

Renormalization of symmetric bimodal maps with low smoothness

Rohit Kumar¹, V.V.M.S. Chandramouli²

^{1,2} Department of Mathematics, Indian Institute of Technology Jodhpur, Rajasthan, India-342037.

E-mail: ¹ kumar.30@iitj.ac.in, ² chsarma@iitj.ac.in.

Abstract. This paper deals with the renormalization of symmetric bimodal maps with low smoothness. We prove the existence of the renormalization fixed point in the space C^{1+Lip} symmetric bimodal maps. Moreover, we show that the topological entropy of the renormalization operator defined on the space of C^{1+Lip} symmetric bimodal maps is infinite. Further we prove the existence of a continuum of fixed points of renormalization. Consequently, this proves the non-rigidity of the renormalization of symmetric bimodal maps.

Keywords: Renormalization fixed point, symmetric bimodal maps, low smoothness, non-rigidity.

1. Introduction

Renormalization is a technique to analyze maps having the property that the first return map to small part of the phase space resembles the original map itself. Period doubling renormalization operator was introduced by M. Feigenbaum [1], [2] and by P. Couillet and C. Tresser [3], to study asymptotic small scale geometry of the attractor of one dimensional systems which are at the transition from simple to chaotic dynamics.

With a relatively complete understanding of the period doubling renormalization of unimodal maps, recent research in dynamical systems has focused on more complicated maps of the real line. Renormalization is very useful tool to describe the dynamics of a map in smaller scale. In particular, Jonker & Rand [4] and V. Strien [5] used renormalization as a natural vehicle to decompose the non-wandering set in a hierarchical manner, for unimodal maps. The multimodal maps are interesting as generalizations of unimodal maps, as well as for their applications. For example, in the case of bimodal maps, they are essential to the understanding of non-invertible circle maps which have been used extensively to model the transitions to chaos in two frequency systems [6]. D. Veitch presented some work on topological renormalization of C^0 bimodal maps with zero and positive entropy [7]. Further, D. Smania developed a combinatorial theory for

certain kind of multimodal maps and proved that for the same combinatorial type the renormalizations of infinitely renormalizable smooth multimodal maps are exponentially close [8].

In this paper, we focus on the construction of renormalization fixed point for the family of symmetric bimodal maps with low smoothness. We show that there exists a sequence of affine pieces which are nested and contract to the critical points of the bimodal map corresponding to a pair of proper scaling data. This helps us to prove that the renormalization operator defined on the space of piece-wise affine infinitely renormalizable maps has a fixed point, denoted by f_{s^*} , corresponding to a pair of proper scaling data s^* . In the next section 3, we explain the extension of the renormalization fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map. In section 4, we describe the topological entropy of renormalization defined on the space of C^{1+Lip} symmetric bimodal maps. Furthermore, we prove the existence of another fixed point of renormalization by considering the small perturbation on the scaling data. Consequently, for two different perturbed scaling data we get two Cantor attractors of renormalization fixed points. This leads to the non-rigidity of the Cantor attractors of renormalizable symmetric bimodal map with low smoothness.

We recall some basic definitions. Let $I = [a, b]$ be a closed interval.

A unimodal map $\mathbf{u} : I \rightarrow I$ is called *period tripling renormalizable map* if there exists a proper subinterval J of I such that

- (1) J , $\mathbf{u}(J)$ and $\mathbf{u}^2(J)$ are pairwise disjoint,
- (2) $\mathbf{u}^3(J) \subset J$.

Then $\mathbf{u}^3 : J \rightarrow J$ is called a renormalization of \mathbf{u} .

A map $\mathbf{u} : I \rightarrow I$ is *period tripling infinitely renormalizable map* if there exists an infinite sequence $\{I_n\}_{n=0}^{\infty}$ of nested intervals such that $\mathbf{u}^3|_{I_n} : I_n \rightarrow I_n$ are renormalizations of \mathbf{u} and the length of I_n tends to zero as $n \rightarrow \infty$.

Let \mathcal{U} be the collection of unimodal maps and $\mathcal{U}_{\infty}(\subset \mathcal{U})$ be the collection of period tripling infinitely renormalizable unimodal maps.

An interval map f is *piece-wise monotone* if there exists a partition of I into finitely many subintervals on each of which f is strictly monotonic.

If three is the minimal number of such subintervals, we say f is a *bimodal map*.

Definition 1.1. Let $f : I \rightarrow I$ be a map with two subsets J_l and J_r such that $J_l^{\circ} \cap J_r^{\circ} = \emptyset$. If $f|_{J_l}$ and $f|_{J_r}$ are unimodal maps which are concave up and concave down respectively, their *join*, denoted by $f|_{J_l} \oplus f|_{J_r}$, is a bimodal map whose graph is obtained by joining $f(\max(J_l))$ and $f(\min(J_r))$ by a C^{1+Lip} curve.

Definition 1.2. A bimodal map $b : I \rightarrow I$, having two critical points c_l and c_r , is said to

be *renormalizable* if there exists two disjoint intervals I_l containing c_l and I_r containing c_r such that

- (i) $b^i(I_l) \cap b^j(I_l) = \emptyset$, for each $i \neq j$ and $i, j \in \{0, 1, 2\}$,
 $b^i(I_r) \cap b^j(I_r) = \emptyset$, for each $i \neq j$ and $i, j \in \{0, 1, 2\}$,
- (ii) $b^3(I_l) \subset I_l$ and $b^3(I_r) \subset I_r$,
- (iii) After applying suitable reflections and rescalings on the unimodal maps $b|_{I_l}$ and $b|_{I_r}$, the unimodal pieces b'_l and b'_r are joined to generate a bimodal map.

The renormalization of a symmetric bimodal map is illustrated in Figure 1.

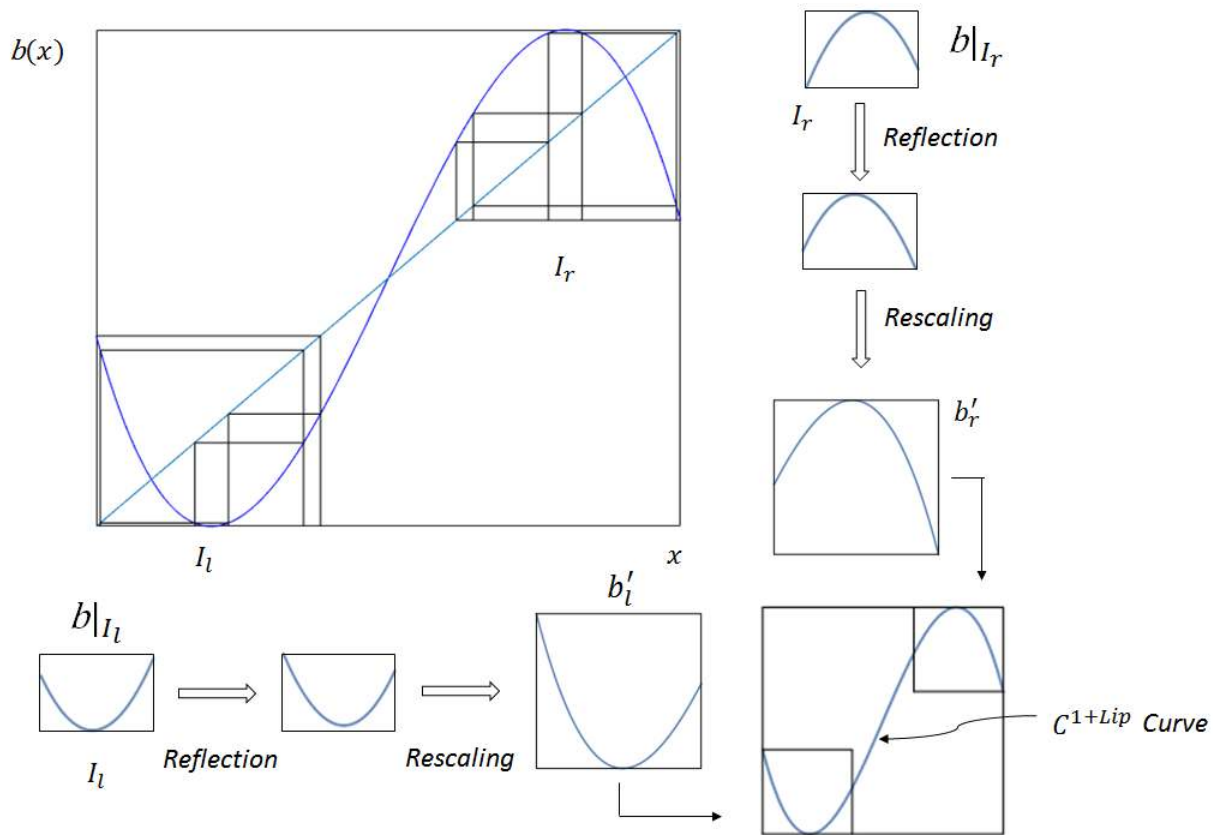


Figure 1: Renormalization of a bimodal map

In the next section, we construct the renormalization operator defined on the space of piece-wise affine maps which are infinitely renormalizable maps.

2. Piece-wise affine renormalizable maps

A symmetric bimodal map $b : [0, 1] \rightarrow [0, 1]$ of the form $b(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, for $a_3 < 0$, is a C^1 map with the following conditions

- $b(0) = 1 - b(1)$,

- $b(\frac{1}{2}) = \frac{1}{2}$,
- let c_l and c_r be the two critical points of $b(x)$, then $b(c_l) = 0$ and $b(c_r) = 1$.

