Reliability study of a coherent system with single general standby component

Pradip Kundu[1], Nil Kamal Hazra[2] and Asok K. Nanda[1*]
[1] Department of Mathematics and Statistics, IISER Kolkata
Mohanpur 741246, India
[2] Department of Mathematical Statistics and Actuarial Science
University of the Free State, 339 Bloemfontein 9300, South Africa

Abstract

The properties of a coherent system with a single general standby component is investigated. Here three different switch over viz. perfect switching, imperfect switching and random worm up period of the standby component are considered with some numerical examples.

Keywords & Phrases: General standby system, System signature.

1 Introduction

Standby allocation is one of the widely used techniques to improve the reliability of a system. Standby components are mostly of three types - cold standby, warm standby and hot standby. Cold standby means that the redundant component is inactive and has zero failure rate while in standby, and it starts to function at the time when the system/component fails. Hot standby describes the scenario where the redundant component and the corresponding system undergoes the same operational environment. In case of warm standby, the redundant component undergoes two operational environments, namely, usual environment (the environment in which the system is running) and milder environment (where the redundant component has less failure rate than that in the usual environment). Initially the warm standby component functions in milder environment and then it switches over to the usual environment at the time of system/component failure. Papageorgiou and Kokolakis [9] derived the reliability function of a two component

*Corresponding author, e-mail: asok.k.nanda@gmail.com

parallel system with $(n-2)$ warm standby components, where two units start their operation simultaneously and any one of them is replaced instantaneously upon its failure by one of the $(n-2)$ warm standbys. Cha et al. [1] introduced a general standby model for a single-component system with a single standby component, and derived system performance measures. The cold and the hot standby models are derived as special cases of the general standby model. Li et al. [12, 13] investigated some general standby systems and derived some stochastic comparison results on the lifetimes of the systems. Hazra and Nanda [6] discussed some standby models with one and two general standby components, and compared some different series and parallel systems corresponding to the models with respect to the usual stochastic and the stochastic precedence orders. Eryilmaz [2] derived reliability function of a coherent system equipped with a cold standby component such that the coherent system may fail at the time of the first component failure. Recently Franko et al. [5] generalized this case by considering that the coherent system may fail at the time of sth component failure so that the standby component may be put into operation actively at the time of s-th component failure. In case of warm standby redundancy, Eryilmaz [3] investigated the reliability properties of k-out-of-n system equipped with a single warm standby component.

In this paper, we investigate the reliability properties of a coherent system equipped with a general standby component. It is to be mentioned here that the general standby studied in this paper has three different states. To be more specific, the standby component starts to work in cold state, it is switched over to warm state (from cold state) after a specified time u (≥ 0) and put into operation in active state in usual environment at the time of sth component failure which might cause the system to fail. Switching from cold state to warm state of the standby component takes place after a specified time u (> 0) only if it is known a priori that sth component failure does not occur before time u (> 0); otherwise, u must be zero, i.e., standby component starts to work in warm state from the beginning. We also consider the perfectness of the switching from one state to another state of the standby component.

The rest of the paper is organized as follows. Section 2 discusses briefly general standby model and representation of survival function of a coherent system using system signature. In Section 3, we obtain reliability of a coherent system equipped with a single general standby component in different switch over cases, namely perfect switching case, imperfect switching case and the case with random warm-up period. Numerical example is presented in this section. Finally the paper is concluded in Section 4.

For two random variables X and Y, $X =^{st} Y$ means that X and Y have the same distribution.

2 General standby model

The concept of accelerated life tests and that of virtual age (see also Kijima [7], Finkelstein [4]) have been used by Cha et al. [1] for modeling general standby system in case of a single active component system. Let X be the lifetime of an active component with cumulative distribution function (c.d.f.) $F(\cdot)$ and let Y be the lifetime of a standby component in the usual environment with c.d.f. $G(\cdot)$. Further, let X and Y be independent. Write $\bar{F}(\cdot) = 1 - F(\cdot)$ and $\bar{G}(\cdot) = 1 - G(\cdot)$. For the standby unit in warm state, the component operates in an environment which is milder than the usual level of environment. Thus, the lifetime of the standby component in warm state will have the c.d.f. $G(\gamma(\cdot))$ where $\gamma(\cdot)$ is a non-decreasing function satisfying $\gamma(t) \leq t$, for all $t \geq 0$ with $\gamma(0) = 0$. Now, suppose that the standby unit has worked during $(0, t]$ without failure in a warm state, and is activated under usual environment at time t. Then the virtual age $\omega(t)$ of the standby component is non-decreasing satisfying $\omega(t) \leq t$, for all $t \geq 0$ and $\omega(0) = 0$. Let Y^* denote the remaining lifetime of the standby component after the failure of the active unit at time $X = x$. Then (cf. Cha et al. [1])

$$P\{Y^* > t, I = 1 \mid X = x\} = \frac{\bar{G}(\omega(x) + t)}{\bar{G}(\omega(x))}\bar{G}(\gamma(x))$$

and the survival function of the standby system is

$$R(t) = \bar{F}(t) + \int_0^t \frac{\bar{G}(\omega(x) + t - x)}{\bar{G}(\omega(x))}\bar{G}(\gamma(x))dF(x), \tag{2.1}$$

where $\{I = 1\}$ indicates that the standby component survives at least up to the failure time of the active component. The cold and the hot standby models can be derived as special cases by substituting $\gamma(t) = \omega(t) = 0$ and $\gamma(t) = \omega(t) = t$, respectively.

