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RESEARCH ARTICLE

Robust Non-overshooting Tracking Control for Fractional Order Systems

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Summary

A robust tracking control is proposed for the fractional order systems to achieve a tracking response with no overshoot, even in the presence of a class of disturbances. The control proposed makes use of a newly designed integral sliding mode technique for fractional order systems, which is capable of rejecting the bounded disturbances acting through the input channel. The proposed integral sliding mode control design has two components, a nominal control component and a discontinuous control component. The overshoot in the system response is avoided by the nominal control designed with the use of Moore's eigenstructure assignment algorithm. The sliding mode technique is used for the design of discontinuous part of the control which imparts the desired robustness properties.

KEYWORDS:

Fractional order systems, tracking control, robust control, integral sliding mode control

1 | INTRODUCTION

The problem of ensuring the system response tracks a known reference output faithfully remains an important area of control systems design. It is particularly important when the system is operating with very low tolerance on the output. Avoiding overshoot while tracking is preferred in several applications such as servo- control mechanisms in robotics, minimizing the spill off in tanks while filling, temperature control mechanisms in machines working with inflammable substances and so on[1,2,3]. Several design techniques have been proposed in the literature for the overshoot reduction in the system response. In most of the design techniques fast response and reduction in overshoot appear as a contradictory design criteria and the control designer must seek a trade off[2] between these conflicting objectives. Many works are available for reduction of overshoot in single-input single-output (SISO) systems[4,5,6]. The composite nonlinear feedback technique is discussed in[2] for fast tracking with reduced overshoot. The non-overshooting tracking control (NOTC) technique was proposed in[8,10] which has the advantage of a fast response without overshoot. This technique is applicable to multi-input multi-output(MIMO) systems, where the overshoot is avoided by eigenstructure assignment, using Moore's technique to obtain the required feedback matrix[9].

Fractional order derivatives include both integer order derivatives and non-integer order derivatives, in that sense the fractional order calculus is considered as a generalisation of the integer order calculus[13]. The behaviour of systems can be more faithfully captured by fractional order representation. The historical aspect of the development of the fractional order calculus is described in[11,12]. The improvement in computing facilities and the advances of the fractional order calculus has lead to the increased application of fractional order calculus to various engineering problems[14,15]. Various physical phenomena and plants are represented as fractional order system(FOS) in[16,17]. Stability of linear time invariant (LTI) FOS are discussed in[23,24].

Sliding mode control (SMC) is known for its ability to provide robustness to the system by completely rejecting matched disturbances. The early works of Utkin[28] made the research community aware of the potential of SMC. The capability of SMC to reject the disturbance completely has encouraged lot of research on SMC for past few decades. SMC is applied in wide range of applications such as mobile robots, robotic manipulators, electric drives, power converters and so on, to get a robust performance against the disturbances[29,30,31]. The SMC is applied to the FOS for the robust stabilization of the FOS in[19]. The integral sliding mode control(ISMC) is a particular kind of the SMC where an integral sliding surface is designed such that the system always starts from the sliding surface. Hence, the sliding mode is induced from the initial time. So, the ISMC is able to reject the disturbance from initial time[32,33]. Also the ISMC can be easily combined with other control techniques to impart the robustness against the matched and bounded disturbances. ISMC is used to achieve robust NOTC for integer order systems (IOS) in[34,35].

In this paper, initially a NOTC is derived for MIMO FOS to achieve tracking without any overshoot. Secondly, an ISMC is proposed for FOS to achieve robustness for the proposed NOTC. The NOTC is designed by eigenstructure assignment using Moore's algorithm. ISMC technique is designed to reject the effect of the matched and bounded disturbances completely.

The organisation of the paper is as follows: Section 2 contains a review of fractional order calculus. In Section 3 the problem statement is introduced. NOTC for FOS is discussed in Section 4. Section 5 contains the main result, where the robust NOTC with ISMC technique for FOS is discussed. In Section 6 the proposed result is verified using a simulation example. Section 7 contains the conclusions.

2 | REVIEW OF FRACTIONAL CALCULUS

Preliminaries on fractional order calculus are introduced briefly in this section. In this paper the Caputo fractional-order derivative is used to represent the FOS dynamics[13,18]. The definitions of fractional-order integral and derivatives are given below.

Definition 1. The αth order Riemann-Liouville fractional integral is defined as

$$ {}_{t_0}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}} \mathrm{d}\tau $$

where $\alpha \in \mathbb{R}^+$.

Definition 2. The αth order Riemann-Liouville fractional derivative is defined as

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} f(t) &:= {}_{t_0}D_t^{m}\, {}_{t_0}I_t^{(m-\alpha)} f(t) \\ &:= \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(\alpha-m+1)}} \mathrm{d}\tau \right]. \end{aligned} $$

where $\alpha \in \mathbb{R}^+$, and $m-1 < \alpha < m$ and $m \in \mathbb{N}$.

Definition 3. The αth order Caputo fractional derivative is defined as

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\alpha} f(t) &:= {}_{t_0}I_t^{(m-\alpha)}\, {}_{t_0}D_t^{m} f(t) \\ &:= \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{(\alpha-m+1)}} \mathrm{d}\tau. \end{aligned} $$

where $\alpha \in \mathbb{R}^+$ and $m-1 < \alpha < m$ and $m \in \mathbb{N}$.

The relationship between Caputo fractional order derivative and Riemann-Liouville fractional order derivative is given by

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) = {}_{t_0}D_t^{\alpha} f(t) - \frac{f(t_0)}{\Gamma(1-\alpha)} (t)^{-\alpha}. \tag{1} $$

The Caputo and Riemann-Liouville fractional order derivatives are equal when the initial condition $f(t_0) = 0$.

Definition 4. For $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}^+$, the one-parameter Mittag-Leffler function is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{2}$$

where $\Gamma(\cdot)$ is Gamma function. For $\alpha, \beta \in \mathbb{R}^+$, the two-parameter Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{3}$$

The Mittag-Leffler function can be viewed as a generalized form of the exponential function. When $\alpha = \beta = 1$, we obtain

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{4}$$

For $0 < \alpha < 1$ and $t \in \mathbb{R}^+$, the single parameter Mittag-Leffler function $E_\alpha(-t^\alpha)$ is monotonically decreasing[21].

Lemma 1. Given $0 < \alpha < 1$, $\lambda_1 < \lambda_2 < 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, the function

$$f(t) = \gamma_1 E_\alpha(\lambda_1 t^\alpha) + \gamma_2 E_\alpha(\lambda_2 t^\alpha) \tag{5}$$

has a root in $\mathbb{R}^+$ if and only if

$$\frac{\lambda_2}{\lambda_1} < \frac{-\gamma_2}{\gamma_1} < 1 \tag{6}$$

Proof: See the Appendix. We note that as f is a continuous function, if it does not have a root, then it cannot change sign.

Lemma 2. [18],[23],[24] Let the commensurate LTI Caputo FOS dynamics, with $\alpha > 0$, be given by

$$ {}_{t_0}^{c}D_t^\alpha x(t) = Ax(t) \tag{7}$$

where A is a constant real matrix. The solution is given by $x(t) = E_\alpha(A(t-t_0)^\alpha)x_0$, where $x_0 = x(t_0)$ denotes the initial condition. The system dynamics are asymptotically stable if $|\arg(spec(A))| > \alpha\pi/2$.

Remark 1. We denote the stability region of (7) by $\bar{\mathbb{C}}_\alpha$. When $\alpha = 1$, the FOS is of integer order, and we refer to it as IOS. For $\alpha \geq 2$, the system is unstable for all A, and $\bar{\mathbb{C}}_\alpha = \emptyset$. While the roots of $det(v^\alpha I - A) = 0$ are the poles of the commensurate FOS, it is only required to consider the equation $det(\lambda I - A) = 0$, with $\lambda = v^\alpha$, to determine the system stability. So, for the sake of simplicity the term 'poles of FOS' is abused. From here onwards, by 'poles of FOS', we mean the roots of the equation $det(\lambda I - A) = 0$. For the FOS with input

$$\begin{aligned} {}_{0}^{c}D_t^\alpha x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{8}$$

the concept of system controllability was defined in[22]. It was shown that, as for IOS, controllability is assured if the controllability matrix has full rank. As for IOS, the invariant zeros of FOS are the values of $\lambda \in \mathbb{C}$ for which the Rosenbrock matrix $\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix}$ loses rank. We say that an invariant zero of (8) is stable if it belongs to $\bar{\mathbb{C}}_\alpha$.

