

Quantum code for quantum error characterization

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A quantum error correcting code is a subspace \mathcal{C} such that allowed errors acting on any state in \mathcal{C} can be corrected. A quantum code for which state recovery is only required up to a logical rotation within \mathcal{C} , can be used for detection of errors, but not for quantum error correction. Such a code with stabilizer structure, which we call an “ambiguous stabilizer code” (ASC), can nevertheless be useful for the characterization of quantum dynamics (CQD). The use of ASCs can help lower the size of CQD probe states used, but at the cost of increased number of operations.

PACS numbers: 03.67.Pp, 03.65.Wj

I. INTRODUCTION

In quantum information processing, a quantum error correcting code (QECC) is a subspace \mathcal{C} , carefully selected to protect from certain noise, any initial state $|\Psi\rangle \in \mathcal{C}$ [1]. Let $\{|j_L\rangle\}$ be a n -qubit basis for \mathcal{C} , encoding k -qubit states $|j\rangle$, with $0 \leq j \leq 2^k - 1$. Such a code is a $[[n, k]]$ QECC, where k is the code rate. In this work, we will represent the noise using the error basis given by elements E_k of the Pauli group \mathcal{P}_n , the set of all possible tensor products of n Pauli operators, with and without factors $\pm 1, \pm i$. Thus $E_k^\dagger = E_k$ and $(E_k)^2 = I_n$, the identity operator over n qubits.

A. Stabilizer formalism

A stabilizer description of error correction is connected to classical error correcting codes over GF(4) [2]. Where applicable, the stabilizer formalism is advantageous in focussing attention on measurement operators, which can be compact, rather than on code words, which can be large. A state $|\psi_L\rangle$ is said to be stabilized by an operator S if $S|\psi_L\rangle = |\psi_L\rangle$. Let \mathcal{G} be a subset of $n - k$ independent, commuting elements from \mathcal{P}_n . A $[[n, k]]$ QECC is the 2^k -dimensional simultaneous $+1$ -eigenspace \mathcal{C} of the elements of \mathcal{G} . A basis for \mathcal{C} are the code words $|j_L\rangle$. The set of 2^{n-k} operators generated by \mathcal{G} constitute the stabilizer \mathcal{S} . The centralizer of \mathcal{S} is the set of all elements of \mathcal{P}_n that commute with each member of \mathcal{S} :

$$\mathcal{Z} = \{P \in \mathcal{P}_n \mid \forall S \in \mathcal{S}, [P, S] = 0\}, \quad (1)$$

while the normalizer of \mathcal{S} is the set of all elements of \mathcal{P}_n that conjugate the stabilizer to itself:

$$\mathcal{N} = \{P \in \mathcal{P}_n \mid PSP^\dagger = S\}. \quad (2)$$

We note that $\mathcal{S} \subseteq \mathcal{N}$ because the elements of \mathcal{S} are unitary and mutually commute. Similarly, $\mathcal{Z} \subseteq \mathcal{N}$ because elements of the \mathcal{Z} are unitary and commute with all elements of \mathcal{S} . To see that $\mathcal{N} \subseteq \mathcal{Z}$, we note that if $N \in \mathcal{N}$ then given any $S \in \mathcal{S}$, $NSN^\dagger = S'$, for some $S' \in \mathcal{S}$, or $NS = S'N$. For Pauli operators, $NS = \pm SN$, meaning $S' = \pm S$. But if $S' = -S$, then $NSN^\dagger = -S$, which would require that both S and $-S$ are in \mathcal{S} . However if $S \in \mathcal{S}$, then $-S$ is not in the stabilizer, so the only possibility is $S' = S$, and we obtain that for each N and any S , $[N, S] = 0$, i.e., $\mathcal{N} \subseteq \mathcal{Z}$. It thus follows that here $\mathcal{Z} = \mathcal{N}$. We have $SN|j_L\rangle = NS|j_L\rangle = N|j_L\rangle$, which shows that the action of N maps code words to code words, and thus has the action of a logical Pauli operation on code words.

A set of operators $E_j \in \mathcal{P}_n$ constitutes a basis for correctable errors if one of the following conditions hold:

$$E_j E_k \in \mathcal{S} \quad (3a)$$

$$\exists G \in \mathcal{G} : [E_j E_k, G] \neq 0. \quad (3b)$$

The case (3a) corresponds to *degeneracy*. Here $\langle \psi_L | E_j E_k | \psi_L \rangle = \langle \psi_L | \psi_L \rangle = 1$, meaning that both errors produce the same effect, and the code space is indifferent as to which of them happened. Thus either error operator can be applied as a recovery operation when one of them occurs. The case (3b) corresponds to $E_j E_k \notin \mathcal{N}$. In that case, $\exists G \in \mathcal{G} : E_j E_k G = -G E_j E_k$, which ensures that G anti-commutes with precisely one of the operators E_j and E_k . Thus the noisy logical states $E_j |\psi_L\rangle$ and $E_k |\psi_L\rangle$ will yield distinct eigenvalues (one being $+1$ and the other -1) when G is measured. The set of $n - k$ eigenvalues ± 1 obtained by measuring the generators G forms the error syndrome. The consolidated error correcting condition (3) can be stated as the requirement

$$E_j E_k \notin \mathcal{N} - \mathcal{S}, \quad (4)$$

for every pair of error basis elements, with $j \neq k$.

B. Noise characterization and stabilizer codes

Characterization of the quantum dynamics (CQD) of a quantum system is vital for practical applications

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in quantum information processing, communication and computation, on account of environmental decoherence. If σ represents the quantum state of the system at time $t = 0$, then it evolves under the action of the noise to

$$\mathcal{E}(\sigma) = \sum_{m,n} \chi_{m,n} E_m \sigma E_n^\dagger, \quad (5)$$

where χ is the *process matrix*, a Hermitian operator satisfying the properties $\sum_{j,k} \chi_{j,k} E_j^\dagger E_k = \mathbb{I}$, and $\sum_m \chi_{m,m} = 1$ [3]. The (positive) elements $\chi_{j,j}$ are probabilities for errors E_i to occur. The terms $\chi_{j,k}$ ($j \neq k$) refer to the coherence between two distinct errors.

Quantum process tomography (QPT) denotes a CQD technique where for selected input states σ_j , complete state tomographic data $\lambda_{j,k} = \text{Tr}(\sigma_j \mathcal{E}(\sigma_k))$ is obtained. The process matrix is derived by inversion of this experimental data. There have been several QPT techniques, like standard quantum process tomography (SQPT) [4, 5] and ancilla-assisted process tomography (AAPPT) [6]. Other CQD methods, which bypass state tomography, include “direct characterization of quantum dynamics” (DCQD) [7] and “quantum error correction based characterization of dynamics” (QECCD), introduced by the present authors [3]. Other developments include an efficient method for estimating diagonal terms of the process matrix using twirling [8], which is useful for determining QECCs [9]. Other related works on channel estimation include Ref. [10], a technique like that in Ref. [9] extended to cover off-diagonal χ_{jk} terms, and Refs. [11].

QECCD brings about a twist to the theme of using QECCs in that the codes are used not just for protecting quantum states, but also for CQD. This permits one to implement CQD concurrently with quantum computation, making QECCD work “online”. The intuition is that the statistics of errors detected during the error correction process are used to characterize the noise. QECCD makes use of the properties of a class of stabilizer codes in which the allowed Pauli error operators form a group.

C. Code ambiguity

A natural extension of the concept behind QECCD would be to adapt stabilizer techniques purely or primarily for CQD, rather than for quantum error correction. Freed from correction duty, codes are no longer bound to obey (3). This can be exploited to design codes with code lengths smaller than permissible under (3), thereby making code words easier to implement experimentally.

However the price to pay for violating (3) is that some errors will be indistinguishable, making stabilizer measurement outcomes ambiguous. The new class of stabilizer codes that we introduce here are therefore called *ambiguous stabilizer codes* (ASCs). In an ASC, the final state after recovery may contain a residual logical Pauli operation with respect to the initial logical state. An ASC generalizes the concept of a degenerate code,

which is the special case where the only residual logical operation after recovery is the trivial one.

Thus, the purpose of invoking ambiguity—indeed the principal motivation behind the construction of ASC’s—is to exploit the stabilizer formalism and structure for the construction of codes that are better suited for error characterization rather than for error correction.

Code ambiguity entails, as detailed below, that more state preparations involving other ASCs are required to unambiguously determine the process matrix. Later we will find that the number of ASCs required for full characterization scales linearly with ambiguity, in a way made precise later. Thus there is a trade-off between spatial resources (length of code words) and temporal resources (number of operations). We call this ambiguous extension of QECCD as “quantum ASC-based characterization of dynamics” (QASCD). From an experimental viewpoint, the above trade-off means QASCD helps simplify quantum state preparation at the cost of increased number of trials and classical post-processing.

