particular DM. It has been shown in [12, 30] that the probability P_i that an alternative $\boldsymbol{a}_i$ yields the highest utility to the DM, and thus is chosen, is given by

$$\begin{aligned} P_i &= \frac{exp(V_i)}{\sum_{k=1}^{K} exp(V_k)} \\ &= \frac{exp\left(\sum_{m=1}^{M} \beta_m a_i^{(m)}\right)}{\sum_{k=1}^{K} exp\left(\sum_{m=1}^{M} \beta_m a_k^{(m)}\right)} \end{aligned} \tag{8}$$

B. Intuitionistic Fuzzy Sets

In addition to the usual membership and non-membership grades, an intuitionistic fuzzy value (IFV) is devised to have a hesitancy degree. An IFV to IFS is what a membership grade is to a fuzzy set. An IFV $\tilde{a}$, comprises of three grades: membership $t_{\tilde{a}}$, non membership $f_{\tilde{a}}$, and hesitancy $\pi_{\tilde{a}}$, where $t_{\tilde{a}}, f_{\tilde{a}}, \pi_{\tilde{a}} \in [0, 1]$, and $t_{\tilde{a}} + f_{\tilde{a}} + \pi_{\tilde{a}} = 1$. Since, $\pi_{\tilde{a}} = 1 - (t_{\tilde{a}} + f_{\tilde{a}})$, we omit $\pi_{\tilde{a}}$ and IFV $\tilde{a}$ is represented as $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$. The following operational laws [34, 3] are valid for IFVs $\tilde{a}$ and $\tilde{b}$:-

$$\tilde{a} \oplus \tilde{b} = (t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}} t_{\tilde{b}}, f_{\tilde{a}} f_{\tilde{b}}) \quad (9)$$

$$\tilde{a} \otimes \tilde{b} = (t_{\tilde{a}} t_{\tilde{b}}, f_{\tilde{a}} + f_{\tilde{b}} - f_{\tilde{a}} f_{\tilde{b}}, f_{\tilde{a}} f_{\tilde{b}}) \quad (10)$$

$$\lambda \tilde{a} = (1 - (1 - t_{\tilde{a}})^{\lambda}, (f_{\tilde{a}})^{\lambda}), \lambda > 0 \quad (11)$$

$$(\tilde{a})^{\lambda} = ((\tilde{a})^{\lambda}, (1 - (1 - f_{\tilde{a}})^{\lambda}) \quad (12)$$

$$\tilde{a} \oslash \tilde{b} = \left(t_{\tilde{a} \oslash \tilde{b}}, f_{\tilde{a} \oslash \tilde{b}}\right), \text{ where} \quad (13)$$

$$t_{\tilde{a} \oslash \tilde{b}} = \begin{cases} \frac{t_{\tilde{a}}}{t_{\tilde{b}}}, & \begin{array}{l} \text{if } t_{\tilde{a}} \leq t_{\tilde{b}} \text{ and } f_{\tilde{a}} \geq f_{\tilde{b}} \\ \text{and } t_{\tilde{b}} > 0 \\ \text{and } t_{\tilde{a}} \pi_{\tilde{b}} \leq \pi_{\tilde{a}} t_{\tilde{b}} \end{array} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad (14)$$

$$f_{\tilde{a} \oslash \tilde{b}} = \begin{cases} \frac{t_{\tilde{a}} - t_{\tilde{b}}}{1 - t_{\tilde{b}}}, & \begin{array}{l} \text{if } t_{\tilde{a}} \leq t_{\tilde{b}} \text{ and } f_{\tilde{a}} \geq f_{\tilde{b}} \\ \text{and } t_{\tilde{b}} > 0 \\ \text{and } t_{\tilde{a}} \pi_{\tilde{b}} \leq \pi_{\tilde{a}} t_{\tilde{b}} \end{array} \\ 1, & \text{otherwise} \end{cases} \quad (15)$$

where $\oplus$, $\otimes$, and $\oslash$ are the intuitionistic fuzzy counterparts of additive, multiplicative, and division operations.