3 | PROBLEM STATEMENT

In this paper we consider the commensurate Caputo FOS with fractional-order derivative $0 < \alpha < 1$, represented by

$$\begin{aligned} {}_{0}^{c}D_t^\alpha x(t) &= Ax(t) + B\big(u(t) + d(t)\big) \\ y(t) &= Cx(t) \end{aligned} \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $d(t) \in \mathbb{R}^m$ represent the states of the system, control input, system output and the unknown disturbance entering the system, respectively. The matrices A, B and C are constant matrices of appropriate dimensions. A full column rank is assumed for matrix B and a full row rank is assumed for C matrix. The system is assumed to be controllable. The uncertainty entering the system through the input channel is considered to be bounded, i.e. $\|d(t)\| \leq d_{\max}$.

Also, the system is considered to be in the initial state x_0 with input u_0, yielding an output y_0. The reference signal is represented by $r \in \mathbb{R}^p$, which is taken as a step signal. The tracking control input

$$u(t) = F(x(t) - x_s) + u_s \tag{10}$$

where $u_s \in \mathbb{R}^m$, $x_s \in \mathbb{R}^n$ are input and state at the steady state. The feedback matrix F is to be designed. The steady state vectors $u_s \in \mathbb{R}^m$, $x_s \in \mathbb{R}^n$ are computed from

$$0 = Ax_s + Bu_s \tag{11}$$

$$r = Cx_s \tag{12}$$

The closed-loop FOS dynamics with the tracking control input u given by (10) is

$$\begin{aligned} {}_0^cD_t^\alpha x(t) &= Ax(t) + B\big(F(x(t) - x_s) + u_s\big) \\ y(t) &= Cx(t) \end{aligned} \tag{13}$$

With a coordinate change $\zeta = x - x_s$, the FOS dynamics is computed as

$$\begin{aligned} {}_0^cD_t^\alpha \zeta(t) &= (A + BF)\zeta(t), \\ y(t) &= C\zeta(t) + r \end{aligned} \tag{14}$$

where $\zeta_0 = x_0 - x_s$ represents the initial condition. The feedback matrix F designed such that the closed loop system dynamics (14) is stable, i.e. $\zeta \to 0$ as $t \to \infty$, and hence the tracking error by $\epsilon(t) = r - y(t) = C\zeta(t)$ approaches zero as $t \to \infty$. The reference signal r is tracked by the output $y(t)$, when the control (10) is applied to the FOS (9). But, this design cannot guarantee a non-overshooting tracking response for FOS.

In this paper we address the problem of obtaining a robust NOTC which ensures the FOS output track the step reference signal without overshoot in all output components, while also rejecting the effect of bounded disturbances acting through the input channel. This requires the tracking error, $\epsilon(t) = r - y(t)$ to converge to zero without changing sign in any of its output components[8]. A NOTC for a square FOS with $n - p$ stable invariant zeros has been discussed in[20]. But, the control introduced in[20] does not guarantee a non-overshooting response in the presence of the disturbances. Motivated from these results, initially a NOTC is designed for FOS with $n - 2p$ stable invariant zeros, without disturbance. Then it is extended to a general case without disturbance for $n - lp$ stable invariant zeros, $l \in \mathbb{N}$. Then the ISMC is used to make the proposed NOTC robust against the matched and bounded disturbances.

4 | DESIGN OF NON-OVERSHOOTING TRACKING CONTROL FOR FOS

In this section, a NOTC is designed for FOS without any disturbance. The overshoot in the output response can be avoided through a proper design of the feedback gain matrix F. For IOS without any disturbance, a NOTC is computed using a proper design of F as in[8]. To obtain the NOTC, the assumptions listed below are made on the FOS (9).

Assumption 1. FOS (9) is right invertible and controllable. Also, there is no invariant zero at the origin.

Assumption 2. The number of inputs and outputs are equal for FOS (9), i.e. $m = p$.

Assumption 3. No disturbance is entering the FOS (9), i.e. $d(t) = 0$.

The following lemma from[8] will be used for the design of a state feedback F matrix to achieve NOTC.

Lemma 3. [8]. Let $\mathcal{L} = \{\lambda_1, \lambda_2, \dots \lambda_n\}$ be a self-conjugate set of n distinct complex numbers. Let $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$ be a set of n vectors in $\mathbb{R}^p$, not necessarily distinct. Assume that, for each $i \in \{1, 2, \dots, n\}$, the matrix equation.

$$\begin{bmatrix} A - \lambda_i I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ s_i \end{bmatrix} \tag{15}$$

has solutions $\{v_1, v_2, \dots v_n\} \subset \mathbb{C}^n$ and $\{w_1, w_2, \dots w_n\} \subset \mathbb{C}^p$. Then, provided the vectors $\{v_1, v_2, \dots v_n\}$ are linearly independent, a unique real feedback matrix F exists such that, for all $i \in \{1, 2, \dots, n\}$,

$$(A + BF)v_i = \lambda_i v_i \tag{16}$$

Using $\mathcal{W} = \{w_1, \dots, w_n\}$ and $\mathcal{V} = \{v_1, \dots, v_n\}$, the matrix F can be constructed using Moore's algorithm[9].

4.1 | Systems having $n-2p$ stable invariant zeros

Consider the system under the following assumption

Assumption 4. The system (9) has at least $n-2p$ stable and distinct invariant zeros in $\bar{\mathbb{C}}_\alpha$.

For the FOS (9) with the control input (10), satisfying assumptions 1-4, the feedback matrix F using the following algorithm.

Algorithm 4.1.

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1. Let the closed-loop poles of FOS be given by the self-conjugate set $\mathcal{L} = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \in \mathbb{C}$, where λ_i for $i \in \{1, 2, \dots, n-2p\}$ are placed at the distinct $n-2p$ stable invariant zeros of the FOS. For λ_i, $i \in \{n-2p+1, n-2p+2, \dots, n\}$ are placed on the negative real axis such that they are distinct.
2. Choose a set of vectors $\mathcal{S} = \{s_1, s_2, \dots, s_n\} \in \mathbb{R}^p$, such that
$$s_i = \begin{cases} 0 & \text{for } i \in \{1, 2, \dots, n-2p\}; \\ \mathbf{e_1} & \text{for } i \in \{n-2p+1, n-2p+2\}; \\ \mathbf{e_2} & \text{for } i \in \{n-2p+3, n-2p+4\}; \\ \vdots & \\ \mathbf{e_p} & \text{for } i \in \{n-1, n\}; \end{cases} \tag{17}$$
where $\mathbf{e_i}$, $i \in \{1, 2, 3, \dots, p\}$ denote the canonical basis vectors of $\mathbb{R}^p$.
3. Solve (15) by taking $\lambda_i \in \mathcal{L}$ and $s_i \in \mathcal{S}$ to obtain $\begin{bmatrix} v_i \\ w_i \end{bmatrix}$. Let $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, and $\mathcal{W} = \{w_1, w_2, \dots, w_n\}$.
4. By Lemma 3, provided $\mathcal{V}$ is linearly independent, a feedback matrix F satisfying (16) exists. The matrix F can be computed from $\mathcal{V}$ and $\mathcal{W}$ using Moore's algorithm.

The following notations help in stating Theorem 1.

Notation 4.1

.

1. For each $k \in \{1, 2, \dots p\}$, let $\lambda_{k,1}$ and $\lambda_{k,2}$ denote the eigenvalues in $\mathcal{L}$ associated with $s_i = \mathbf{e_k}$, ordered as $\lambda_{k,1} < \lambda_{k,2}$.
2. Let the eigenvectors $v_{k,1}$ and $v_{k,2}$ correspond to eigenvalues $\lambda_{k,1}$ and $\lambda_{k,2}$, respectively.
3. Introduce coordinates $\zeta(t) = x(t) - x_s$, $\zeta_0 = \zeta(0)$.
4. Let matrix V be formed from $\mathcal{V}$ as
$V := [v_1|v_2|\dots|v_{n-2p}|v_{1,1}|v_{1,2}|\dots|v_{p,1}|v_{p,2}]$
5. Let $\psi := V^{-1}\zeta_0$, with the elements denoted as
$\psi = [\psi_1\ \psi_2\ \cdots\ \psi_{n-2p}\ \psi_{1,1}\ \psi_{1,2} \cdots\ \psi_{p,l}\ \psi_{p,2}]^T$
6. $H_k := \text{span}\{v_{k,1}, v_{k,2}\}$
7. Let $J_k \subseteq H_k$ and for $\gamma_{k,1}, \gamma_{k,2} \in \mathbb{R}$
$J_k := \left\{\gamma_{k,1}v_{k,1} + \gamma_{k,2}v_{k,2} : \frac{-\gamma_{k,2}}{\gamma_{k,1}} \notin (\frac{\lambda_{k,2}}{\lambda_{k,1}}, 1)\right\}$,
8. For each $x \in \mathbb{R}^n$, let x_k denote the orthogonal projection of x onto H_k, and let $J \subseteq \mathbb{R}^n$ consists of all points $x \in \mathbb{R}^n$ for which $x_k \in J_k$ for all $k \in \{1, 2, \dots, p\}$.

