Some Reliability Properties of Transformed-Transformer Family of Distributions

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Abstract

The Transformed-Transformer family of distributions are the resulting family of distributions as transformed from a random variable T through another transformer random variable X using a weight function ω of the cumulative distribution function of X. In this paper, we study different stochastic ageing properties, as well as different stochastic orderings of this family of distributions. We discuss the results with several well known distributions.

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1 Introduction

Different probability distributions have been developed in the literature by different researcher to give more flexibility in modelling and data analysis. Sometimes these developments are data-driven, and sometimes they are theory-driven. Most of the methods developed to generate new family of distributions generally fall into two categories - combining existing distributions into new distributions and adding new parameters to an existing distribution. Although, in general, more number of parameters in a probability distribution gives rise to more flexibility in modelling, after sometime the increase in the number of parameters leads to quite marginal improvement as far as flexibility in the analysis is concerned. It is observed that although at least three parameters are required for a probability distribution to have some practical

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usefulness, the optimum number of parameters is four in the sense that the increase in the number of parameters beyond four does not give any substantial improvement (Johnson et al., 1994; Alzaatreh et al., 2013a). System of probability distributions developed by Karl Pearson (Elderton and Johnson, 1969), and Burr family of distributions developed by Burr (1942) are two such well known family of distributions. Though generalized lamda distribution (Ramberg and Schmeiser, 1974; Ramberg et al., 1979), skew-normal family of distributions (Azzalini, 1985), beta-generated distributions (Akinsete et al., 2012; Alshawarbeh et al., 2012; Barreto-Souza et al., 2011, 2010; Cordeiro et al., 2013; Eugene et al., 2002; Famoye et al., 2005; Kong et al., 2007; Lee et al., 2013; Nadarajah and Kotz, 2006), Kumaraswamy-generated distribution (Jones, 2009; Cordeiro and de Castro, 2011) among others are the recent addition in the family of distributions, a more recent development is the Transformed-Transformer family of distributions studied by Alzaatreh et al. (2013a,b, 2014), who call it T-X family of distributions. This family of distributions is generated by transformation from a random variable T through another random variable X using weight function ω of the cumulative distribution function of X. This is the reason to call it Transformed-Transformer family of distributions. Different choices of T, X and ω lead to different families of distributions. Below we give a brief discussion on this.

Let T be an absolutely continuous random variable with support $[a, b]$, where $-\infty < a < b < \infty$ and let X be another random variable with support $[c, d]$, where $-\infty < c < d < \infty$. Further, let $\omega_1 : [0, 1) \to [a, b]$ be a continuous function such that

(*i*) $\omega_1(\cdot)$ is differentiable and monotonically increasing;

(*ii*) $\omega_1(0) = a$ and $\lim_{x \to 1-} \omega_1(x) = b$.

For a random variable Z, we denote the probability density function (p.d.f.) of Z by f_Z with cumulative distributive function (c.d.f.) F_Z and survival function $\bar{F}_Z$. Then the c.d.f. $F(\cdot)$ of the Transformed-Transformer family of distributions is defined, for $x \in [c, d]$, as

$$\begin{aligned} F(x) &= \int_{a}^{\omega_1(F_X(x))} f_T(u)du \\ &= F_T[\omega_1(F_X(x))]. \end{aligned} \tag{1.1}$$

Let the corresponding random variable be denoted by R. The reliability function of R is given by

$$\begin{aligned} \bar{F}(x) &= \int_{\omega_1(F_X(x))}^{b} f_T(u)du \\ &= \bar{F}_T[\omega_1(F_X(x))] \\ &= \bar{F}_T[\omega_2(\bar{F}_X(x))], \end{aligned} \tag{1.2}$$

where $\omega_2(x) = \omega_1(1-x)$.

Alzaatreh et al. (2013a) obtained different distributions for different choice of the distributions of T and X based on different weight function $\omega(\cdot)$. It is interesting to note (Alzaatreh et al., 2014) that for any random variable with support in (a,b), $\omega(\cdot)$ can be taken as the quantile function of distribution of that random variable.