III. Intuitionistic Fuzzy MNL Model

A. Motivation

It has been shown in [29] that humans show a utility maximizing decision behaviour. A decision maker (DM) sees an alternative as a bundle of desired criteria, and the DM chooses the alternative with the greatest aggregated score of the utility values, corresponding to the given criteria. In the existing choice models, the product of the actual value of a criterion, and the corresponding utility coefficient is considered as the utility derived by a DM from the given criterion value. With this approach, all of the values that a criterion assumes for the given alternatives are scaled up (or down) in accordance with the corresponding utility coefficient. The utility coefficient vector therefore models a DM's decision behaviour.

In real decision making, however, quite often, the crisp criteria values are not known precisely. For instance, in a supplier selection problem, various alternatives are evaluated against different criteria, say brand value, past customer experiences etc. In such scenarios, the criteria values are best described by fuzzy values. Besides, in practice, a criterion value is perceived differently by different individuals as per their own background. The perceived values can be easily represented through fuzzy evaluations, in contrast to the crisp values.

To the best of our knowledge there is no model in the literature to give the choice probability of an alternative, based on the fuzzy evaluations. This forms our main motivation behind this work. We propose a choice model that could predict the choice probabilities for the alternatives on the basis of the vague (fuzzy) criteria values. In the real world decision-making, fuzzy criteria values are quite common, because the crisp values are often inaccessible, or it requires a lot of effort to collect them. In comparison, the fuzzy values are far more easy to assess than the crisp values with precision.

Therefore, a choice model that is able to process fuzzy criteria values holds a substantial potential in the representation of the real world decision-making situations. More specifically, we propose an intuitionistic-fuzzy values based logit choice model that also considers the hesitancy of the DM, while determining the membership grade. The proposed model reduces to a fuzzy logit model when hesitancy is nil.

B. The Proposed Model

In the real world MCDM, due to time constraints or the nature of the problem, the DMs often need to resort to imprecise evaluations. In this regard, it is easy to conceptualize a fuzzy MNL model on the lines of the conventional MNL model as shown in (8). The fuzzy MNL model is a special case of MNL model, with fuzzy criteria values, i.e. $a_i^{(m)} \in [0, 1]$. However, such a fuzzy MNL lacks a provision to take on record the DM's hesitancy that is often encountered in determining a membership grade. In this section, we introduce an intuitionistic fuzzy variant of MNL model, which we term as intuitionistic fuzzy MNL (IF-MNL). It helps to extend the abilties of fuzzy MNL by also considering the agent's hesitancy associated with a membership degree. Besides, IF-MNL model reduces to fuzzy MNL in the case of nil hesitancy.

We denote each alternative $\boldsymbol{a}_i$ in terms of the intuitionistic fuzzy criteria values as:

$$\tilde{\boldsymbol{a}}_i = \left(\tilde{a}_i^{(1)}, \ldots, \tilde{a}_i^{(M)}\right), \quad (16)$$

where each of the $\tilde{a}_i^{(m)}$, $m = 1, \ldots, M$ values is a IFV, shown as $\left(t_i^{(m)}, f_i^{(m)}\right)$. The observable utility corresponding to IFV $\tilde{a}_i^{(m)}$ is computed by applying (10) as:

$$\tilde{v}_i^{(m)} = \beta_m \otimes \tilde{a}_i^{(m)} = \left(1 - \left(1 - t_i^{(m)}\right)^{\beta_m}, \left(f_i^{(m)}\right)^{\beta_m}\right) \quad (17)$$

The representative utility $\tilde{V}_i$ is determined as aggregating the utility values $\tilde{v}_i^{(m)}$, $m = 1, \ldots, M$, so obtained as:

$$\begin{aligned} \tilde{V}_i &= \bigoplus_{m=1}^{M} \beta_m \otimes \tilde{a}_i^{(m)} \\ &= \left[1 - \prod_{m=1}^{M} \left(1 - t_i^{(m)}\right)^{\beta_m}, \prod_{m=1}^{M} \left(f_i^{(m)}\right)^{\beta_m}\right], \end{aligned} \quad (18)$$

where $\beta_m \in [0, 1]$. The intuitionistic fuzzy value $\tilde{a}_i^{(m)}$ in conjunction with the utility coefficient β_m can be seen as the DM's "*taste*" for c_m. The vector of β_m values characterize the unique choice behaviour of the DM.

