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BRANCHING RULES AND COMMUTING PROBABILITIES FOR TRIANGULAR AND UNITRIANGULAR MATRICES

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Abstract. This paper concerns the enumeration of simultaneous conjugacy classes of k-tuples of commuting matrices in the upper triangular group $GT_n(\mathbf{F}_q)$ and unitriangular group $UT_m(\mathbf{F}_q)$ over the finite field $\mathbf{F}_q$ of odd characteristic. This is done for $n = 2, 3, 4$ and $m = 3, 4, 5$, by computing the branching rules. Further, using the branching matrix thus computed, we explicitly get the commuting probabilities cp_k for $k \leq 5$ in each case.

1. Introduction

Simultaneous conjugacy of commuting k-tuples in a group is understood by computing its branching matrix. In [Sh1] and [SS], the branching table/matrix of finite general linear, unitary and symplectic groups of small rank is computed. In this paper, we continue the work for certain solvable groups, namely, upper triangular matrices. Since, this work is continuation of that in [SS], we refer a reader to the same for definition of branching and other related notation. We work with the groups of upper-triangular invertible matrices, $GT_n(\mathbf{F}_q)$, and the groups of upper unitriangular matrices $UT_n(\mathbf{F}_q)$, over a finite field $\mathbf{F}_q$ of odd characteristic. We compute the branching matrix for $GT_2(\mathbf{F}_q)$ (Theorem 2.1), $GT_3(\mathbf{F}_q)$ (Theorem 3.1), $GT_4(\mathbf{F}_q)$ (Theorem 4.1), $UT_3(\mathbf{F}_q)$ (Theorem 5.1), $UT_4(\mathbf{F}_q)$ (Theorem 6.1) and $UT_5(\mathbf{F}_q)$ (Theorem 7.1).

Further, for a group G, the relation of branching matrix B_G to commuting probabilities $cp_k(G)$ was explored in [SS, Theorem 1.1]. This relation is further explored in the survey article [SS2], where commuting probabilities $cp_k(G)$ up to $k \leq 5$ is computed for $G = GL_2(\mathbf{F}_q)$, $GL_3(\mathbf{F}_q)$, $U_2(\mathbf{F}_q)$ and $U_3(\mathbf{F}_q)$. It was also proved that $cp_k(GL_2(\mathbb{F}_q)) = cp_k(U_2(\mathbf{F}_q))$ for all k even though the branching matrices of the two groups are not same (see Proposition 3.3 [SS2]). In [GR] (see Theorem 12) bounds for commuting probability cp_2, when G is a solvable group or p-group, is computed. Using the branching matrix we compute the commuting probabilities cp_k, up to $k \leq 5$, for each of the groups $GT_n(\mathbf{F}_q)$ and $UT_n(\mathbf{F}_q)$ for which we have branching matrix (see Section 8).

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For this work, we need *conjugacy class types* or *z-classes* (as defined in [SS] and also dealt in [Bh]). This is defined as follows: two matrices are said to be of the same conjugacy class type/z-class, if the centralizers of two elements are conjugate. However, a further weaker version is enough for our purpose here. We say that two matrices are of same type if their centralizers are isomorphic. This helps us reduce the size of computation (and size of branching matrix) and causes no loss of generality. Throughout, we assume q is odd. We hope our computation throws some light on the subject of commuting probability and will help us understand the groups better.

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2. Branching rules for $GT_2(\mathbf{F}_q)$

There are four conjugacy class types in $GL_2(\mathbf{F}_q)$ given by the following partitions (as in [Sh1]) $(1,1)_2,\ (2)_1,\ (1)_1(1)_1,$ and $(1)_2$. We use this to get the same for $GT_2(\mathbf{F}_q)$. Since we are looking at $GT_2(\mathbf{F}_q)$, the last one, $(1)_2$ doesn't exist in $GT_2(\mathbf{F}_q)$. In this paper, we shall not use the partition based nomenclature for the conjugacy class types. Instead we use alphanumeric nomenclature as follows (similar to the pattern in [SS]).

Canonical Form	No. of Classes	Centralizer	Name of Class Type
$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbf{F}_q^*$	$q-1$	$GT_2(\mathbf{F}_q)$	C
$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $a \in \mathbf{F}_q^*$.	$q-1$	$\left\{\begin{pmatrix} x_0 & x_1 \\ & x_0 \end{pmatrix} \mid x_0 \in \mathbf{F}_q^*\right\}$	R_1
$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbf{F}_q^*,\ a \neq b$	$(q-1)(q-2)$	$\left\{\begin{pmatrix} x_0 & \\ & z_0 \end{pmatrix} \mid x_0, z_0 \in \mathbf{F}_q^*\right\}$	R_2

Theorem 2.1. *The branching rules are summarized in the table below given by the branching matrix:*

$$B_{GT_2(\mathbf{F}_q)} = \begin{pmatrix} q-1 & 0 & 0 \\ q-1 & q(q-1) & 0 \\ (q-1)(q-2) & 0 & (q-1)^2 \end{pmatrix}.$$

We mention the branching rules below.