In the above discussion, the standby component starts to work in warm state at the beginning. However, for a general standby component, it can be assumed that the main (active) component starts to work in active state, and the standby component starts to work in cold state, and is switched over to warm state after a pre-specified time u up to which the active component certainly does not fail. In this case, if the standby unit operates during $(u, x]$ without failure in warm state, then obviously the virtual age at time x would be $\omega(x - u)$. Now to survive the system up to time t, the following cases may arise. The active component survives up to time t, or fails in $(u, t]$ and the standby component survives for the remaining time. Clearly, if $t \leq u$, then the reliability of the system is $\bar{F}(t)$; otherwise (cf. Yun and Cha [14]),

$$R(t) = \bar{F}(t) + \int_u^t \frac{\bar{G}(\omega(x - u) + t - x)}{\bar{G}(\omega(x - u))}\bar{G}(\gamma(x - u))dF(x). \tag{2.2}$$

3 Main results

Let T denote the lifetime of a binary coherent system without a standby and let T^{gs} denote the lifetime of the same system with a general standby component whose lifetime is Y. Suppose F is the common absolutely continuous c.d.f. of $X_1, X_2, \ldots, X_n$ having probability density function (p.d.f.) f, and G is the absolutely continuous c.d.f. of Y having p.d.f. g, where X_i denoteS the lifetime of the ith component, $i = 1, 2, \ldots, n$. Here, Y and $X_1, X_2, \ldots, X_n$ are independent. Eryilmaz [3] studied the reliability properties of the k-out-of-n system with a single warm standby component. Franko et al. [5] studied coherent systems equipped with a cold standby component which may be put into operation at the time of the sth component failure, $s = k_\phi, k_\phi + 1, \ldots, z_\phi + 1$, where k_ϕ is the minimum number of failed components that causes the system failure whereas z_ϕ is the maximum number of failed components with which the system can still operate. In this paper, we consider coherent system equipped with a general standby component which may start to work in cold state, and is switched over to warm state after a specified time $u(\geq 0)$; after that it may be put to work actively in the usual environment at the time of the sth component failure which may cause the system failure. It is clear that the standby component operated under warm standby state may be put to work in the usual environment if the system has a positive probability of failure at the time of the sth component failure. This means that $P\{T = X_{s:n}\} > 0$, for $s = k_\phi, k_\phi + 1, \ldots, z_\phi + 1$.

Now, after putting the warm standby component into operation in the usual environment at the time when the system fails upon the failure of the sth component, the remaining lifetime of the system consisting of $(s-1)$ failed components (0's), a standby component, and $(n-s)$ working components, can be represented as

$$\phi_s(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{s-1}}, Y^{(u,s)}_{V_s}, X^{(s)}_{R_1}, X^{(s)}_{R_2}, \ldots, X^{(s)}_{R_{n-s}}),$$

where V_s is the discrete random variable representing the index of the standby component indicating that it is put into operation from warm state to active state when sth component fails, $Y^{(u,s)}_{V_s}$ is the remaining lifetime of the general standby component after failure of the sth component (where u indicates the time after which the standby component starts to work in warm state), $B_i \in \{1, 2, \ldots, n\}$, $i = 1, 2, \ldots, s-1$ and $R_j \in \{1, 2, \ldots, n\}$, $j = 1, 2, \ldots, n-s$ are the discrete random variables representing respectively the index of the ith failed component and the jth surviving component at the time of sth component failure, and consequently $X^{(s)}_l$ represents the remaining lifetime of the lth component after failure of the sth component, i.e., $X^{(s)}_l =^{st} (X_l - X_{s:n} \mid X_l > X_{s:n})$, $1 \leq l \leq n-s$, $X_{s:n}$ is the sth order statistic from $X_1, X_2, \ldots, X_n$. It is to be noted that $V_s = c$ if and only if $(X_c = X_{s:n} \mid T = X_{s:n}), c = 1, 2, \ldots, n$. The reliability properties of the remaining lifetime of the system is computed based on $(n-s+1)$ functioning components (including the standby component). Note that, here only the places of the $(s-1)$ failed components are taken into consideration in the structure function, and not their lifetimes (as they have already failed).

Our main goal is to investigate the reliability characteristics of T^{gs} which is the lifetime of the system with general standby component, i.e.,

$$T^{gs} = T + \sum_{s=k_\phi}^{z_\phi+1} \phi_s(0_{B_1}, 0_{B_2}, \ldots, 0_{B_{s-1}}, Y_{V_s}^{(u,s)}, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \ldots, X_{R_{n-s}}^{(s)}).$$

The following proposition will be used to calculate the survival functions of the system with a general standby.