Theorem 1. Under the Assumptions 1-4, let F be a feedback matrix obtained from Algorithm 4.1, and let the control input (10) be applied to the FOS (9). Then the reference signal r will be tracked by the output $y(t)$ of (9) without overshoot, if and only if $\zeta_0 \in J$.

Proof. Consider the FOS (9) under the Assumption 1-4, given by

$$\begin{aligned} {}_0^cD_t^\alpha x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{18}$$

When the control input (10) is applied to the system (18) where the feedback matrix F is obtained as in Theorem 1, the closed-loop dynamics of the FOS (18) is given by

$$\begin{aligned} {}_0^cD_t^\alpha x(t) &= Ax(t) + B\big(F(x(t) - x_s) + u_s\big) \\ y(t) &= Cx(t) \end{aligned} \tag{19}$$

Applying the a change of coordinates $\zeta(t) = x(t) - x_s$ and using (11), the closed-loop system dynamics and output equation are given by

$$\begin{aligned} {}_0^cD_t^\alpha \zeta(t) &= {}_0^cD_t^\alpha x(t) \\ &= Ax(t) + B\big(F(x(t) - x_s) + u_s\big) \\ &= Ax(t) + BF(x(t) - x_s) + Bu_s \\ &= Ax(t) + BF\zeta(t) + Bu_s \\ &= A(\zeta(t) + x_s) + BF\zeta(t) + Bu_s \\ &= (A + BF)\zeta(t) + \big(Ax_s + Bu_s\big) \\ &= (A + BF)\zeta(t) \end{aligned} \tag{20}$$

$$y(t) = C\zeta(t) + r \tag{21}$$

F is designed such that first $n - 2p$ poles of the closed-loop system are placed at the stable invariant zeros of the FOS located in $\bar{\mathbb{C}}_\alpha$. The remaining $2p$ poles are placed on the negative real axis. From Lemma 2, the FOS (20) is asymptotically stable, and the tracking error is given by

$$\begin{aligned} \epsilon(t) &= r - y(t) \\ &= -C\zeta(t) \\ &= -CE_\alpha((A + BF)t^\alpha)\zeta_0 \\ &= -C\sum_{k=0}^{\infty} \frac{(A + BF)^k (t^\alpha)^k}{\Gamma(\alpha k + 1)}\zeta_0 \end{aligned}$$

From the diagonal decomposition of the matrix $A+BF$ given by $A+BF = V\Lambda V^{-1}$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, one can compute $\epsilon(t)$ as

$$\begin{aligned} \epsilon(t) &= -C\sum_{k=0}^{\infty} \frac{(V\Lambda V^{-1})^k (t^\alpha)^k}{\Gamma(\alpha k + 1)}\zeta_0 \\ &= -CV\sum_{k=0}^{\infty} \frac{(\Lambda)^k (t^\alpha)^k}{\Gamma(\alpha k + 1)}V^{-1}\zeta_0 \end{aligned} \tag{22}$$

Introducing $\psi = V^{-1}\zeta_0 \in \mathbb{R}^n$, the above equation can be written as

$$\begin{aligned} \epsilon(t) &= -CV\sum_{k=0}^{\infty} \frac{(\Lambda)^k (t^\alpha)^k}{\Gamma(\alpha k + 1)}\psi \\ &= -CVE_\alpha(\Lambda t^\alpha)\psi \end{aligned}$$

From the equation (15), $Cv_i = s_i$, where v_i is the i-th column vector of V, and ψ_i as the i-th element of ψ, one can write

$$\epsilon(t) = -\sum_{i=1}^{n} s_i E_\alpha(\lambda_i t^\alpha)\psi_i \tag{23}$$

By using the Notation 4.1: $1-5$ and substituting for s_i from (17),

$$\epsilon(t) = -\begin{bmatrix} \psi_{1,1}E_\alpha(\lambda_{1,1}t^\alpha) + \psi_{1,2}E_\alpha(\lambda_{1,2}t^\alpha) \\ \psi_{2,1}E_\alpha(\lambda_{2,1}t^\alpha) + \psi_{2,2}E_\alpha(\lambda_{2,2}t^\alpha) \\ \vdots \\ \psi_{p,1}E_\alpha(\lambda_{p,1}t^\alpha) + \psi_{p,2}E_\alpha(\lambda_{p,2}t^\alpha) \end{bmatrix} \tag{24}$$

Assuming $\zeta_0 \in \boldsymbol{J}$, the projection of ζ_0 onto each $\boldsymbol{H}_k$ lies in $\boldsymbol{J}_k$. So by Lemma 1, each component of the tracking error has no root in $\mathbb{R}^+$, and hence converges to zero without changing sign as $t \to \infty$. Hence, a tracking response without overshoot is achieved and the response converges to the reference value monotonically as time $t \to \infty$. Conversely, if there is a $k \in \{1, \dots, p\}$ such that the projection of ζ_0 onto $\boldsymbol{H}_k$ does not lie in $\boldsymbol{J}_k$, then again by Lemma 1, the system response will be overshooting in the k-th output component. Thus the condition $\zeta_0 \in \boldsymbol{J}$ is both necessary and sufficient for a non-overshooting response. □

Remark 2. The system response can be made arbitrarily fast since the modes appearing in the output are the freely chosen poles placed on the negative real axis.

Remark 3. For the FOS with at least $n-p$ stable invariant zeros by using the proposed method of NOTC, it can be shown that only one Mittag-Leffler function will appear in each component of the error term. Hence, one can obtain a tracking response that is non-overshooting from all initial conditions.[20].

Remark 4. In the case of at least $n-2p$ stable invariant zeros, to get the maximum advantage of the available stable invariant zeros in the system, instead of considering any of the $n-2p$ stable invariant zeros as in Theorem 1 one can place the closed loop poles at all of the available stable invariant zeros. By doing so one can get a non-overshooting response from a wider set of initial conditions. For example if the system has $n-(p+q)$ stable invariant zeros where $q < p$ and $q \in \mathbb{N}$, then system come under the category of at least $n-2p$ distinct stable invariant zero case. But, if we are placing $n-(p+q)$ distinct stable invariant zeros instead of $n-2p$ stable invariant zeros as the closed loop poles of the system and place rest of the $p+q$ closed loop poles on the negative real axis and select s_i in equation (17) as

$$s_i = \begin{cases} 0 & for\ \ i \in \{1, 2, \dots, n-2p\}; \\ \mathbf{e_1} & for\ \ i \in \{n-2p+1, n-2p+2\}; \\ \mathbf{e_2} & for\ \ i \in \{n-2p+3, n-2p+4\}; \\ \vdots & \\ \mathbf{e_q} & for\ \ i \in \{n-2p+2q-1, n-2p+2q\}; \\ \mathbf{e_{q+1}} & for\ \ i \in \{n-2p+2q+1\}; \\ \vdots & \\ \mathbf{e_{p-1}} & for\ \ i \in \{n-1\}; \\ \mathbf{e_p} & for\ \ i \in \{n\}; \end{cases} \tag{25}$$

Then out of p output components, q output components have the sum of two Mittag-Leffler functions appearing in the output and the remaining e $p-q$ output components can have only a single Mittag-Leffler function appearing in the output. In this way one can get a larger set of initial conditions from which non-overshooting response can be obtained. While computing the set $\boldsymbol{J}$ given in Notation 4.1 (8), it is only necessary check the projection of ζ_0 onto $\boldsymbol{H}_k$ belongs to $\boldsymbol{J}_k$ for $k \in \{1, 2, 3 \dots, q\}$.

4.2 | Systems with fewer than $n-2p$ stable invariant zeros

In this case we relax the Assumption 4 with following assumption

Assumption 5. The system (9) has at least $n-lp$ distinct stable invariant zeros in $\bar{\mathbb{C}}_\alpha$, where $l \in \mathbb{N}$.