One may notice that a large number of distributions appear as special cases of the Transformed-Transformer family of distributions, viz. beta-generated family of distributions, exponential, gamma, Weibull, Lomax, Rayleigh, generalized gamma, exponentiated-Weibull, gamma-Pareto, exponentiated-exponential and many more distributions. Different properties of distributions have been separately studied in the literature by different researchers. The Transformed-Transformer family of distributions, being quite general, the properties studied here will hold true for all those distributions which are members of this family (as particular cases).

In this article we study different reliability properties of the Transformed-Transformer family of distributions taking a general weight function ω_1 or ω_2. The paper is organized as follows. Different stochastic ageing properties of this family of distributions are studied in Section 2. Section 3 deals with different stochastic orders. We give a brief conclusion in Section 4.

For any function g, we write $g'(x)$ to denote the first order derivative of g with respect to x. The word increasing and decreasing are not used in strict sense.

2 Stochastic Ageing properties

In this section we throw some light on how different ageing properties of T and X are transmitted to the random variable R through the weight function ω_2 (or ω_1).

Below we see that, under certain condition on ω_2, the IFR (increasing failure rate) property of T and X is transmitted to the random variable R. It is to be mentioned here that a random variable Z is said to be IFR if the failure rate function defined as the ratio of the p.d.f. to its survival function, is increasing, i.e., if $r_Z(t) = f_Z(t)/\bar{F}_Z(t)$ is increasing in t.

Theorem 2.1 *Let $x\omega_2'(x)$ be increasing in $x \in (0,1]$. If X and T are IFR then R is IFR.*

Proof: From (1.2), the density function $f(\cdot)$ of R is given by

$$f(x) = f_T[\omega_2(\bar{F}_X(x))]\frac{d}{dx}\left(\omega_2(\bar{F}_X(x))\right),$$

which gives the corresponding failure rate as

$$r(x) = -r_T[\omega_2(\bar{F}_X(x))]r_X(x)\bar{F}_X(x)\omega_2'(\bar{F}_X(x)).$$

Since T is IFR, we have that $r_T[\omega_2(\bar{F}_X(x))]$ is increasing in $x \in [c,d]$. Thus, to prove that R is IFR, it suffices to show that

$$r_X(x)\bar{F}_X(x)\left(-\omega_2'(\bar{F}_X(x))\right) \text{ is increasing in } x \in [c,d]. \tag{2.1}$$

Since X is IFR, we have that $r_X(x)$ is increasing in $x \in [c, d]$. Thus, (2.1) holds if

$$\bar{F}_X(x)\omega_2'(\bar{F}_X(x)) \text{ is decreasing in } x \in [c, d],$$

or equivalently,

$$x\omega_2'(x) \text{ is increasing in } x \in (0, 1].$$

Hence, the result follows.

Remark 2.1 It is easy to see that $x\omega_2'(x)$ is increasing in $x \in (0, 1]$ if, and only if, $(1-x)\omega_1'(x)$ is increasing in $x \in [0, 1)$.

Remark 2.2 It is to be noted that (i) $\omega_2(x) = -\ln x$, (ii) $\omega_2(x) = (1-x)^\alpha/x^{\alpha-1}$, for $\alpha \geq 1$, (iii) $\omega_2(x) = (1-x)/x$, (iv) $\omega_2(x) = (1-x^2)/x$, (v) $\omega_2(x) = (1-x)^2/(1-(1-x)^2)$ satisfy the condition given in Theorem 2.1.

Following counterexample shows that the condition '$x\omega_2'(x)$ is increasing in $x \in (0, 1]$' in the above theorem cannot be dropped.

Counterexample 2.1 Take $\omega_2(x) = (1-x^3)/\sqrt{x}$, $x \in (0, 1]$ so that $x\omega_2'(x)$ is neither increasing nor decreasing in $x \in (0, 1]$. Let X follow Weibull distribution with distribution function $F_X(x) = 1 - e^{-(x/\beta)^k}$, $x \geq 0$, $\beta > 0$, $k \geq 1$, and let T follow exponential distribution with distribution function $F_T(x) = 1 - e^{-\gamma x}$, $x \geq 0$, $\gamma > 0$. Then the failure rate function of the random variable R is given by

$$r(x) = \frac{\gamma k x^{k-1}(5e^{-3(x/\beta)^k+1})}{2\beta^k\sqrt{e^{-(x/\beta)^k}}}.$$

Now, it can be easily verified that $r(x)$ is neither increasing nor decreasing in $x \geq 0$, for $\gamma = 2$, $\beta = 0.4$, $k = 2$. This shows that R is not IFR. □

Using Theorem 2.1 and different $\omega_2(x)$ as given in Remark 2.2, one can generate large number of IFR distributions taking different T and X having IFR property. Below we discuss some distributions where IFR property of gamma distribution is transmitted to R.