Proposition 3.1 *For* $t \geq x \geq u \geq 0$ *and* $s = k_\phi, k_\phi + 1, \ldots, z_\phi + 1$,

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c^{(u,s)}, X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}) > t \mid X_{s:n} = x\}$$
$$= \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}^{n-s}(x)} \int \cdots \int_{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, y+u, x_{r_1}, x_{r_2}, \ldots, x_{r_{n-s}}) > t} g(\omega(x-u) + y + u)$$
$$\prod_{m=1}^{n-s} f(x_{r_m} + x) dx_{r_1} dx_{r_2} \ldots dx_{r_{n-s}} dy.$$

Proof: Because Y and $X_1, X_2, \ldots, X_n$ are independent,

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c^{(u,s)}, X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}) > t \mid X_{s:n} = x\}$$
$$= \int \cdots \int_{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, y+u, x_{r_1}, x_{r_2}, \ldots, x_{r_{n-s}}) > t} g(y + u \mid X_{s:n} = x) \times$$
$$f_J(x_{r_1}, x_{r_2}, \ldots, x_{r_{n-s}} \mid X_{s:n} = x) dx_{r_1} dx_{r_2} \ldots dx_{r_{n-s}} dy,$$

where f_J denotes the joint p.d.f. of $X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}$ given that $X_{s:n} = x$. Note that

$$f_J(x_{r_1}, x_{r_2}, \ldots, x_{r_{n-s}} \mid X_{s:n} = x) = \frac{1}{\bar{F}^{n-s}(x)} \prod_{m=1}^{n-s} f(x_{r_m} + x),$$

since the random variables $X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}$ are conditionally independent given that $X_{s:n} = x$, and

$$P\{X_{r_1}^{(s)} > x_{r_1}, X_{r_2}^{(s)} > x_{r_2}, \ldots, X_{r_{n-s}}^{(s)} > x_{r_{n-s}} \mid X_{s:n} = x\} = \prod_{m=1}^{n-s} \frac{\bar{F}(x_{r_m} + x)}{\bar{F}(x)}.$$

Also

$$\begin{aligned} G(y+u \mid X_{s:n}=x) &= 1 - P\{Y_c^{(u,s)} > y+u \mid X_{s:n}=x\} \\ &= 1 - \frac{\bar{G}(\omega(x-u)+y+u)}{\bar{G}(\omega(x-u))}\bar{G}(\gamma(x-u)). \end{aligned}$$

Now the result follows after simplification. □

3.1 Reliability of a coherent system with perfect switching

Consider a coherent system having n components and one general standby component. Initially, n components start working in active state and the standby component starts to work in cold state. As we have mentioned earlier, the standby component is switched over to the warm state (from the cold state) after a specified time u (≥ 0) up to which the system certainly does not fail, and it starts to work in active state in the usual environment at the time of sth component failure which may cause the system failure. Obviously, for $u = 0$, the system becomes a coherent system equipped with a warm standby component. Here we assume that, for the standby component, switching from cold state to warm state and from warm state to active state are perfect, i.e., the standby component does not fail at the time of switch over from one state to another, and that it is instantaneous. The reliability of such a system is given by the following theorem. Before going to the theorem, we briefly discuss the representation of survival function of a coherent system using system signature. Let T denote the lifetime of a coherent system consisting of n components having lifetimes $X_1, X_2, \ldots, X_n$. If X_i's are independent and have common distribution, then the survival function can be given by

$$P\{T > t\} = \sum_{i=1}^{n} p_i P\{X_{i:n} > t\},$$

where $p_i = P\{T = X_{i:n}\}$ is the probability that the ith component failure causes the system to fail. The n-dimensional probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is called system signature (cf. Kocher et al. [8], Samaniego ([10],[11])).

Let $\mathbf{B}_{s,c}$ represent the discrete multivariate random variable representing the indices of the failed component given that $V_s = c$; $s, c \in \{1, 2, \ldots, n\}$, i.e., $\mathbf{B}_{s,c} = (B_1 = b_1, B_2 = b_2, \ldots B_{s-1} = b_{s-1} \mid V_s = c)$ if and only if $(0_{B_1} = 0_{b_1}, 0_{B_2} = 0_{b_2}, \ldots, 0_{B_{s-1}} = 0_{b_{s-1}} \mid X_c = X_{s:n}, T = X_{s:n})$, where $B_i \in \{1, 2, \ldots, n\}$, $i = 1, 2, \ldots, s-1$, is the discrete random variable representing the index of the ith component failure. The following theorem gives the reliability function of a coherent system with a general standby component in case of perfect switching. The calculation has been illustrated in Example 3.1.