For the LTI FOS system with at least $n-lp$ stable invariant zeros, the following lemma 4 is useful to prove that the components of tracking errors do not change sign.

Lemma 4. Let $0 < \alpha < 1$, let $\lambda_1 < \lambda_2 < \dots < \lambda_l < 0$, and let $\gamma_1, \gamma_2, \dots, \gamma_l \in \mathbb{R}$. Then the function

$$f(t) = \gamma_1 E_\alpha(\lambda_1 t^\alpha) + \gamma_2 E_\alpha(\lambda_2 t^\alpha) + \dots + \gamma_l E_\alpha(\lambda_l t^\alpha) \tag{26}$$

does not have a root in $\mathbb{R}^+$ if
(a) γ_i, $i \in \{1, 2, \dots, l\}$ are all of same sign, or

(b) For some $k \in \{1, 2, \ldots, l-1\}$, the elements of $\{\gamma_1, \ldots, \gamma_k\}$ are all of same sign, the elements of $\{\gamma_{k+1}, \ldots, \gamma_l\}$ are all of the same sign, and $|\sum_1^k \gamma_i| < |\sum_{k+1}^l \gamma_i|$.

Proof. Case (a) follows directly from the fact that $E_\alpha(-\kappa)$ is positive and monotonically decreasing as κ increases, for $0 < \alpha < 1$ and $\kappa > 0$[21]. Case (b) can be proved from Lemma 1. □

Under Assumption 5, one can include any commensurate LTI FOS by considering an sufficiently large value of l. For the FOS (9) under the control input (10), satisfying Assumptions 1-3 and 5, a NOTC can be achieved by designing the feedback matrix F using the following algorithm.

Algorithm 4.2

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1. Let the closed-loop poles of the FOS be at $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \bar{\mathbb{C}}_\alpha$. The first $n - lp$ closed-loop poles of the FOS, λ_i for $i \in \{1, 2, \ldots, n - lp\}$, are placed at any of the $n - lp$ distinct stable invariant zeros of the FOS. The remaining poles, λ_i for $i \in \{n - lp + 1, n - lp + 2, \ldots, n\}$ are placed on the negative real axis such that all the poles are distinct.

2. Choose a set of vectors $\mathcal{S} = \{s_1, s_2, \ldots, s_n\} \in \mathbb{R}^p$ as follows
$$s_i = \begin{cases} 0 & for \;\; i \in \{1, 2, \ldots, n - lp\}; \\ \mathbf{e_1} & for \;\; i \in \{n - lp + 1, n - lp + 2, \ldots, n - lp + l\}; \\ \mathbf{e_2} & for \;\; i \in \{n - l(p-1) + 1, n - l(p-1) + 2, \ldots, n - l(p-1) + l\}; \\ \vdots & \\ \mathbf{e_p} & for \;\; i \in \{n - l + 1, n - l + 2, \ldots, n\}; \end{cases} \tag{27}$$
where $\mathbf{e_i}$, $i \in \{1, 2, 3, \ldots, p\}$ denote the canonical basis vectors of $\mathbb{R}^p$.

3. Compute the vector $\begin{bmatrix} v_i \\ w_i \end{bmatrix}$ by solving (15), considering $\lambda_i \in \mathcal{L}$ and $s_i \in \mathcal{S}$. Hence obtain $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, and $\mathcal{W} = \{w_1, w_2, \ldots, w_n\}$.

4. From Lemma 3, if $\mathcal{V}$ is linearly independent, then the feedback matrix F exists. The matrix F can be computed by Moore's algorithm.

Notation 4.2

.

The following notations are considered to state Theorem 2.

1. For each $k \in \{1, 2, \ldots p\}$, let $\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,l}$ denote eigenvalues in $\mathcal{L}$ that are associated with $s_i = \mathbf{e_k}$, ordered as $\lambda_{k,1} < \lambda_{k,2} < \ldots \lambda_{k,l}$,

2. Let $v_{k,1}, v_{k,2}, \ldots, v_{k,l}$ denote the eigenvectors corresponding to the eigenvalues $\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,l}$, respectively.

3. Define $\zeta(t) = x(t) - x_s$ and let $\zeta_0 = \zeta(0)$.

4. Form matrix V with the columns as the elements in the set $\mathcal{V}$,
$V := [v_1 | v_2 | \ldots | v_{n-lp} | v_{1,1} | v_{1,2} | \ldots | v_{1,l} | v_{2,1} | \ldots | v_{p,l}]$

5. Define $\psi := V^{-1}\zeta_0 \in \mathbb{R}^p n$, denoted as
$\psi = [\psi_1 \;\; \psi_2 \;\; \ldots \;\; \psi_{n-lp} \;\; \psi_{1,1} \;\; \psi_{1,2} \cdots \psi_{1,l} \;\; \psi_{2,1} \;\; \cdots \;\; \psi_{p,l}]^T$

6. For each $k \in \{1, 2, \ldots, p\}$, define
$$\epsilon_k(t) := -(\psi_{k,1} E_\alpha(\lambda_{k,1} t^\alpha) + \psi_{k,2} E_\alpha(\lambda_{k,2} t^\alpha) + \cdots + \psi_{k,l} E_\alpha(\lambda_{k,l} t^\alpha)), \tag{28}$$

Theorem 2. Assume that FOS (9) satisfies Assumptions 1-3 and 5. Let control input (10) is applied to the FOS, where the matrix F designed from the Algorithm 4.2 by considering $\mathcal{L}$ as the closed loop poles and $\mathcal{V}$ as the associated eigenvectors. Then, the response $y(t)$ of FOS (9) with control input (10) will track the reference signal r with no overshoot, if and only if the functions $\epsilon_k(t)$ do not change sign, for all $k \in \{1, 2, \dots, p\}$.

Proof. The proof follows the similar lines of the proof of Theorem 1. The tracking error ϵ_k in the k-th output component has the form of (28). □

Remark 5. NOTC can be computed for any commensurate LTI FOS by considering the case of least $n-lp$ stable invariant zeros with appropriate value of $l \in \mathbb{N}$. But, as the value of l increases the non-overshooting response can be achieved only form a reduced set of initial conditions by using the proposed NOTC.

Remark 6. For the case of $n-lp$ stable invariant zeros, Lemma 4 only provides a sufficient condition for the tracking errors to not change sign, and thus underestimates the set of initial conditions from which a non-overshooting response can be achieved. In the case of $n-2p$ zeros, the Lemma 1 gives a necessary and sufficient condition for the tracking errors to not change sign, and hence one can find the complete set of initial conditions from which a non-overshooting response can be achieved.

5 | DESIGN OF ROBUST NON-OVERSHOOTING TRACKING CONTROL FOR FOS

In this Section, Assumption 3 is relaxed by Assumption 6 stated below:

Assumption 6. Matched and bounded disturbance act on FOS (9), i.e $\|d(t)\| \leq d_{max}$, where d_{max} is a known quantity.

When there is a disturbance acting on the FOS, the NOTC discussed Section 4 does not yield a non-overshooting response. So, the ISMC technique is exploited in this section to make the FOS response robust towards the matched and bounded disturbances. A robust NOTC for integer order system is discussed in [34]. ISMC technique is used with NOTC to achieve the robust NOTC for IOS in [34]. The ISMC for FOS , $u(t)$, introduced in this section has two parts

$$u(t) = u_n(t) + u_i(t) \tag{29}$$

where $u_i(t)$ is discontinuous control part and $u_n(t)$ is the nominal control part. The discontinuous control, $u_i(t)$, is designed from the ISMC technique for FOS, which make the FOS response invariant to the matched and bounded disturbance. Nominal control part, $u_n(t)$, is designed by NOTC discussed in Section 4.

In order to compute $u_i(t)$, define an fractional order integral sliding surface, $\sigma(t) = 0$, where the sliding variable, $\sigma(t)$, is designed as

$$\sigma(t) = {}_0I_t^{1-\alpha}(Gx(t)) - G\int_0^t \frac{x_0}{\Gamma(1-\alpha)}(\tau)^{-\alpha} d\tau - G\int_0^t (Ax(\tau) + Bu_n(\tau))d\tau \tag{30}$$

where $G \in \mathbb{R}^{m\times n}$ is the projection matrix, chosen such that the $(GB)^{-1}$ exists. x is the state of FOS, x_0 is the initial condition and u_n is the nominal control. The discontinuous control part, u_i, is given by

$$u_i(t) = -k(GB)^{-1}\frac{\sigma(t)}{\|\sigma(t)\|} \tag{31}$$

where k is the gain to be computed.