We emphasize that the representative utility obtained in (18) is a intuitionistic fuzzy value, as it is obtained through the aggregation of intuitionistic fuzzy criteria values (See [1]). Replacing (18) in (8), we obtain the choice probability P_i for alternative $\boldsymbol{a}_i$ to be chosen as :

$$
\begin{aligned}
P_i &= \left[1-\prod_{m=1}^{M}\left(1-t_i^{(m)}\right)^{\beta_m}, \prod_{m=1}^{M}\left(f_i^{(m)}\right)^{\beta_m}\right] \oslash \left[\bigoplus_{k=1}^{K}\left(1-\prod_{m=1}^{M}\left(1-t_k^{(m)}\right)^{\beta_m}, \prod_{m=1}^{M}\left(f_i^{(m)}\right)^{\beta_m}\right)\right] \\
&= \left[1-\prod_{m=1}^{M}\left(1-t_i^{(m)}\right)^{\beta_m}, \prod_{m=1}^{M}\left(f_i^{(m)}\right)^{\beta_m}\right] \oslash \\
&\qquad \left[K-\sum_{k=1}^{K}\prod_{m=1}^{M}\left(1-t_k^{(m)}\right)^{\beta_m}-\prod_{k=1}^{K}\left(1-\prod_{m=1}^{M}\left(1-t_k^{(m)}\right)^{\beta_m}\right), \prod_{k=1}^{K}\left(\prod_{m=1}^{M}\left(f_i^{(m)}\right)^{\beta_m}\right)\right]
\end{aligned} \tag{19}
$$

Unlike the conventional MNL model, the proposed IF-MNL model considers the agent's perceived enjoyment values, weighted by the relative importance that the agent associates with each criterion. Hence, IF-MNL model considers a greater degree of individualism through both intuitionistic fuzzy evaluations as well as the utility coefficients. We summarize the main features of the proposed IF-MNL model as:

- IF-MNL model considers the individualistic utility value derived by th DM from a criterion value along with the relative importance he/she attaches to the criterion.
- IF-MNL model implies a proportional substitution[1] across alternatives, with the given model's specification of representative utility.
- Along with the inconsistencies in the DM's evaluation of the criteria values, IML models can capture the dynamics of repeated choice.

IV. Case-study

We devote this section to illustrate the proposed choice model in a real application on the selection of the most suitable car by a prospective buyer. Typically, in such decisions, a decision-maker (DM) evaluates each of the alternatives against a set of criteria. Often, the DM wants to determine the best choice quickly and also does not have access to the crisp values, in which case the DM vaguely evaluates the criteria values in his/her cognition. For example, in a car-buying situation, a prospective buyer considers a large number of alternatives, each with multiple criteria such as length, height, brand value, luxury. In such situations, the prospective buyer arrives at his/her choice based on his/her perceptions of the criteria values. Given such perceptions of a DM, we illustrate the usefulness of the proposed model in predicting a DM's best choice, along with the probability of choosing the same.

A. Selection of the Best Car

We consider a case-study that is about a prospective buyer's selection of the car that suits him the most. The buyer evaluates multiple alternatives against a set of desirable criteria :- c_1: length (mm), c_2: width (mm), c_3: height (mm), and c_4: engine capacity (cc). The utility coefficients for the given criteria are: $(0.35, 0.60, 0.06, 0.15)$.

[1]Independence from irrelevant alternatives (IIA)

For the sake of the case-study, and drawing out comparison with the conventional model, we collected the actual criteria values for the latest car models, available in the Indian markets. The real identities of the car models have been withheld. We convert these values to IFVs as following:

$$
\begin{aligned}
t_i^{(m)} &= \frac{a_i^{(m)} - a_{\min}^{(m)}}{a_{\max}^{(m)} - a_{\min}^{(m)}} \\
f_i^{(m)} &= \frac{2*(1-t_i^{(m)})}{3} \\
\pi_i^{(m)} &= 1 - t_i^{(m)} - f_i^{(m)},
\end{aligned}
$$

where $a_i^{(m)}$ refers to the actual value that alternative $\boldsymbol{a}_i$ takes for c_m, $a_{\max}^{(m)}$ and $a_{\min}^{(m)}$ are the maximum and the minimum values among the collection of values for c_m against the given alternatives. The corresponding IFV for $a_i^{(m)}$ is thus shown as: $\tilde{a}^{(i)} = \left(t_i^{(m)}, f_i^{(m)}, \pi_i^{(m)}\right)$.