Let us consider a family of symmetric bimodal maps $B_c : [0, 1] \rightarrow [0, 1]$ which are increasing on the interval between the critical points and decreasing elsewhere. then, we obtained a family of bimodal maps as

$$B_c(x) = \begin{cases} 1 - \frac{1-6c+9c^2-4c^3+6cx-6c^2x-3x^2+2x^3}{(1-2c)^3}, & \text{if } c \in [0, \frac{1}{4}] \\ 1 - \frac{4c^3-3c^2+6cx-6c^2x-3x^2+2x^3}{(2c-1)^3}, & \text{if } c \in [\frac{3}{4}, 1] \end{cases} \quad (2.1)$$

$$\equiv \begin{cases} b_c(x), & \text{if } c \in [0, \frac{1}{4}] \\ \tilde{b}_c(x), & \text{if } c \in [\frac{3}{4}, 1] \end{cases}$$

Note that the bimodal maps b_c and \tilde{b}_c are identical maps.

Let us define an open set

$$\Delta^3 = \left\{ (s_0, s_1, s_2) \in \mathbb{R}^3 : s_0, s_1, s_2 > 0, \sum_{i=0}^2 s_i < 1 \right\}.$$

Each element (s_0, s_1, s_2) of Δ^3 is called a scaling tri-factor. A pair of scaling tri-factors $(s_{0,l}, s_{1,l}, s_{2,l})$ and $(s_{0,r}, s_{1,r}, s_{2,r})$ induces two sets of affine maps $(F_{0,l}, F_{1,l}, F_{2,l})$ and $(F_{0,r}, F_{1,r}, F_{2,r})$ respectively. For each $i = 0, 1, 2$,

$$F_{i,l} : I_L = [0, b_c(0)] \rightarrow I_L \quad \text{and} \quad F_{i,r} : I_R = [\tilde{b}_c(1), 1] \rightarrow I_R$$

are defined as

$$\begin{aligned} F_{0,l}(t) &= b_c(0) - s_{0,l} \cdot t; & F_{0,r}(t) &= \tilde{b}_c(1) + s_{0,r} \cdot (1 - t) \\ F_{1,l}(t) &= b_c^2(0) - s_{1,l} \cdot t; & F_{1,r}(t) &= \tilde{b}_c^2(1) + s_{1,r} \cdot (1 - t) \\ F_{2,l}(t) &= s_{2,l} \cdot t; & F_{2,r}(t) &= 1 - s_{2,r} \cdot (1 - t). \end{aligned}$$

Note that $I_L^\circ \cap I_R^\circ = \emptyset$, for $c \in [0, \frac{3-\sqrt{3}}{6}]$.

The functions $s_l : \mathbb{N} \rightarrow \Delta^3$ and $s_r : \mathbb{N} \rightarrow \Delta^3$ are said to be a scaling data. We set scaling tri-factors

$s_l(n) = (s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in \Delta^3$ and $s_r(n) = (s_{0,r}(n), s_{1,r}(n), s_{2,r}(n)) \in \Delta^3$, so that $s_l(n)$ and $s_r(n)$ induce the triplets of affine maps $(F_0^l(n)(t), F_1^l(n)(t), F_2^l(n)(t))$ and $(F_0^r(n)(t), F_1^r(n)(t), F_2^r(n)(t))$ as described above.

For $i = 0, 1, 2$, let us define the intervals

$$I_{i,l}^n = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ \dots \circ F_{1,l}(n-1) \circ F_{i,l}(n)([0, b_c(0)]).$$

Also,

$$I_{i,r}^n = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ \dots \circ F_{1,r}(n-1) \circ F_{i,r}(n)([\tilde{b}_c(1), 1]).$$

Definition 2.1. A scaling data $s(n)$ is said to be proper if

$$d(s(n), \partial\Delta^3) \geq \epsilon, \quad \text{for some } \epsilon > 0.$$

A pair of proper scaling data $s_l : \mathbb{N} \rightarrow \Delta^3$ and $s_r : \mathbb{N} \rightarrow \Delta^3$ induce the sets $D_{s_l} = \bigcup_{n \geq 1} (I_{0,l}^n \cup I_{2,l}^n)$ and $D_{s_r} = \bigcup_{n \geq 1} (I_{0,r}^n \cup I_{2,r}^n)$, respectively. Consider a map

$$f_s : D_{s_l} \cup D_{s_r} \rightarrow [0, 1]$$

defined as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

where $f_{s_l}|_{I_{0,l}^n}$ and $f_{s_l}|_{I_{2,l}^n}$ are the affine extensions of $b_c|_{\partial I_{0,l}^n}$ and $b_c|_{\partial I_{2,l}^n}$ respectively. Similarly, $f_{s_r}|_{I_{0,r}^n}$ and $f_{s_r}|_{I_{2,r}^n}$ are the affine extensions of $b_c|_{\partial I_{0,r}^n}$ and $b_c|_{\partial I_{2,r}^n}$ respectively. These affine extensions are shown in Figure 2.

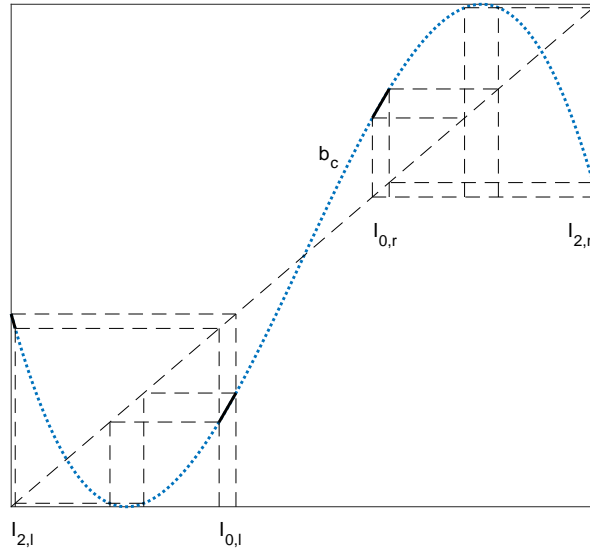


Figure 2

The end points of the intervals at each level are labeled by

$$y_0 = 0, \quad z_0 = b_c(0), \quad I_{1,l}^0 = I_L = [0, b_c(0)]$$

and for $n \geq 1$

$$\begin{aligned} x_n &= \partial I_{0,l}^n \setminus \partial I_{1,l}^{n-1} \\ y_{2n-1} &= \max\{\partial I_{1,l}^{2n-1}\} \\ y_{2n} &= \min\{\partial I_{1,l}^{2n}\} \\ z_{2n-1} &= \min\{\partial I_{1,l}^{2n-1}\} \\ z_{2n} &= \max\{\partial I_{1,l}^{2n}\} \\ w_n &= \partial I_{2,l}^n \setminus \partial I_{1,l}^{n-1}. \end{aligned}$$

These points are illustrated in Figure 3.

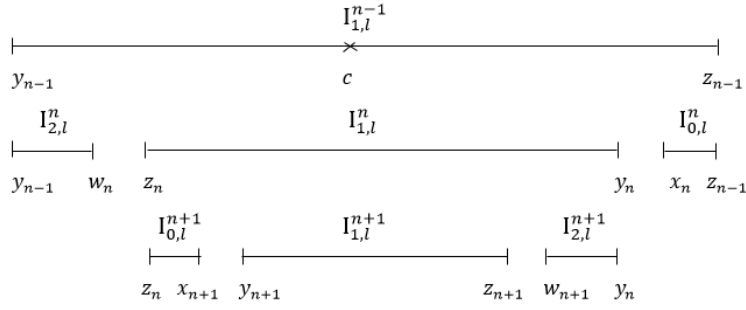


Figure 3: Intervals of next generations

Also, the end points of the intervals at each level are labeled by

$$z'_0 = b_c(1), y'_0 = 1, I_{1,r}^0 = I_R = [\tilde{b}_c(1), 1]$$

and for $n \geq 1$

$$\begin{aligned} x'_n &= \partial I_{0,r}^n \setminus \partial I_{1,r}^{n-1} \\ y'_{2n-1} &= \min\{\partial I_{1,r}^{2n-1}\} \\ y'_{2n} &= \max\{\partial I_{1,r}^{2n}\} \\ z'_{2n-1} &= \max\{\partial I_{1,r}^{2n-1}\} \\ z'_{2n} &= \min\{\partial I_{1,r}^{2n}\} \\ w'_n &= \partial I_{2,r}^n \setminus \partial I_{1,r}^{n-1}. \end{aligned}$$

These points are illustrated in Figure 4.