Example 2.1 Take $\omega_1(x) = -\ln(1-x)$ for $x \in [0, 1)$, i.e. $\omega_2(x) = -\ln\ x$ for $x \in (0, 1]$. Let T follow gamma distribution with p.d.f. given by

$$f_T(t) = \frac{1}{\Gamma(\alpha)\lambda^\alpha} t^{\alpha-1} e^{-t/\lambda},\ t > 0,\ \alpha, \lambda > 0. \tag{2.2}$$

Note that T is IFR if $\alpha \geq 1$. From (1.1), we have the p.d.f. of R as

$$\begin{aligned} f(x) &= \frac{f_X(x)}{1 - F_X(x)} f_T[-\ln(1 - F_X(x))] \\ &= \frac{1}{\Gamma(\alpha)\lambda^\alpha} f_X(x)[-\ln(1 - F_X(x))]^{\alpha-1}(1 - F_X(x))^{\frac{1}{\lambda}-1}. \end{aligned} \tag{2.3}$$

For different distributions of X, using Theorem 2.1, we get the IFR property of the gamma-X family of distributions with p.d.f. given in (2.3), as presented in Table 1.

We can generate more gamma-X family of distributions with IFR property by taking different $\omega_2(x)$. For instance, taking T to be a gamma random variable with p.d.f. as given in (2.2) with $\alpha \geq 1$, and X, a Weibull distribution with $k \geq 1$ as given in Table 1, we present different gamma-Weibull family of distributions satisfying IFR property for different $\omega_2(x)$ in Table 2. One can similarly generate gamma-exponential, gamma-half normal, gamma-Gompertz, gamma-Makeham, gamma-Rayleigh family of distributions each of which is IFR. The newly generated distributions can be seen as a generalization of the many well-known distributions. For example, consider the gamma-Weibull distribution generated by using $\omega_2(x) = -\ln x$ with p.d.f. given in Table 2. Then putting $\lambda = 1$ we get the generalized gamma distribution discussed by Stacy (1962) (also see Khodabin and Ahmadabadi, 2010); for $\beta = k = 1$ or $\lambda = k = 1$, we get gamma distribution; for $k = 2$, $\alpha = 1/2$, $\lambda = 1$, setting $\beta^2 = 2\sigma^2$ we get half-normal distribution; for $k = 2$, $\alpha = \lambda = 1$, setting $\beta^2 = 2\sigma^2$ we get Rayliegh distribution. □

Table 1: Ageing property of gamma-X family of distributions

Distribution of X	Ageing property of R
Exponential $F_X(x) = 1 - e^{-\beta x}$, $x \geq 0$, $\beta > 0$	IFR for $\alpha \geq 1$
Weibull $F_X(x) = 1 - e^{-(x/\beta)^k}$, $x \geq 0$, $k, \beta > 0$	IFR for $\alpha \geq 1$, $k \geq 1$
Half Normal⋆ $F_X(x) = erf(\frac{x}{\sigma\sqrt{2}})$, $x > 0$, $\sigma > 0$	IFR for $\alpha \geq 1$
Gompertz $F_X(x) = 1 - e^{-B(c^x-1)/\ln c}$, $x, c \geq 0$, $B > 0$	IFR for $\alpha \geq 1$, $c \geq 1$
Makeham $F_X(x) = 1 - e^{[-\beta x + (\gamma/\eta)(e^{\eta x}-1)]}$, $x \geq 0$, $\beta, \gamma, \eta > 0$	IFR for $\alpha \geq 1$
Rayleigh $F_X(x) = 1 - e^{-x^2/2\sigma^2}$, $x \geq 0$	IFR for $\alpha \geq 1$

⋆ $erf(\cdot)$ is the error function defined as $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Below we see how NBU (new better than used) property of T and X is transmitted to the random variable R. It is to be noted that a random variable Z is said to be NBU if $\bar{F}_Z(x+t) \leq \bar{F}_Z(x)\bar{F}_Z(t)$, for all x, t.

