Symmetry-noise interplay in quantum walk on an n-cycle

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arXiv:0803.4453v2 [quant-ph] 24 Nov 2008

Augmenting the unitary transformation which generates a quantum walk by a generalized phase gate G is a symmetry for both noisy and noiseless quantum walk on a line, in the sense that it leaves the position probability distribution invariant. However, this symmetry breaks down in the case of a quantum walk on an n-cycle, and hence can be regarded as a probe of the walk topology. Noise, modelled here as phase flip and generalized amplitude damping channels, tends to restore the symmetry because it classicalizes the walk. However, symmetry restoration happens even in the regime where the walker is not entirely classical, because noise also has the effect of desensitizing the operation G to the walk topology. We discuss methods for physical implementation, and talk about the wider implications to condensed matter systems.

I. INTRODUCTION

Classical random walks (CRWs) has found broad applications—from randomized algorithms to the understanding of condensed matter systems [1]. Unlike CRW, quantum walks (QWs) [2] involve a superposition of states, being unitary, thereby simultaneously exploring multiple possible paths of a walker, and the amplitudes corresponding to different paths are then interfered via a measurement to arrive at a probabilistic result. This makes QW spread quadratically faster than CRW. The quadratic advantage of QW, for instance, is exploited to speedup the spatial search variant of Grover's search algorithm [3]. A single-particle quantum lattice gas automata (QLGA) can also be shown to be equivalent to QW [4]. Experimental implementation of QW has been reported [5], and various other schemes have been proposed for its physical realization [6, 7], which has motivated us to study QW in an open quantum system [8, 9]: since environmental effects (noise) will necessarily tend to destroy the coherent superposition of states central to QW, thus transforming it to CRW.

This paper studies QW on an n-cycle, which is subjected to the following noise processes: the phase flip (decoherence without dissipation) [10], and generalized amplitude damping channels (decoherence with dissipation) [8, 11, 12, 13], which are of relevance to studies in quantum optics and condensed matter systems. The latter type of noise in the QW context has been studied by us in Ref. [8]. We study the transition of a noisy QW on an n-cycle to a CRW in a different way: we identify a certain symmetry operation (defined below) that is sensitive to the walk topology, in the sense that the symmetry holds for QW on a line but not for that on a cycle. The difference arises due to the fact that unlike QW in a line, the walk on an n-cycle involves interference between forward and backward propagating wavefunctions. Noise tends to restore symmetry both by classicalizing the walk and also desensitizing the symmetry operation as a topology probe for the QW.

This paper is organized as follows. In Sec. II, we define the discrete time QW on an n-cycle and introduce a generalized phase gate which is a symmetry operation and show the breakdown in symmetry when the QW is implemented on an n-cycle. In Sec. III, the effect of noise on the QW and its influence on the restoration of symmetry is discussed. The implication of this work to condensed matter systems is discussed in Sec. IV before conclusions are made in Sec. V.

II. QUANTUM WALK ON AN n-CYCLE WITH GENERALIZED PHASE GATE

Consider a particle (a qubit) which is executing a discrete time QW in one dimension, and its internal states $|0\rangle$ and $|1\rangle$ span $\mathcal{H}_c$, which is referred as the coin Hilbert space. The allowed position states of the particle are $|x\rangle_p$, which spans $\mathcal{H}_x$, where $x \in \mathbf{I}$, the set of integers (the subscript p in the ket is used to distinguish the position kets from the internal states, which are represented subscriptless). In an n-cycle walk, there are n allowed positions, and in addition the periodic boundary condition $|x\rangle_p = |x \bmod n\rangle_p$ is imposed. A t step coined QW is generated by iteratively applying a unitary operation W which acts on the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_x$:

$$|\psi_t\rangle = W^t|\psi_0\rangle \;, \tag{1}$$

where $|\psi_0\rangle = (\cos(\theta_0/2)|0\rangle + \sin(\theta_0/2)e^{i\phi_0}|1\rangle)|0\rangle_p$ is an arbitrary initial state of the particle and $W \equiv U\ B(\xi,\theta,\zeta)$. The $B(\xi,\theta,\zeta)$ is an arbitrary $SU(2)$ coin

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toss operation which acts on the coin space given by

$$B(\xi,\theta,\zeta) = \begin{pmatrix} e^{i\xi}\cos(\theta) & e^{i\zeta}\sin(\theta) \\ e^{-i\zeta}\sin(\theta) & -e^{-i\xi}\cos(\theta) \end{pmatrix}. \quad (2)$$