Theorem 3.1 *Let* $\boldsymbol{p}$ *be the signature vector of a coherent system which is equipped with* *a general standby component. Then, for $t > u$,*

$$P\{T^{gs} > t\} = \sum_{s=k_\phi}^{z_\phi+1} \Big(p_s P\{X_{s:n} > t\} + p_s \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{B_{s,c} = (b_1,\ldots,b_{s-1})\}\Big) \times \int_u^t P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t-x \mid X_{s:n}=x\}dF_{s:n}(x)\Big).$$

Proof: To derive the system reliability the following cases may arise in which the system survives at time $t > u$. Under the condition $P\{T = X_{s:n}\} > 0$, for $s = k_\phi, k_\phi+1,\ldots,z_\phi+1$, any coherent system operating with n active components may fail at the time of sth component failure. If the system failure occurs due to the sth component failure in $(u,t]$, then the warm standby component is switched over to usual environment. Thus, the survival function of the system can be written as

$$\begin{aligned}
P\{T^{gs} > t\} &= P\{T + \phi_{k_\phi}(0_{B_1},0_{B_2},\ldots,0_{B_{k_\phi-1}},Y_{V_{k_\phi}}^{(u,k_\phi)},X_{R_1}^{(k_\phi)},X_{R_2}^{(k_\phi)},\ldots,X_{R_{n-k_\phi}}^{(k_\phi)}) > t, T = X_{k_\phi:n}\} \\
&+P\{T + \phi_{k_\phi+1}(0_{B_1},0_{B_2},\ldots,0_{B_{k_\phi}},Y_{V_{k_\phi+1}}^{(u,k_\phi+1)},X_{R_1}^{(k_\phi+1)},X_{R_2}^{(k_\phi+1)},\ldots,X_{R_{n-k_\phi-1}}^{(k_\phi+1)}) > t, T = X_{k_\phi+1:n}\} + \ldots \\
&+P\{T + \phi_{z_\phi+1}(0_{B_1},0_{B_2},\ldots,0_{B_{z_\phi}},Y_{V_{z_\phi+1}}^{(u,z_\phi+1)},X_{R_1}^{(z_\phi+1)},X_{R_2}^{(z_\phi+1)},\ldots,X_{R_{n-z_\phi-1}}^{(z_\phi+1)}) > t, T = X_{z_\phi+1:n}\} \\
&+P\{T > t, T > X_{z_\phi+1:n}\}. \\
&= p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1},0_{B_2},\ldots,0_{B_{k_\phi-1}},Y_{V_{k_\phi}}^{(u,k_\phi)},X_{R_1}^{(k_\phi)},X_{R_2}^{(k_\phi)},\ldots,X_{R_{n-k_\phi}}^{(k_\phi)}) > t \mid T = X_{k_\phi:n}\} + p_{k_\phi+1} \\
&\quad P\{T + \phi_{k_\phi+1}(0_{B_1},0_{B_2},\ldots,0_{B_{k_\phi}},Y_{V_{k_\phi+1}}^{(u,k_\phi+1)},X_{R_1}^{(k_\phi+1)},X_{R_2}^{(k_\phi+1)},\ldots,X_{R_{n-k_\phi-1}}^{(k_\phi+1)}) > t \mid T = X_{k_\phi+1:n}\} + \ldots \\
&+p_{z_\phi+1} P\{T + \phi_{z_\phi+1}(0_{B_1},0_{B_2},\ldots,0_{B_{z_\phi}},Y_{V_{z_\phi+1}}^{(u,z_\phi+1)},X_{R_1}^{(z_\phi+1)},X_{R_2}^{(z_\phi+1)},\ldots,X_{R_{n-z_\phi-1}}^{(z_\phi+1)}) > t \mid T = X_{z_\phi+1:n}\},
\end{aligned} \tag{3.1}$$

since $P\{T > t, T > X_{z_\phi+1:n}\} = 0$. Now, write $\mathbf{0}_B$ as the event that $(0_{B_1} = 0_{b_1},\ldots,0_{B_{s-1}} = 0_{b_{s-1}})$. Then, for $s = k_\phi, k_\phi+1,\ldots,z_\phi+1$ and $t > u(\ge 0)$, we have

$$\begin{aligned}
&P\{T + \phi_s(0_{B_1},0_{B_2},\ldots,0_{B_{s-1}},Y_{V_s}^{(u,s)},X_{R_1}^{(s)},X_{R_2}^{(s)},\ldots,X_{R_{n-s}}^{(s)}) > t \mid T = X_{s:n}\} \\
= &\sum_{c=1}^{n} \frac{P\{X_c + \phi_s(0_{B_1},0_{B_2},\ldots,0_{B_{s-1}},Y_c^{(u,s)},X_{R_1}^{(s)},X_{R_2}^{(s)},\ldots,X_{R_{n-s}}^{(s)}) > t, X_{s:n} = X_c, T = X_{s:n}\}}{P\{T = X_{s:n}\}} \\
= &\sum_{c=1}^{n} P\{X_c + \phi_s(0_{B_1},0_{B_2},\ldots,0_{B_{s-1}},Y_c^{(u,s)},X_{R_1}^{(s)},X_{R_2}^{(s)},\ldots,X_{R_{n-s}}^{(s)}) > t \mid X_{s:n} = X_c, T = X_{s:n}\} \\
&\times P\{X_{s:n} = X_c \mid T = X_{s:n}\} \\
= &(P\{X_{s:n} = X_c, T = X_{s:n}\})^{-1} \Bigg[\sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{\mathbf{0}_B, X_{s:n} = X_c, T = X_{s:n}\} \times \\
&P\{X_c + \phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t \mid \mathbf{0}_B, X_{s:n} = X_c, T = X_{s:n}\}\Big] \\
= &\sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{\mathbf{0}_B \mid X_{s:n} = X_c, T = X_{s:n}\} \\
&\times P\{X_c + \phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t \mid \mathbf{0}_B, X_{s:n} = X_c, T = X_{s:n}\}
\end{aligned}$$