5.1 | Systems having $n-2p$ stable invariant zeros

Consider the case with the FOS having at least $n-2p$ stable invariant zeros.

Theorem 3. Assume that the FOS (9) satisfies the Assumptions 1,2,4 and 6. Let control input (29) be designed such that $u_n(t)$ is obtained using Theorem 1 and the $u_i(t)$ is computed by (31) with gain $k > \|GB\|d_{max}$. When the control (29) is applied to FOS (9), the response $y(t)$ will track the reference r without overshoot by rejecting the effect of the disturbances, if the initial condition satisfies $\zeta_0 \in J$.

Proof. Consider the sliding variable, $\sigma(t)$, defined in (30). The sliding variable $\sigma(t)$ is designed such that $\sigma(0) = 0$. Also, the derivative of the sliding variable is computed as

$$\begin{aligned}\dot{\sigma}(t) &= {}_0I_t^{-\alpha}(Gx(t)) - G\frac{x_0}{\Gamma(1-\alpha)}(t)^{-\alpha} \\ &\quad - G(Ax(t) + Bu_n(t)) \\ &= {}_0D_t^{\alpha}(Gx(t)) - G\frac{x_0}{\Gamma(1-\alpha)}(t)^{-\alpha} \\ &\quad - G(Ax(t) + Bu_n(t)) \\ &= G{}_0^cD_t^{\alpha}x(t) - G(Ax(t) + Bu_n(t)) \\ &= G(Ax(t) + B(u_n(t) + u_i(t) + d(t))) \\ &\quad - G(Ax(t) + Bu_n(t)))\end{aligned} \tag{32}$$

After substituting for $u_i(t)$ from (31), we get $\dot{\sigma}(t)$ as

$$\dot{\sigma}(t) = -k\frac{\sigma(t)}{\|\sigma(t)\|} + GBd(t) \tag{33}$$

The gain k, is selected as $k > \|GB\|d_{max}$, so the η-reachability condition is satisfied and the surface remains attractive[30]. Hence the system trajectories remain on the sliding surface. Since $\sigma(0) = 0$, the system trajectories start from siding surface and then remains on the sliding surface. For the discontinuous control $u_i(t)$, one can compute an equivalent value, $u_{ieq}(t)$, by considering $\sigma(t) \equiv 0$ for $t \geq 0$ and equating the derivative of the sliding variable to zero.

$$\begin{aligned}\dot{\sigma}(t) &= G(Ax(t) + B(u_n(t) + u_i(t) + d(t))) \\ &\quad - G((Ax(t) + Bu_n(t))) \\ &= GB(u_i(t) + d(t)) \\ 0 &= GB(u_{ieq}(t) + d(t))\end{aligned} \tag{34}$$

Without loss of generality, if GB be selected as identity matrix $I^{m\times m}$, one can conclude

$$u_{ieq}(t) = -d(t) \tag{35}$$

Hence $u_{ieq}(t)$, the equivalent value of $u_i(t)$ applied to the FOS always negates the disturbance(if GB is not taken as $I^{m\times m}$, then $u_{ieq}(t)$ will be scaled by $(GB)^{-1}$). So the effect of the disturbance get cancelled by the $u_i(t)$ applied to the FOS and the system trajectories always remain on the sliding surface. The states of (9) will evolve according to the nominal system given by

$${}_0^cD_t^{\alpha}x(t) = Ax(t) + Bu_n(t) \tag{36}$$

Since the $u_n(t)$ is designed using NOTC from Theorem 1, the response of (9) will track the reference r without any overshoot. The disturbance, $d(t)$, is rejected by the discontinuous control, $u_i(t)$, by forcing the FOS trajectories to remain on the sliding surface.

□

5.2 | Systems having $n - lp$ stable invariant zeros

Consider the case with the system having at least $n - lp$ stable invariant zeros, where $l \in \mathbb{N}$. The design of a robust NOTC can be done using Theorem 4 as follows:

Theorem 4. Let FOS (9) satisfies the Assumptions 1,2,5 and 6. Let control input (29) be designed such that the $u_n(t)$ is obtained from Theorem 2 and the $u_i(t)$ computed from (31) with gain $k > \|GB\|d_{max}$. Control input (29), when applied to the FOS (9), will make $y(t)$ to track the signal r with no overshoot by rejecting the disturbances, if and only if the functions $\epsilon_k(t)$ in (28) do not change sign for all $k \in \{1, 2, \ldots, p\}$.

Proof. The proof of Theorem 4 follows the similar lines of the proof for Theorem 3. □

6 | SIMULATION RESULT

The robust NOTC for FOS proposed in Section 5 is verified using two simulation examples.

6.1 | Example 1

Consider the LTI commensurate FOS in (9) where

$$A = \begin{bmatrix} -9 & -9 & 5 & 0 & -3 \\ -8 & 0 & 0 & -7 & 0 \\ 10 & -9 & -5 & 0 & -8 \\ 10 & 0 & 8 & -5 & 0 \\ -1 & 0 & 0 & 0 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 2 & -10 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -4 \end{bmatrix}$$

and $\alpha = 0.9$. The disturbance entering the system is taken as $d(t) = [0.2\sin(2t) \;\; 0.1\cos(2t)]^T$. The initial condition is given by $x_0 = [2.5 \;\; -6.5 \;\; -7 \;\; -8 \;\; 1]^T$. The step reference signal to be tracked is assumed to be $r = [5 \;\; -5]^T$.

The steady state values x_s and u_s are obtained by solving (11) and (12) as $x_s = [7.1795 \;\; 0.2564 \;\; 14.8462 \;\; -4.7436 \;\; 2.4359]^T$, $u_s = [24.2308 \;\; 26.2744]^T$.

The initial condition x_0 considered in the example satisfies the condition $\zeta_0 = x_0 - x_s \in J$ given in Notation 4.1. Hence when the control input, $u(t)$, is designed from Theorem 1 in the absence of disturbance or from Theorem 3 in the presence of matched and bounded disturbance is acting on the system will not show any overshoot in the response.

The robust NOTC, $u(t)$, proposed in Section 5.1 is given by

$$u(t) = u_n(t) + u_i(t)$$

The FOS considered has number of states $n = 5$, the number of outputs $p = 2$ and the number of inputs $m = 2$. Also, the given system is square, i.e. $m = p$. The invariant zeros of the system considered are found to be at 8.0462 and −12.1176. To compute the nominal control part $u_n(t)$, the system considered comes under the category of at least $n - 2p$ stable invariant zeros, since $n - 2p = 1$ and system has only one stable invariant zero. For computing nominal part, $u_n(t)$, choose the first closed-loop pole as the stable invariant zero of the FOS $\lambda_1 = -12.1176$. The remaining $2p = 4$ closed-loop poles are placed at $\lambda_2 = -4.5$, $\lambda_3 = -5.5$, $\lambda_4 = -5$, $\lambda_5 = -6$. The vectors $s_i, i = 1, 2, \dots, 5$ are chosen according to (17) given by $s_1 = [0 \;\; 0]^T$, $s_2 = [1 \;\; 0]^T$, $s_3 = [1 \;\; 0]^T$, $s_4 = [0 \;\; 1]^T$, $s_5 = [0 \;\; 1]^T$. To compute the feedback matrix F, first one need to solve for v_i and w_i, $i \in \{1, 2, \dots, 5\}$, from the equation (15), which are obtained as

$$v_1 = \begin{bmatrix} 0.2673 \\ -0.1213 \\ -0.2974 \\ -0.5841 \\ 0.1460 \end{bmatrix}, v_2 = \begin{bmatrix} 0.5154 \\ 0.5254 \\ 1.4340 \\ -0.1631 \\ 0.0408 \end{bmatrix}, v_3 = \begin{bmatrix} 0.4081 \\ 0.5805 \\ 1.3238 \\ 0.0456 \\ -0.0114 \end{bmatrix}, v_4 = \begin{bmatrix} -0.2019 \\ -0.0673 \\ -0.4442 \\ 0.0769 \\ -0.2692 \end{bmatrix}, v_5 = \begin{bmatrix} -0.1978 \\ -0.0756 \\ -0.4190 \\ 0.0939 \\ -0.2735 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} -0.4799 \\ -0.4823 \end{bmatrix}, w_2 = \begin{bmatrix} 0.6174 \\ 1.7942 \end{bmatrix}, w_3 = \begin{bmatrix} 0.3910 \\ 1.5475 \end{bmatrix}, w_4 = \begin{bmatrix} -0.7404 \\ -0.7054 \end{bmatrix}, w_5 = \begin{bmatrix} -0.4713 \\ -0.6179 \end{bmatrix}.$$