QASCD is, unlike traditional methods of process tomography but like the techniques presented in Refs. [3, 7], a *direct* method in that it does not require the full state tomography of probe states used for CQD. At the same time, QASCD may require smaller codes than these direct techniques. In particular, to characterize m -qubit noise, the above direct techniques require probes to be $2m$ -qubit states or larger. By contrast, with ASCs one can beat this bound. For example, to characterize 2-qubit noise, one can in principle use (a family of) 3-qubit ASCs.

The remaining article is structured as follows. After first developing a theory of ASCs in Section II, we study in Section III their specific group theoretic properties as would be useful for CQD. In Section IV, we detail the protocol that would be used for CQD by employing (a family of) ASCs. The resources, in terms of number of ASCs and operations required for CQD, are discussed in Section V. A trade-off between the space resources (length of codes used) vs time resources (required configurations) is discussed here. After illustrating our new method as applied to a toy 2-qubit noise in Section VI, we finally present conclusions in Section VII.

II. AMBIGUOUS STABILIZER CODES

A. Definition and basic features

A 2^k -dimensional subspace \mathcal{C}' of n qubits, together with an allowed set \mathbb{E} of Pauli error operators, is *ambiguous* when two or more errors cannot be distinguished via syndrome measurements on the logical state. The indistinguishable errors may require different recovery operations. Thus ambiguity generalizes the concept of degeneracy, and in general prevents error correction.

An *ambiguous set* $A^{(p)}$ is a collection of allowed Pauli errors that cannot be mutually distinguished by a syn-

drome measurement. Let $A^{(p)} \equiv \{E_1^{(p)}, E_2^{(p)}, \dots, E_{v(p)}^{(p)}\}$, where $E_j^{(p)}$ is the j th error in the p th ambiguous set, and there are $v(p)$ elements in the p th ambiguous set. Then any two errors $E_j^{(p)}$ and $E_k^{(p)}$, where $j \neq k$, produce the same error syndrome. Thus, Pauli errors that are elements of the same ambiguous set are mutually indistinguishable. However, they may not be degenerate, and thus may require different recovery operations, making an ambiguous code unsuitable for error correction.

Ambiguity of the code can be represented by partitioning \mathbb{E} into ambiguous sets. The collection of all ambiguous sets is the ambiguous class $\mathcal{A} = \{A^{(1)}, A^{(2)}, \dots, A^{(\sigma)}\}$, with $\bigcup_p A^{(p)} = \mathbb{E}$. The *order of ambiguity* of the code is σ , the number of ambiguous sets in the ambiguous class. The *degree of ambiguity*, denoted γ , is the largest number of Pauli errors in \mathbb{E} that can be mutually ambiguous, i.e.,

$$\gamma \equiv \max_p |A^{(p)}|. \quad (6)$$

A conventional (unambiguous) stabilizer code is characterized by $\gamma = 1$, whereas for an ASC, $\gamma > 1$. Any set of up to s known errors drawn from distinct ambiguous sets $A^{(p)}$ can be detected, and if the errors are known, they can be corrected.

Within an ambiguous set $A^{(p)}$, the error elements produce the same error syndrome. This means that the action of two ambiguous errors $E_n^{(p)}$ and $E_m^{(p)}$ must be related by

$$E_n^{(p)}|j_L\rangle = N E_m^{(p)}|j_L\rangle, \quad (7)$$

where N is a normalizer element. Note that $N E_m^{(p)}|j_L\rangle = \pm E_m^{(p)} N|j_L\rangle$. Thus,

$$\begin{aligned} \sum_{j=0}^{2^k-1} E_n^{(p)}|j_L\rangle \langle j_L| E_n^{(p)} &= \sum_{j=0}^{2^k-1} E_m^{(p)} N|j_L\rangle \langle j_L| N E_m^{(p)} \\ &= \sum_{j'=0}^{2^k-1} E_m^{(p)}|j'_L\rangle \langle j'_L| E_m^{(p)} \\ &\equiv \Pi^{(p)}, \end{aligned} \quad (8)$$

where j' is simply a re-ordering of j . In other words, every element $E_m^{(p)}$ generates the same erroneous subspace, with projector $\Pi^{(p)}$. However, individual code words are not necessarily mapped to the same erroneous code word, in view of (7). Further, from Eq. (7), we have $N = E_n^{(p)} E_m^{(p)}$. If $E_m^{(p)}|j_L\rangle = N' E_n^{(p)}|j_L\rangle$, then $N' = E_m^{(p)} E_n^{(p)}$. Thus, $N^\dagger = N'$.

Note that if $[E_n^{(p)}, E_m^{(p)}] = 0$, then $N^\dagger = N$ (Hermiticity condition) and thus $N = N'$. Conversely, if $N = N'$, then $E_n^{(p)} E_m^{(p)} = E_m^{(p)} E_n^{(p)}$, and thus $[E_n^{(p)}, E_m^{(p)}] = 0$. If $\{E_n^{(p)}, E_m^{(p)}\} = 0$, then $N^\dagger = -N$ (anti-Hermiticity) and thus $N = -N'$. Conversely, if $N = -N'$, then $E_n^{(p)} E_m^{(p)} = -E_m^{(p)} E_n^{(p)}$, and thus $\{E_n^{(p)}, E_m^{(p)}\} = 0$.

In contrast to the case with subspaces generated by ambiguous errors, projectors to distinct unambiguous erroneous subspaces are orthogonal:

$$\Pi^{(p)} \Pi^{(q)} = 0, \quad (9)$$

if $p \neq q$. Thus two or more errors belonging to distinct ambiguous sets can always be disambiguated.

B. Ambiguously detectable errors

Ambiguous errors $E_m^{(p)}$ and $E_n^{(p)}$ that are linked in Eq. (7) with $N = I_L$, where I_L is the logical Pauli identity operator, require the same recovery operation. Ambiguous errors related by non-trivial logical Pauli operations will require distinct recovery operations. Thus, an ambiguous code cannot be used for quantum error correction.

For ASCs, the error correcting condition (3) becomes:

$$p \neq q \Rightarrow E_m^{(p)} E_n^{(q)} \notin \mathcal{N}, \quad (10a)$$

$$p = q \Rightarrow E_m^{(p)} E_n^{(q)} \in \mathcal{N}. \quad (10b)$$

Eq. (10a) implies that quantum error correction can be implemented for any collection of *known* errors which belong to distinct ambiguous sets. Eq. (10b) implies that any pair of errors belonging to the same ambiguous set will produce the same syndrome, and thus be indistinguishable. In particular, if $E_m^{(p)} E_n^{(p)} \in \mathcal{S}$, then $\langle \psi_L | E_m^{(p)} E_n^{(p)} | \psi_L \rangle = \langle \psi_L | \psi_L \rangle$, meaning that the two errors are mutually degenerate, and the ambiguity is harmless in the sense that the recovery operation for any one of them works for the other, too. On the other hand, if $E_m^{(p)} E_n^{(p)} \in \mathcal{N} - \mathcal{S}$, then the erroneous code words they produce are related by non-trivial logical Pauli operations Eq. (7), and the error correcting conditions (4) are violated. If one implements a recovery operation favoring a single error in each ambiguous set, this will in general produce a mixture of states within the code space \mathcal{C}' , which are logical Pauli rotated versions of each other.

In \mathcal{A} , each ambiguous set $A^{(p)}$ corresponds to the same error syndrome, so that order $\sigma \leq 2^{n-k}$. By definition, the set $A^{(0)}$ will contain the element I and, by virtue of Eq. (10b), only elements of the normalizer \mathcal{N} . The remaining sets $A^{(1)}, A^{(2)}, \dots$ will contain Pauli operators not present in \mathcal{N} , since they will fail to commute with at least one stabilizer generator.

For unambiguous (and non-degenerate) recovery using a linear QECC, the dimension of the code space and the volume $|\mathbb{E}|$ must satisfy the quantum Hamming bound, $2^k |\mathbb{E}| \leq 2^n$, or

$$\log(|\mathbb{E}|) \leq n - k. \quad (11)$$

An ambiguous code may violate (11), though not necessarily. A QECC that saturates Eq. (11) is called *perfect*. The 5-qubit code of Ref. [12] is such an example.

C. Constructing ASC's

The simplest way to produce an ASC is by *error overloading* a stabilizer code. This involves allowing additional errors in violation of condition (3), such that instead condition (10) holds. Ambiguity produced by error overloading a perfect code will result in a violation of the quantum Hamming bound (11), while for an imperfect QECC, a sufficiently large amount of error overloading would be required to violate Ineq. (11). For example, consider the (perfect) 5-qubit code of Ref. [12]

$$\begin{aligned}
|0_L\rangle_5 &= \frac{1}{2\sqrt{2}} (-|00000\rangle + |0111\rangle - |10011\rangle + |11100\rangle \\
&\quad + |00110\rangle + |01001\rangle + |10101\rangle + |11010\rangle) \\
|1_L\rangle_5 &= \frac{1}{2\sqrt{2}} (-|11111\rangle + |10000\rangle + |01100\rangle - |00011\rangle \\
&\quad + |11001\rangle + |10110\rangle - |01010\rangle - |00101\rangle), \quad (12)
\end{aligned}$$

which corrects an arbitrary single-qubit error on any qubit. The code space is stabilized by generators $IXXXY, IYYXX, XIYZY, YXYIZ$. They can each take values ± 1 , thereby determining 16 syndromes, corresponding to the 16 allowed errors $\mathbb{E} \equiv \{I, X_i, Y_i, Z_i\}$ where $i = 1, \dots, 5$. By allowing any more errors into the error set \mathbb{E} , we introduce ambiguity, and also violate (11). In Table I, we present a partial listing of the ambiguous class \mathcal{A} for this code. In all, it has $1 + \binom{5}{1} \cdot 3 + \binom{5}{2} \cdot 3^2 = 106$ arbitrary 1-qubit and 2-qubit errors, of which 49 are displayed. The errors are partitioned into their respective ambiguous sets, labelled by the corresponding error syndrome. Set $A^{(0)}$ has only 1 element, I , since all other elements of \mathcal{N} have a Hamming weight greater than 2.