The matrix $B(\xi,\theta,\zeta)$, whose elements are written as B_{jk}, controls the evolution of the walk, with the Hadamard walk corresponding to $B(0^\circ, 45^\circ, 0^\circ)$. The U is a unitary controlled-shift operation:

$$U \equiv |0\rangle\langle 0| \otimes \sum_x |x-1\rangle_p \langle x|_p + |1\rangle\langle 1| \otimes \sum_x |x+1\rangle_p \langle x|_p. \quad (3)$$

The probability to find the particle at site x after t steps is given by

$$p(x,t) = \langle x|_p \mathrm{tr}_c(|\psi_t\rangle\langle\psi_t|)|x\rangle_p. \quad (4)$$

Now, given an element from the 2-parameter group

$$G(\alpha,\beta) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \quad (5)$$

which represents a generalized phase gate acting on the $\mathcal{H}_c$, we find that the operation $W \longrightarrow GW$ leaves the probability distribution $p(x,t)$ of the particle on the line invariant; hence the walk is symmetric under the operation

$$G(\alpha,\beta) : |j\rangle \mapsto e^{i(\overline{j}\alpha + j\beta)}|j\rangle \quad (6)$$

for $|j\rangle$ in the computational basis (eigenstates of the Pauli operator σ_z) and $j = 0, 1$. The physical significance of G is that it helps identify a family of QWs that are equivalent from the viewpoint of physical implementation, which can sometimes allow a significant practical simplification [8]. For example, suppose the application of the conditional shift is accompanied by a phase gate. The walk symmetry implies that this gate need not be corrected for, thereby resulting in a saving of experimental resources. The inclusion of a phase gate on the coin operator is equivalent to a phase gate at each lattice site in the sense of QLGA, with the physical meaning of a constant potential. The evolution rules for single-particle QLGA can be classified into gauge equivalent classes, there being a difference between the class of rules for periodic (n-cycle) and non-periodic 1-dimensional lattice and this feature can be exploited to distinguish between these two spatial topologies [14].

It turns out that in the case of QW on an n-cycle, this symmetry breaks down. To see this, we note that the t-fold application of the operation GUB on a particle with initial state $|\psi_0\rangle$ on the line and on an $n-$cycle produces, respectively, the states

$$(GUB)^t|\psi_0\rangle = \sum_{j_1,j_2,\cdots,j_t} e^{i(\overline{J}_t\alpha + J_t\beta)} B_{j_t,j_{t-1}} \cdots B_{j_2,j_1}(B_{j_1 0}a + B_{j_1 1}b)|j_t\rangle|2J_t - t\rangle_p, \quad (7a)$$

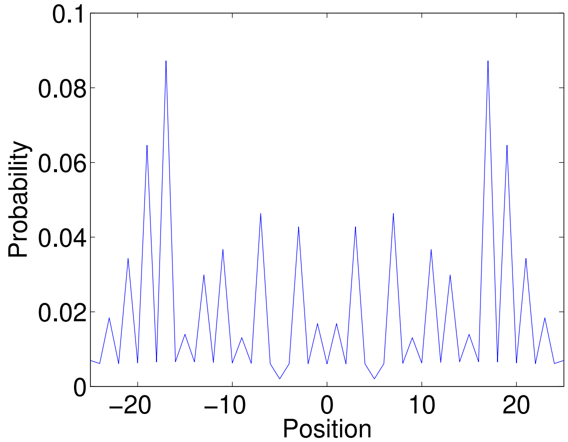
$$(GUB)^t|\psi_0\rangle = \sum_{j_1,j_2,\cdots,j_t} e^{i(\overline{J}_t\alpha + J_t\beta)} B_{j_t,j_{t-1}} \cdots B_{j_2,j_1}(B_{j_1 0}a + B_{j_1 1}b)|j_t\rangle|2J_t - t \mod n\rangle_p, \quad (7b)$$

where $J_t = j_1 + j_2 + \cdots + j_t$ and $\overline{J}_t$ is a bitwise complement of J_t. All terms in superposition (7a) contributing to the probability to detect the walker at a given position $x = 2J_t - t$ have the *same* phase factor, $e^{i(\overline{J}_t\alpha + J_t\beta)}$, which is fixed by $J_t = (x+t)/2$ (where, it may be noted, x and t are both even or both odd). Thus, this factor does not affect the probability to detect the walker at x, whence the symmetry. In the case of QW on an n-cycle the breakdown of the symmetry [8], see Fig. 1, can be attributed to the topology of the cycle, which introduces a periodicity in the walker position (determined by a congruence relation with modulus given by the number of sites), but not in the phase of the superposition terms. As a result, fixing x fixes $J_t \mod n = (x+t)/2 \mod n$, but not J_t itself, so that the phase terms in the superposition Eq. (7b) do not factor out globally. Thus if α or β is non-vanishing and $\alpha \neq \beta$, then in general the symmetry G is absent in the cyclic case.