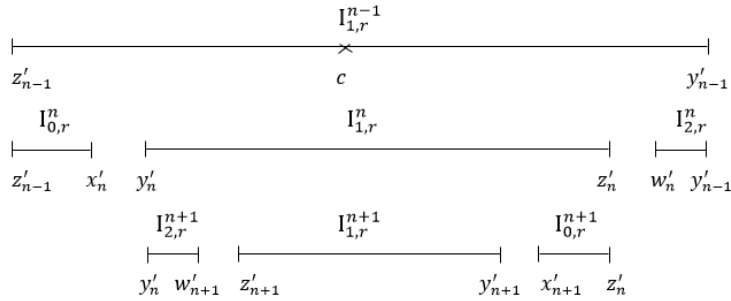


Figure 4: Intervals of next generations

Definition 2.2. For a given pair of proper scaling data $s_l, s_r : \mathbb{N} \rightarrow \Delta^3$, a map f_s is said to be *infinitely renormalizable* if for $n \geq 1$,

- (1) $[0, f_{s_l}(y_n)]$ is the maximal domain containing 0 on which $f_{s_l}^{3^n-1}$ is defined affinely, $[f_{s_l}^2(y_n), f_{s_l}(0)]$ is the maximal domain containing $f_{s_l}(0)$ on which $f_{s_l}^{3^n-2}$ is defined affinely, $[f_{s_r}(y'_n), 1]$ is the maximal domain containing 1 on which $f_{s_r}^{3^n-1}$ is defined affinely and $[f_{s_r}(1), f_{s_r}^2(y'_n)]$ is the maximal domain containing $f_{s_r}(1)$ on which $f_{s_r}^{3^n-2}$ is defined affinely,

$$\begin{aligned}
(2) \quad & f_{s_l}^{3^n-1}([0, f_{s_l}(y_n)]) = I_{1,l}^n, \\
& f_{s_l}^{3^n-2}([f_{s_l}^2(y_n), f_{s_l}(0)]) = I_{1,l}^n, \\
& f_{s_r}^{3^n-1}([f_{s_r}(y'_n), 1]) = I_{1,r}^n, \\
& f_{s_r}^{3^n-2}([f_{s_r}(1), f_{s_r}^2(y'_n)]) = I_{1,r}^n.
\end{aligned}$$

The combinatorics for renormalization of f_{s_l} and f_{s_r} are shown in the following Figures 5a and 5b.

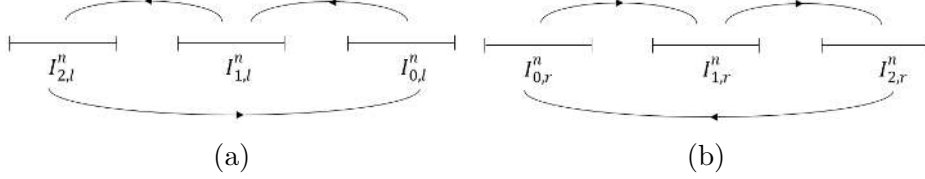


Figure 5: The combinatorics: (a) corresponding to f_{s_l} , $(I_{1,l}^n \rightarrow I_{2,l}^n \rightarrow I_{0,l}^n \rightarrow I_{1,l}^n)$ and (b) corresponding to f_{s_r} , $(I_{1,r}^n \rightarrow I_{2,r}^n \rightarrow I_{0,r}^n \rightarrow I_{1,r}^n)$.

2.1. Renormalization on $I_L = [0, b_c(0)]$

Let $f_{s_l} \in \mathcal{U}_\infty$ be given by the proper scaling data $s_l : \mathbb{N} \rightarrow \Delta^3$ and define

$$\tilde{I}_{1,l}^n = [\max\{b_c^{-1}(z_n)\}, b_c(0)] = [\max\{f_{s_l}^{-1}(z_n)\}, f_{s_l}(0)],$$

where $b_c^{-1}(x)$ denotes the preimage(s) of x under b_c and

$$\hat{I}_{1,l}^n = [0, b_c(y_n)] = [0, f_{s_l}(y_n)].$$

Let

$$h_{s_l,n} : [0, b_c(0)] \rightarrow [0, b_c(0)]$$

be defined by

$$h_{s_l,n} = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ \dots \circ F_{1,l}(n)$$

Furthermore, let

$$\tilde{h}_{s_l,n} : [0, b_c(0)] \rightarrow \tilde{I}_{1,l}^n \quad \text{and} \quad \hat{h}_{s_l,n} : [0, b_c(0)] \rightarrow \hat{I}_{1,l}^n$$

be the affine orientation preserving homeomorphisms. Then define

$$R_n^l f_s : h_{s_l,n}^{-1}(D_{s_l}) \rightarrow [0, b_c(0)]$$

by

$$R_n^l f_{s_l}(x) = \begin{cases} R_n^- f_{s_l}(x), & \text{if } x \in h_{s_l,n}^{-1}(I_{0,l}^n) \\ R_n^+ f_{s_l}(x), & \text{if } x \in h_{s_l,n}^{-1}(I_{2,l}^n) \end{cases}$$

where,

$$R_n^- f_{s_l} : h_{s_l,n}^{-1}(\cup_{n \geq 1} I_{0,l}^n) \rightarrow [0, b_c(0)]$$

and

$$R_n^{l+} f_{s_l} : h_{s_l, n}^{-1} \left(\bigcup_{n \geq 1} I_{2, l}^n \right) \rightarrow [0, b_c(0)]$$

are defined by

$$\begin{aligned} R_n^{l-} f_{s_l}(x) &= \tilde{h}_{s_l, n}^{-1} \circ f_{s_l}^{-1} \circ h_{s_l, n}(x) \\ R_n^{l+} f_{s_l}(x) &= \hat{h}_{s_l, n}^{-1} \circ f_{s_l} \circ h_{s_l, n}(x), \end{aligned}$$

which are illustrated in Figure 6.

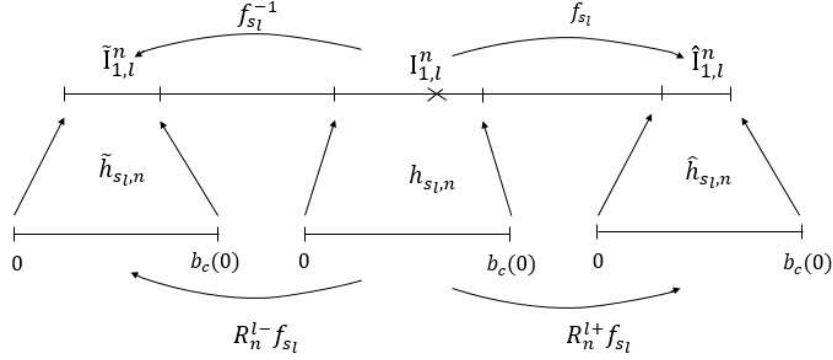


Figure 6

Let $\sigma : (\Delta^3)^\mathbb{N} \rightarrow (\Delta^3)^\mathbb{N}$ be the shift map defined as $\sigma(s_1^1 s_1^2 s_1^3 s_1^4 \dots) = (s_1^2 s_1^3 s_1^4 \dots)$, where $s_i^j \in \Delta^3$ for all $i \in \mathbb{N}$.

Lemma 2.1. *Let $s_l : \mathbb{N} \rightarrow \Delta^3$ be proper scaling data such that f_{s_l} is infinitely renormalizable. Then*

$$R_n^l f_{s_l} = f_{\sigma^n(s_l)}.$$

Let f_{s_l} be infinitely renormalization, then for $n \geq 0$, we have

$$f_{s_l}^{3^n} : D_{s_l} \cap I_{1, l}^n \rightarrow I_{1, l}^n$$

is well defined.

Define the renormalization $R^l : \mathcal{U}_\infty \rightarrow \mathcal{U}_\infty$ by

$$R^l f_{s_l} = h_{s_l, 1}^{-1} \circ f_{s_l}^3 \circ h_{s_l, 1}.$$

The maps $f_{s_l}^{3^n-2} : \tilde{I}_{1, l}^n \rightarrow I_{1, l}^n$ and $f_{s_l}^{3^n-1} : \hat{I}_{1, l}^n \rightarrow I_{1, l}^n$ are the affine homeomorphisms whenever $f_{s_l} \in \mathcal{U}_\infty$. Then

Lemma 2.2. *We have $(R^l)^n f_{s_l} : D_{\sigma^n(s_l)} \rightarrow [0, b_c(0)]$ and $(R^l)^n f_{s_l} = R_n^l f_{s_l}$.*

The lemma 2.1 and lemma 2.2 give the following result.

Proposition 2.3. *There exists a map $f_{s_l^*} \in \mathcal{U}_\infty$, where s_l^* is characterized by*

$$R^l f_{s_l^*} = f_{s_l^*}.$$

Proof. Consider $s_l : \mathbb{N} \rightarrow \Delta^3$ be proper scaling data such that f_{s_l} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_l)}$. Then

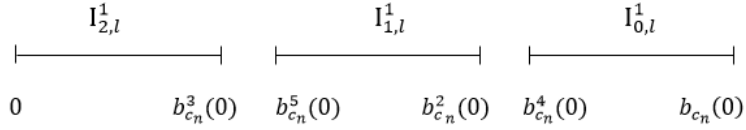


Figure 7: Length of intervals

we have the following scaling ratios which are illustrated in Figure 7

$$s_{0,l}(n) = \frac{b_{c_n}(0) - b_{c_n}^4(0)}{b_{c_n}(0)} \quad (2.2)$$

$$s_{1,l}(n) = \frac{b_{c_n}^2(0) - b_{c_n}^5(0)}{b_{c_n}(0)} \quad (2.3)$$

$$s_{2,l}(n) = \frac{b_{c_n}^3(0)}{b_{c_n}(0)} \quad (2.4)$$

$$c_{n+1} = \frac{b_{c_n}^2(0) - c_n}{s_{1,l}(n)} \equiv \mathcal{R}(c_n). \quad (2.5)$$

Since $(s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in \Delta^3$, this implies the following conditions

$$s_{0,l}(n), s_{1,l}(n), s_{2,l}(n) > 0 \quad (2.6)$$

$$s_{0,l}(n) + s_{1,l}(n) + s_{2,l}(n) < 1 \quad (2.7)$$

As the intervals $I_{i,l}^n$, for $i = 0, 1, 2$, are mutually disjoint, we denote the gap ratios as $g_{0,l}^n$ and $g_{1,l}^n$ which are in between $I_{0,l}^n$ & $I_{1,l}^n$ and $I_{1,l}^n$ & $I_{2,l}^n$ respectively. The gap ratios are defined as,

for $n \in \mathbb{N}$,

$$g_{0,l}^n = \frac{b_{c_n}^4(0) - b_{c_n}^2(0)}{b_{c_n}(0)} \equiv G_{0,l}(c_n) > 0 \quad (2.8)$$

$$g_{1,l}^n = \frac{b_{c_n}^5(0) - b_{c_n}^3(0)}{b_{c_n}(0)} \equiv G_{1,l}(c_n) > 0 \quad (2.9)$$

$$0 < c_n < \frac{3 - \sqrt{3}}{6} \quad (2.10)$$

We use Mathematica for solving the equations (2.2), (2.3) and (2.4), then we get the expressions for $s_{0,l}(n)$, $s_{1,l}(n)$ and $s_{2,l}(n)$.