Another way to create an ASC from a stabilizer code is by *syndrome coarse-graining*: dropping one or more syndrome measurements. For example consider *not* to measure the last stabilizer of the code (12). From the first entry corresponding to syndromes (the *un-error-overloaded* case) of the Table I it can be seen that $|\mathcal{A}| = 8$, $A^{(0)} = \{I, X_1\}$ corresponding to syndrome $(+++)$, $A^{(1)} = \{Y_1, Z_1\}$ corresponding to syndrome $(++-)$, and so on. The order of ambiguity is halved and the degree of ambiguity is doubled.

A final method to obtain an ASC begins by constructing a stabilizer code that corrects arbitrary errors on m known coordinates. An ASC may then be obtained by allowing noise to act on m' known coordinates, where $m' > m$. A detailed description of this method and its application to the characterization of quantum dynamics [3] are considered below.

III. AMBIGUOUS GROUP

An arbitrary error on l qubits can be expressed as a linear combination of 4^l Pauli operators. Suppose these l -

++++	+++ -	++- +	+-+-
I	X_1 Y_2Y_3 X_3Y_4	Y_1 Z_2Z_3 Y_3X_4	Z_1 X_2X_3 Z_3Z_4
+ - + +	+ - + -	+ - - +	+ - - -
X_2 Z_1X_3 Y_3Z_4	Y_5 X_1X_2 Z_2Y_3	Y_4 Y_1X_2 Y_2Z_3	X_3 Z_1X_2 Z_2X_4
- + + +	- + + -	- + - +	- + - -
Y_3 X_1Y_2 X_2Z_4	Y_2 X_1Y_3 Z_3Y_4	X_4 Z_1Y_2 Z_2X_3	X_5 Y_1Y_2 X_2Z_3
- - + +	- - + -	- - - +	- - - -
Z_4 X_1Z_2 X_2Y_3	Z_2 Y_1Z_3 X_3X_4	Z_5 Z_1Z_2 X_1Z_3	Z_3 Y_1Z_2 Y_2Y_4

TABLE I. Ambiguous class (partial listing) for the ASC obtained by error-overloading the code (12), to allow arbitrary errors on any two qubits. Each error syndrome labels an ambiguous set. The first error row in each column corresponds to arbitrary single-qubit errors allowed in the original QECC. Inclusion of the two-qubit errors (second and third rows of the table) to the list turns the QECC into an ASC. In all, there are 106 elements in the ambiguous class, with $|A^{(0)}| = 1$ and $|A^{(p)}| = 7$ for $p = 1, 2, \dots, 15$. Thus the degree of ambiguity is 7. For example, the full ambiguous set, corresponding to the syndrome $+++-$ has four more elements $E_3^{(1)} \equiv X_4X_5, E_4^{(1)} \equiv Z_3Z_5, E_5^{(1)} \equiv X_2Y_5$, and $E_6^{(1)} \equiv Z_2Z_4$. The normalizers between $E_0^{(1)} \equiv X_1$ and other elements in the set are $XYYYI \rightarrow Z_L, XIXYI \rightarrow -Y_L, XIIXX \rightarrow Z_L, XIZIZ \rightarrow -X_L, XXIIY \rightarrow -Y_L$ and $XZIZI \rightarrow -X_L$. Any set of sixteen elements, with one drawn from each ambiguous set will satisfy condition Eq. (10a), while any pair of errors within a column satisfy Eq. (10b) and thus are ambiguous. Further note that the product of ambiguous errors linked by the same logical Pauli are mutually degenerate (e.g., $E_4^{(1)}E_6^{(1)} \in \mathcal{S}$), and are correctable by the same recovery operation, while those linked by different logical Pauli operators are not (e.g., $E_4^{(1)}E_5^{(1)} \in \mathcal{N} - \mathcal{S}$).

qubits form a subsystem of n qubits prepared in a $[[n, k]]$ stabilizer code. Setting $|\mathbb{E}| := 4^l$ in Ineq. (11) we find:

$$l \leq \lfloor \frac{n-k}{2} \rfloor \quad (13)$$

This means that a 5-qubit code can correct all possible errors on at most 2 fixed coordinates. An example of a perfect code of this kind will be presented later. We thus obtain a $[[n, k]]$ ASC by allowing m noisy coordinates, where $m > l$ in Ineq. (13). The order σ of the code is just the number of syndromes, 2^{n-k} , while the degree of ambiguity $\gamma = 4^m / 2^{n-k} = 2^{2m-n+k}$.

Suppose we are given a $[[n, k]]$ ASC with errors allowed on m known coordinates. It is worth noting here that the set of errors (including the factors $\pm 1, \pm i$) forms a group, i.e., $\mathbb{E} = \mathcal{P}_m$. Furthermore, the subset \mathfrak{B} of \mathcal{P}_m that is ambiguous with I_m (the trivial error on the m qubits) constitutes a group, the *ambiguous group*, as shown below.

Theorem 1 Given a $[[n, k]]$ ASC with $\mathbb{E} = \mathcal{P}_m$, the subset \mathfrak{B} of allowed errors that correspond to the no-error syndrome forms a normal subgroup.

Proof. Note that if $B_j, B_k \in \mathfrak{B}$, then B_j and B_k both commute with all stabilizers, by virtue of Eq. (10b). (Note that this doesn't imply that $[B_j, B_k] = 0$. Thus the subgroup is not Abelian.) For any element $G \in \mathcal{G}$, then $[B_j B_k, G] = B_j B_k G - G B_j B_k = 0$, meaning that $B_j B_k \in \mathfrak{B}$. This guarantees closure of the set. By definition, I_m is an element of this set, and a Pauli operator is its own inverse. Thus all required group properties are satisfied. Normalcy of the subgroup (the equality of the left and right cosets) is guaranteed because we implicitly include Pauli operators with and without factors ± 1 and $\pm i$. ■

For an ASC obtained in this way, the ambiguous class \mathcal{A} has a simple structure. It corresponds to a partition of \mathcal{P}_m , determined by the quotient or factor group

$$\mathcal{Q} \equiv \frac{\mathcal{P}_m}{\mathfrak{B}}. \quad (14)$$

This means that any element E in \mathcal{P}_m is either in \mathfrak{B} or can be expressed as the product of an element in \mathfrak{B} and an element not in \mathfrak{B} .

A. Example of a $[[3, 1]]$ ASC

A $[[3, 1]]$ perfect QECC that unambiguously corrects errors on the first qubit is [3]:

$$\begin{aligned} |0_L\rangle_3 &= \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle) \\ |1_L\rangle_3 &= \frac{1}{2}(|110\rangle - |101\rangle + |011\rangle - |000\rangle), \end{aligned} \quad (15)$$

whose stabilizer generators are given by the set $\mathcal{G}_3 \equiv \{XIX, YYZ\}$. The stabilizer is thus the set of four elements, $\mathcal{S}_3 = i^4 \times 2^{\mathcal{G}} \equiv i^4 \times \{I, XIX, YYZ, ZYY\}$, where the pre-factor indicates possible factors $\pm 1, \pm i$. The normalizer \mathcal{N}_3 is the set of all elements of \mathcal{P}_3 that commute with the elements of \mathcal{S}_3 . (We note that a Pauli operator P commutes with every element of \mathcal{S}_3 iff P commutes with every element of \mathcal{G}_3 .)

For code (15), the normalizer \mathcal{N}_3 is given in Table II. Since there are only four logical Pauli operator, various normalizer elements map to the same logical Pauli operator by virtue of their effect on the code words (15). The subset \mathcal{S}_3 (the first column) corresponds to the identity logical Pauli operation I_L , while the elements of $\mathcal{N}_3 - \mathcal{S}_3$ correspond to non-trivial logical Pauli operations, as tabulated in the remaining columns of Table II.