The matrix F, can be obtained as

$$\begin{aligned} F &= [w_1 \; w_2 \; w_3 \; w_4 \; w_5][v_1 \; v_2 \; v_3 \; v_4 \; v_5]^{-1}. \\ &= \begin{bmatrix} 11.1134 & -9.3696 & 0.7204 & 6.8703 & -2.4684 \\ 1.4120 & -2.3561 & 1.7298 & 0.9741 & -0.4258 \end{bmatrix} \end{aligned}$$

The nominal control input is obtained as

$$u_n(t) = F(x(t) - x_s) + u_s$$

To compute the discontinuous part, $u_i(t)$, the sliding surface must be designed. The projection matrix, G, to compute sliding surface is taken as follows so that GB will be an identity matrix.

$$G = \begin{bmatrix} 1.0 & 1.0 & -1.0 & 0 & 1.0 \\ 1.0 & 0.2 & 1.0 & -0.1 & -1.0 \end{bmatrix}.$$

The sliding variable is computed using the expression

$$\sigma(t) = {}_0I_t^{1-\alpha}(Gx(t)) - G\int_0^t \frac{x_0}{\Gamma(1-\alpha)}(\tau)^{-\alpha}d\tau - G\int_0^t (Ax(\tau) + Bu_n(\tau))d\tau \tag{37}$$

The gain, $k = .5$, satisfies the condition $k \geq \|GB\|d_{max}$. The discontinuous control, $u_i(t)$, is obtained as

$$u_i(t) = -.5\frac{\sigma(t)}{\|\sigma(t)\|} \tag{38}$$

The ISMC applied to the FOS, $u(t)$, is given as

$$u(t) = u_n(t) + u_i(t)$$

Figure 1 a shows the output plot of the FOS with NOTC without ISMC and Figure 1 b shows the output plot of the FOS with robust NOTC with ISMC. Comparing the two output plots of the FOS from Figure 1 a and Figure 1 b, one can clearly observe that the robust NOTC proposed in Section 5 reject the effect of the disturbances acting on the FOS and make all output components to track the reference r avoiding overshoot. Figure 1 c shows the plots of the control input designed by NOTC without ISMC. Figure 1 d shows robust NOTC with ISMC applied to FOS. From the Figure 1 d it can be observed that the proposed robust NOTC has a high frequency switching and hence it will result in chattering in the system but it is able to reject the disturbance and give the desired non overshooting response. Figure 1 e and 1 f show the plots of the states of the FOS with NOTC and with robust NOTC, respectively. The low frequency disturbance acting on the system is plotted in Figure 1 g. The disturbance acting through the input channel is causing oscillations in the output and states in the case of NOTC but the robust NOTC is able to nullify the effect of the disturbances. The sliding variable plotted in Figure 1 h shows that the system trajectory starts from the sliding surface and is maintained on it. Since the system trajectory is on the sliding surface from the initial time, the robust NOTC is able to neutralize the effect of disturbance form the initial time. From the Figures 1 a and 1 b, one can conclude that the proposed ISMC performs better in the presence of disturbances. Since the nominal control is designed from NOTC, there is no overshoot in both output components. The speed of the output of the FOS can be made faster by choosing closed-loop poles further to the left on the negative real axis.

6.2 | Example 2

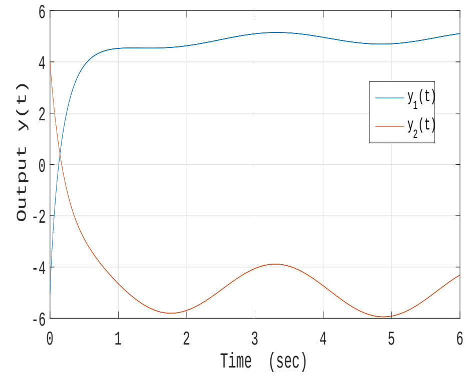
Here we consider an example FOS falling into the category of $n - lp$ stable invariant zeros, as discussed in Section 4.2. Let the LTI commensurate FOS in (9) have $\alpha = 0.9$, and

$$A = \begin{bmatrix} 0 & -9 & 5 & 0 & -3 & 8 \\ 2 & 0 & 0 & 7 & 0 & 1 \\ 1 & -9 & 1 & 0 & -8 & 0 \\ -2 & 0 & 8 & 5 & 0 & 7 \\ -1 & 0 & 0 & 0 & 7 & -2 \\ 1 & 0 & 9 & 2 & 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 2 & -10 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

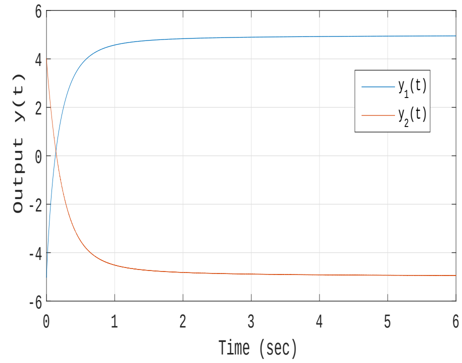
The matched disturbance entering the system is taken as high frequency signal given by $d(t) = [5\sin(100t)\ \ 4\cos(100t)]^T$. The initial condition is given by $x_0 = [34.57\ 15.18\ -2.24\ -28.15\ -2.73\ 14.42]^T$. The reference signal is taken to be $r = [6\ -6]^T$.

The values of the state and input at the steady state x_s and u_s, respectively are obtained by solving (11) and (12) as $x_s = [30.0000\ \ 10.7419\ \ -0.3871\ \ -18.8710\ \ -0.8710\ \ 12.0000]^T$, $u_s = [60.0968\ \ 4.6742]^T$.

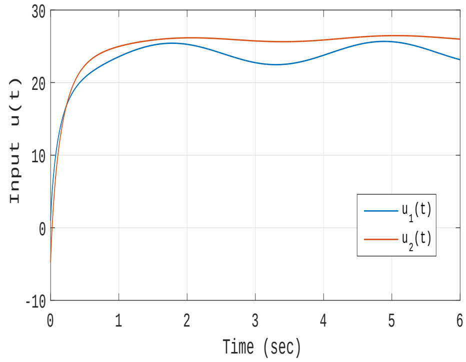
The initial condition x_0 satisfies the condition $x_0 - x_s \in J$ given in Notation 4.2. The ψ, obtained as $\psi = V^{-1}\zeta_0$ are all negative and hence $\zeta_0 = x_0 - x_s \in J$. So by Theorem 2, the NOTC designed will generate a non-overshooting tracking response in the absence of disturbance or from Theorem 3 the robust NOTC designed will generate a non-overshooting tracking response in the presence of matched and bounded disturbances acting on the system.



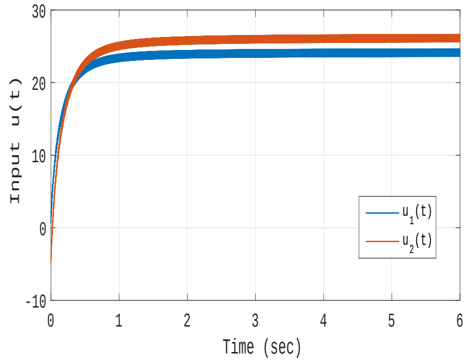
(a) Response of the FOS for NOTC without ISMC



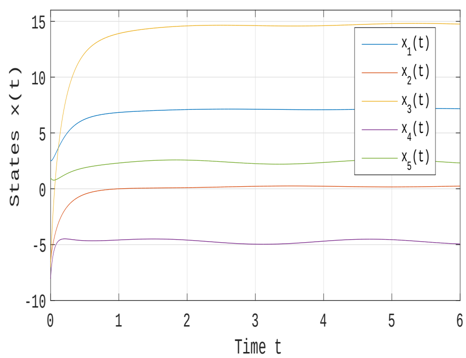
(b) Response of the FOS for Robust NOTC with ISMC