Let $s_{i,l}(n) \equiv S_{i,l}(c_n)$ for $i = 0, 1, 2$. The graphs of $S_{i,l}(c)$ are shown in Figures 8a, 8b and 8c.

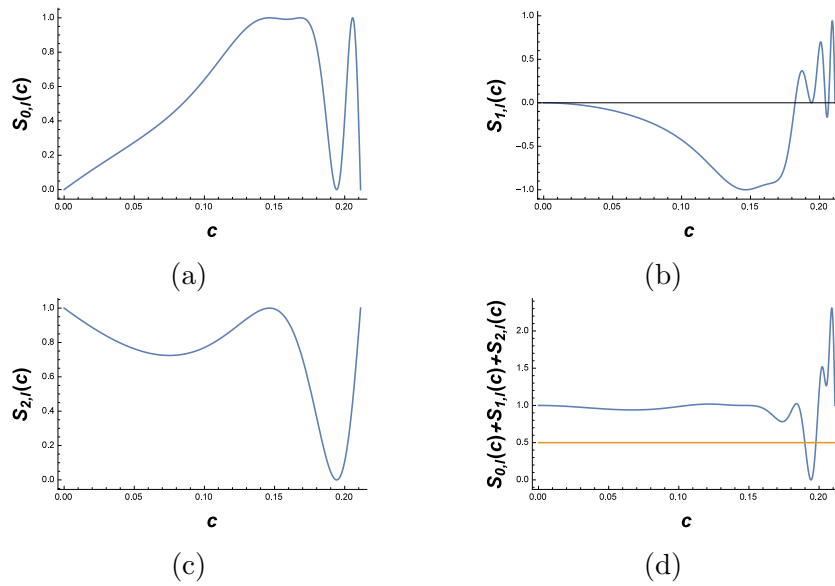


Figure 8: (a), (b), (c) and (d) show the graphs of $S_{0,l}(c)$, $S_{1,l}(c)$, $S_{2,l}(c)$ and $(S_{0,l} + S_{1,l} + S_{2,l})(c)$.

Note that the conditions (2.6), (2.8) and (2.9) give the condition (2.7)

$$0 < \sum_{i=0}^2 s_{i,l}(n) < 1.$$

The conditions (2.6) together with (2.8) to (2.10) define the feasible domain F_d^l is to be:

$$F_d^l = \left\{ c \in \left(0, \frac{3 - \sqrt{3}}{6} \right) : S_{i,l}(c) > 0 \text{ for } i = 0, 1, 2, G_{0,l}(c) > 0, G_{1,l}(c) > 0 \right\}. \quad (2.11)$$

To compute the feasible domain F_d^l , we need to find subinterval(s) of $\left(0, \frac{3 - \sqrt{3}}{6} \right)$ which satisfies the conditions of (2.11). By using Mathematica software, we employ the following command to obtain the feasible domain

$$\text{N[Reduce[\{ } S_{0,l}(c) > 0, S_{1,l}(c) > 0, S_{2,l}(c) > 0, G_{0,l}(c) > 0, G_{1,l}(c) > 0, 0 < c < \frac{3 - \sqrt{3}}{6} \}, c]].$$

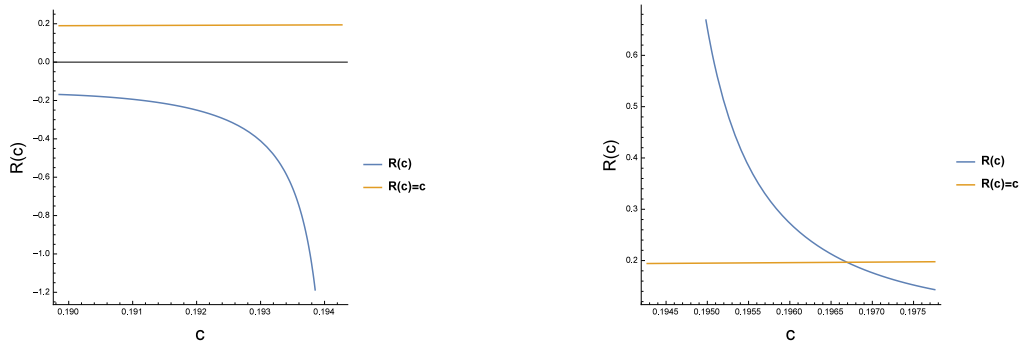
This yields:

$$F_d^l = (0.188816\dots, 0.194271\dots) \cup (0.194271\dots, 0.199413\dots) \equiv F_{d_1}^l \cup F_{d_2}^l.$$

From the Eqn.(2.5), the graphs of $\mathcal{R}(c)$ are plotted in the sub-domains $F_{d_1}^l$ and $F_{d_2}^l$ of F_d^l which are shown in Figure 9.

The map $\mathcal{R} : F_d^l \rightarrow \mathbb{R}$ is expanding in the neighborhood of fixed point c_l^* which is illustrated in Figure 9b. By Mathematica computations, we get an unstable fixed points $c_l^* = 0.196693\dots$ in F_d^l such that

$$\mathcal{R}(c_l^*) = c_l^*$$

(a) \mathcal{R} has no fixed point in $F_{d_1}^l$.(b) \mathcal{R} has only one fixed point in $F_{d_2}^l$.Figure 9: The graph of $\mathcal{R} : F_d^l \rightarrow \mathbb{R}$ and the diagonal $\mathcal{R}(c) = c$.

corresponds to an infinitely renormalizable maps $f_{s_l^*}$. We observe that the map $f_{s_l^*}$ corresponding to c_l^* has the following property

$$\{c_l^*\} = \bigcap_{n \geq 1} I_{1,l}^n.$$

In other words, consider the scaling data $s_l^* : \mathbb{N} \rightarrow \Delta^3$ with

$$\begin{aligned} s_l^*(n) &= (s_{0,l}^*(n), s_{1,l}^*(n), s_{2,l}^*(n)) \\ &= \left(\frac{b_{c_l^*}^4(0) - b_{c_l^*}^4(0)}{b_{c_l^*}^4(0)}, \frac{b_{c_l^*}^2(0) - b_{c_l^*}^5(0)}{b_{c_l^*}^4(0)}, \frac{b_{c_l^*}^3(0)}{b_{c_l^*}^4(0)} \right). \end{aligned}$$

Then $\sigma(s_l^*) = s_l^*$ and using Lemma 2.1 we have

$$R^l f_{s_l^*} = f_{s_l^*}.$$

□

2.2. Renormalization on $I_R = [\tilde{b}_c(1), 1]$

In subsection 2.1, the bimodal map $b_c(x)$ has two critical points $c \in I_L$ and $1 - c \in I_R$ and we define the piece-wise renormalization on I_L . In similar fashion, to define the renormalization on I_R with $c \in I_R$, from Equation 2.1, we consider

$$\tilde{b}_c(x) = 1 - \frac{4c^3 - 3c^2 + 6cx - 6c^2x - 3x^2 + 2x^3}{(2c - 1)^3}$$

where $x \in [0, 1]$ and $c \in [\frac{3}{4}, 1]$.

Note that $I_L^\circ \cap I_R^\circ = \emptyset$, for $c \in [\frac{3+\sqrt{3}}{6}, 1]$.

Let $f_{s_r} \in \mathcal{U}_\infty$ be given by the proper scaling data $s_r : \mathbb{N} \rightarrow \Delta^3$ and define

$$\tilde{I}_{1,r}^n = [\tilde{b}_c(1), \min\{\tilde{b}_c^{-1}(z'_n)\}] = [f_{s_r}(1), \min\{f_{s_r}^{-1}(z'_n)\}],$$

where $\tilde{b}_c^{-1}(x)$ denotes the preimage(s) of x under \tilde{b}_c and

$$\hat{I}_{1,r}^n = [\tilde{b}_c(y'_n), 1] = [f_{s,r}(y'_n), 1].$$

Let

$$h_{s_r,n} : [\tilde{b}_c(1), 1] \rightarrow [\tilde{b}_c(1), 1]$$

be defined by

$$h_{s_r,n} = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ \dots \circ F_{1,r}(n).$$

Furthermore, let

$$\tilde{h}_{s_r,n} : [\tilde{b}_c(1), 1] \rightarrow \tilde{I}_{1,r}^n \quad \text{and} \quad \hat{h}_{s_r,n} : [\tilde{b}_c(1), 1] \rightarrow \hat{I}_{1,r}^n$$

be the affine orientation preserving homeomorphisms. Then define

$$R_n^r f_s : h_{s_r,n}^{-1}(D_{s_r}) \rightarrow [\tilde{b}_c(1), 1]$$

by

$$R_n^r f_s(x) = \begin{cases} R_n^- f_{s_r}(x), & \text{if } x \in h_{s_r,n}^{-1}(I_{0,r}^n) \\ R_n^+ f_{s_r}(x), & \text{if } x \in h_{s_r,n}^{-1}(I_{2,r}^n) \end{cases}$$

where,

$$R_n^- f_{s_r} : h_{s_r,n}^{-1}\left(\bigcup_{n \geq 1} I_{0,r}^n\right) \rightarrow [\tilde{b}_c(1), 1]$$

and

$$R_n^+ f_{s_r} : h_{s_r,n}^{-1}\left(\bigcup_{n \geq 1} I_{2,r}^n\right) \rightarrow [\tilde{b}_c(1), 1]$$

are defined by

$$R_n^- f_{s_r}(x) = \tilde{h}_{s_r,n}^{-1} \circ f_{s_r}^{-1} \circ h_{s_r,n}(x)$$

$$R_n^+ f_{s_r}(x) = \hat{h}_{s_r,n}^{-1} \circ f_{s_r} \circ h_{s_r,n}(x),$$

which are illustrated in Figure 10.