We create an ASC for the code (15) by allowing errors, in addition to the first coordinate, also on the second coordinate. There are four elements in Table II that have no non-trivial operator on the last qubit, i.e., they are elements of $\mathcal{P}_2 \otimes \mathbb{I}_3$, where \mathbb{I}_3 is the identity operator on the third qubit. They are $\{III, XZI, IYI, XXI\}$, which

I_L	$-X_L$	Y_L	Z_L
III	XZI	IYI	XXI
XIX	IZX	XYX	IXX
YYZ	YXY	YIZ	YZY
ZYY	ZXZ	ZIY	ZZZ

TABLE II. Normalizer for the $[[3, 1]]$ stabilizer code (15) and the logical Pauli operations they map to. All elements commute with the elements of \mathcal{S}_3 , while their commutation properties amongst themselves reflect the logical operation they map to. Thus, an element in the column Y_L commutes with all elements in the same column and those in the column I_L , but will anti-commute with every element in the columns $-X_L$ and Z_L . On the other hand, the elements in the column I_L , which are precisely those of \mathcal{S}_3 , commute with every other element in the normalizer.

$++$	$+ -$	$- +$	$--$	Normalizer
I	X_1	Y_1	Z_1	I_L
Y_2	$X_1 Y_2$	$Y_1 Y_2$	$Z_1 Y_2$	Y_L
$X_1 X_2$	X_2	$Z_1 X_2$	$Y_1 X_2$	Z_L
$X_1 Z_2$	Z_2	$Z_1 Z_2$	$Y_1 Z_2$	$-X_L$

TABLE III. Ambiguous class \mathcal{A}_3 for errors on the first 2 qubits of 3-qubit code (15), depicting the quotient group (16). The first column is the ambiguous group \mathfrak{B}_3 , drawn from Table II, subject to the requirement that the operator on the third qubit is identity. The remaining three columns are its cosets $X_1 \mathfrak{B}_3, Y_1 \mathfrak{B}_3$ and $Z_1 \mathfrak{B}_3$, which represent ambiguous sets. The last column lists the normalizer element with respect to first element in the column, in the sense of Eq. (7). For example, the normalizer element that maps error $Z_1 Z_2$ to error Y_1 or vice versa is $N_{Z_1 Z_2, Y_1} = N_{Y_1, Z_1 Z_2} = -X_L$, while the normalizer element which maps error $Y_1 Y_2$ to error $Z_1 X_2$ is $N_{Z_1 X_2, Y_1 Y_2} = Z_L Y_L = iX_L$, while $N_{Y_1 Y_2, Z_1 X_2} = Y_L Z_L = -iX_L$.

constitute the ambiguous group \mathfrak{B}_3 . The partitioning of \mathcal{A} for ASC (15) with $\mathbb{E} = \mathcal{P}_2$ can be represented by the quotient group:

$$\mathcal{Q}_3 \equiv \frac{\mathcal{P}_2}{\mathfrak{B}_3}. \quad (16)$$

This is depicted in Table III, where the first column is the ambiguous subgroup \mathfrak{B}_3 , and the other columns are its cosets and the other ambiguous errors.

IV. APPLICATION TO NOISE CHARACTERIZATION

While code ambiguity makes ASCs not useful for quantum error correction, they can be used for experimentally studying noise. In this Section, we elaborate on the intuition presented earlier, of extending QECCD by replacing the use of stabilizer codes with that of ASCs.

A. Ambiguity and QEC channel-state isomorphism

The basis of QECCD is the quantum error correction (QEC) isomorphism, qualitatively similar to the Choi-Jamiokowski channel-state isomorphism, which associates a correctable noise channel with the unique erroneous logical state corresponding to a given input logical state. This clearly is necessary if complete information about the channel is to be extracted via measurements. In the presence of ambiguity, for any initial logical state, it can be shown that one can always construct two or more noise channels such that they produce the same erroneous logical state. Thus QEC isomorphism no longer holds.

In QECCD, the QEC isomorphism is leveraged through some state manipulations to yield full noise data. The basic idea is that the syndrome obtained from the stabilizer measurement is used to correct the noisy state, while the experimental probabilities of syndromes will characterize the noisy quantum channel. While direct syndrome measurements yield the diagonal terms of the process matrix, for off-diagonal terms preprocessing via suitable unitaries is required. For the purpose of noise characterization, the code qubits are divided into two parts; (a) the qubits on which the elements of \mathbb{E} act non-trivially; (b) the remaining qubits.

The former qubits constitute the principal system \mathbf{P} , whose unknown dynamics is to be determined. The latter qubits constitute the CQD ancilla \mathbf{A} , and are assumed to be clean, i.e., noiseless. Suppose the full system $\mathbf{P} + \mathbf{A}$ is in the state

$$|\psi_L\rangle \equiv \sum_{j=0}^{2^k-1} \alpha_j |j_L\rangle, \quad (17)$$

where $\{|j_L\rangle\}$ denotes a logical basis for the code space of a $[[p+q, k]]$ ASC (which encodes k qubits into $n \equiv p+q$ qubits) such that allowed errors in the p known coordinates of \mathbf{P} can be ambiguously detected.

The main difference QASCD has with respect to QECCD is that QASCD employs more than one code to fully characterize the noise. Herebelow, we present details of the QASCD protocol, which has a quantum part, which is experimental, and a classical part, which concerns post-processing data from experiments. The quantum part involves using state preparations and syndrome measurements of different ASCs to determine $\chi_{m,n}$ ambiguously. The classical part involves simultaneous equations to disambiguate the ambiguous experimental probabilities.

B. Direct measurement

Let Q be an ASC that can detect noise \mathcal{E} , with associated process matrix χ . Let E_{α_j} ($j = 0, 1, 2, \dots, \gamma - 1$)

be the elements of an ambiguous set in Q , with E_x denoting any one of these α_j 's. It is convenient to employ the notation $|j_L^{(\alpha)}\rangle \equiv E_{\alpha}|j_L\rangle$. The probability that one of these ambiguous errors occur:

$$\begin{aligned} \xi \left(\bigwedge_j \alpha_j \right) &= \text{Tr} \left(\mathcal{E} (|\psi_L\rangle\langle\psi_L|) \left[\sum_{j=0}^{2^k-1} |j_L^x\rangle\langle j_L^x| \right] \right) \\ &= \sum_{j=0}^{2^k-1} \langle j_L^x | \left[\dots + \chi_{\alpha_1, \alpha_1} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_1, \alpha_2} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_2)}| \right. \\ &\quad \left. + \chi_{\alpha_2, \alpha_1} |\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_2, \alpha_2} |\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_2)}| + \dots \right] |j_L^x\rangle \\ &= \dots + \chi_{\alpha_1, \alpha_1} + \chi_{\alpha_1, \alpha_2} \langle\psi_L^{(\alpha_2)}|\psi_L^{(\alpha_1)}\rangle + \chi_{\alpha_2, \alpha_1} \langle\psi_L^{(\alpha_1)}|\psi_L^{(\alpha_2)}\rangle \\ &\quad + \chi_{\alpha_2, \alpha_2} + \dots \\ &= \dots + \chi_{\alpha_1, \alpha_1} + \chi_{\alpha_1, \alpha_2} \langle\psi_L|N_{1,2}|\psi_L\rangle + \chi_{\alpha_2, \alpha_1} \langle\psi_L|N_{2,1}|\psi_L\rangle \\ &\quad + \chi_{\alpha_2, \alpha_2} + \dots \\ &= \sum_j \chi_{\alpha_j, \alpha_j} + 2 \sum_{j \neq k} \text{Re} (\chi_{\alpha_j, \alpha_k} \langle N_{j,k} \rangle_L), \end{aligned} \quad (18)$$

where $N_{m,n} \equiv E_m E_n$. Note that because E_m and E_n produce the same syndrome by virtue of ambiguity, $N_{m,n}$ so defined will commute with all elements of the stabilizer.

Let $D \equiv 2^p$, the dimension of \mathbf{P} . In an unambiguous code, the D^2 diagonal terms of χ would appear as probabilities of syndrome measurements [3]. Now, however, any measurement outcome probability will contain contributions from the probabilities of γ ambiguous errors plus $\binom{\gamma}{2}$ off-diagonal terms between these ambiguous errors. Of the 4^k can be disambiguated by using as many different initial state preparations, by exploiting the fact that the χ terms have factors given by expectation values of different normalizer elements (logical Pauli operations). However, the problem of disambiguation would still remain *within* each such 'logical Pauli class', i.e., different pairs of ambiguous errors (E_j, E_k) such that the normalizers $E_j E_k$ correspond to the same logical Pauli operation. This is related to limits imposed by the ambiguity and can be sorted out by using other ASCs. For accessing cross terms terms of χ across ambiguous sets, we use the unitary pre-processing described below in Section IV C and IV D, based on the method introduced by us in Ref. [3].