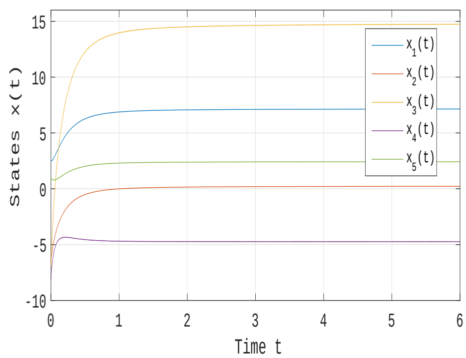
(c) NOTC input for FOS without ISMC



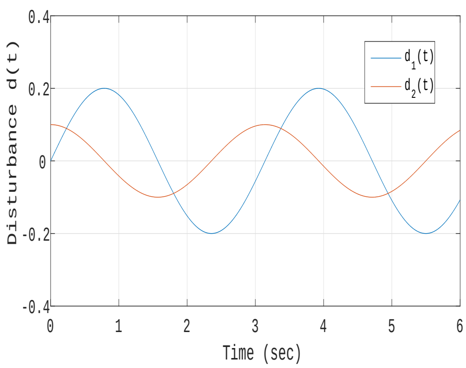
(d) Robust NOTC input for FOS with ISMC



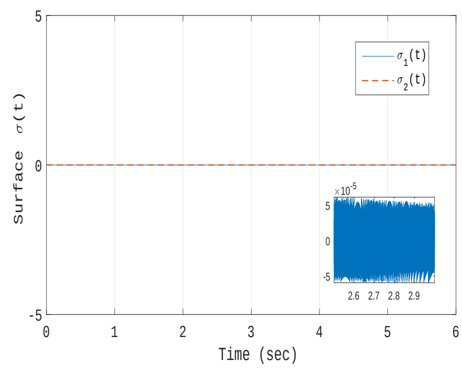
(e) State of the FOS with NOTC



(f) State of the FOS with robust NOTC

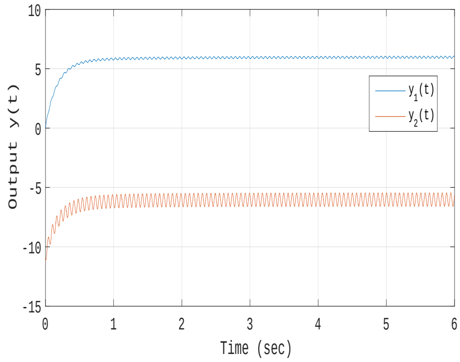


(g) Disturbance acting on the system

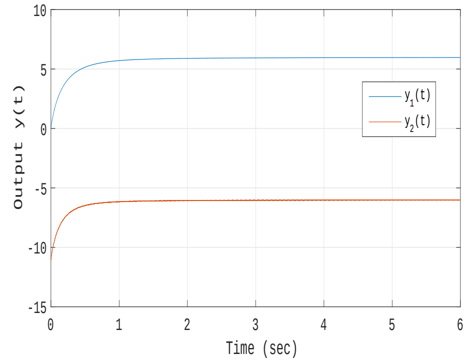


(h) Sliding surface for the robust NOTC input

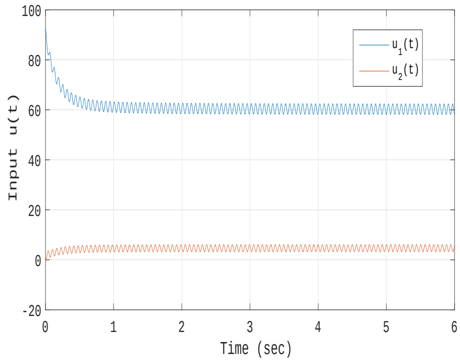
FIGURE 1 Plots for Example 1



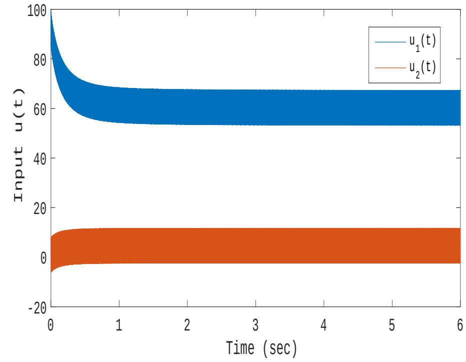
(a) Response of the FOS for NOTC without ISMC



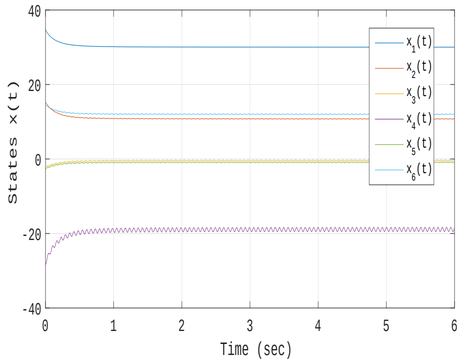
(b) Response of the FOS for Robust NOTC with ISMC



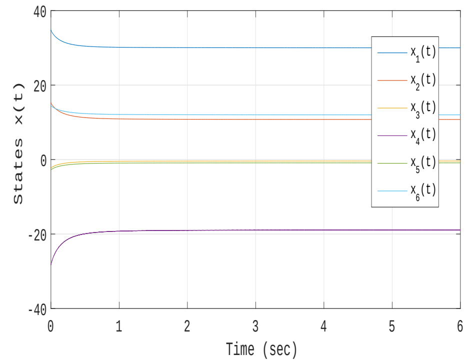
(c) NOTC input for FOS without ISMC



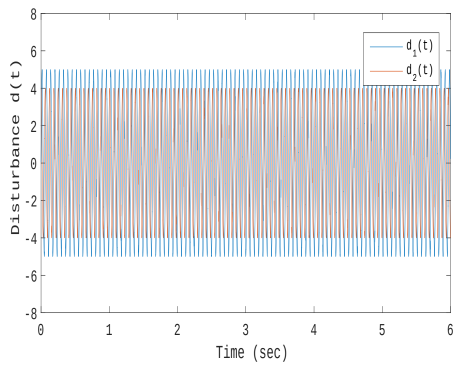
(d) Robust NOTC input for FOS with ISMC



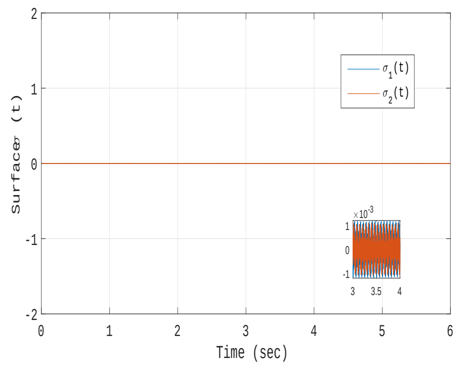
(e) State of the FOS with NOTC



(f) State of the FOS with robust NOTC



(g) Disturbance acting on the system



(h) Sliding surface for the robust NOTC input

FIGURE 2 Plots for Example 2

The robust NOTC, $u(t)$, proposed in Section 5.2 is given by

$$u(t) = u_n(t) + u_i(t)$$

The FOS considered has number of states $n = 6$, the number of outputs and inputs are same $p = m = 2$. The invariant zeros of the system are $2.2624 \pm 11.6318i$ and 1.2986. So there are no stable invariant zeros for the system. To compute the nominal control part $u_n(t)$, the system considered comes under the category of at least $n - lp$ stable invariant zeros, where $l = 3$. For computing nominal part, $u_n(t)$, the closed-loop poles are placed at $\lambda_1 = -4$, $\lambda_2 = -4.5$, $\lambda_3 = -5$, $\lambda_4 = -5.5$, $\lambda_5 = -6$, $\lambda_5 = -6.5$. The vectors $s_i, i = 1, 2, \dots, 6$ are chosen according to (17) given by $s_1 = [1\ \ 0]^T$, $s_2 = [1\ \ 0]^T$, $s_3 = [1\ \ 0]^T$, $s_4 = [0\ \ 1]^T$, $s_5 = [0\ \ 1]^T$ and $s_6 = [0\ \ 1]^T$. From equation (15), v_i and w_i, for $i = \{1, 2, \dots, 6\}$, are obtained as

$$v_1 = \begin{bmatrix} 0.0672 \\ -0.3011 \\ 0.1069 \\ 0.5248 \\ 0.1370 \\ -0.3878 \end{bmatrix}, v_2 = \begin{bmatrix} 0.0895 \\ -0.2715 \\ 0.1243 \\ 0.5147 \\ 0.1301 \\ -0.3846 \end{bmatrix}, v_3 = \begin{bmatrix} 0.1137 \\ -0.2167 \\ 0.1670 \\ 0.5026 \\ 0.1221 \\ -0.3806 \end{bmatrix}, v_4 = \begin{bmatrix} -1.0715 \\ -0.5728 \\ 0.2385 \\ 1.2601 \\ 0.2180 \\ -0.0421 \end{bmatrix}, v_5 = \begin{bmatrix} -0.8648 \\ -0.5102 \\ 0.1866 \\ 1.1680 \\ 0.2017 \\ 0.0337 \end{bmatrix}, v_6 = \begin{bmatrix} -0.7814 \\ -0.4876 \\ 0.1598 \\ 1.1339 \\ 0.1926 \\ 0.0587 \end{bmatrix}.$$

$$w_1 = \begin{bmatrix} -2.2156 \\ -0.1702 \end{bmatrix}, w_2 = \begin{bmatrix} -2.1758 \\ -0.1338 \end{bmatrix}, w_3 = \begin{bmatrix} -2.1732 \\ -0.0624 \end{bmatrix}, w_4 = \begin{bmatrix} -3.7714 \\ 0.8815 \end{bmatrix}, w_5 = \begin{bmatrix} -3.4193 \\ 0.9467 \end{bmatrix}, w_6 = \begin{bmatrix} -3.2645 \\ 0.9764 \end{bmatrix}.$$