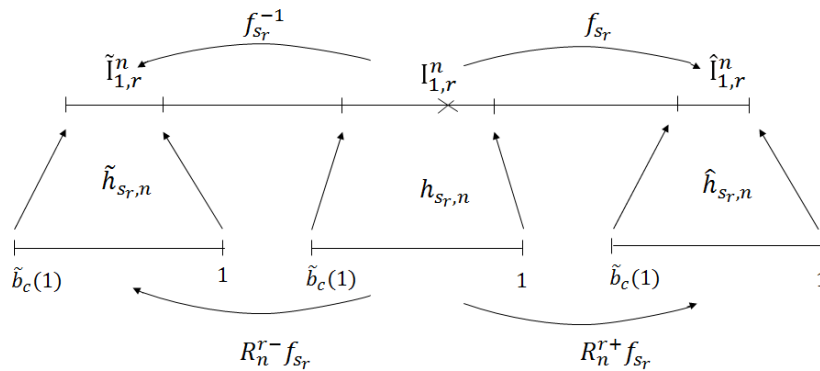


Figure 10

Let $\sigma : (\Delta^3)^\mathbb{N} \rightarrow (\Delta^3)^\mathbb{N}$ be the shift map which is defined as $\sigma(s_r^1 s_r^2 s_r^3 s_r^4 \dots) = (s_r^2 s_r^3 s_r^4 \dots)$, where $s_r^i \in \Delta^3$ for all $i \in \mathbb{N}$.

Lemma 2.4. *Let $s_r : \mathbb{N} \rightarrow \Delta^3$ be proper scaling data such that f_{s_r} is infinitely renormalizable. Then*

$$R_n^r f_{s_r} = f_{\sigma^n(s_r)}.$$

Let f_{s_r} be infinitely renormalization, then for $n \geq 0$, we have

$$f_{s_r}^{3^n} : D_{s_r} \cap I_{1,r}^n \rightarrow I_{1,r}^n$$

is well defined.

Define the renormalization $R^r : \mathcal{U}_\infty \rightarrow \mathcal{U}_\infty$ by

$$R^r f_{s_r} = h_{s_r,1}^{-1} \circ f_{s_r}^3 \circ h_{s_r,1}.$$

The maps $f_{s_r}^{3^{n-2}} : \tilde{I}_{1,r}^n \rightarrow I_{1,r}^n$ and $f_{s_r}^{3^{n-1}} : \hat{I}_{1,r}^n \rightarrow I_{1,r}^n$ are the affine homeomorphisms whenever $f_{s_r} \in \mathcal{U}_\infty$. Then

Lemma 2.5. *We have $(R^r)^n f_{s_r} : D_{\sigma^n(s_r)} \rightarrow [\tilde{b}_c(1), 1]$ and $(R^r)^n f_{s_r} = R_n^r f_{s_r}$.*

The lemma 2.1 and lemma 2.2 give the following result.

Proposition 2.6. *There exists a map $f_{s_r^*} \in \mathcal{U}_\infty$, where s_r^* is characterized by*

$$R^r f_{s_r^*} = f_{s_r^*}.$$

Proof. Consider $s_r : \mathbb{N} \rightarrow \Delta^3$ be proper scaling data such that f_{s_r} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_r)}$. Then

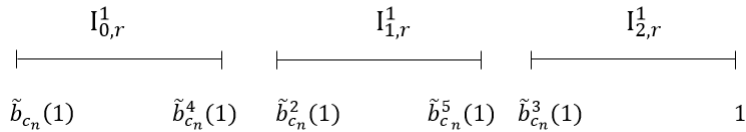


Figure 11: Length of intervals

from Figure 11, we have the following scaling ratios

$$s_{0,r}(n) = \frac{\tilde{b}_{c_n}^4(1) - \tilde{b}_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \quad (2.12)$$

$$s_{1,r}(n) = \frac{\tilde{b}_{c_n}^5(1) - \tilde{b}_{c_n}^2(1)}{1 - \tilde{b}_{c_n}(1)} \quad (2.13)$$

$$s_{2,r}(n) = \frac{1 - \tilde{b}_{c_n}^3(1)}{1 - \tilde{b}_{c_n}(1)} \quad (2.14)$$

$$c_{n+1} = 1 - \frac{c_n - \tilde{b}_{c_n}^2(1)}{s_{1,r}(n)} \equiv \mathcal{R}(c_n). \quad (2.15)$$

Since $(s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in \Delta^3$, this implies the following conditions

$$s_{0,r}(n), s_{1,r}(n), s_{2,r}(n) > 0 \quad (2.16)$$

$$s_{0,r}(n) + s_{1,r}(n) + s_{2,r}(n) < 1 \quad (2.17)$$

As the intervals $I_{i,r}^n$, for $i = 0, 1, 2$, are mutually disjoint, we will introduce the gap ratios $g_{0,r}^n$ and $g_{1,r}^n$ in between $I_{0,r}^n$ & $I_{1,r}^n$ and $I_{1,r}^n$ & $I_{2,r}^n$ respectively. The gap ratios are defined as,
for $n \in \mathbb{N}$,

$$g_{0,r}^n = \frac{\tilde{b}_{c_n}^2(1) - \tilde{b}_{c_n}^4(1)}{1 - \tilde{b}_{c_n}(1)} \equiv G_{0,r}(c_n) > 0 \quad (2.18)$$

$$g_{1,r}^n = \frac{\tilde{b}_{c_n}^3(1) - \tilde{b}_{c_n}^5(1)}{1 - \tilde{b}_{c_n}(1)} \equiv G_{1,r}(c_n) > 0 \quad (2.19)$$

$$\frac{3 + \sqrt{3}}{6} < c_n < 1 \quad (2.20)$$

We use Mathematica for solving the equations (2.12), (2.13) and (2.14), we have the expressions for $s_{0,r}(n)$, $s_{1,r}(n)$ and $s_{2,r}(n)$. Let $s_{i,r}(n) \equiv S_{i,r}(c_n)$ for $i = 0, 1, 2$. Note that the conditions (2.16), (2.18) and (2.19) give the condition (2.17)

$$0 < \sum_{i=0}^2 s_{i,r}(n) < 1.$$

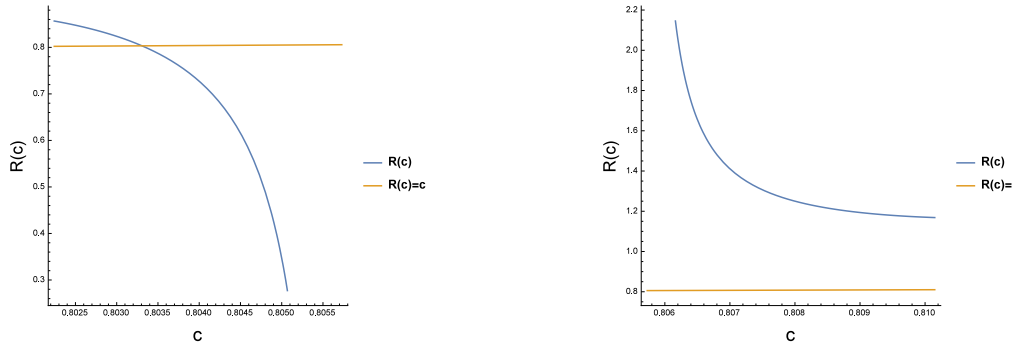
The conditions (2.16) together with (2.18) to (2.20) define the feasible domain F_d^r is to be:

$$F_d^r = \left\{ c \in \left(\frac{3 + \sqrt{3}}{6}, 1 \right) : S_{i,r}(c) > 0 \text{ for } i = 0, 1, 2, G_{0,r}(c) > 0, G_{1,r}(c) > 0 \right\} \quad (2.21)$$

One can compute feasible domain F_d^r as described in subsection 2.1. This yields:

$$F_d^r = (0.800587\dots, 0.805729\dots) \cup (0.805729\dots, 0.811184\dots) \equiv F_{d_1}^r \cup F_{d_2}^r.$$

From the Eqn.(2.5), the graphs of $\mathcal{R}(c)$ are plotted in the sub-domains $F_{d_1}^r$ and $F_{d_2}^r$ of F_d^r which are shown in Figure 12.



(a) \mathcal{R} has only one fixed point in $F_{d_1}^r$.

(b) \mathcal{R} has no fixed point in $F_{d_2}^r$.

Figure 12: The graph of $\mathcal{R} : F_d^r \rightarrow \mathbb{R}$ and the diagonal $\mathcal{R}(c) = c$.