Theorem 2.2 *Let $\omega_2(xy) \geq \omega_2(x) + \omega_2(y)$ for all $x, y \in (0, 1]$. If X and T are NBU then R is NBU.*

Table 2: Gamma-Weibull family of distributions having IFR property

$\omega_2(x)$	p.d.f. of gamma-Weibull family of distributions with $\alpha \geq 1$, $k \geq 1$
$-\ln x$	$\frac{k}{\Gamma(\alpha)\lambda^\alpha\beta}\left(\frac{x}{\beta}\right)^{k\alpha-1} e^{-\frac{1}{\lambda}\left(\frac{x}{\beta}\right)^k}$
$(1-x)/x$	$\frac{k}{\Gamma(\alpha)\lambda^\alpha\beta} e^{\left(\frac{x}{\beta}\right)^k}\left(\frac{x}{\beta}\right)^{k-1}\left(e^{\left(\frac{x}{\beta}\right)^k}-1\right)^{\alpha-1} e^{-\frac{1}{\lambda}\left(e^{\left(\frac{x}{\beta}\right)^k}-1\right)}$
$(1-x)^2/x$	$\frac{k\left(2-e^{-\left(\frac{x}{\beta}\right)^k}\right)}{\Gamma(\alpha)\lambda^\alpha\beta}\left(\frac{x}{\beta}\right)^{k-1}\left(\frac{1-e^{-\left(\frac{x}{\beta}\right)^k}}{\sqrt{e^{-\left(\frac{x}{\beta}\right)^k}}}\right)^{2(\alpha-1)} e^{-\frac{1}{\lambda}\left(\frac{1-e^{-\left(\frac{x}{\beta}\right)^k}}{\sqrt{e^{-\left(\frac{x}{\beta}\right)^k}}}\right)^2}$
$(1-x^2)/x$	$\frac{k\left(e^{\left(\frac{x}{\beta}\right)^k}+e^{-\left(\frac{x}{\beta}\right)^k}\right)}{\Gamma(\alpha)\lambda^\alpha\beta}\left(\frac{x}{\beta}\right)^{k-1}\left(e^{\left(\frac{x}{\beta}\right)^k}-e^{-\left(\frac{x}{\beta}\right)^k}\right)^{\alpha-1} e^{-\frac{1}{\lambda}\left(e^{\left(\frac{x}{\beta}\right)^k}-e^{-\left(\frac{x}{\beta}\right)^k}\right)}$

Proof: Since X is NBU, we have, for all $x, t \in [c, d]$,

$$\omega_2(\bar{F}_X(x+t)) \quad \geq \quad \omega_2(\bar{F}_X(x)) + \omega_2(\bar{F}_X(t)),$$

which, on using the fact that T is NBU, gives

$$\bar{F}_T\left(\omega_2(\bar{F}_X(x+t))\right) \quad \leq \quad \bar{F}_T\left(\omega_2(\bar{F}_X(x))\right)\bar{F}_T\left(\omega_2(\bar{F}_X(t))\right).$$

Hence, the result follows.

Remark 2.3 The condition of Theorem 2.2 is satisfied by (i) $\omega_2(x) = -\ln x$, (ii) $\omega_2(x) = (1-x)/x$, (iii) $\omega_2(x) = (1-x)^2/x$.

Remark 2.4 Since IFR implies NBU, the distribution of R given in Table 1 and Table 2 are NBU. Consider X having the following distribution.

$$F_X(x) = 1 - e^{-h(x)},\ x \geq 0,$$

where

$$h(x) = \begin{cases} \sin^2 x, & \text{if } x \in (0, \frac{\pi}{2}]; \\ \frac{1}{2}\pi(x - \frac{1}{2}\pi) + 1, & \text{if } x \in (\frac{\pi}{2}, \infty). \end{cases}$$

Then X is NBU (but not IFR). So, if T is NBU, then the resulting family of distributions is NBU, for any $\omega_2(x)$ given in Remark 2.3. For instance, if T follows gamma distribution with density function as in (2.2) with $\alpha \geq 1$, then, for $\omega_2(x) = -\ln x$, we have a NBU distribution with p.d.f. $f_X(x) = \frac{1}{\Gamma(\alpha)\lambda^\alpha}(h(x))^{\alpha-1}h'(x)e^{-\frac{1}{\lambda}h(x)}$. □

The following counterexample shows that the condition '$\omega_2(xy) \geq \omega_2(x) + \omega_2(y)$ for all $x, y \in (0, 1]$' in Theorem 2.2 cannot be dropped.