As an example of result (18), for the data in Table III,

the probabilities to obtain all the outcomes are

$$\begin{aligned}
p(++ &= \chi_{I,I} + \chi_{Y_2,Y_2} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1Z_2,X_1Z_2} \\
&+ 2 \times [(\text{Re}(\chi_{I,X_1Z_2}) + \text{Im}(\chi_{Y_2,X_1X_2}))\langle X_L \rangle \\
&- (\text{Re}(\chi_{I,Y_2}) - \text{Im}(\chi_{X_1X_2,X_1Z_2}))\langle Y_L \rangle \\
&+ (\text{Im}(\chi_{Y_2,X_1Z_2}) + \text{Re}(\chi_{I,X_1X_2}))\langle Z_L \rangle], \\
p(+ - &= \chi_{X_1,X_1} + \chi_{X_1Y_2,X_1Y_2} + \chi_{X_2,X_2} + \chi_{Z_2,Z_2} \\
&+ 2 \times [(\text{Re}(\chi_{X_1,Z_2}) - \text{Im}(\chi_{X_1Y_2,X_2}))\langle X_L \rangle \\
&+ (\text{Re}(\chi_{X_1,X_1Y_2}) - \text{Im}(\chi_{X_2,Z_2}))\langle Y_L \rangle \\
&+ (\text{Im}(\chi_{X_1Y_1,Z_2}) + \text{Re}(\chi_{X_1,X_2}))\langle Z_L \rangle], \\
p(- + &= \chi_{Y_1,Y_1} + \chi_{Y_1Y_2,Y_1Y_2} + \chi_{Z_1X_2,Z_1X_2} + \chi_{Z_1Z_2,Z_1Z_2} \\
&+ 2 \times [(\text{Re}(\chi_{Y_1,Z_1Z_2}) + \text{Im}(\chi_{Y_1Y_2,Z_1X_2}))\langle X_L \rangle \\
&+ (\text{Im}(\chi_{Z_1X_2,Z_1Z_2}) - \text{Re}(\chi_{Y_1,Y_1Y_2}))\langle Y_L \rangle \\
&+ (\text{Im}(\chi_{Y_1Y_2,Z_1Z_2}) - \text{Re}(\chi_{Y_1,Z_1X_2}))\langle Z_L \rangle], \\
p(- - &= \chi_{Z_1,Z_1} + \chi_{Z_1Y_2,Z_1Y_2} + \chi_{Y_1X_2,Y_1X_2} + \chi_{Y_1Z_2,Y_1Z_2} \\
&+ 2 \times [(\text{Im}(\chi_{Z_1Y_2,Y_1X_2}) - \text{Re}(\chi_{Z_1,Y_1Z_2}))\langle X_L \rangle \\
&+ (\text{Re}(\chi_{Z_1,Z_1Y_2}) - \text{Im}(\chi_{Y_1X_2,Y_1Z_2}))\langle Y_L \rangle \\
&+ (\text{Im}(\chi_{Z_1Y_2,Y_1Z_2})\text{Re}(\chi_{Z_1,Y_1X_2}))\langle Z_L \rangle]. \quad (19)
\end{aligned}$$

By choosing input $|0\rangle_L$, one finds $p(++) = \chi_{I,I} + \chi_{Y_2,Y_2} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1Z_2,X_1Z_2} + 2\text{Re}(\chi_{I,X_1X_2}) + 2\text{Im}(\chi_{Y_2,X_1Z_2}) \equiv C + 2\text{Re}(\chi_{I,X_1X_2}) + 2\text{Im}(\chi_{Y_2,X_1Z_2})$. By choosing input $|+\rangle_L \equiv \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$, one finds $p(++) = C + 2\text{Re}(\chi_{I,X_1Z_2}) + 2\text{Im}(\chi_{Y_2,X_1X_2})$. By choosing input $|\uparrow\rangle_L \equiv \frac{1}{\sqrt{2}}(|0\rangle_L + i|1\rangle_L)$, one finds $P(++) = C + 2\text{Re}(\chi_{I,Y_2}) - 2\text{Im}(\chi_{X_1X_2,X_1Z_2})$. We thus have four unknowns, given by C (the diagonal contributions), and the coefficients of $\langle X_L \rangle$, $\langle Y_L \rangle$ and $\langle Z_L \rangle$. One more input, say $\cos(\theta)|0\rangle_L + \sin(\theta)|1\rangle_L$ will suffice to determine these 4 quantities. It will thus suffice to determine C . More generally, 4^k (the number of logical Pauli operations) preparations are needed to solve for C . When C is extracted for each outcome, then each code gives $D^2/\gamma = 2^{n-k}$ equations. Note that we have ignored the off-diagonal terms for ambiguous errors, since they will be dealt with in other ASCs, where they correspond to off-diagonal terms that are unambiguous.

C. Preprocessing with U

For a given ASC, to derive off-diagonal terms between errors in different ambiguous sets, we preprocess the system by applying a suitable unitary U , based on the idea we introduced in Ref. [3]. However, even this may allow one to access only the real or imaginary part of these terms. To access the other part, one uses a further preprocessing described in the next Subsection.

The unitary U will be in one of two forms. In the first form, $U = \frac{1}{\sqrt{2}}(E_a + E_b)$, in case $[E_a, E_b] \neq 0$. In the second form, $U = \frac{1}{\sqrt{2}}(E_a + iE_b)$, in case $[E_a, E_b] = 0$. We require E_a and E_b to be mutually unambiguous in the given ASC because otherwise, as explained later, we obtain a situation similar to not using U , as far as noise characterization is concerned.

Let us consider the first case. Suppose the pre-processed noisy logical state produces an ambiguous outcome E_j . Let $g_{A_j}E_j = E_aE_{\alpha_j}$, where the E_{α_j} 's constitute an ambiguous set, and $g_{A_j} \in \{\pm 1, \pm i\}$ is the *Pauli factor*. Similarly, let $g_{B_j}E_j = E_bE_{\beta_j}$, where the E_{β_j} 's constitute an ambiguous set, and $g_{B_j} \in \{\pm 1, \pm i\}$ is a Pauli factor.

When $U(a, b)$ is applied to the noisy logical state, and an outcome x has been observed, then one of the E_j must have been detected, and thus the only contributing terms of $\mathcal{E}(\rho_L)$ will be those restricted to $|\psi_L^{\alpha_j}\rangle$ and $|\psi_L^{\beta_j}\rangle$. Denoting by Π_C the projector to the code space \mathcal{C} of the ASC, the probability to observe x when $U(a, b)$ has been applied is:

$$\xi(a, b, x) \equiv \text{Tr} [U [\mathcal{E}(|\psi_L\rangle\langle\psi_L|)] U^\dagger (E_x \Pi_C E_x)]. \quad (20)$$

The terms within the square bracket in Eq. (20) that would make a contribution to the probability of obtaining ambiguous outcome E_x are:

$$\begin{aligned}
&\dots + \chi_{\alpha_1, \alpha_1} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_1, \alpha_2} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_2)}| \\
&+ \chi_{\alpha_2, \alpha_1} |\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_2, \alpha_2} |\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_2)}| + \dots \\
&+ \chi_{\alpha_1, \beta_1} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\beta_1)}| + \chi_{\beta_1, \alpha_1} |\psi_L^{(\beta_1)}\rangle\langle\psi_L^{(\alpha_1)}| + \dots \\
&+ \chi_{\alpha_1, \beta_2} |\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\beta_2)}| + \chi_{\beta_2, \alpha_1} |\psi_L^{(\beta_2)}\rangle\langle\psi_L^{(\alpha_1)}| \\
&+ \dots \quad (21)
\end{aligned}$$

When the expression in Eq. (21) is left- and right-multiplied by $U(a, b)$, then the only resulting terms that contribute to the lhs of Eq. (20) are:

$$\begin{aligned}
&\dots + \chi_{\alpha_1, \alpha_1} |\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_1, \alpha_2} g_{A_1} g_{A_2}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| \\
&+ \chi_{\alpha_2, \alpha_1} g_{A_2} g_{A_1}^* |\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_2, \alpha_2} |\psi_L^{(2)}\rangle\langle\psi_L^{(2)}| + \dots \\
&+ \chi_{\alpha_1, \beta_1} g_{A_1} g_{B_1}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \chi_{\beta_1, \alpha_1} g_{B_1} g_{A_1}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \dots \\
&+ \chi_{\alpha_1, \beta_2} g_{A_1} g_{B_2}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| + \chi_{\beta_2, \alpha_1} g_{B_2} g_{A_1}^* |\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| \\
&+ \dots \quad (22)
\end{aligned}$$

The contribution of the first term in Eq. (22) to the probability in Eq. (20) would be:

$$\begin{aligned}
\epsilon_{\alpha_1, \alpha_1} &\equiv \chi_{\alpha_1, \alpha_1} \sum_{j=1}^{2^k} \langle j_L^{(x)} | \psi_L^{(1)} \rangle \langle \psi_L^{(1)} | j_L^{(x)} \rangle \\
&= \chi_{\alpha_1, \alpha_1}, \quad (23)
\end{aligned}$$

since the traced out quantity has support only in the erroneous code space $E_x \mathcal{C}'$ (i.e., the code space \mathcal{C}' shifted by the ambiguous error). Analogously, the contribution of the fourth term in Eq. (22) to Eq. (20) would be $\epsilon_{\alpha_2, \alpha_2} = \chi_{\alpha_2, \alpha_2}$. In like fashion, the contribution of the fifth and sixth terms in Eq. (22) to Eq. (20) would be $\epsilon_{\alpha_1, \beta_1} = \chi_{\alpha_1, \beta_1}$ and $\epsilon_{\beta_1, \alpha_1} = \chi_{\beta_1, \alpha_1}$.