The criteria values, in terms of IFVs, for the given car models are shown in Table I. We give the corresponding utility values in Table II each of the models, we compute IF-MNL choice probability, shown in (19). The choice probabilities, so obtained are populated in Table **??**. The choice probabilities, along with the alternatives, are shown in the descending order of their magnitudes. We observe that the $\boldsymbol{a}_3$ is the most likely alternative to be chosen on the basis of intuitionistic-fuzzy representative utility. We note that $\boldsymbol{a}_3$ is only the most probable alternative by the prospective buyer to be chosen, and the real choice could be different. This is because of the presence of unobservable component of the utility. When the observable or representative utility forms a significant portion of the total utility, then the alternative with the highest choice probability is quite likely to be actually chosen. If the unobservable utility is nil, or all possible criteria have been considered, then it is possible to determine the best choice of the prospective buyer with certainty.

We also redo the case-study, and compute the choice probability with the conventional MNL model. The choice probabilities, in the descending order, are shown as the last column of Table III. We notice some differences in the alternative rankings obtained with the proposed IF-MNL and the conventional MNL model. One of the main reasons for the same is the fact that MNL model lacks a provision to take account of the hesitancy values that may lead to significant difference in the choice probabilities.

TABLE I: Intuitionistic Fuzzy Criteria Values

$\boldsymbol{a}_i$	c_1	c_2	c_3	c_4
1	(0.73,0.18)	(0.58,0.28)	(0.09,0.61)	(0.63,0.25)
2	(0.59,0.27)	(0.37,0.42)	(0.25,0.50)	(0.19,0.54)
3	(0.92,0.05)	(1.00,0.00)	(0.63,0.25)	(0.73,0.18)
4	(0.53,0.31)	(0.32,0.45)	(0.26,0.49)	(0.20,0.53)
5	(0.86,0.09)	(0.88,0.08)	(0.19,0.54)	(0.72,0.19)
6	(0.57,0.29)	(0.36,0.43)	(0.24,0.51)	(0.17,0.55)
7	(1.00,0.00)	(0.64,0.24)	(0.34,0.44)	(0.47,0.35)
8	(0.68,0.21)	(0.39,0.41)	(0.27,0.49)	(0.20,0.53)
9	(0.55,0.30)	(0.71,0.19)	(0.05,0.63)	(0.72,0.19)
10	(0.82,0.12)	(0.48,0.35)	(0.30,0.47)	(0.19,0.54)
11	(0.77,0.15)	(0.59,0.27)	(0.22,0.52)	(0.73,0.18)
12	(0.81,0.13)	(0.81,0.13)	(0.31,0.46)	(0.33,0.45)
13	(0.59,0.27)	(0.67,0.22)	(0.00,0.67)	(1.00,0.00)