$$= \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{\mathbf{B}_{s,c} = (b_1,\ldots,b_{s-1})\}$$
$$\times \int P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t-x \mid X_{s:n}=x\} dF_{s:n}(x)$$
$$= \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{\mathbf{B}_{s,c} = (b_1,\ldots,b_{s-1})\} \times \left[\int_t^{\infty} dF_{s:n}(x) + \right.$$
$$\left. \int_u^t P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t-x \mid X_{s:n}=x\} dF_{s:n}(x)\right]$$

$$= P\{X_{s:n} > t\} + \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{B_{s,c} = (b_1,\ldots,b_{s-1})\}$$
$$\times \int_u^t P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t-x \mid X_{s:n}=x\} dF_{s:n}(x). \tag{3.2}$$

Now using (3.2), from (3.1) we have the required result. □

Below we give the result corresponding to the coherent system with single warm standby component.

Corollary 3.1 *If the standby component starts to work in the warm state at time zero, then we get the reliability function of a coherent system equipped with single warm standby component from Theorem 3.1 by taking* $u = 0$. □

Corollary 3.2 *Reliability function of a* k*-out-of-*n *system equipped with a general standby component is given by*

$$P\{T^{gs} > t\} = P\{X_{n-k+1:n} > t\} + \frac{\bar{F}^{k-1}(t)}{B(n-k+1,k)} \times \int_u^t \frac{\bar{G}(\omega(x-u)+t-x)}{\bar{G}(\omega(x-u))} \bar{G}(\gamma(x-u)) F^{n-k}(x) dF(x).$$

Proof: For a k-out-of-n system, we have $s = k_\phi = z_\phi + 1 = n-k+1$. Because the signature of a k-out-of-n system is the n-dimensional vector $\mathbf{p} = (0,\ldots,0,1,0,\ldots,0)$ with 1 is in the $(n-k+1)$th place, we have, from Theorem 3.1,

$$P\{T^{gs} > t\} = P\{X_{n-k+1:n} > t\} +$$
$$\int_u^t P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{n-k}},Y_{n-k+1}^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{k-1}}^{(s)}) > t-x \mid X_{n-k+1:n}=x\} dF_{n-k+1:n}(x),$$

with $s = n-k+1$. Now, from Proposition 3.1, we have

$$P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{n-k}},Y_{n-k+1}^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{k-1}}^{(s)}) > t-x \mid X_{n-k+1:n}=x\}$$

$$= \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}^{k-1}(x)} \times$$

$$\int \cdots \int_{\min(y+u, x_{r_1}, x_{r_2}, \ldots, x_{r_{k-1}}) > t-x} g(\omega(x-u)+y+u) \prod_{m=1}^{k-1} f(x_{r_m}+x) dx_{r_1} dx_{r_2} \ldots dx_{r_{k-1}} dy$$

$$= \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}^{k-1}(x)} \bar{G}(\omega(x-u)+t-x)\bar{F}^{k-1}(t),$$

so that

$$\begin{aligned} P\{T^{gs} > t\} &= P\{X_{n-k+1:n} > t\} + \\ &\quad \bar{F}^{k-1}(t) \int_u^t \frac{\bar{G}(\omega(x-u)+t-x)}{\bar{G}(\omega(x-u))} \bar{G}(\gamma(x-u)) \frac{1}{\bar{F}^{k-1}(x)} dF_{n-k+1:n}(x) \\ &= P\{X_{n-k+1:n} > t\} + \frac{\bar{F}^{k-1}(t)}{B(n-k+1,k)} \\ &\quad \int_u^t \frac{\bar{G}(\omega(x-u)+t-x)}{\bar{G}(\omega(x-u))} \bar{G}(\gamma(x-u)) F^{n-k}(x) dF(x), \end{aligned} \tag{3.3}$$

where $B(a,b) = \frac{\Gamma a \Gamma b}{\Gamma(a+b)}$. Now, for $u = 0$, (3.3) gives the reliability function of a k-out-of-n system equipped with a warm standby component as shown in Eryilmaz [3].

Remark 3.1 Taking $n = k = 1$ in Corollary 3.2, we get the reliability of a single (active) component system with a general standby unit as given in (2.2). In particular, for $u = 0$, we obtain (2.1).

Remark 3.2 By taking $u = 0$ and $\gamma(x) = \omega(x) = 0$ in Theorem 3.1, we obtain the reliability function of a coherent system equipped with a cold standby component given in Franko et al. [5]. □

The result of Theorem 3.1 can also be represented as follows.