The contribution of the second term in Eq. (22) to the

probability in Eq. (20) would be:

$$\begin{aligned}\epsilon_{\alpha_1, \alpha_2} &\equiv \chi_{\alpha_1, \alpha_2} g_{A_1} g_{A_2}^* \sum_{j=1}^{2^k} \langle j_L^{(x)} | \psi_L^{(1)} \rangle \langle \psi_L^{(2)} | j_L^{(x)} \rangle \\ &= \chi_{\alpha_1, \alpha_2} g_{A_1} g_{A_2}^* \langle \psi_L^{(2)} | \psi_L^{(1)} \rangle \\ &= \chi_{\alpha_1, \alpha_2} g_{A_1} g_{A_2}^* \langle N_{2,1} \rangle_L,\end{aligned}\quad (24)$$

where $N_{2,1}$ is the normalizer element that propagates error E_{A_1} on a logical ket to E_{A_2} . The contribution of the third term in Eq. (22) to the probability in Eq. (20) would be, analogously to Eq. (24), namely, $\epsilon_{\alpha_2, \alpha_1} = \chi_{\alpha_2, \alpha_1} g_{A_2} g_{A_1}^* \langle N_{1,2} \rangle_L$. In like fashion, the contribution of the seventh and eighth terms in Eq. (22) to Eq. (20) would be $\epsilon_{\alpha_1, \beta_2} = \chi_{\alpha_1, \beta_2} g_{A_1} g_{B_2}^* \langle N_{1,2} \rangle_L$ and $\epsilon_{\beta_2, \alpha_1} = \chi_{\beta_2, \alpha_1} g_{B_2} g_{A_1}^* \langle N_{2,1} \rangle_L$.

Putting together all these $\epsilon_{\alpha_j, \alpha_k}$, $\epsilon_{\alpha_j, \beta_k}$, etc., terms into Eq. (20), we obtain:

$$\begin{aligned}\xi(a, b, x) &= \frac{1}{2} \left(\sum_{j=1}^{\gamma} \chi_{\alpha_j, \alpha_j} + \chi_{\beta_j, \beta_j} + \chi_{\alpha_j, \beta_j} + \chi_{\beta_j, \alpha_j} \right) \\ &+ \sum_{j < k} [\text{Re}(\chi_{\alpha_j, \alpha_k} g_{A_j} g_{A_k}^* \langle N_{j,k} \rangle_L) \\ &\quad + \text{Re}(\chi_{\beta_j, \beta_k} g_{B_j} g_{B_k}^* \langle N_{j,k} \rangle_L)] \\ &+ \sum_{j \neq k} [\text{Re}(\chi_{\alpha_j, \beta_k} g_{A_j} g_{B_k}^* \langle N_{j,k} \rangle_L)],\end{aligned}\quad (25)$$

where the $\langle N \rangle$ terms, being always real, can be removed out of the argument of Re or Im.

In constructing $U(a, b)$, the errors E_a and E_b should not be mutually ambiguous. Otherwise, the result is effectively the same as that direct measurement without preprocessing using $U(a, b)$. To see this, consider an application of this method to Eq. (19), with $U(X_1 X_2, Y_2) \equiv \frac{1}{\sqrt{2}}(X_1 X_2 + Y_2)$. We find:

	α_j, g_A	β_k, g_B	
I	$X_1 X_2, 1$	$Y_2, 1$	
Y_2	$X_1 Z_2, i$	$I, 1$	
$X_1 X_2$	$I, 1$	$X_1 Z_2, i$	
$X_1 Z_2$	Y_2, i	$X_1 X_2, -i$	

(26)

From (25), it follows that with pre-processing by $U(X_1 X_2, Y_2)$, the probability expressions in the example (19) are altered altered, e.g.,

$$\begin{aligned}p(++) &= 2 \times [\text{Re}(\chi_{X_1 X_2, Y_2}) + \text{Re}(\chi_{I, X_1 Z_2})] \langle X_L \rangle \\ &+ 2 \times [\text{Re}(\chi_{I, Y_2}) + -\text{Im}(\chi_{X_1 X_2, X_1 Z_2})] \langle Z_L \rangle \\ &+ 2 \times [\text{Re}(\chi_{I, X_1 X_2}) + \text{Im}(\chi_{Y_2, X_1 Z_2})] \langle Y_L \rangle \\ &- 2 \times [\text{Im}(\chi_{Y_2, X_1 X_2}) + \text{Im}(\chi_{I, X_1 Z_2})].\end{aligned}\quad (27)$$

Thus, the ambiguous errors in the coefficients of normalizer expectation values remain the same even though the particular normalizer element changes.

Now, let $U = \frac{1}{\sqrt{2}}(X_1 + Z_1)$, where X_1 and Z_1 are seen to be unambiguous for code (15). Set the outcome to be

‘++’. This fixes E_j . Thus:

$$\begin{bmatrix} E_j \\ I \\ Y_2 \\ X_1 X_2 \\ X_1 Z_2 \end{bmatrix}; \begin{bmatrix} E_{\alpha}, g_A \\ X_1, 1 \\ X_1 Y_2, 1 \\ X_2, 1 \\ Z_2, 1 \end{bmatrix}; \begin{bmatrix} E_{\beta}, g_B \\ Z_1, 1 \\ Z_1 Y_2, 1 \\ Y_1 X_2, -i \\ Y_1 Z_2, -i \end{bmatrix}; \begin{bmatrix} N \\ I_L \\ Y_L \\ Z_L \\ -X_L \end{bmatrix};\quad (28)$$

The coefficient $\langle X_L \rangle$ to $\xi(X_1, Z_1, ++)$ can be read off (28), using (25), by forming cross-terms between elements of the second and third columns, such that their corresponding logical Pauli operators multiply to X_L up to a sign g_A . In the present case, this is seen to be

$$\langle X_L \rangle [\text{Im}(\chi_{X_1, Y_1 Z_2} - \chi_{X_1 Y_2, Y_1 X_2}) + \text{Re}(\chi_{X_2, Z_1 Y_2} + \chi_{Z_2, Z_1})].\quad (29)$$

We can thus form cross-terms between all ambiguous sets using suitable U .

In the second case, $[E_a, E_b] = 0$ and we set $U = \frac{E_a + i E_b}{\sqrt{2}}$. As a result, instead of Eq. (22), one gets:

$$\begin{aligned}&\dots + \chi_{\alpha_1, \alpha_1} |\psi_L^{(1)} \rangle \langle \psi_L^{(1)}| + \chi_{\alpha_1, \alpha_2} g_{A_1} g_{A_2}^* |\psi_L^{(1)} \rangle \langle \psi_L^{(2)}| \\ &+ \chi_{\alpha_2, \alpha_1} g_{A_2} g_{A_1}^* |\psi_L^{(2)} \rangle \langle \psi_L^{(1)}| + \chi_{\alpha_2, \alpha_2} |\psi_L^{(2)} \rangle \langle \psi_L^{(2)}| + \dots \\ &- i \chi_{\alpha_1, \beta_1} g_{A_1} g_{B_1}^* |\psi_L^{(1)} \rangle \langle \psi_L^{(1)}| + i \chi_{\beta_1, \alpha_1} g_{B_1} g_{A_1}^* |\psi_L^{(1)} \rangle \langle \psi_L^{(1)}| + \dots \\ &- i \chi_{\alpha_1, \beta_2} g_{A_1} g_{B_2}^* |\psi_L^{(1)} \rangle \langle \psi_L^{(2)}| + i \chi_{\beta_2, \alpha_1} g_{B_2} g_{A_1}^* |\psi_L^{(2)} \rangle \langle \psi_L^{(1)}| \\ &+ \dots\end{aligned}\quad (30)$$

Consequently, one obtains in place of Eq. (25):

$$\begin{aligned}\xi(a, b, x) &= \frac{1}{2} \left(\sum_{j=1}^{\gamma} \chi_{\alpha_j, \alpha_j} + \chi_{\beta_j, \beta_j} + \chi_{\alpha_j, \beta_j} + \chi_{\beta_j, \alpha_j} \right) \\ &+ \sum_{j < k} [\text{Re}(\chi_{\alpha_j, \alpha_k} g_{A_j} g_{A_k}^* \langle N_{j,k} \rangle_L) \\ &\quad + \text{Re}(\chi_{\beta_j, \beta_k} g_{B_j} g_{B_k}^* \langle N_{j,k} \rangle_L)] \\ &+ \sum_{j \neq k} [\text{Im}(\chi_{\alpha_j, \beta_k} g_{A_j} g_{B_k}^* \langle N_{j,k} \rangle_L)],\end{aligned}\quad (31)$$

where, like before, the $\langle N \rangle$ terms, which are always real, can be removed out of the argument of Re or Im. It is worth noting that in Eqs. (25) or (31), in the terms that contain Pauli factors, the matter of whether the real or imaginary part of the process element of the process matrix contributes to the measured probability, depends on whether the Pauli factors are of same type (real/imaginary).