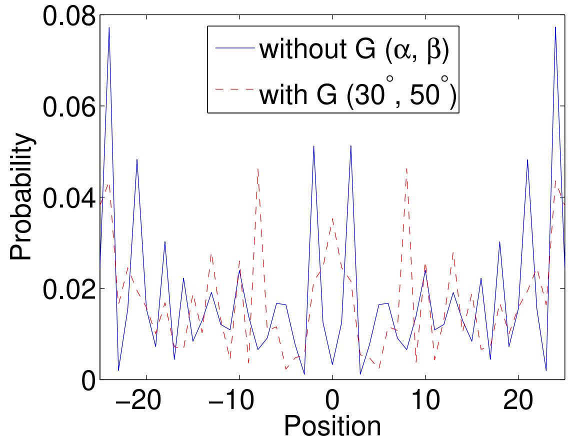
We quantify the breakdown in symmetry by means of the Kolmogorov distance (or, trace distance [12]), given by

$$d(t) = (1/2)\sum_x |p(x,t) - q(x,t)|, \quad (8)$$

between the walker position distributions obtained without and with the symmetry operation, given by $p(x,t)$ and $q(x,t)$, respectively. The breakdown in symmetry for a noiseless cyclic QW is depicted by the bold curve in Fig. 2 as a function of the number of turns τ (where $t = \tau s$, with $n = 2s+1$).



(a)



(b)

FIG. 1: (Color online) Position probability distribution for a Hadamard walk, $B(0°, 45°, 0°)$ on (a) a line, and (b) an n-cycle, with $(n = 51)$ and initial state $(1/\sqrt{2})(|0\rangle + |1\rangle)$, for the unitary case $(\gamma_0 = 0)$. Each figure presents the distribution with and without being subjected to the phase operation $G(30°, 50°)$. In (a), there is perfect symmetry, since both distributions coincide. In (b) the two plots do not overlap, indicating the breakdown of the symmetry.

III. EFFECT OF NOISE AND SYMMETRY RESTORATION

We now describe the n-cycle quantum walk on the particle, a two level system, when subjected to noise. The situation is modeled as an interaction with a thermal bath, characterized by phase damping [8, 12] or a generalized amplitude damping channel, the latter process being represented by the following Kraus operators [13]:

$$E_0 \equiv \sqrt{\kappa}\begin{bmatrix} \sqrt{1-\lambda(\Delta)} & 0 \\ 0 & 1 \end{bmatrix}; \qquad E_1 \equiv \sqrt{\kappa}\begin{bmatrix} 0 & 0 \\ \sqrt{\lambda(\Delta)} & 0 \end{bmatrix};$$
$$E_2 \equiv \sqrt{1-\kappa}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda(\Delta)} \end{bmatrix}; \; E_3 \equiv \sqrt{\frac{1-\kappa}{\kappa}}E_1^\dagger,$$

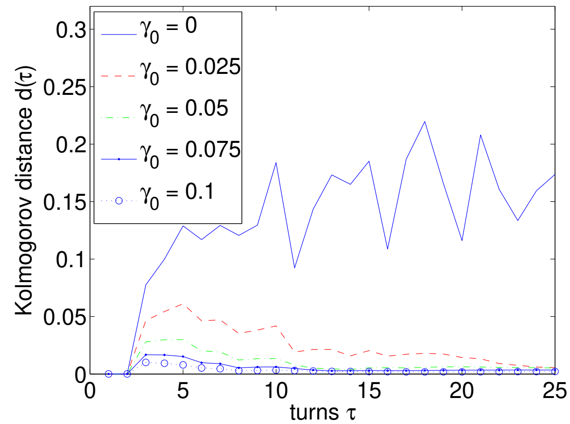


FIG. 2: (Color online) Kolmogorov distance $d(\tau)$ against the number of turns (τ) of the cyclic QW in the noiseless and noisy case with $n = 51$. For the unitary case ($\lambda = 0$; bold line) the walk becomes increasingly asymmetric as the number of turns is increased, until about 7–10 turns, after which it fluctuates around $d \approx 0.15$. The plots represent generalized amplitude damping noise at different noise levels at temperature $T = 3.5$ (in units where $\hbar \equiv k_B \equiv 1$), $\Delta = 0.1$ and $\theta = 30°$. The walker is evolved with the initial state parameters (in degrees) $\theta_0 = 30°$, $\phi_0 = 40°$, with $B(20°, 10°, 30°)$ and $G(40°, 50°)$.