The map $\mathcal{R} : F_d^r \rightarrow \mathbb{R}$ is expanding in the neighborhood of fixed point c_r^* which is illustrated in Figure 12a. By Mathematica computations, we get an unstable fixed

points $c_r^* = 0.803307\dots$ in F_d^r such that

$$\mathcal{R}(c_r^*) = c_r^*$$

corresponds to an infinitely renormalizable maps $f_{s_r^*}$. We observe that the map $f_{s_r^*}$ corresponding to c_r^* has the following property

$$\{c_r^*\} = \bigcap_{n \geq 1} I_{1,r}^n.$$

In other words, consider the scaling data $s_r^* : \mathbb{N} \rightarrow \Delta^3$ with

$$\begin{aligned} s_r^*(n) &= (s_{0,r}^*(n), s_{1,r}^*(n), s_{2,r}^*(n)) \\ &= \left(\frac{\tilde{b}_{c_r^*}^4(1) - \tilde{b}_{c_r^*}(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{\tilde{b}_{c_r^*}^5(1) - \tilde{b}_{c_r^*}^2(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{1 - \tilde{b}_{c_r^*}^3(1)}{1 - \tilde{b}_{c_r^*}(1)} \right). \end{aligned}$$

Then $\sigma(s_r^*) = s_r^*$ and using Lemma 2.4 we have

$$R^r f_{s_r^*} = f_{s_r^*}.$$

□

For a given pair of proper scaling data $s = (s_l, s_r)$, we defined a map

$$f_s : D_{s_l} \cup D_{s_r} \rightarrow [0, 1]$$

as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

Then, the renormalization of f_s is defined as

$$Rf_s(x) = \begin{cases} R^l f_{s_l}(x), & \text{if } x \in D_{s_l} \\ R^r f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

From proposition 2.3 and 2.6, we conclude that $f_{s_l^*}$ and $f_{s_r^*}$ are period tripling infinitely renormalizable maps corresponding to the proper scaling data s_l^* and s_r^* , respectively. Then, for a given pair of scaling data $s^* = (s_l^*, s_r^*)$, we have

$$\begin{aligned} Rf_{s^*}(x) &= \begin{cases} R^l f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ R^r f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= \begin{cases} f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= f_{s^*}(x) \end{aligned}$$

The above construction will lead to the following theorem,

Theorem 2.7. *There exists a map $f_{s^*} \in B_\infty$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

In particular, $B_\infty = \{f_{s^}\}$.*

Remark 1. *If f_{s^*} is the map with a pair of proper scaling data $s^* = (s_l^*, s_r^*)$ then the scaling data holds the following properties,*

$$(i) \quad s_{2,l}^* \leq (s_{1,l}^*)^2$$

$$(ii) \quad s_{2,r}^* \leq (s_{1,r}^*)^2$$

Remark 2. *The invariant Cantor set of the map f_{s^*} is next in complexity to the invariant doubling Cantor set of piece-wise affine renormalizable map [9] in the following sense,*

(i) *like the both Cantor set, on each scale and everywhere the same scaling ratio are used.*

(ii) *But unlike the doubling Cantor set, there are now the pair of three ratios at each scale.*

Furthermore, the geometry of the invariant Cantor set of f_{s^} is different from the geometry of the invariant Cantor set of piece-wise affine period tripling renormalizable map because the Cantor set of f_{s^*} has 2-copy of Cantor set of [10].*

3. C^{1+Lip} extension of f_{s^*}

In Section 2, we have constructed a piece-wise affine infinitely renormalizable map f_{s^*} corresponding to the pair of scaling data $s^* = (s_l^*, s_r^*)$. Let us define a pair of scaling functions

$$S_l : [0, b_{c_l^*}(0)]^2 \rightarrow [0, b_{c_l^*}(0)]^2$$

$$S_r : [\tilde{b}_{c_r^*}(1), 1]^2 \rightarrow [\tilde{b}_{c_r^*}(1), 1]^2$$

as

$$S_l \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{c_l^*}^2(0) - s_{1,l}^* \cdot x \\ s_{2,l}^* \cdot y \end{pmatrix}; \quad S_r \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{b}_{c_r^*}^2(1) + s_{1,r}^* \cdot (1-x) \\ 1 - s_{2,r}^* \cdot (1-y) \end{pmatrix}.$$

Let G be the graph of g_{s^*} which is an extension of f_{s^*} where $f_{s^*} : D_{s_l^*} \cup D_{s_r^*} \rightarrow [0, 1]$. Let G_l^1 and G_l^2 are the graphs of $g_{s^*}|_{[y_1, z_0]}$ and $g_{s^*}|_{[y_0, z_1]}$ respectively. Also, G_r^1 and G_r^2 are the graphs of $g_{s^*}|_{[z'_0, y'_1]}$ and $g_{s^*}|_{[z'_1, y'_0]}$ respectively which are shown in Figure 13. Also, note that G_r^1 and G_r^2 are the reflections of G_l^1 and G_l^2 across the point $(\frac{1}{2}, \frac{1}{2})$ respectively. Define

$$G_l = \cup_{n \geq 1} S_l^n(G_l^1 \cup G_l^2) \quad \text{and} \quad G_r = \cup_{n \geq 1} S_r^n(G_r^1 \cup G_r^2).$$

Then, G_l is the graph of a unimodal map $g_{s_l^*}$ which extends $f_{s_l^*}$ and G_r is the graph of a unimodal map $g_{s_r^*}$ which extends $f_{s_r^*}$. Consequently, G is the graph of $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$. We claim that g_{s^*} is a C^{1+Lip} bimodal map.

Let $B_l^0 = [0, b_{c_l^*}(0)] \times [0, b_{c_l^*}(0)]$ and $B_r^0 = [\tilde{b}_{c_r^*}(1), 1] \times [\tilde{b}_{c_r^*}(1), 1]$. For $n \in \mathbb{N}$, define

$$B_l^n = S_l^n(B_l^0) \quad \text{and} \quad B_r^n = S_r^n(B_r^0)$$

as

$$B_l^n = \begin{cases} [z_n, y_n] \times [0, \hat{y}_n], & \text{if } n \text{ is odd} \\ [y_n, z_n] \times [0, \hat{y}_n], & \text{if } n \text{ is even} \end{cases}$$

and

$$B_r^n = \begin{cases} [y'_n, z'_n] \times [\hat{y}'_n, 1], & \text{if } n \text{ is odd} \\ [z'_n, y'_n] \times [\hat{y}'_n, 1], & \text{if } n \text{ is even.} \end{cases}$$

Let p_l^n and p_r^n be the points on the graph of the bimodal map $b_{c_l^*}(x)$ and $b_{c_r^*}(x)$ respectively. For all $n \in \mathbb{N}$, p_l^n and p_r^n are defined as

$$p_l^n = \begin{cases} (y_{\frac{n+1}{2}}, \hat{y}_{\frac{n+1}{2}}), & \text{if } n \text{ is odd} \\ (z_{\frac{n}{2}}, \hat{z}_{\frac{n}{2}}), & \text{if } n \text{ is even} \end{cases}$$

$$p_r^n = \begin{cases} (y'_{\frac{n+1}{2}}, \hat{y}'_{\frac{n+1}{2}}), & \text{if } n \text{ is odd} \\ (z'_{\frac{n}{2}}, \hat{z}'_{\frac{n}{2}}), & \text{if } n \text{ is even} \end{cases}$$

where $\hat{y}_n = b_{c_l^*}(y_n)$, $\hat{z}_n = b_{c_l^*}(z_n)$, $\hat{y}'_n = \tilde{b}_{c_r^*}(y'_n)$ and $\hat{z}'_n = \tilde{b}_{c_r^*}(z'_n)$.

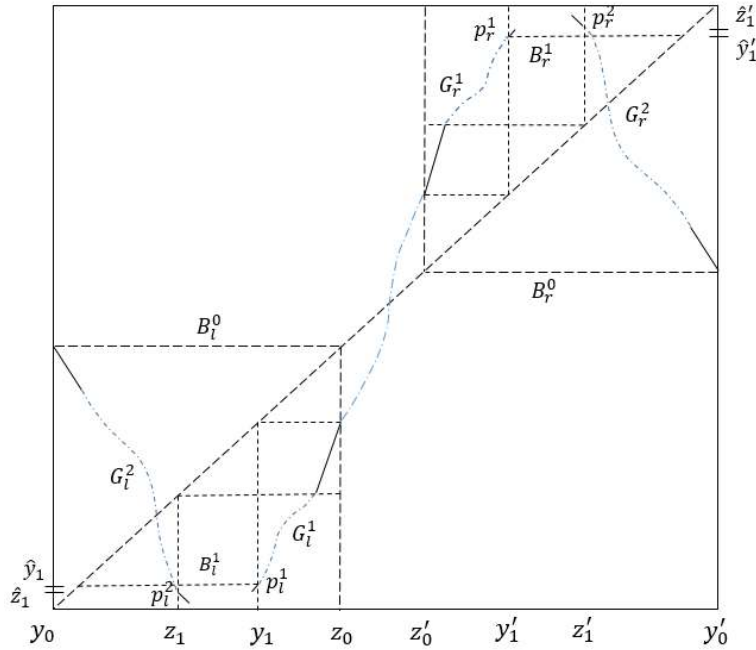


Figure 13: Extension of f_{s^*}

Then the above construction will lead to following proposition,

Proposition 3.1. G is the graph of g_{s^*} which is a C^1 extension of f_{s^*} .