14	(0.20,0.53)	(0.07,0.62)	(0.37,0.42)	(0.06,0.63)
15	(0.27,0.49)	(0.13,0.58)	(0.33,0.45)	(0.09,0.61)
16	(0.64,0.24)	(0.62,0.25)	(0.00,0.67)	(0.20,0.53)
17	(0.00,0.67)	(0.05,0.63)	(0.77,0.15)	(0.00,0.67)
18	(0.58,0.28)	(0.19,0.54)	(0.66,0.23)	(0.27,0.49)
19	(0.09,0.61)	(0.01,0.66)	(0.40,0.40)	(0.04,0.64)
20	(0.65,0.23)	(0.60,0.27)	(0.00,0.67)	(0.53,0.31)
21	(0.00,0.67)	(0.05,0.63)	(0.77,0.15)	(0.00,0.67)
22	(0.58,0.28)	(0.19,0.54)	(0.66,0.23)	(0.27,0.49)
23	(0.09,0.61)	(0.01,0.66)	(0.40,0.40)	(0.04,0.64)
24	(0.65,0.23)	(0.60,0.27)	(0.00,0.67)	(0.53,0.31)
25	(0.65,0.23)	(0.53,0.31)	(0.14,0.57)	(0.45,0.37)
26	(0.67,0.22)	(0.58,0.28)	(0.09,0.61)	(0.77,0.15)
27	(0.81,0.13)	(0.60,0.27)	(0.21,0.53)	(0.77,0.15)
28	(0.65,0.23)	(0.60,0.27)	(0.01,0.66)	(0.45,0.37)
29	(0.20,0.53)	(0.11,0.59)	(0.33,0.45)	(0.11,0.59)
30	(0.37,0.42)	(0.20,0.53)	(0.36,0.43)	(0.11,0.59)
31	(0.37,0.42)	(0.23,0.51)	(0.40,0.40)	(0.10,0.60)
32	(0.66,0.23)	(0.27,0.49)	(0.34,0.44)	(0.11,0.59)
33	(0.79,0.14)	(0.34,0.44)	(0.80,0.13)	(0.22,0.52)
34	(0.37,0.42)	(0.40,0.40)	(1.00,0.00)	(0.28,0.48)
35	(0.37,0.42)	(0.22,0.52)	(0.38,0.41)	(0.09,0.61)
36	(0.37,0.42)	(0.32,0.45)	(0.59,0.27)	(0.13,0.58)
37	(0.80,0.13)	(0.46,0.36)	(0.75,0.17)	(0.23,0.51)
38	(0.32,0.45)	(0.22,0.52)	(0.38,0.41)	(0.09,0.61)
39	(0.75,0.17)	(0.61,0.26)	(0.22,0.52)	(0.59,0.27)
40	(0.32,0.45)	(0.22,0.52)	(0.38,0.41)	(0.13,0.58)
41	(0.82,0.12)	(0.44,0.37)	(0.31,0.46)	(0.20,0.53)
42	(0.37,0.42)	(0.19,0.54)	(0.36,0.43)	(0.13,0.58)
43	(0.59,0.27)	(0.27,0.49)	(0.57,0.29)	(0.13,0.58)
44	(0.56,0.29)	(0.20,0.53)	(0.47,0.35)	(0.13,0.58)
45	(0.56,0.29)	(0.20,0.53)	(0.47,0.35)	(0.13,0.58)
46	(0.58,0.28)	(0.22,0.52)	(0.35,0.43)	(0.13,0.58)
47	(0.63,0.25)	(0.40,0.40)	(0.57,0.29)	(0.20,0.53)
48	(0.35,0.43)	(0.21,0.53)	(0.40,0.40)	(0.13,0.58)
49	(0.50,0.33)	(0.34,0.44)	(0.50,0.33)	(0.12,0.59)
50	(0.65,0.23)	(0.37,0.42)	(0.31,0.46)	(0.14,0.57)
51	(0.36,0.43)	(0.27,0.49)	(0.36,0.43)	(0.12,0.59)
52	(0.13,0.58)	(0.00,0.67)	(0.35,0.43)	(0.04,0.64)
53	(0.26,0.49)	(0.16,0.56)	(0.37,0.42)	(0.09,0.61)
54	(0.17,0.55)	(0.07,0.62)	(0.41,0.39)	(0.08,0.61)
55	(0.37,0.42)	(0.31,0.46)	(0.42,0.39)	(0.12,0.59)
56	(0.70,0.20)	(0.49,0.34)	(0.57,0.29)	(0.23,0.51)
57	(0.21,0.53)	(0.25,0.50)	(0.51,0.33)	(0.09,0.61)
58	(0.60,0.27)	(0.45,0.37)	(0.54,0.31)	(0.20,0.53)
59	(0.55,0.30)	(0.22,0.52)	(0.32,0.45)	(0.14,0.57)
60	(0.37,0.42)	(0.16,0.56)	(0.37,0.42)	(0.08,0.61)