where $0 \leq \kappa \leq 1$, $\lambda(\Delta) = 1 - e^{-\gamma_0(2N_{\rm th}+1)\Delta}$, and $\kappa \equiv \frac{N_{\rm th}+1}{2N_{\rm th}+1}$, with $N_{\rm th} \equiv (\exp(\hbar\omega/k_BT) - 1)^{-1}$, T is temperature, γ_0 is a measure of the strength of coupling to the environment, and Δ is the duration for which the environment is modeled to interact with the coin. The density operator ρ_c of the coin evolves according to $\rho_c \to \sum_j E_j \rho_c E_j^\dagger$. The full evolution of the walker, described by density operator $\rho(t)$, is given by $\sum_j E_j(W\rho(t-1)W^\dagger)E_j^\dagger$, where the E_j's are understood to act only in the coin space.

The curves in Fig. 2 plot $d(\tau)$ as a function of turns in the case of unitary and noisy QW (parametrized by γ_0), and demonstrate the gradual restoration of symmetry with time on account of the noise. Although the figure employs generalized amplitude damping noise, qualitatively the same behavior can be seen for a phase damping noise. Here, a general feature is that $d(\tau)$ is non-zero when $\tau < 2$, being then equivalent to (noisy) QW on a line. Thereafter, $d(\tau)$ at first increases with increasing turns, being dominated by unitary evolution, and eventually falls down, being dominated by noise. It is observed that for sufficiently low noise levels, the time at which this turnover in slope happens remains constant, for given θ. This is depicted in Fig. 2 for the case of a generalized amplitude damping channel corresponding to a fixed temperature and varying γ_0. However, we note that for strong enough noise, the turnover happens earlier.

Typical noisy probability distributions are depicted in Fig. 3(a),(b) at an instant where the symmetry has been almost fully restored even when the walk is well within

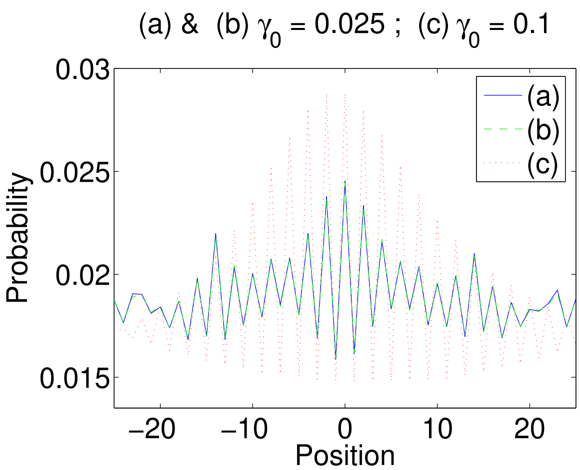


FIG. 3: (Color online) Position probability distribution for a Hadamard walk $B(0, 45°, 0)$ on an n-cycle ($n = 51$) with initial state $(1/\sqrt{2})(|0\rangle + |1\rangle)$ when subjected to generalized amplitude damping noise with $\Delta = 0.1$ and finite T $(= 6.0)$ for $\tau = 11$. (a) and (b) $\gamma_0 = 0.025$, (b) is the distribution for the QW augmented by operation $G(30°, 50°)$; note the walk remains quantum yet with symmetry almost completely restored; (c) classicalized pattern (indicated by the regular envelope) obtained with larger noise level corresponding to $\gamma_0 = 0.1$.

the quantum regime. Fig. 3(c) represents a classicalized distribution, indicated by the regular envelope (that will eventually turn into a uniform distribution).