Proof. Since G_l^1 and G_r^1 are the graph of $f_{s_l^*}|_{[y_1, z_0]}$ and $f_{s_r^*}|_{[y_0, z_1]}$, respectively, and G_l^2 and G_r^2 are the graph of $f_{s_l^*}|_{[z'_0, y'_1]}$ and $f_{s_r^*}|_{[z'_1, y'_0]}$, respectively, we obtain $G_l^{2n+1} = S_l^n(G_l^1)$

and $G_l^{2n+2} = S_l^n(G_l^2)$ for each $n \in \mathbb{N}$. Note that G_l^n is the graph of a C^1 function defined

$$\begin{aligned} & \text{on } [z_{\frac{n-1}{2}}, y_{\frac{n+1}{2}}] \text{ if } n \in 4\mathbb{N} - 1, \\ & \text{on } [z_{\frac{n}{2}}, y_{\frac{n}{2}-1}] \text{ if } n \in 4\mathbb{N}, \\ & \text{on } [y_{\frac{n+1}{2}}, z_{\frac{n-1}{2}}] \text{ if } n \in 4\mathbb{N} + 1, \\ & \text{and on } [y_{\frac{n}{2}-1}, z_{\frac{n}{2}}] \text{ if } n \in 4\mathbb{N} + 2. \end{aligned}$$

Also, we have $G_r^{2n+1} = S_r^n(G_r^1)$ and $G_r^{2n+2} = S_r^n(G_r^2)$ for each $n \in \mathbb{N}$. Note that G_r^n is the graph of a C^1 function defined

$$\begin{aligned} & \text{on } [y'_{\frac{n+1}{2}}, z'_{\frac{n-1}{2}}] \text{ if } n \in 4\mathbb{N} - 1, \\ & \text{on } [y'_{\frac{n}{2}-1}, z'_{\frac{n}{2}}] \text{ if } n \in 4\mathbb{N}, \\ & \text{on } [z'_{\frac{n-1}{2}}, y'_{\frac{n+1}{2}}] \text{ if } n \in 4\mathbb{N} + 1, \\ & \text{and on } [z'_{\frac{n}{2}}, y'_{\frac{n}{2}-1}] \text{ if } n \in 4\mathbb{N} + 2. \end{aligned}$$

To prove the proposition, we have to check continuous differentiability at the points p_l^n and p_r^n . Consider the neighborhoods $(y_1 - \epsilon, y_1 + \epsilon)$ around y_1 and $(z_1 - \epsilon, z_1 + \epsilon)$ around z_1 , the slopes are given by an affine pieces of $f_{s_l^*}$ on the subintervals $(y_1 - \epsilon, y_1)$ and $(z_1, z_1 + \epsilon)$ and the slopes are given by the chosen C^1 extension on $(y_1, y_1 + \epsilon)$ and $(z_1 - \epsilon, z_1)$. This implies, G_l^1 and G_l^2 are C^1 at p_l^1 and p_l^2 , respectively.

Let $\gamma_1 \subset G_l$ be the graph over the interval $(y_1 - \epsilon, y_1 + \epsilon)$ and $\gamma_2 \subset G_l$ be the graph over the interval $(z_1 - \epsilon, z_1 + \epsilon)$,

then the graph G_l locally around p_l^n is equal to $\begin{cases} S_l^{\frac{n-1}{2}}(\gamma_1) & \text{if } n \text{ is odd} \\ S_l^{\frac{n-2}{2}}(\gamma_2) & \text{if } n \text{ is even} \end{cases}$.

This implies, for $n \in \mathbb{N}$, G_l^{2n-1} is C^1 at p_l^{2n-1} and G_l^{2n} is C^1 at p_l^{2n} .

Hence G_l is a graph of a C^1 function on $[0, b_{c_l^*}(0)] \setminus \{c_l^*\}$.

We note that the horizontal contraction of S_l is smaller than the vertical contraction. This implies that the slope of G_l^n tends to zero when n is large. Therefore, G_l is the graph of a C^1 function $g_{s_l^*}$ on $[0, b_{c_l^*}]$. In similar way, one can prove that G_r is the graph of a C^1 function $g_{s_r^*}$ on $[b_{c_r^*}, 1]$. Therefore, $G = G_l \oplus G_r$ is the graph of a C^1 bimodal map $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$ which is a C^1 extension of f_{s^*} . \square

Proposition 3.2. *Let g_{s^*} be the function whose graph is G then g_{s^*} is a C^{1+Lip} bimodal map.*

Proof. As the function g_{s^*} is a C^1 extension of f_{s^*} . We have to show that, for $i \in \{l, r\}$, G_i^n is the graph of a C^{1+Lip} function

$$g_{s_i^*}^n : \text{Dom}(G_i^n) \rightarrow [0, 1]$$

with an uniform Lipschitz bound.

That is, for $n \geq 1$,

$$\text{Lip}((g_{s_i^*}^{n+1})') \leq \text{Lip}((g_{s_i^*}^n)')$$

let us assume that $g_{s_l^*}^n$ is C^{1+Lip} with Lipschitz constant λ_n for its derivatives. We show that $\lambda_{n+1} \leq \lambda_n$.

For given (u, v) on the graph of $g_{s_l^*}^n$, there is $(\tilde{u}, \tilde{v}) = S_l(u, v)$ on the graph of $g_{s_l^*}^{n+1}$, this implies

$$g_{s_l^*}^{n+1}(\tilde{u}) = s_{2,l}^* \cdot g_{s_l^*}^n(u)$$

Since $u = \frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*}$, we have

$$g_{s_l^*}^{n+1}(\tilde{u}) = s_{2,l}^* \cdot g_{s_l^*}^n \left(\frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*} \right)$$

Differentiate both sides with respect to \tilde{u} , we get

$$\left(g_{s_l^*}^{n+1} \right)'(\tilde{u}) = -\frac{s_{2,l}^*}{s_{1,l}^*} \cdot \left(g_{s_l^*}^n \right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*} \right)$$

Therefore,

$$\begin{aligned} \left| \left(g_{s_l^*}^{n+1} \right)'(\tilde{u}_1) - \left(g_{s_l^*}^{n+1} \right)'(\tilde{u}_2) \right| &= \left| \frac{s_{2,l}^*}{s_{1,l}^*} \right| \cdot \left| \left(g_{s_l^*}^n \right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}_1}{s_{1,l}^*} \right) - \left(g_{s_l^*}^n \right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}_2}{s_{1,l}^*} \right) \right| \\ &\leq \frac{s_{2,l}^*}{(s_{1,l}^*)^2} \cdot \lambda \left(g_{s_l^*}^n \right)' |\tilde{u}_1 - \tilde{u}_2| \end{aligned}$$

From remark 1, we have $(s_{1,l}^*)^2 \geq s_{2,l}^*$. Then,

$$\lambda \left(g_{s_l^*}^{n+1} \right)' \leq \lambda \left(g_{s_l^*}^n \right)' \leq \lambda \left(g_{s_l^*}^1 \right)'.$$

Similarly, one can show that

$$\lambda \left(g_{s_r^*}^{n+1} \right)' \leq \lambda \left(g_{s_r^*}^n \right)' \leq \lambda \left(g_{s_r^*}^1 \right)'.$$

Therefore, choose $\lambda = \max\{\lambda(g_{s_l^*}^1)', \lambda(g_{s_r^*}^1)'\}$ is the uniform Lipschitz bound. This completes the proof. \square

Note that for a given pair of proper scaling data $s^* = (s_l^*, s_r^*)$, the piece-wise affine map f_{s^*} is infinitely renormalizable and g_{s^*} is a C^{1+Lip} extension of f_{s^*} . This implies g_{s^*} is also renormalizable map. Further, we observe that Rg_{s^*} is an extension of Rf_{s^*} . Therefore Rg_{s^*} is renormalizable. Hence, g_{s^*} is infinitely renormalizable map which is not a C^2 map. Then we have the following theorem,

Theorem 3.3. *There exists an infinitely renormalizable C^{1+Lip} bimodal map g_{s^*} such that*

$$Rg_{s^*} = g_{s^*}.$$

4. Topological entropy of renormalization

In this section, we calculate the topological entropy of the renormalization operator defined on the space of C^{1+Lip} bimodal maps.

Let us consider three pairs of C^{1+Lip} maps $\phi_i : [0, z_1] \cup [y_1, b_{c_i^*}(0)] \rightarrow [0, b_{c_i^*}(0)]$ and $\psi_i : [\tilde{b}_{c_i^*}(1), y'_1] \cup [z'_1, 1] \rightarrow [\tilde{b}_{c_i^*}(1), 1]$, for $i = 0, 1, 2$, which extend f_{s^*} . Because of symmetricity, $\psi_i(x) = 1 - \phi_i(1 - x)$. For a sequence $\alpha = \{\alpha_n\}_{n \geq 1} \in \Sigma_3$, where $\Sigma_3 = \{\{x_n\}_{n \geq 1} : x_n \in \{0, 1, 2\}\}$ is called full 3-Shift.

Now define

$$G_l^n(\alpha) = S_l^n(\text{graph } \phi_{\alpha_n}) \quad \text{and} \quad G_r^n(\alpha) = S_r^n(\text{graph } \psi_{\alpha_n}),$$

we have

$$G_l(\alpha) = \bigcup_{n \geq 1} G_l^n(\alpha) \quad \text{and} \quad G_r(\alpha) = \bigcup_{n \geq 1} G_r^n(\alpha).$$

Therefore, we conclude that $G(\alpha) = G_l(\alpha) \oplus G_r(\alpha)$ is the graph of a C^{1+Lip} bimodal map b_α by using the same facts of Section 3.