Counterexample 2.2 Take $\omega_2(x) = \left(\frac{1-x^3}{\sqrt{x}}\right)^{1/2}$, $x \in (0,1]$. It can be seen that $\omega_2(xy) \ngeq \omega_2(x) + \omega_2(y)$, $\forall\ x \in (0,1]$. Let X follow Gompertz distribution with distribution function $F_X(x) = 1 - e^{-B(c^x-1)/\ln c}$, $x \geq 0$, $B > 0$, $c \geq 1$, and let T follow Weibull distribution with distribution function $F_T(x) = 1 - e^{-(x/\beta)^k}$, $x \geq 0$, $\beta > 0$, $k \geq 1$. Then X and T both are NBU. For $k = 2$, $\beta = 1$, $B = 1$, $c = 2$, we have

$$\bar{F}_T\left(\omega_2(\bar{F}_X(x))\right) = \exp\left\{-\left(\frac{1 - e^{-3B(c^x-1)/\ln c}}{\sqrt{e^{-B(c^x-1)/\ln c}}}\right)\right\}.$$

Now, for $x = 0.4$ and $t = 0.3$, we have

$$\bar{F}_T\left(\omega_2(\bar{F}_X(x+t))\right) = 0.2313256 \text{ and } \bar{F}_T\left(\omega_2(\bar{F}_X(x))\right) \cdot \bar{F}_T\left(\omega_2(\bar{F}_X(t))\right) = 0.1844626.$$

This shows that R is not NBU.

3 Stochastic Orderings

Let the random variable R_1 be derived from the random variables T_1 and X_1, and let R_2 be another random variable derived from the random variables T_2 and X_2. In both the cases we take the same weight function ω_1 (or equivalently ω_2). In this section we study how different ordering properties between T_1 and T_2, and those between X_1 and X_2 are transformed to the ordering properties between R_1 and R_2.

Below we show that if T_2 dominates T_1, and X_2 dominates X_1 in usual stochastic order, then R_2 dominates R_1 in usual stochastic order. It is to be mentioned here that, for random variable Z_i having survival function $\bar{F}_i$, $i = 1, 2$, Z_2 is said to dominate Z_1 in stochastic order, written as $Z_1 \leq_{st} Z_2$, if $\bar{F}_1(t) \leq \bar{F}_2(t)$ for all t.

Theorem 3.1 *If $T_1 \leq_{st} T_2$ and $X_1 \leq_{st} X_2$, then $R_1 \leq_{st} R_2$.*

Proof: $X_1 \leq_{st} X_2$ gives that, for all $x \in [c,d]$, $\omega_2(\bar{F}_{X_1}(x)) \geq \omega_2(\bar{F}_{X_2}(x))$, which further gives that

$$\begin{aligned} \bar{F}_{T_1}[\omega_2(\bar{F}_{X_1}(x))] &\leq \bar{F}_{T_1}[\omega_2(\bar{F}_{X_2}(x))] \\ &\leq \bar{F}_{T_2}[\omega_2(\bar{F}_{X_2}(x))], \end{aligned}$$

giving $R_1 \leq_{st} R_2$. □

In the following theorem we give conditions on T_1, T_2, X_1 and X_2, under which R_2 dominates R_1 in up shifted hazard rate order. For two random variables Z_1 and Z_2 having respective survival functions $\bar{F}_1$ and $\bar{F}_2$, Z_2 is said to dominate Z_1 in up shifted hazard rate order, written as $Z_1 \leq_{hr\uparrow} Z_2$, if $\bar{F}_2(x)/\bar{F}_1(x+t)$ is increasing in x for all t.