The matrix F, can be obtained as

$$\begin{aligned} F &= [w_1\ w_2\ w_3\ w_4\ w_5\ w_6][v_1\ v_2\ v_3\ v_4\ v_5\ v_6]^{-1}. \\ &= \begin{bmatrix} 0.7257 & -4.4439 & 2.0635 & -2.5304 & -12.6647 & 1.9595 \\ -0.3498 & 3.5949 & -1.9480 & 2.2354 & 1.0685 & 0.4525 \end{bmatrix} \end{aligned}$$

The nominal control input is obtained as

$$u_n(t) = F(x(t) - x_s) + u_s$$

To compute the discontinuous part, $u_i(t)$, sliding surface must be designed. The projection matrix to compute sliding surface is taken as

$$G = \begin{bmatrix} 1.0 & 1.0 & -1.0 & 0 & 1.0 & 0 \\ 1.0 & 0.2 & 1.0 & -0.1 & -2.0 & 1.0 \end{bmatrix}.$$

The sliding variable is computed using the expression

$$\sigma(t) = {}_0I_t^{1-\alpha}(Gx(t)) - G\int_0^t \frac{x_0}{\Gamma(1-\alpha)}(\tau)^{-\alpha}d\tau - G\int_0^t (Ax(\tau) + Bu_n(\tau))d\tau \tag{39}$$

The gain, $k = 7$, satisfies the condition $k \geq \|GB\|d_{max}$. The discontinuous control is computed as

$$u_i(t) = -7\frac{\sigma(t)}{\|\sigma(t)\|} \tag{40}$$

The robust NOTC with ISMC applied to the FOS, $u(t)$, is given as

$$u(t) = u_n(t) + u_i(t)$$

Figure 2 a and Figure 2 b show the output plots of the FOS with NOTC without ISMC and robust NOTC with ISMC. As in the Example 6.1, here also the robust NOTC is able to reject the effect of the disturbances and perform better without causing any overshoot and oscillations in the output. Figure 2 c shows the plots of the control input designed by NOTC without ISMC. Figure 2 d shows robust NOTC with ISMC applied to FOS. Here also the high frequency switching in the control can be observed from the Figure 2 d where as the input applied in the case of NOTC in Figure 2 c is continuous and does not have any high frequency switching. The high frequency switching will cause chattering in the system which will results in wear and tear in the system and actuator. But even the effect of the high frequency disturbance considered in the system is neutralised by the robust NOTC. Figure 2 e shows the plot of the states of the FOS with NOTC and 2 f shows the states of FOS with robust NOTC. The high frequency disturbance acting on the system is plotted in Figure 2 g. The sliding variable plotted in Figure 2 h.

In Example 6.1 a low frequency disturbance was considered and in Example 6.2 a high frequency disturbance was considered, but the robust NOTC proposed in the Section 5 was able to successfully reject the disturbances and ensure a non-overshooting tracking response in both the cases.

7 | CONCLUSION AND FUTURE WORKS

NOTC is designed for MIMO FOS to achieve an non-overshooting output response that tracks the reference step signal without any overshoot. By this technique, rise time can be reduced arbitrarily without any overshoot in the tracking response. NOTC can be computed for any square MIMO FOS by using the assumption of least $n-lp$ stable invariant zeros, by suitably choosing l in assumption. Secondly, a robust NOTC for MIMO FOS is designed by using ISMC technique. A fractional integral sliding surface has been designed for the ISMC. The nominal control part of ISMC is designed using Moore's eigenstructure assignment technique so that there is no overshoot in the response. The discontinuous control part of ISMC provides robustness towards the effect of the disturbances.

The proposed ISMC for FOS is able to reject the disturbances, but it causes chattering phenomenon in the system due to the discontinuous control. The in future work the possibility of using the higher order sliding mode control techniques which are capable of reducing the chattering in the system can be investigated. The effect of delays and measurement noise are not studied in the present paper which also can be considered in the future works since both may affect the stability and transient response of the system.

APPENDIX

To prove Lemma 1, we firstly prove

Lemma 5. Let $0 < \alpha < 1$ and let $\lambda_1 < \lambda_2 < 0$. Then for all $t > 0$,

$$\sup_{t>0} \frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_2 t^\alpha)} = 1 \quad \text{and} \quad \inf_{t>0} \frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_2 t^\alpha)} = \frac{\lambda_2}{\lambda_1} \tag{41}$$

Proof. For brevity we introduce

$$\rho(t) = \frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_2 t^\alpha)} \tag{42}$$

As $E_\alpha(0) = 1$ and $E_\alpha(\lambda_1 t^\alpha) < E_\alpha(\lambda_2 t^\alpha)$ for all $t > 0$, we have the first equation in (41).

First derivative of Mittag-Leffler function[26] for a complex z and $\beta = 1$, is given by

$$\left(\frac{d}{dz}\right)[E_\alpha(z)] = (\alpha z)^{-1} E_{\alpha,0}(z) \tag{43}$$

If we introduce $z = \lambda t^\alpha$, we may obtain the integer derivative (with respect to t) as

$$\begin{aligned} E_\alpha^{(1)}(\lambda t^\alpha) &= \left(\frac{d}{dz}\right)[E_\alpha(z(t))]\frac{dz}{dt} \\ &= (\alpha z)^{-1} E_{\alpha,0}(z)\frac{dz}{dt} \\ &= \frac{1}{\alpha\lambda t^\alpha} E_{\alpha,0}(\lambda t^\alpha)(\alpha\lambda t^{\alpha-1}) \\ &= t^{-1} E_{\alpha,0}(\lambda t^\alpha) \end{aligned} \tag{44}$$

Hence, form the equation (44) one can obtain that, for all $t > 0$,

$$E_\alpha^{(1)}(\lambda_1 t^\alpha) = t^{-1} E_{\alpha,0}(\lambda_1 t^\alpha) < t^{-1} E_{\alpha,0}(\lambda_2 t^\alpha) = E_\alpha^{(1)}(\lambda_2 t^\alpha) < 0 \tag{45}$$

so ρ is monotonically decreasing. Finally we note the well-known result[36] that, for $0 < \alpha < 1$, $\tau > 0$ and $t \gg 0$,

$$E_\alpha(-(t/\tau)^\alpha) \simeq \frac{\sin(\alpha\pi)\Gamma(\alpha)}{\pi(t/\tau)^\alpha} \tag{46}$$

To obtain the second equation in (41), we introduce $\tau_1, \tau_2 > 0$ such that $\tau_1^\alpha = 1/|\lambda_1|$ and $\tau_2^\alpha = 1/|\lambda_2|$. Then

$$\begin{aligned}\inf_{t>0} \rho(t) &= \lim_{t\to\infty} \rho(t) \\ &= \lim_{t\to\infty} \frac{(t/\tau_2)^\alpha}{(t/\tau_1)^\alpha} \\ &= \frac{\lambda_2}{\lambda_1} \end{aligned} \tag{47}$$

□

Now we can prove Lemma 1. Firstly assume that f has a root in $\mathbb{R}^+$; then there exists $t_* > 0$ such that $f(t_*) = 0$, and we obtain

$$\frac{E_\alpha(\lambda_1 t_*^\alpha)}{E_\alpha(\lambda_2 t_*^\alpha)} = \frac{-\gamma_2}{\gamma_1} \tag{48}$$

Applying Lemma 5, we obtain (6). Next assume (6); then by the continuity of ρ in (42), there exists $t_* > 0$ satisfying (48), and hence $f(t_*) = 0$.

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