D. Toggling

The method of Section IV C gives only the real or imaginary parts of the cross-terms. Using an idea we proposed in [3], we solve this problem by pre-processing the noisy state even before applying U . Consider a density operator $\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$ subjected to the phase operator

given by the diagonal $T \equiv e^{i\theta_0}|0\rangle\langle 0| + e^{i\theta_1}|1\rangle\langle 1|$. Then, if $\theta_0 = -\theta_1 = \pi/4$, one finds $T\rho T^\dagger = \begin{pmatrix} a & ib \\ -ib^* & 1-a \end{pmatrix}$, meaning that the imaginary and real parts of the off-diagonal terms have been interchanged or ‘toggled’ (apart from a possible sign change).

Similarly, now we construct

$$T \equiv \sum_{m=0}^{\sigma-1} e^{i\theta_m} \Pi_L^m, \quad (32)$$

where σ is the number of ambiguous sets (order of ambiguity), Π_L^m is the projector to the erroneous logical space given by $E_m \mathcal{C}'$ (E_m being any one error from each ambiguous set), and $\theta_m \in \{\pm \frac{\pi}{4}\}$, with equal entries with both signs. Prior to U , we apply the operation $T^+ = T \oplus \mathbb{I}'$, where \mathbb{I}' acts trivially outside the correctable space, i.e., the code space \mathcal{C}' plus the erroneous code spaces.

For example, suppose we construct the toggler T^+ having $\theta_{E_\alpha} = -\theta_{E_\beta} = \pm \frac{\pi}{4}$, then in place of (28) we have:

$$\langle X_L \rangle [\text{Re}(\chi_{X_1, Y_1 Z_2} - \chi_{X_1 Y_2, Y_1 X_2}) + \text{Im}(\chi_{X_2, Z_1 Y_2} + \chi_{Z_2, Z_1})], \quad (33)$$

i.e., cross-term $\chi_{\mu, \nu}$, where μ and ν come, respectively, from ambiguous set E_α and E_β , get their real and imaginary parts toggled.

The tools described in this and the preceding two subsections, as well as the different ASCs, form our repertoire for characterizing the noise in the method of ASCs.

V. RESOURCES

We may begin by supposing that data from γ ASCs will suffice, giving the required D^2 equations to solve for the D^2 variables. These D^2 equations will correspond to an adjacency matrix, wherein the D^2/γ rows corresponding to each code will sum to a *unit row*, i.e., one with 1’s in all columns. Thus there are (at least) $\gamma - 1$ constraints among the D^2 equations. Adding one more code will introduce D^2/γ equations and one more constraint i.e., $2^{n-k} - 1$ constraints. If there are no other constraints in the first D^2 rows, and if $2^{n-k} - 1 \geq \gamma - 1$, i.e., $n - k \geq p$, then the remaining required linearly independent equations can be found from the last code. Thus, in general, with γ -fold full degeneracy, the necessary number of preparations is $\gamma + 1$.

More generally, because of the failure of QEC isomorphism with ambiguous codes, of the $O(4^m \times 4^m)$ terms in the process matrix, only $O\left(\frac{4^m}{\gamma} \times \frac{4^m}{\gamma}\right)$ independent terms can be determined per ASC, implying that a full characterization would require $\mu = O(\gamma^2)$ different ASC’s. Also, syndrome measurements on each ASC yields $D^2/\gamma = 4^m/\gamma$ outcomes. We may thus estimate that the number of configurations required is $c = O\left(\frac{16^m}{\gamma^2} / \frac{4^m}{\gamma}\right) = O\left(\frac{4^m}{\gamma}\right)$ per ASC. Thus in all, counting each ASC as a separate configuration, we require $\mu \times c$

configurations, i.e., $O(\gamma 4^m)$, meaning that there is a factor γ excess when using ambiguous codes. (Moreover each code would require up to 4^k state preparations for disambiguation of the Pauli logical classes.) This can be considered as a time cost to pay for the saving in ‘space’, i.e., in terms of number of entangled qubits used.

Now we present an example of applying QASCD, with three 4-qubit ASCs being used to characterize a 2-qubit noise.

VI. ILLUSTRATION USING A FAMILY OF THREE 4-QUBIT AMBIGUOUS CODES

Consider the $[[4, 1]]$ ASC \mathbf{C}^1 for arbitrary errors on the first two qubits, constructed by dropping the last qubit of the $[[5, 1]]$ QECC of Ref. [12]:

$$\begin{aligned} |0_L^1\rangle_4 &= \frac{1}{2\sqrt{2}} (-|0000\rangle + |0010\rangle + |0101\rangle + |0111\rangle \\ &\quad - |1001\rangle + |1011\rangle + |1100\rangle + |1110\rangle) \\ |1_L^1\rangle_4 &= \frac{1}{2\sqrt{2}} (-|1111\rangle + |1101\rangle + |1010\rangle + |1000\rangle \\ &\quad - |0110\rangle + |0100\rangle + |0011\rangle + |0001\rangle), \quad (34) \end{aligned}$$

whose stabilizer generators are $XIX, YIXY$ and $YYZZ$. The following equation presents two other such codes \mathbf{C}^2 and \mathbf{C}^3 which are two fold amiguous:

$$\begin{aligned} |0_L^2\rangle_4 &= H_{ZY}^{\otimes 4} |0_L\rangle, \quad |1_L^2\rangle_4 = H_{ZY}^{\otimes 4} |1_L\rangle_4, \\ |0_L^3\rangle_4 &= H_{YX}^{\otimes 4} |0_L\rangle_4, \quad |1_L^3\rangle_4 = H_{YX}^{\otimes 4} |1_L\rangle_4, \quad (35) \end{aligned}$$

where $H_{ZY} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$, $H_{YX} = \frac{1}{2}((1+i)|0\rangle\langle 0| + (1+i)|0\rangle\langle 1| - (1-i)|1\rangle\langle 0| + (1-i)|1\rangle\langle 1|)$. The corresponding stabilizer generators and the error syndromes are given in Table IV.

By method described in Sec. IV, the statistics of syndrome outcomes on QECs \mathbf{C}^1 , \mathbf{C}^2 and \mathbf{C}^3 , can completely determine the process matrix $\chi_{m,n}$ corresponding to an arbitrary 2-qubit noise \mathcal{E} . It can be noticed from the Table IV, that the normalizer corresponds to logical Y_L . By the direct measurement, as in Eq. (18), we get $\chi_{\alpha_1, \alpha_1} + \chi_{\alpha_2, \alpha_2} + 2\text{Re}(\chi_{\alpha_1, \alpha_2} \langle Y_L \rangle)$. By choosing any state of \mathbf{C}^1 other than $|\uparrow\rangle$, $\langle Y_L \rangle$ vanishes. The direct syndrome measurements on suitably prepared \mathbf{C}^1 yields the following expressions,

$$\begin{aligned} \chi_{I, I} + \chi_{Y_2, Y_2} &= a_1, \quad \chi_{X_1, X_1} + \chi_{X_1 Y_2, X_1 Y_2} = b_1, \\ \chi_{X_2, X_2} + \chi_{Z_2, Z_2} &= c_1, \quad \chi_{X_1 X_2, X_1 X_2} + \chi_{X_1 Z_2, X_1 Z_2} = d_1, \\ \chi_{Y_1 X_2, Y_1 X_2} + \chi_{Y_1 Z_2, Y_1 Z_2} &= e_1, \quad \chi_{Y_1, Y_1} + \chi_{Y_1 Y_2, Y_1 Y_2} = f_1, \\ \chi_{Z_1 Z_2, Z_1 Z_2} + \chi_{Z_1 Z_2, Z_1 X_2} &= g_1, \quad \chi_{Z_1, Z_1} + \chi_{Z_1 Y_2, Z_1 Y_2} = h_1. \quad (36) \end{aligned}$$

Similarly procedure followed on \mathbf{C}^2 yields

$$\begin{aligned} \chi_{I, I} + \chi_{Z_2, Z_2} &= a_2, \quad \chi_{X_1, X_1} + \chi_{X_1 Z_2, X_1 Z_2} = b_2, \\ \chi_{Y_1, Y_1} + \chi_{Y_1 Z_1, Y_1 Z_2} &= c_2, \quad \chi_{Z_1, Z_1} + \chi_{Z_1 Z_2, Z_1 Z_2} = d_2. \quad (37) \end{aligned}$$

\mathbf{C}^1	II	X_1	X_2	Y_1	Z_1	XX	YX	ZX
	Y_2	XY	Z_2	YY	ZY	XZ	YZ	ZZ
$XIIX$	+	+	+	-	-	+	-	-
$YIXY$	+	-	+	+	-	-	+	-
$YYZZ$	+	-	-	+	-	+	-	+
\mathbf{C}^2	II	X_1	X_2	Y_1	Z_1	XX	YX	ZX
	Z_2	XZ	Y_2	YZ	ZZ	XY	YY	ZY
$IZZX$	+	+	-	+	+	-	-	-
$XIIX$	+	+	+	-	-	+	-	-
$YZYZ$	+	-	-	+	-	+	-	+
\mathbf{C}^3	II	X_1	Y_1	Y_2	Z_1	XY	YY	ZY
	X_2	XX	YX	Z_2	ZX	XZ	YZ	ZZ
$IXXZ$	+	+	+	-	+	-	-	-
$XIXZ$	+	+	-	+	-	+	-	-
$YXYX$	+	-	+	-	-	+	-	+

TABLE IV. Ambiguous class for the three 4-qubit codes. The Hadamard operation H_{ZY} (H_{YX}) toggles errors Z and Y (errors Y and X) while keeping error X (Z) fixed, and the above syndromes are corresponding toggled versions of each other.