Theorem 3.2 *Let $x\omega_2'(x)$ be increasing in $x \in (0,1]$. Suppose that the following conditions hold:*

(i) $T_1 \le_{hr} T_2$*, and* T_1 *or* T_2 *is IFR;*

(ii) $X_1 \le_{hr\uparrow} X_2$*.*

Then $R_1 \le_{hr\uparrow} R_2$*.*

Proof: $R_1 \le_{hr\uparrow} R_2$ holds if, and only if, for all $t \ge 0$,

$$r_{T_1}[\omega_2(\bar{F}_{X_1}(x+t))]\frac{d}{dx}\{\omega_2(\bar{F}_{X_1}(x+t))\} \ge r_{T_2}[\omega_2(\bar{F}_{X_2}(x))]\frac{d}{dx}\{\omega_2(\bar{F}_{X_2}(x))\}.$$

This holds if

$$r_{T_1}[\omega_2(\bar{F}_{X_1}(x+t))] \ge r_{T_2}[\omega_2(\bar{F}_{X_2}(x))] \tag{3.1}$$

and

$$\frac{d}{dx}\{\omega_2(\bar{F}_{X_1}(x+t))\} \ge \frac{d}{dx}\{\omega_2(\bar{F}_{X_2}(x))\}. \tag{3.2}$$

Since up hazard rate order is stronger than usual stochastic order, (ii) gives, for all $t \ge 0$ and for all $x \in [c,d]$,

$$\omega_2(\bar{F}_{X_1}(x+t)) \ge \omega_2(\bar{F}_{X_2}(x)). \tag{3.3}$$

Let us consider the following two cases.
Case I: Let T_1 be IFR. Then, from (3.3), we have, for all $t \ge 0$ and for all $x \in [c,d]$,

$$\begin{aligned} r_{T_1}\left(\omega_2(\bar{F}_{X_1}(x+t))\right) &\ge r_{T_1}\left(\omega_2(\bar{F}_{X_2}(x))\right) \\ &\ge r_{T_2}\left(\omega_2(\bar{F}_{X_2}(x))\right). \end{aligned} \tag{3.4}$$

Case II: Let T_2 be IFR. Then, from (3.3), we have, for all $t \ge 0$ and for all $x \in [c,d]$,

$$\begin{aligned} r_{T_2}\left(\omega_2(\bar{F}_{X_2}(x))\right) &\le r_{T_2}\left(\omega_2(\bar{F}_{X_1}(x+t))\right) \\ &\le r_{T_1}\left(\omega_2(\bar{F}_{X_1}(x+t))\right). \end{aligned} \tag{3.5}$$

Thus, from (3.4) and (3.5), (3.1) holds. Note that, (3.2) holds if, and only if, for all $t \ge 0$ and, for all $x \in [c,d]$,

$$r_{X_1}(x+t)\left[-\bar{F}_{X_1}(x+t)\omega_2'\left(\bar{F}_{X_1}(x+t)\right)\right] \ge r_{X_2}(x)\left[-\bar{F}_{X_2}(x)\omega_2'\left(\bar{F}_{X_2}(x)\right)\right]. \tag{3.6}$$

Since $X_1 \le_{hr\uparrow} X_2$, to prove (3.6), it suffices to show that

$$\bar{F}_{X_1}(x+t)\omega_2'\left(\bar{F}_{X_1}(x+t)\right) \le \bar{F}_{X_2}(x)\omega_2'\left(\bar{F}_{X_2}(x)\right), \tag{3.7}$$

which holds from the fact that $x\omega_2'(x)$ is increasing in $x \in (0,1]$. Hence, the result is proved.□

With the following counterexample, we show that the condition '$x\omega_2'(x)$ is increasing in $x \in (0,1]$' in Theorem 3.2 cannot be dropped.

Counterexample 3.1 Take $\omega_2(x) = \left(\frac{1-x^3}{\sqrt{x}}\right)^{1/2}$, $x \in (0,1]$. It can be seen that $x\omega_2'(x)$ is not monotone. Let X_1 and X_2 follow exponential distribution with failure rates β_1 and β_2 respectively. Then $X_1 \leq_{hr\uparrow} X_2$ if $\beta_1 \geq \beta_2$. Let T_1 and T_2 follow Rayleigh distribution with distribution function $F_{T_1}(x) = 1-e^{-x^2/2\sigma_1^2}$, $x \geq 0$ and $F_{T_2}(x) = 1-e^{-x^2/2\sigma_2^2}$, $x \geq 0$, respectively. Here T_1 and T_2 are IFR, and $T_1 \leq_{hr} T_2$ for $\sigma_1^2 \leq \sigma_2^2$. It can be easily verified that, for $\beta_1 = 2$, $\beta_2 = 1.8$, $\sigma_1 = 0.6$, $\sigma_2 = 0.65$ and $t = 0.9$, $\bar{F}_{T_2}[\omega_2(\bar{F}_{X_2}(x))]/\bar{F}_{T_1}[\omega_2(\bar{F}_{X_1}(x+t))]$ is neither increasing nor decreasing in x. Thus, R_2 does not always dominate R_1 with respect to the up shifted hazard rate order. □