We define coherence $\mathbf{C}$ as the sum of the off-diagonal terms of states in $\mathcal{H}_c \otimes \mathcal{H}_p$, where $\mathcal{H}_c$ and $\mathcal{H}_p$ are the internal and position Hilbert space of the quantum walker, respectively. If the state of the quantum walker is $\rho = \sum_{ab;jk} \alpha_{ab;jk} |a\rangle|j\rangle_p\langle b|\langle k|_p$, where $|a\rangle$, $|b\rangle \in \mathcal{H}_c$ and $|j\rangle_p$, $|k\rangle_p \in \mathcal{H}_p$, then

$$\mathbf{C} \equiv \left(\sum_{a\neq b, j=k} + \sum_{a,b,j\neq k} + \sum_{a\neq b, j\neq k} \right) |\alpha_{ab;jk}|, \qquad (9)$$

the sum of the absolute values of all off-diagonal terms of ρ in the computational-position basis. The coherence function is defined as the quantity $C(m)$, where $m \in \{1, 2, \cdots, M\}$, obtained by partitioning $\mathbf{C}$ into M intervals of size s/M, such that for the mth interval $(m-1)(s/M) \leq |j-k| < m(s/M)$. Physically, $C(m)$ is a measure of coherence between two points on a (in general, noisy) quantum walker, as a function of their mutual separation. Let $C_0(m)$ represent the coherence function of the corresponding noiseless walk. At any turn τ, we define the normalized coherence function by $c(m) \equiv C(m)/C_0(m)$, and, analogously, normalized Kolmogorov distance by $D(\tau) \equiv d(\tau)/d_0(\tau)$.

Since noise tends to destroy superpositions, and the breakdown in symmetry is essentially a phenomenon of superposition of the forward and backward waves, noise tends to restore symmetry, as seen from Fig. 3. This is brought out by Fig. 4 for two possible values of G. In the figure, in spite of its considerable spikiness, the bold curve, representing $c(m = M)$, shows an overall fall. A similar trend as depicted in this figure, has been numerically checked for various other values of G. This raises the question whether symmetry restoration of the cyclic QW can be considered as a good indicator of classicalization. Here we note that from Fig. 3(a),(b) the probability distribution pattern is seen to be clearly quantum, even though symmetry has been almost fully restored. This is suggestive of the notion that that symmetry tends to be restored *even* in the regime where the walk still possesses some quantum features.

The reason $D(t)$ is not a faithful indicator of classicalization of the walk has to do with the effect of noise on the sensitivity of the symmetry operation $G(\alpha, \beta)$ to the topology of the path. Since this operation senses the closure of the path through the superposition of the forward and backward waves, the suppression of superposition through noise will also have the effect of desensitizing the operation to the closure of the path, thereby moving the noisy cyclic QW towards a noisy QW on a line from the perspective of this operation, before further classicalization transforms it into a cyclic CRW. And as shown in Ref. [8], all the above symmetries are respected by a (noisy) quantum walk on a line, both in the case of phase damping noise, which in NMR nomenclature [15] is called a T_2 process, and generalized amplitude damping, which is a T_1, T_2 process [15]. This brings out the point that decoherence (T_2 process) is the principal mechanism responsible for the restoration of symmetries. It also highlights the interplay between topology and noise in a quantum walk on an $n-$cycle. A similar interplay may be expected also in the case of QW with other nontrivial topologies.

The experimental study of the decoherence and decay of quantum states of a trapped atomic ion's harmonic motion subjected to engineered reservoirs, both of the phase damping and amplitude damping kind, have been reported in [11]. The phase reservoir is simulated by random variation of the trap frequency ω without changing its energy (non-dissipative), while the amplitude reservoir is simulated by random electric field along the axis of the trap (dissipative). Coupling the reservoirs reported in Ref. [11] to the scheme presenting the combination of pulses required to implement a QW on a line and on a cycle in an ion trap [6] provides a convenient set up to demonstrate the symmetry-noise interplay.

The interplay between geometry and decoherence has been noted before in the case of delocalized bath modes [16], as against localized bath modes [16, 17]. This is of relevance as the noise processes considered here [8, 13] are described by the interaction of the system with delocalized bath modes.