The shift map $\sigma : \Sigma_3 \rightarrow \Sigma_3$ is defined as

$$\sigma(\alpha_1 \alpha_2 \alpha_3 \dots) = (\alpha_2 \alpha_3 \alpha_4 \dots).$$

Proposition 4.1. *The restricted maps $b_\alpha^3 : [y_1, z_1] \rightarrow [y_1, z_1]$ and $b_\alpha^3 : [y'_1, z'_1] \rightarrow [y'_1, z'_1]$ are the unimodal maps for all $\alpha \in \Sigma_3$. In particular, b_α is a renormalizable map and $Rb_\alpha = b_{\sigma(\alpha)}$.*

Proof. We know that $b_\alpha : [y_1, z_1] \rightarrow I_{2,l}^1$ is a unimodal and onto, $b_\alpha : I_{2,l}^1 \rightarrow I_{0,l}^1$ is onto and affine and also $b_\alpha : I_{0,l}^1 \rightarrow [y_1, z_1]$ is onto and affine. Therefore b_α^3 is a unimodal map on $[y_1, z_1]$. Analogously, b_α^3 is a unimodal map on $[y'_1, z'_1]$. The above construction implies

$$Rb_\alpha = b_{\sigma(\alpha)}.$$

□

This gives us the following theorem.

Theorem 4.2. *The renormalization operator R acting on the space of C^{1+Lip} bimodal maps has unbounded topological entropy.*

Proof. From the above construction, we conclude that $\alpha \mapsto b_\alpha \in C^{1+Lip}$ is injective. The domain of R contains two copies, namely Λ_1 and Λ_2 , of the full 3-shift. As topological entropy h_{top} is an invariant of topological conjugacy. Hence $h_{top}(R|_{\Lambda_1 \cup \Lambda_2}) > \ln 3$. In fact, if we choose n different pairs of C^{1+Lip} maps, say, $\phi_0, \phi_1, \phi_2, \dots, \phi_{n-1}$ and $\psi_0, \psi_1, \psi_2, \dots, \psi_{n-1}$, which extends f_{s^*} , then it will be embedded two copies of the full n -shift in the domain of R . Hence, the topological entropy of R on C^{1+Lip} bimodal maps is unbounded. □

5. An ϵ perturbation of the scaling data

In this section, we use an ϵ perturbation on the construction of the scaling data as presented in Section 2, to obtain the following theorem

Theorem 5.1. *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} bimodal maps.*

Proof. Consider an ϵ variation on scaling data and we modify the construction which is described in section 2.

Let us define the neighborhoods N_ϵ^l and N_ϵ^r about the respective points $(b_c^3(0), b_c^4(0))$ and $(b_c^3(1), b_c^4(1))$ as

$$N_\epsilon^l(b_c^3(0), b_c^4(0)) = \{(b_c^3(0), \epsilon \cdot b_c^4(0)) : \epsilon > 0 \text{ and } \epsilon \text{ close to } 1\}$$

$$N_\epsilon^r(b_c^3(1), b_c^4(1)) = \{(b_c^3(1), \epsilon \cdot b_c^4(1)) : \epsilon > 0 \text{ and } \epsilon \text{ close to } 1\}$$

Case (i) The perturbed scaling data on I_0^l , then the scaling ratios are defined as

$$\begin{aligned} s_{2,l}(c, \epsilon) &= \frac{b_c^3(0)}{b_c(0)} \\ s_{0,l}(c, \epsilon) &= \frac{b_c(0) - \epsilon b_c^4(0)}{b_c(0)} \\ s_{1,l}(c, \epsilon) &= \frac{b_c^2(0) - b_c(\epsilon b_c^4(0))}{b_c(0)}, \end{aligned}$$

where $c \in (0, \frac{3-\sqrt{3}}{6})$. Also, we define

$$\mathcal{R}(c, \epsilon) = \frac{b_c^2(0) - c}{s_{1,l}(c, \epsilon)}.$$

From subsection 2.1, we know that the map \mathcal{R} which is defined in Eqn. 2.5, has unique fixed point c^* . Consequently, for a given ϵ close to 1, $\mathcal{R}(c, \epsilon)$ has only one unstable fixed point, namely c_ϵ^* . Therefore, we consider the perturbed scaling data $s_{l,\epsilon}^* : \mathbb{N} \rightarrow \Delta^3$ with

$$s_{l,\epsilon}^* = \left(\frac{b_{c_\epsilon^*}(0) - \epsilon b_{c_\epsilon^*}^4(0)}{b_{c_\epsilon^*}(0)}, \frac{b_{c_\epsilon^*}^2(0) - b_{c_\epsilon^*}(\epsilon b_{c_\epsilon^*}^4(0))}{b_{c_\epsilon^*}(0)}, \frac{b_{c_\epsilon^*}^3(0)}{b_{c_\epsilon^*}(0)} \right).$$

Then $\sigma(s_{l,\epsilon}^*) = s_{l,\epsilon}^*$ and using Lemma 2.1, we have

$$R^l f_{s_{l,\epsilon}^*} = f_{s_{l,\epsilon}^*}.$$

Case (ii) Considering the perturbed scaling data on I_0^r , one have the scaling data $s_{r,\epsilon}^* : \mathbb{N} \rightarrow \Delta^3$ with

$$s_{r,\epsilon}^* = \left(\frac{\epsilon b_{c_\epsilon^*}^4(1) - b_{c_\epsilon^*}(1)}{1 - b_{c_\epsilon^*}(1)}, \frac{b_{c_\epsilon^*}(\epsilon b_{c_\epsilon^*}^4(1)) - b_{c_\epsilon^*}^2(1)}{1 - b_{c_\epsilon^*}(1)}, \frac{1 - b_{c_\epsilon^*}^3(1)}{1 - b_{c_\epsilon^*}(1)} \right).$$

Then $\sigma(s_{r,\epsilon}^*) = s_{r,\epsilon}^*$ and using Lemma 2.4, we have

$$R^r f_{s_{r,\epsilon}^*} = f_{s_{r,\epsilon}^*}.$$

Moreover, $f_{s_{l,\epsilon}^*}$ and $f_{s_{r,\epsilon}^*}$ are the piece-wise affine maps which are infinitely renormalizable. For a given pair of proper scaling data $s_\epsilon^* = (s_{l,\epsilon}^*, s_{r,\epsilon}^*)$, we have

$$Rf_{s_\epsilon^*} = f_{s_\epsilon^*}$$

Now we use similar extension described in section 3, then we get $g_{s_\epsilon^*}$ is the C^{1+Lip} extension of $f_{s_\epsilon^*}$. This implies that $g_{s_\epsilon^*}$ is a renormalizable map. As $Rg_{s_\epsilon^*}$ is an extension of $Rf_{s_\epsilon^*}$. Therefore $Rg_{s_\epsilon^*}$ is renormalizable. Hence, for each ϵ close to 1, $g_{s_\epsilon^*}$ is a fixed point of the renormalization. This proves the existence of a continuum of fixed points of the renormalization. \square

Remark 3. *In particular, for two different perturbed scaling data $s_{\epsilon_1^*}$ and $s_{\epsilon_2^*}$, one can construct two infinitely renormalizable maps $g_{s_{\epsilon_1^*}}$ and $g_{s_{\epsilon_2^*}}$. Therefore, the respective Cantor attractors will have different scaling ratios. Consequently, it shows the non-rigidity for low smooth symmetric bimodal maps.*

6. Conclusions

In this paper, we have investigated the existence of fixed point of the renormalization operator which is defined on the space of symmetric bimodal maps with low smoothness. For a given pair of proper scaling data $s^* = (s_l^*, s_r^*)$, we have first constructed the piece-wise affine infinitely renormalizable map f_{s^*} which is the only fixed point of the renormalization. We observe that the geometry of invariant Cantor set is more complex than the geometry of the Cantor set of piece-wise affine period doubling renormalizable map[9]. Further, we have extended this fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map. Moreover, we proved that the renormalization operator acting on the space of C^{1+Lip} symmetric bimodal maps has infinite topological entropy. Finally, we proved the existence a continuum of fixed points of renormalization by considering a small perturbation on the scaling data. Consequently, it showed the non-rigidity of the Cantor attractors of infinitely renormalizable symmetric bimodal maps with low smoothness.

References

- [1] Feigenbaum M J 1978, Quantitative universality for a class of non-linear transformations, *J. Stat. Phys.*, **19** 25-52.
- [2] Feigenbaum M J 1979, The universal metric properties of nonlinear transformations, *J. Stat. Phys.*, **21** 669-706.
- [3] Couillet P, Tresser C 1978, Itération d'endomorphismes et groupe de renormalisation, *J. Phys. Colloque* , **C5** 25-28.
- [4] Jonkar L, Rand D 1980, Bifurcations in one dimension I. The nonwandering set, *Invent Math*, **62** 347-365.
- [5] Strien S V 1988, Smooth dynamics on the interval, in : New Directions in Dynamical Systems, eds. T. Bedford and J. Swift, *Cambridge Univ. Press, Cambridge* , 57-119.
- [6] Mackay R S, Tresser C 1986, Transition to topological chaos for circle maps, *Physica D: Nonlinear Phenomena* , **19** 206-237.
- [7] Veitch D 1994, Renormalization of C^0 bimodal maps, *Physica D*, **71** 269-284.

- [8] Smiana D 2005, Phase space universality for multimodal maps, *Bull. Braz. Math. Soc.*, **36** 225-274.
- [9] Chandramouli V V M S, Martens M, Melo W de, Tresser C P 2009, Chaotic period doubling, *Ergodic Theory and Dynamical Systems*, **29** 381-418.
- [10] Kumar R, Chandramouli V V M S 2020, Period tripling and quintupling renormalizations below C^2 space, preprint, *arXiv:2010.01293 [math.DS]*.
- [11] Tresser C 1991, Fine Structure of Universal Cantor Sets, *Instabilities and Nonequilibrium Structures III*, E. Tirapegui and W. Zeller Eds., (Kluwer, Dordrecht/Boston/London . 27-42.
- [12] Welington de Melo & Sebastian van Strien, *One-Dimensional dynamics*, (Springer Verlag, Berlin; 1993).