The following theorem gives a result similar to that given in Theorem 3.2 for the hazard rate order. The proof follows in the same line as that of Theorem 3.2. If $\bar{F}_1$ and $\bar{F}_2$, the respective survival functions of random variables Z_1 and Z_2, be such that $\bar{F}_1(x)/\bar{F}_2(x)$ is decreasing in x, we say that Z_2 dominates Z_1 in hazard rate order, and we write $Z_1 \leq_{hr} Z_2$.

Theorem 3.3 *Let $x\omega_2'(x)$ be increasing in $x \in (0,1]$. Suppose that the following conditions hold:*

(i) $T_1 \leq_{hr} T_2$, and T_1 or T_2 is IFR;

(ii) $X_1 \leq_{hr} X_2$.

Then $R_1 \leq_{hr} R_2$. □

For two random variables Z_1 and Z_2 having respective reversed hazard rates $\tilde{r}_1$ and $\tilde{r}_2$ (and respective distribution functions F_1 and F_2), Z_2 is said to dominate Z_1 in reversed hazard rate order, written as $Z_1 \leq_{rhr} Z_2$, if $\tilde{r}_1(x) \leq \tilde{r}_2(x)$ for all x, or equivalently, $F_1(x)/F_2(x)$ is decreasing in x. Below we give conditions under which Z_2 dominates Z_1 in reversed hazard rate order.

Theorem 3.4 *Let $x\omega_1'(x)$ be decreasing in $x \in [0,1)$. Suppose that the following conditions hold:*

(i) $T_1 \leq_{rhr} T_2$, and T_1 or T_2 is DRHR;

(ii) $X_1 \leq_{rhr} X_2$.

Then $R_1 \leq_{rhr} R_2$.

Proof: $R_1 \leq_{rhr} R_2$ holds if, and only if,

$$\tilde{r}_{T_2}[\omega_1(F_{X_2}(x))]\frac{d}{dx}\{\omega_1(F_{X_2}(x))\} \geq \tilde{r}_{T_1}[\omega_1(F_{X_1}(x))]\frac{d}{dx}\{\omega_1(F_{X_1}(x))\}.$$

This holds if

$$\tilde{r}_{T_2}[\omega_1(F_{X_2}(x))] \geq \tilde{r}_{T_1}[\omega_1(F_{X_1}(x))] \tag{3.8}$$

and

$$\frac{d}{dx}\{\omega_1(F_{X_2}(x))\} \geq \frac{d}{dx}\{\omega_1(F_{X_1}(x))\}. \tag{3.9}$$

Since reversed hazard rate order is stronger than usual stochastic order, we have, for all $x \in [c, d]$,

$$\omega_1(F_{X_2}(x)) \leq \omega_1(F_{X_1}(x)). \tag{3.10}$$

Let us consider the following two cases.
Case I: Let T_1 be DRHR. Then, from (3.10), we have, for all $x \in [c, d]$,

$$\begin{aligned}\tilde{r}_{T_1}(\omega_1(F_{X_1}(x))) &\leq \tilde{r}_{T_1}(\omega_1(F_{X_2}(x))) \\ &\leq \tilde{r}_{T_2}(\omega_1(F_{X_2}(x))).\end{aligned} \tag{3.11}$$

Case II: Let T_2 be DRHR. Then, from (3.10), we have, for all $x \in [c, d]$,

$$\begin{aligned}\tilde{r}_{T_2}(\omega_1(F_{X_2}(x))) &\geq \tilde{r}_{T_2}(\omega_1(F_{X_1}(x))) \\ &\geq \tilde{r}_{T_1}(\omega_1(F_{X_1}(x))).\end{aligned} \tag{3.12}$$

Thus, from (3.11) and (3.12), (3.8) holds. Now, (3.9) holds if, and only if, for all $x \in [c, d]$, $\omega_1'(F_{X_2}(x))f_{X_2}(x) \geq \omega_1'(F_{X_1}(x))f_{X_1}(x)$, which means that

$$F_{X_2}(x)\omega_1'(F_{X_2}(x))\tilde{r}_{X_2}(x) \geq F_{X_1}(x)\omega_1'(F_{X_1}(x))\tilde{r}_{X_1}(x). \tag{3.13}$$

This holds on using the fact that $x\omega_1'(x)$ is decreasing in $x \in [0, 1)$, and $X_1 \leq_{rhr} X_2$.