TABLE II: Intuitionistic Fuzzy Utility Values

$\boldsymbol{a}_i$	c_1	c_2	c_3	c_4	$\tilde{V}_i$
1	(0.37,0.55)	(0.41,0.47)	(0.01,0.97)	(0.14,0.81)	(0.25,0.67)
2	(0.27,0.63)	(0.24,0.60)	(0.02,0.96)	(0.03,0.91)	(0.15,0.76)
3	(0.59,0.35)	(1.00,0.00)	(0.06,0.92)	(0.18,0.77)	(1.00,0.00)
4	(0.23,0.67)	(0.21,0.62)	(0.02,0.96)	(0.03,0.91)	(0.13,0.78)
5	(0.50,0.43)	(0.72,0.22)	(0.01,0.96)	(0.17,0.78)	(0.42,0.52)
6	(0.26,0.64)	(0.24,0.60)	(0.02,0.96)	(0.03,0.91)	(0.14,0.76)
7	(1.00,0.00)	(0.46,0.42)	(0.02,0.95)	(0.09,0.85)	(1.00,0.00)
8	(0.33,0.58)	(0.26,0.58)	(0.02,0.96)	(0.03,0.91)	(0.17,0.74)
9	(0.25,0.65)	(0.52,0.38)	(0.00,0.97)	(0.17,0.78)	(0.26,0.66)
10	(0.45,0.47)	(0.33,0.53)	(0.02,0.96)	(0.03,0.91)	(0.23,0.68)
11	(0.40,0.52)	(0.41,0.46)	(0.02,0.96)	(0.18,0.77)	(0.27,0.65)
12	(0.44,0.49)	(0.63,0.29)	(0.02,0.95)	(0.06,0.89)	(0.34,0.59)
13	(0.27,0.63)	(0.48,0.41)	(0.00,0.98)	(1.00,0.00)	(1.00,0.00)
14	(0.07,0.80)	(0.04,0.75)	(0.03,0.95)	(0.01,0.93)	(0.04,0.85)
15	(0.10,0.78)	(0.08,0.72)	(0.02,0.95)	(0.01,0.93)	(0.05,0.84)
16	(0.30,0.61)	(0.44,0.44)	(0.00,0.98)	(0.03,0.91)	(0.21,0.70)
17	(0.00,0.87)	(0.03,0.76)	(0.08,0.89)	(0.00,0.94)	(0.03,0.86)
18	(0.26,0.64)	(0.12,0.69)	(0.06,0.91)	(0.05,0.90)	(0.13,0.78)
19	(0.03,0.84)	(0.01,0.78)	(0.03,0.95)	(0.01,0.94)	(0.02,0.87)
20	(0.31,0.60)	(0.42,0.46)	(0.00,0.98)	(0.11,0.84)	(0.23,0.69)
21	(0.00,0.87)	(0.03,0.76)	(0.08,0.89)	(0.00,0.94)	(0.03,0.86)
22	(0.26,0.64)	(0.12,0.69)	(0.06,0.91)	(0.05,0.9)	(0.13,0.78)
23	(0.03,0.84)	(0.01,0.78)	(0.03,0.95)	(0.01,0.94)	(0.02,0.87)
24	(0.31,0.60)	(0.42,0.46)	(0.00,0.98)	(0.11,0.84)	(0.23,0.69)
25	(0.31,0.60)	(0.37,0.50)	(0.01,0.97)	(0.09,0.86)	(0.21,0.71)
26	(0.32,0.59)	(0.41,0.46)	(0.01,0.97)	(0.20,0.75)	(0.25,0.67)
27	(0.44,0.49)	(0.42,0.45)	(0.01,0.96)	(0.20,0.75)	(0.29,0.63)
28	(0.30,0.60)	(0.42,0.45)	(0.00,0.98)	(0.09,0.86)	(0.22,0.69)
29	(0.08,0.80)	(0.07,0.73)	(0.02,0.95)	(0.02,0.92)	(0.05,0.85)