Theorem 3.2 *Let* $\mathbf{p}$ *be the signature vector of a coherent system equipped with a general standby component with lifetime distribution* G*. Then*

$$\begin{aligned} P\{T^{gs} > t\} &= \sum_{s=k_\phi}^{z_\phi+1} \Big(p_s P\{X_{s:n} > t\} + p_s \sum_{c=1}^{n} P\{V_s = c\} \sum_{1 \le b_1 < \ldots < b_{s-1} \le n} P\{B_{s,c} = (b_1, \ldots, b_{s-1})\} \\ &\quad \times \int_u^t \frac{\bar{G}(\omega(x-u)+t-x)}{\bar{G}(\omega(x-u))} \bar{G}(\gamma(x-u)) \left[\sum_{k=1}^{n-s} p_k^{c,(b_1,\ldots,b_{s-1})} P\{X_{k:n-s}^{(s)} > t-x \mid X_{s:n} = x\} \right. \\ &\quad \left. + p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})} \right] dF_{s:n}(x) \Big), \end{aligned}$$

where $\{p_k^{c,(b_1,\ldots,b_{s-1})},\ k = 1, 2, \ldots, n-s+1\}$ *represents the signature vector corresponding to the* $(n-s)$ *surviving components and the standby component at place* c *with the understanding that when* $p_k^{c,(b_1,\ldots,b_{s-1})}$ *is non-zero for at least one* $k \in \{1, 2, \ldots, n-s\}$*, then* $p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})} = 0$.

Proof: Let the standby component switch over to the usual environment from the warm state at the time $x \in [u,t]$, for $u \geq 0$. Then the system may fail due to the failure of any of the $(n-s)$ surviving components (other than the standby) or due to the failure of the standby component. Clearly, if $p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})}$ is the probability that the system fails due to the failure of the standby component, then $p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})}$ will be zero if and only if at least one of $p_k^{c,(b_1,\ldots,b_{s-1})}$, $k \in \{1,2,\ldots,n-s\}$ is non-zero, otherwise $p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})}$ will be unity. Then, we have

$$
\begin{aligned}
& P\{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},Y_c^{(u,s)},X_{r_1}^{(s)},X_{r_2}^{(s)},\ldots,X_{r_{n-s}}^{(s)}) > t-x \mid X_{s:n}=x\} \\
= \; & P\{Y_c^{(u,s)} > t-x \mid X_{s:n}=x\}\sum_{k=1}^{n-s} p_k^{c,(b_1,\ldots,b_{s-1})} P\{X_{k:n-s}^{(s)} > t-x \mid X_{s:n}=x\} \\
& + p_{n-s+1}^{c,(b_1,\ldots,b_{s-1})} P\{Y_c^{(u,s)} > t-x \mid X_{s:n}=x\}.
\end{aligned}
$$

Thus, the result follows from Theorem 3.1. □

Below we consider an example to demonstrate the result of the theorems.

Example 3.1 Let us consider the following coherent system having lifetime

$$T = \min(X_1, \max(X_2, X_3))$$

formed by three independent components C_1, C_2, C_3 with lifetimes X_1, X_2, X_3, respectively and a standby component Y which starts to work in cold state and switches over to warm state after a pre-specified time $u \geq 0$. The signature of this system is $\mathbf{p} = (\frac{1}{3}, \frac{2}{3}, 0)$, and $k_\phi = z_\phi = 1$ so that $s = 1$ or 2. It is to be noted that, for $s=1$, $B_{1,1} = \phi$ (as there is no previously failed component) and $P\{V_1 = 1\} = 1$, $P\{V_1 = 2\} = P\{V_1 = 3\} = 0$. Now, from Proposition 3.1, we have

$$
\begin{aligned}
& P\{\phi_1(Y_1^{(u,1)}, X_2^{(1)}, X_3^{(1)}) > t-x \mid X_{1:3}=x\} \\
= \; & \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))(\bar{F}(x))^2} \iiint\limits_{\min(y+u,\max(x_2,x_3))>t-x} g(\omega(x-u)+y+u)f(x_2+x)f(x_3+x)dx_2dx_3dy \\
= \; & \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))(\bar{F}(x))^2}\bar{G}(\omega(x-u)+t-x)\big[(\bar{F}(t))^2 + 2\bar{F}(t)(F(t)-F(x))\big].
\end{aligned}
$$

For $s=2$, previously failed component is either C_2 or C_3. In this case $P\{V_2 = 1\} = \frac{1}{2}$, $P\{V_2 = 2\} = P\{V_2 = 3\} = \frac{1}{4}$ and $P\{B_{2,2} = 3\} = P\{B_{2,3} = 2\} = 1$. Now, from Proposition 3.1, we have

$$
\begin{aligned}
& P\{\phi_2(0_2, Y_1^{(u,2)}, X_3^{(2)}) > t-x \mid X_{2:3}=x\} \\
= \; & \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}(x)} \iint\limits_{\min(y+u,,x_3)>t-x} g(\omega(x-u)+y+u)f(x_3+x)dx_3dy
\end{aligned}
$$