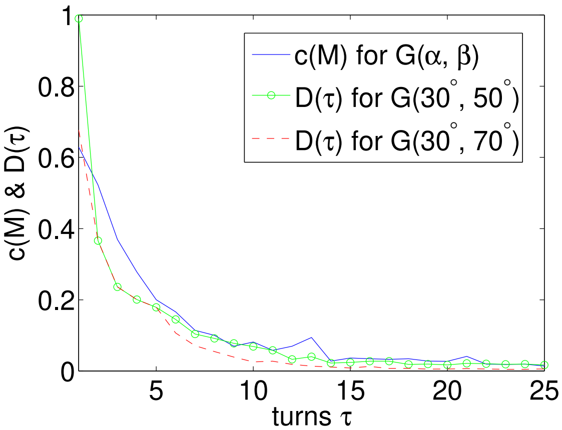


FIG. 4: (Color online) (Color online) Normalized coherence function $c(M)$ (bold) and the normalized Kolmogorov distance $D(\tau)$ (dashed) for a cyclic QW as a function of turns τ. An overall reduction of both $c(M)$ and $D(\tau)$ with time is seen for the generalized amplitude damping noise characterized by temperature $T = 3.5$ and $\Delta = 0.1$. The Hadamard walk is evolved with the initial state parameters (in degrees) $\theta_0 = 45^\circ$, $\phi_0 = 0^\circ$, and $G(\alpha, \beta)$. The line-with-circle plot represents $D(\tau)$ with $G(30^\circ, 50^\circ)$ and dashed line represents for $G(30^\circ, 70^\circ)$. The $c(M)$ (solid line) remains roughly the same for both pairs of α and β. For a clear depiction of the notion of symmetry restoration in the walk even in the quantum regime, cf. Figure 3.

IV. IMPLICATIONS TO CONDENSED MATTER SYSTEMS

Breaking of the symmetry due to the change in walk topology causes long-range correlations to develop, in analogy to the hydrodynamics of ordered systems such as spin waves in: ferromagnets, antiferromagnets (where it is the spin wave of staggered magnetization), second sound in He^3, nematic liquid crystals [18]. Here the correlations may be identified with symmetry-broken terms (whose measurement probability depends on α or β in the walk augmented by $G(\alpha, \beta)$) in the superposition of the quantum walker. One finds that correlations are set up rapidly over large distances with increase in the winding of the walker, until symmetry is broken throughout the cycle. However, as noted above, the randomization produced by noise causes the reappearance of symmetries. The symmetry breaking and the symmetry restoring agents are thus different, the former given by the topological transition from a line to an n-cycle, the latter being the noise-induced randomization.

Coherence is also widely used to understand quantum phase transitions, the transition from superfluid to Mott insulator state in an optical lattice being one specific example [19]. In Ref. [20] the quantum phase transition using QW in a one dimensional optical lattice has been discussed. Using various lattice techniques, desired geometries to trap and manipulate atoms can be created. In most physical situations one deals with closed geometries. The characteristics of the $n-$cycle walk, in particular the re-appearance of the symmetry (implying a family of implementationally equivalent noisy cyclic QWs) while still in the quantum regime, presented here could be of direct relevance to such situations.

The ubiquity of the ideas developed in this paper can be seen from the fact that the quantum dynamics of a particle on a ring (cycle) subject to decoherence along with dissipation finds its place in the physics of quantum dots. The effective action of a quantum dot accounting for the joint effect of charging and coupling to an environment [21] mirrors the behavior of the quantum dynamics of a particle on a ring (cycle) subject to a dissipative damping mechanism describing the dissipation of the energy stored in dynamic voltage fluctuations into the microscopic degrees of freedom of the quasi-particle continuum. In the absence of dissipation, the action describes the ballistic motion of a quantum particle on a ring. The ring topology reflects the 2π-periodicity of the quantum phase, which is in turn related to the quantization of charge, thereby highlighting the point that the main source of charge quantization phenomena, in the approach developed in [21], is the periodicity, of the relevant variable, due to the ring topology. With the increase in the effect of dissipation, the particle begins to forget its ring topology (full traversal of the ring become increasingly unlikely), leading to a suppression of charge quantization phenomena. This behavior is similar to that seen here for the case of quantum walk on a cycle, where with an increase in the effect of the environment, i.e., with increasing noise, the walker in unable to perceive the cyclic structure of the walk space. That the topology-noise interplay studied here has an impact on a concrete condensed matter system, viz. the crossover from strong to weak charge quantization in a dissipative quantum dot, highlights the generality and scope of these ideas.

V. CONCLUSION

We conclude that the symmetry-topology-noise interplay presented here would be of relevance to quantum information processing systems, and have wider implications to the implementation of quantum walks to condensed matter systems.

Acknowledgements

We thank David Meyer for helpful comments. PR would like to thank Manoj Toshniwal for financial support. CMC is thankful to Mike and Ophelia Lazaridis for financial support at IQC.

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