30	(0.15,0.74)	(0.13,0.68)	(0.03,0.95)	(0.02,0.92)	(0.08,0.81)
31	(0.15,0.74)	(0.15,0.67)	(0.03,0.95)	(0.02,0.93)	(0.09,0.81)
32	(0.31,0.60)	(0.17,0.65)	(0.02,0.95)	(0.02,0.92)	(0.14,0.76)
33	(0.42,0.50)	(0.22,0.61)	(0.09,0.89)	(0.04,0.91)	(0.21,0.70)
34	(0.15,0.74)	(0.26,0.58)	(1.00,0.00)	(0.05,0.90)	(1.00,0.00)
35	(0.15,0.74)	(0.14,0.68)	(0.03,0.95)	(0.01,0.93)	(0.08,0.82)
36	(0.15,0.74)	(0.21,0.62)	(0.05,0.92)	(0.02,0.92)	(0.11,0.79)
37	(0.43,0.49)	(0.31,0.54)	(0.08,0.90)	(0.04,0.90)	(0.23,0.68)
38	(0.12,0.76)	(0.14,0.68)	(0.03,0.95)	(0.01,0.93)	(0.08,0.82)
39	(0.38,0.53)	(0.43,0.45)	(0.02,0.96)	(0.13,0.82)	(0.26,0.66)
40	(0.12,0.76)	(0.14,0.68)	(0.03,0.95)	(0.02,0.92)	(0.08,0.82)
41	(0.45,0.47)	(0.30,0.55)	(0.02,0.95)	(0.03,0.91)	(0.22,0.69)
42	(0.15,0.74)	(0.12,0.69)	(0.03,0.95)	(0.02,0.92)	(0.08,0.82)
43	(0.27,0.63)	(0.18,0.65)	(0.05,0.93)	(0.02,0.92)	(0.14,0.77)
44	(0.25,0.65)	(0.12,0.69)	(0.04,0.94)	(0.02,0.92)	(0.11,0.79)
45	(0.25,0.65)	(0.12,0.69)	(0.04,0.94)	(0.02,0.92)	(0.11,0.79)
46	(0.26,0.64)	(0.14,0.68)	(0.03,0.95)	(0.02,0.92)	(0.12,0.79)
47	(0.30,0.61)	(0.26,0.58)	(0.05,0.93)	(0.03,0.91)	(0.17,0.74)
48	(0.14,0.75)	(0.13,0.68)	(0.03,0.95)	(0.02,0.92)	(0.08,0.82)
49	(0.22,0.68)	(0.22,0.61)	(0.04,0.94)	(0.02,0.92)	(0.13,0.77)
50	(0.31,0.60)	(0.24,0.59)	(0.02,0.95)	(0.02,0.92)	(0.16,0.75)
51	(0.15,0.74)	(0.17,0.65)	(0.03,0.95)	(0.02,0.92)	(0.10,0.81)
52	(0.05,0.83)	(0.00,0.78)	(0.03,0.95)	(0.01,0.94)	(0.02,0.87)
53	(0.10,0.78)	(0.10,0.70)	(0.03,0.95)	(0.01,0.93)	(0.06,0.83)
54	(0.06,0.81)	(0.04,0.75)	(0.03,0.95)	(0.01,0.93)	(0.04,0.86)
55	(0.15,0.74)	(0.20,0.63)	(0.03,0.95)	(0.02,0.92)	(0.10,0.80)
56	(0.35,0.57)	(0.33,0.52)	(0.05,0.93)	(0.04,0.90)	(0.21,0.71)
57	(0.08,0.80)	(0.16,0.66)	(0.04,0.94)	(0.01,0.93)	(0.07,0.82)
58	(0.27,0.63)	(0.30,0.55)	(0.05,0.93)	(0.03,0.91)	(0.17,0.74)
59	(0.25,0.65)	(0.14,0.67)	(0.02,0.95)	(0.02,0.92)	(0.11,0.79)
60	(0.15,0.74)	(0.10,0.70)	(0.03,0.95)	(0.01,0.93)	(0.07,0.82)