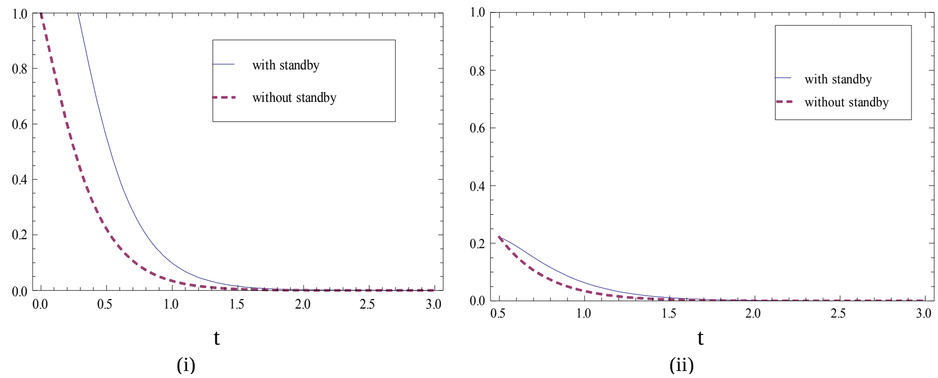

Figure 1: Reliability function of the system with and without standby in case of (i) $u = 0$ and (ii) $u = 0.5$.

$$= \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}(x)}\bar{G}(\omega(x-u)+t-x)\bar{F}(t). \tag{3.4}$$

Because X_1, X_2 and X_3 are independent with common c.d.f. F, the expression for $P\{\phi_2(0_2, Y_3^{(u,2)}, X_1^{(2)}) > t - x \mid X_{2:3} = x\}$, $P\{\phi_2(0_3, Y_1^{(u,2)}, X_2^{(2)}) > t - x \mid X_{2:3} = x\}$, and $P\{\phi_2(0_3, Y_2^{(u,2)}, X_1^{(2)}) > t - x \mid X_{2:3} = x\}$ will be the same as given in (3.4). Thus, from Theorem 3.1, we have

$$\begin{aligned} P\{T^{gs} > t\} &= \frac{1}{3}P\{X_{1:3} > t\} + \frac{2}{3}P\{X_{2:3} > t\} + \\ &\quad \frac{1}{3}\int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))(\bar{F}(x))^2}\bar{G}(\omega(x-u)+t-x)\big[(\bar{F}(t))^2 + 2\bar{F}(t)(F(t)-F(x))\big]dF_{1:3}(x) \\ &\quad + \frac{2}{3}\int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}(x)}\bar{G}(\omega(x-u)+t-x)\bar{F}(t)dF_{2:3}(x) \\ &= \frac{1}{3}P\{X_{1:3} > t\} + \frac{2}{3}P\{X_{2:3} > t\} + \\ &\quad \int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))(\bar{F}(x))^2}\bar{G}(\omega(x-u)+t-x)\big[(\bar{F}(t))^2 + 2\bar{F}(t)(F(t)-F(x))\big]F(x)\bar{F}(x)f(x)dx \\ &\quad + 4\int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}(x)}\bar{G}(\omega(x-u)+t-x)\bar{F}(t)\bar{F}^2(x)f(x)dx. \end{aligned}$$

In Fig. 1, we plot the reliability function of the above coherent system with and without a standby unit when $F(t) = G(t) = 1 - e^{-2t}, t > 0$ and $\omega(t) = \gamma(t) = t/2$.

3.2 Reliability of a coherent system with imperfect switching

Consider the coherent system as discussed in Section 3.1 with all the assumptions remaining the same except the switching case. By imperfect switching we mean that the standby component may fail at the time of switch over from one state to another with certain positive probability. In our discussion, a standby component remains in cold state during the period $[0, u)$ and switches over to the warm state at time u, when it is as good as new. So, we assume that at the time of switch over from cold state to warm state, the standby component does not fail whereas, at the time of switch over from warm state to active state it does not fail with pre-assigned probability $p \in [0,1]$. If a standby component does not fail during switch over, this is known as perfect switching. We also assume that the change over from one state to another is instantaneous. Then the reliability of the system is given by the following theorem. The proof is similar to that of Theorem 3.1, and hence omitted.

Theorem 3.3 *Let* $\boldsymbol{p}$ *be the signature vector of a coherent system as mentioned above. If the probability of perfect switching from warm state to active state is* p*, then, for* $t > u$*,*

$$P\{T^{gs} > t\} = \sum_{s=k_\phi}^{z_\phi+1} \big(p_s P\{X_{s:n} > t\} + p_s \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{B_{s,c} = (b_1,\ldots,b_{s-1})\}$$
$$\times p \int_u^t P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c^{(u,s)}, X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}) > t - x \mid X_{s:n} = x\} dF_{s:n}(x)\big).$$

Corollary 3.3 *Putting* $p = 1$ *in Theorem 3.3 we get Theorem 3.1.*

Remark 3.3 Considering the same coherent system as in Example 3.1, with probability of perfect switching from warm state to active state as p, we have

$$\begin{aligned} P\{T^{gs} > t\} &= \frac{1}{3}P\{X_{1:3} > t\} + \frac{2}{3}P\{X_{2:3} > t\} + p \times \\ &\Big[\frac{1}{3}\int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))(\bar{F}(x))^2}\bar{G}(\omega(x-u)+t-x) \\ &\big[(\bar{F}(t))^2 + 2\bar{F}(t)(F(t)-F(x))\big]dF_{1:3}(x) \\ &+\frac{2}{3}\int_u^t \frac{\bar{G}(\gamma(x-u))}{\bar{G}(\omega(x-u))\bar{F}(x)}\bar{G}(\omega(x-u)+t-x)\bar{F}(t)dF_{2:3}(x)\Big] \end{aligned}$$

3.3 Reliability of a coherent system with Random warm up period

Consider the coherent system as discussed in Section 3.1 with all the assumptions remaining the same except that switching from cold state to warm state is not instantaneous and a random warm up period may be required for state change. Further we consider that warm up starts at a fixed $u_1 \in [0,u]$ so that the distribution function H of the random warm up time has the support $[u_1, u]$. We assume that the warm up time is independent of the life of the standby component. Then the reliability of the system is given by the following theorem. The proof is similar to that of Theorem 3.1.