Remark 3.1 It is to be noted that (i) $\omega_1(x) = -(\ln x)^2$, (ii) $\omega_1(x) = -((1-x)/x)^2$, (iii) $\omega_1(x) = -((1-x)^\alpha/x^{\alpha-1})^2, \alpha > 1$ satisfy the condition in Theorem 3.4. □

With the following counterexample, we show that the condition '$x\omega_1'(x)$ is decreasing in $x \in [0, 1)$' in the above theorem cannot be dropped.

Counterexample 3.2 Take $\omega_1(x) = \frac{x}{1-x}$, $x \in [0, 1)$ which does not satisfy the condition in Theorem 3.4. Let X_1 and X_2 follow exponential distribution with failure rates β_1 and β_2 respectively. Then $X_1 \leq_{rhr} X_2$ if $\beta_1 \geq \beta_2$. Let both T_1 and T_2 follow Weibull distribution with distribution function $F_{T_1}(x) = F_{T_2}(x) = 1 - e^{-x^k}$, $x \geq 0$, $0 < k \leq 1$. Here both T_1 and T_2 are DRHR. It is easy to verify that, for $\beta_1 = 3$, $\beta_2 = 1.5$, and $k = 0.2$, $F_{T_2}[\omega_1(F_{X_2}(x))]/F_{T_1}[\omega_1(F_{X_1}(x))]$ is neither increasing nor decreasing in x. Thus, R_2 does not always dominate R_1 with respect to the reversed hazard rate order. □

It is well known that although a random variable having support $[0, \infty)$ cannot be IRHR (increasing in reversed hazard rate) and most of the well known distributions are DRHR (see Sengupta and Nanda, 1999), a distribution function with finite support or the support of the form $(-\infty, b)$, $0 < b < \infty$, can be IRHR (see Block et al., 1998). A random variable (or equivalently its distribution) is said to have IRHR property if $\tilde{r}(x)$ is increasing in x. Below we give some conditions under which Z_2 dominates Z_1 in reversed hazard rate order. The proof is similar to that of Theorem 3.4, and hence omitted.

Theorem 3.5 *Let $(1-x)\omega_1'(x)$ be increasing in $x \in [0, 1)$. Suppose that the following conditions hold:*

(i) $T_1 \leq_{rhr} T_2$, and T_1 or T_2 is IRHR;

(ii) $X_1 \geq_{hr} X_2$.

Then $R_1 \leq_{rhr} R_2$.

Remark 3.2 It is to be noted that (i) $\omega_1(x) = -\ln(1-x)$, (ii) $\omega_1(x) = x/(1-x)$, satisfy the condition in Theorem 3.5. □

Remark 3.3 A random variable Z, with finite mean μ, having the following distribution function has IRHR property.

$$F_Z(x) = \left(\frac{bp - \mu + x(1-p)}{b - \mu} \right)^{p/(1-p)}, \ x \in (-\infty, b], \ p > 1.$$

4 Conclusion

In this paper, we study different reliability properties of Transformed-Transformer family of distributions, a recently developed method of generating new family of distributions from some existing distributions. A wide range of distributions can be generated by using this method (Alzaatreh et al., 2013a, 2014; Lee et al., 2013). Several well known continuous distributions, e.g., beta-generated family of distributions, Weibull, generalized gamma, exponentiated-Weibull, gamma-Pareto, exponentiated-exponential distributions, etc. (see Alzaatreh et al., 2013a, 2014) are found to be special cases of this newly generated distributions. It is shown that these newly generated distributions are very flexible and are capable of fitting various types of data (Alzaatreh et al., 2014). The results studied here will help to decide which distribution to be used, based on the ageing properties of the newly generated distributions, and also to find the better one in terms of various stochastic orders.

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