TABLE III: Alternative Ranks obtained with IF-MNL and MNL models

Rank	IF-MNL		MNL	
	$\boldsymbol{a}_i$ / i	P_i	$\boldsymbol{a}_i$ / i	P_i
1	3	0.0824	3	0.0300
2	7	0.0824	5	0.0265
3	13	0.0824	12	0.0237
4	34	0.0824	7	0.0235
5	5	0.0346	27	0.0222
6	12	0.028	13	0.0219
7	27	0.0239	11	0.0217
8	11	0.0223	39	0.0213
9	9	0.0214	9	0.0213
10	39	0.0214	26	0.0208
11	1	0.0206	1	0.0207
12	26	0.0206	20	0.0200
13	10	0.019	24	0.0200
14	20	0.019	28	0.0197
15	24	0.019	37	0.0194
16	37	0.019	10	0.0192
17	28	0.0181	25	0.0192
18	41	0.0181	16	0.0191
19	16	0.0173	56	0.0189
20	25	0.0173	41	0.0188
21	33	0.0173	33	0.0180
22	56	0.0173	58	0.0176
23	8	0.014	47	0.0174
24	47	0.014	8	0.0172
25	58	0.014	50	0.0168
26	50	0.0132	34	0.0164
27	2	0.0124	2	0.0164
28	6	0.0115	6	0.0162
29	32	0.0115	32	0.0158
30	43	0.0115	49	0.0158
31	4	0.0107	43	0.0157
32	18	0.0107	4	0.0157
33	22	0.0107	18	0.0153
34	49	0.0107	22	0.0153
35	46	0.0099	36	0.0150
36	36	0.0091	46	0.0149
37	44	0.0091	59	0.0148
38	45	0.0091	44	0.0147
39	59	0.0091	45	0.0147
40	51	0.0082	55	0.0147
41	55	0.0082	51	0.0143
42	31	0.0074	31	0.0139
43	30	0.0066	35	0.0138
44	35	0.0066	48	0.0138
45	38	0.0066	30	0.0137
46	40	0.0066	42	0.0137
47	42	0.0066	40	0.0136
48	48	0.0066	38	0.0135
49	57	0.0058	57	0.0134
50	60	0.0058	60	0.0133
51	53	0.0049	53	0.0129
52	15	0.0041	15	0.0126
53	29	0.0041	29	0.0123
54	14	0.0033	14	0.0118
55	54	0.0033	54	0.0118
56	17	0.0025	17	0.0111
57	21	0.0025	21	0.0111
58	19	0.0016	19	0.0110
59	23	0.0016	23	0.0110
60	52	0.0016	52	0.0110

Intuitively too, most DMs would not like to place much confidence in the fuzzy evaluations, in which they face a high degree of hesitancy. The effect of desirable (say high) membership degree is considerably dampened, if the associated hesitancy is high. These aspects of the real-life decision-making remain unconsidered in the conventional choice models, and thus IF-MNL model may be quite useful in practice.

V. Conclusions

We propose intuitionistic fuzzy multinomial logit (IF-MNL) model to give the choice probabilities of a DM, by taking into consideration the degree of satisfaction derived by the evaluating agent from the criteria values in terms of the intuitionistic fuzzy values. The proposed model is of significance in the real world decision making, where the criteria values are often not precisely known, rendering the conventional models of little applicability. Besides, the proposed model is especially useful in those applications, where various alternatives are evaluated against different criteria vaguely in terms of the satisfaction an alternative-criterion combination provide to a DM. In such situations, the presence of intuitionistic fuzzy values facilitates the representation of the individualistic degree of satisfaction that may significantly vary depending upon the evaluating agent's background, experience, or values.

It is important to note that the proposed IF-MNL model, like MNL model, also considers that the unobservable utility component is identically and independently distributed across the alternatives. In the real world, there might be such situations where this might not hold true. In this regard, the extensions of the proposed model as probit, nested logit, and mixed logit woulf be worthwhile. Besides, it would be interesting to empirically learn the intuitionistic fuzzy evaluations (of an agent) by fitting the proposed logit model to the agent's preference data, through emerging machine learning algorithms. The proposed models find application in the real world decision making probems under uncertainty, such as credit scoring analysis, supplier selection, consumer behaviour, and marketing strategy. These applications are kept for a future study.

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