From \mathbf{C}^3 we obtain the following expressions

$$\begin{aligned} \chi_{I,I} + \chi_{X_2,X_2} &= a_3, \quad \chi_{X_1,X_1} + \chi_{X_1X_2,X_1X_2} = b_3, \\ \chi_{Y_1,Y_1} + \chi_{Y_1X_2,Y_1X_2} &= c_3, \quad \chi_{Z_1,Z_1} + \chi_{Z_1X_2,Z_1X_2} = d_3. \end{aligned} \quad (38)$$

The above 16 expressions suffice to determine the diagonal terms of the process matrix. To demonstrate how the method works for off-diagonal terms, we consider its application to the noise

$$\begin{aligned} \mathcal{E}_A(\rho_L) &= \delta\rho_L + \frac{1-\delta}{5} (X_1\rho_L X_1 + XZ\rho_L XZ + Y_2\rho_L Y_2 \\ &\quad + X_2\rho_L X_2 + XX\rho_L XX) + \frac{1}{6} ((a+ib)X_1\rho_L X_2 \\ &\quad + (c+id)\rho_L XX + (e+if)XZ\rho_L Y_2 + \text{c.c.}) \end{aligned} \quad (39)$$

In the present case, for solving the off-diagonal terms using Eq. (31), the following set of linearly independent equations for off-diagonal terms are obtained by performing unitary operations $U(a, b)$ followed by syndrome measurements on \mathbf{C}^1 , \mathbf{C}^2 and \mathbf{C}^3 respectively

$$\begin{aligned} \xi(I, X_1X_2, I) &= \chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2} \\ &\quad + \text{Im}(I, X_1X_2) + \text{Re}(Y_2, X_1Z_2), \\ \xi(I, X_1X_2, X_1) &= \chi_{X_1,X_1} + \chi_{X_2,X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2} \\ &\quad + \text{Im}(X_1, X_2) - \text{Re}(Y_2, X_1Z_2), \\ \xi(I, X_1X_2, I) &= \chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1,X_1} + \chi_{X_2,X_2} \\ &\quad + \text{Im}(I, X_1X_2) - \text{Im}(X_1, X_2). \end{aligned} \quad (40)$$

In Eq. (40), the diagonal terms are obtained without pre-processing with unitaries using in Eq. (36), (37) and (38). Solving the above set of equations we obtain the off-diagonal terms of the process matrix corresponding to

\mathcal{E}_A :

$$\begin{aligned} \text{Im}(I, X_1X_2) &= \frac{1}{2}(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) = \frac{c}{6}, \\ \text{Im}(X_1, X_2) &= \frac{1}{2}(\mathcal{O}_1 + \mathcal{O}_2 - \mathcal{O}_3) = \frac{a}{6}, \\ \text{Re}(Y_2, X_1Z_2) &= \mathcal{O}_1 - \frac{1}{2}(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) = \frac{f}{6}, \end{aligned} \quad (41)$$

where $\mathcal{O}_1 = \xi(I, X_1X_2, I) - (\chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2})$, $\mathcal{O}_2 = \xi(I, X_1X_2, X_1) - (\chi_{X_1,X_1} + \chi_{X_2,X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2})$ and $\mathcal{O}_3 = \xi(I, X_1X_2, I) - (\chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1,X_1} + \chi_{X_2,X_2})$.

The real or imaginary counterparts of the expressions Eq. (19) are obtained by preprocessing the noisy states with the corresponding toggling operations. For code \mathbf{C}^1 , note that I and Y_2 are ambiguous, and thus cannot have different toggler signs. On the other hand, we want them both to have different toggler signs than X_1X_2 and X_1Z_2 , which are also ambiguous. Thus, one required toggling operation would be:

$$\begin{aligned} T_j^+ &= \frac{1+i}{\sqrt{2}} (\Pi_{\mathbf{C}^1} + X_1\Pi_{\mathbf{C}^1}X_1 + Y_1\Pi_{\mathbf{C}^1}Y_1 + Z_1\Pi_{\mathbf{C}^1}Z_1) \\ &\quad + \frac{1-i}{\sqrt{2}} (X_1X_2\Pi_{\mathbf{C}^1}X_1X_2 + X_2\Pi_{\mathbf{C}^1}X_2 \\ &\quad + Y_1X_2\Pi_{\mathbf{C}^1}Y_1X_2 + Z_1X_2\Pi_{\mathbf{C}^1}Z_1X_2), \end{aligned} \quad (42)$$

and similarly for the codes \mathbf{C}^j ($j \in \{2, 3\}$). The expressions obtained by pre-processing the noisy ASCs with unitary and toggling are

$$\begin{aligned} \xi'(I, X_1X_2, I) &= \chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2} \\ &\quad + \text{Re}(I, X_1X_2) + \text{Im}(Y_2, X_1Z_2), \\ \xi'(I, X_1X_2, X_1) &= \chi_{X_1,X_1} + \chi_{X_2,X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2} \\ &\quad + \text{Re}(X_1, X_2) + \text{Im}(Y_2, X_1Z_2), \\ \xi'(I, X_1X_2, I) &= \chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1,X_1} + \chi_{X_2,X_2} \\ &\quad + \text{Re}(I, X_1X_2) + \text{Re}(X_1, X_2). \end{aligned} \quad (43)$$

Solving the above set of equations we have the real or imaginary parts of the off-diagonal terms of the process matrix that were undetermined by Eq. (19) without toggling:

$$\begin{aligned} \text{Re}(I, X_1X_2) &= \frac{1}{2}(\mathcal{O}'_1 - \mathcal{O}'_2 + \mathcal{O}'_3) = \frac{c}{6}, \\ \text{Re}(X_1, X_2) &= \frac{1}{2}(-\mathcal{O}'_1 + \mathcal{O}'_2 + \mathcal{O}'_3) = \frac{a}{6}, \\ \text{Im}(Y_2, X_1Z_2) &= \frac{1}{2}(\mathcal{O}'_1 + \mathcal{O}'_2 - \mathcal{O}'_3) = \frac{f}{6}, \end{aligned} \quad (44)$$

where $\mathcal{O}'_1 = \xi'(I, X_1X_2, I) - (\chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2})$, $\mathcal{O}'_2 = \xi'(I, X_1X_2, X_1) - (\chi_{X_1,X_1} + \chi_{X_2,X_2} + \chi_{Y_2,Y_2} + \chi_{X_1Z_2,X_1Z_2})$, $\mathcal{O}'_3 = \xi'(I, X_1X_2, I) - (\chi_{I,I} + \chi_{X_1X_2,X_1X_2} + \chi_{X_1,X_1} + \chi_{X_2,X_2})$.

VII. DISCUSSION AND CONCLUSION

We developed the concept of ambiguous stabilizer codes, which exploit the stabilizer formalism for quantum error characterization rather than for quantum error correction. We presented different procedures for constructing an $[[n, k]]$ ASC that ambiguously detects arbitrary errors on m known qubit coordinates ($m < n$). The Pauli operator basis for this set of errors forms a group. The ASC can be characterized as a quotient group $\mathcal{P}_m/\mathfrak{B}$, where \mathfrak{B} is the set of m -qubit Pauli errors ambiguous with the no-error syndrome. The cosets of \mathfrak{B} form other ambiguous sets of errors.

ASCs cannot be used for quantum error correction, except if the basis elements of the noise is known to have at most a single element in each of the ambiguous sets of the ASC. Quite generally, a suitable collection of ASCs can be employed for characterizing noise, and this is the chief

application of ASCs. The code length for an ASC can be smaller than demanded by the requirement of error correction, making state preparations potentially simpler from an experimental perspective than for the techniques of Refs. [3, 7]. We developed a protocol, “quantum ASC-based characterization of dynamics” (QASCD), for this purpose, which, in comparison with the use of conventional stabilizer codes for CQD [3], requires smaller code length, but at the cost of more number of quantum operations and classical post-processing. We illustrated our method using an example of characterization of a toy 2-qubit noise using three 4-qubit ASCs.

ACKNOWLEDGMENTS

OS and RS acknowledge financial support through DST, Govt. of India, for the project SR/S2/LOP-02/2012. OS acknowledges academic support from the Manipal University graduate program.

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