Theorem 3.4 *Let* $\boldsymbol{p}$ *be the signature vector of a coherent system as mentioned above.*

Then, for $t > u$,

$$P\{T^{gs} > t\} = \sum_{s=k_\phi}^{z_\phi+1} \big(p_s P\{X_{s:n} > t\} + p_s \sum_{c=1}^{n} P\{V_s = c\} \sum_{1\le b_1<\ldots<b_{s-1}\le n} P\{B_{s,c} = (b_1, \ldots, b_{s-1})\} \times \int_u^t \int_{u_1}^u P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c^{(r,u_1,u,s)}, X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}) > t - x \mid X_{s:n} = x\} dH(r)dF_{s:n}(x)\big),$$

where $Y_c^{(r,u_1,u,s)}$ is same as $Y_c^{(u,s)}$ with warm up time $r \in [u_1, u]$.

Corollary 3.4 *If warm up time is a degenerate random variable degenerate at u, then Theorem 3.4 gives Theorem 3.1.*

Remark 3.4 *From Proposition 3.1 it can be seen that*

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \ldots, 0_{b_{s-1}}, Y_c^{(r,u_1,u,s)}, X_{r_1}^{(s)}, X_{r_2}^{(s)}, \ldots, X_{r_{n-s}}^{(s)}) > t - x \mid X_{s:n} = x\} = \frac{\bar{G}(\gamma(x-u_1-r))}{\bar{G}(\omega(x-u_1-r))\bar{F}^{n-s}(x)} \int\cdots\int_{\phi_s(0_{b_1},0_{b_2},\ldots,0_{b_{s-1}},y+u_1+r,x_{r_1},x_{r_2},\ldots,x_{r_{n-s}})>t-x} g(\omega(x-u_1-r)+y+u_1+r) \prod_{m=1}^{n-s} f(x_{r_m}+x)dx_{r_1}dx_{r_2}\ldots dx_{r_{n-s}}dy$$

Remark 3.5 Consider the same coherent system as in Example 3.1. Now if switching from cold state to warm state requires warm up time and H is the c.d.f. of warming up time, then we have

$$P\{T^{gs} > t\} = \frac{1}{3}P\{X_{1:3} > t\} + \frac{2}{3}P\{X_{2:3} > t\} + \frac{1}{3}\int_u^t \int_{u_1}^u \frac{\bar{G}(\gamma(x-u_1-r))}{\bar{G}(\omega(x-u_1-r))(\bar{F}(x))^2}\bar{G}(\omega(x-u_1-r)+t-x) \big[(\bar{F}(t))^2 + 2\bar{F}(t)(F(t)-F(x))\big]dH(r)\ dF_{1:3}(x) + \frac{2}{3}\int_u^t \int_{u_1}^u \frac{\bar{G}(\gamma(x-u_1-r))}{\bar{G}(\omega(x-u_1-r))\bar{F}(x)}\bar{G}(\omega(x-u_1-r)+t-x)\bar{F}(t)dH(r)\ dF_{2:3}(x).$$

4 Conclusion

Although the coherent system with a cold or a hot standby component has been studied in the literature, to the best of our knowledge, the coherent system with general standby component has not been investigated. In this paper, we study the reliability of a coherent system equipped with a general standby component. From practical experience, sometimes we get a system, which tells that at least for some initial period of time, say u, the system will never fail. Keeping this in mind, we allow the standby component to be in cold state initially for a period $[0, u]$, where u is fixed and it switches over to the warm state at time u. Because at time u, the standby component is as good as new, it is logical to assume that at the time of switch over from cold state to warm state it does not fail. We also consider the case where the standby component may fail at the time of switch over from warm state to active state (at the failure of a component which makes the system to fail). This is called imperfect switch over. In both the cases we get the explicit expression for the reliability of the system. Since the results in this paper are more general, the mathematical expressions look complicated. We have given one example to demonstrate how one can calculate the reliability of the system in practice. This will surely help people to use the results in practice.

Further, in some cases, the standby component may require some warm up period before switching over from the cold state to the warm state so that once the system fails due to failure of a particular component the standby component may replace the failed component in order to keep the system functioning. This warm up period may not be known in general. Also the warming up time will vary depending on the nature of the standby components. To handle this case, we assume that the warm up time is a random variable having distribution function H with support $[u_1, u]$, $0 \leq u_1 \leq u$.

It is to be noted that from our general results, the results related to a k-out-of-n system and the results related to cold standby and hot standby available in the literature are obtained as particular cases.

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