Proposition 2.2. *For an upper triangular matrix of type C, the branching rules are as mentioned in the table above.*

Proof. The result follows, as this type is central. □

Proposition 2.3. *For matrices of any of the two regular types:*

- *A matrix of type R_1 has $q(q-1)$ branches of type R_1, and*

- *A matrix of type R_2 has $(q-1)^2$ branches of type R_2.*

Proof. The centralizer of a matrix of any of the above mentioned regular types is commutative, hence each element of the centralizer is a branch. $\square$

Proof of Theorem 2.1. The branching rules stated in the above propositions, are summarised in the the branching matrix, as mentioned in the statement of the thoerem. $\square$

3. Branching in $GT_3(q)$

Now, we compute the branching table for $GT_3(\mathbf{F}_q)$. The table for the conjugacy classes and their types are as follows:

Class Representative	Number of Classes	Centralizer size	Name of Type
aI_3, $a \neq 0$	$q-1$	$(q-1)^3q^3$	C
$\begin{pmatrix} a & 1 & \\ & a & \\ & & a \end{pmatrix}, \begin{pmatrix} a & & \\ & a & 1 \\ & & a \end{pmatrix}$, $a \neq 0$	$2(q-1)$	$(q-1)^2q^2$	A_1
$\begin{pmatrix} a & & 1 \\ & a & \\ & & a \end{pmatrix}, a \neq 0$	$q-1$	$(q-1)^2q^3$	A_2
$\begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix}, \begin{pmatrix} a & & \\ & b & \\ & & a \end{pmatrix}$, $\begin{pmatrix} b & & \\ & a & \\ & & a \end{pmatrix}, 0 \neq a \neq b \neq 0$	$3(q-1)(q-2)$	$(q-1)^3q$	B_1
$\begin{pmatrix} a & 1 & \\ & a & 1 \\ & & a \end{pmatrix}, a \neq 0$	$q-1$	$(q-1)q^2$	R_1
$\begin{pmatrix} a & 1 & \\ & a & \\ & & b \end{pmatrix}, \begin{pmatrix} a & & 1 \\ & b & \\ & & a \end{pmatrix}$, $\begin{pmatrix} b & & \\ & a & 1 \\ & & a \end{pmatrix}, a \neq b$	$3(q-1)(q-2)$	$(q-1)^2q$	R_2
$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$ $a \neq b \neq c \neq a$	$(q-1)(q-2)(q-3)$	$(q-1)^3$	R_3

The branching rules are described by the branching matrix as follows.

Theorem 3.1. *The branching matrix for the group $GT_3(\mathbf{F}_q)$ with types written in the order $\{C, A_1, A_2, B_1, R_1, R_2, R_3\}$ is $B_{GT_3(\mathbf{F}_q)}$*

$$= \begin{pmatrix} q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2(q-1) & q(q-1) & 0 & 0 & 0 & 0 & 0 \\ q-1 & 0 & q(q-1) & 0 & 0 & 0 & 0 \\ 3(q-1)(q-2) & 0 & 0 & (q-1)^2 & 0 & 0 & 0 \\ q-1 & q(q-1) & q^2-1 & 0 & (q-1)q^2 & 0 & 0 \\ 3(q-1)(q-2) & q(q-1)(q-2) & q(q-1)(q-2) & (q-1)^2 & 0 & (q-1)^2q & 0 \\ (q-1)(q-2)(q-3) & 0 & 0 & (q-1)^2(q-2) & 0 & 0 & (q-1)^3 \end{pmatrix}.$$

Proposition 3.2. *For an upper triangular matrix of type C, the branches are as in the second column of the table in the the opening paragraph of this section.*

Proof. The result follows, since the matrices of type C are central. $\square$

Proposition 3.3. *An upper triangular matrix of type A_1 has $q(q-1)$ branches of type A_1, $q(q-1)$ branches of type R_1, and $q(q-1)(q-2)$ branches of type R_2.*

Proof. Let $A = \begin{pmatrix} a & 1 & \\ & a & \\ & & a \end{pmatrix}$, a matrix of type A_1. The centralizer of A is: $Z_{GT_3(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 \\ & x_0 & \\ & & z_0 \end{pmatrix} \mid x_0, z_0 \neq 0 \right\}$. Let $X = \begin{pmatrix} x_0 & x_1 & x_2 \\ & x_0 & \\ & & z_0 \end{pmatrix}$, be an invertible member of $Z_{GT_3(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} a_0 & a_1 & a_2 \\ & a_0 & \\ & & c_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & a_1' & a_2' \\ & a_0 & \\ & & c_0 \end{pmatrix} = XBX^{-1}$. Thus equating $XB = B'X$ leads us to the following equations:

$$a_1' = a_1 \tag{3.1}$$
$$x_0 a_2 + x_2 c_0 = x_2 a_0 + z_0 a_2' \tag{3.2}$$

Case: $a_0 = c_0$. Here, equation 3.2 becomes $x_0 a_2 = z_0 a_2'$. When $a_2 = 0$, then, we have B reduced to $\begin{pmatrix} a_0 & a_1 & \\ & a_0 & \\ & & a_0 \end{pmatrix}$, with $Z_{GT_3(\mathbf{F}_q)}(A, B) = Z_{GT_3(\mathbf{F}_q)}(A)$. Thus (A, B) is of type A_1, and there are $q(q-1)$ such branches.

When $a_2 \neq 0$, choose z_0 so that $a_2 = 1$. Then B is reduced to $\begin{pmatrix} a_0 & a_1 & 1 \\ & a_0 & \\ & & a_0 \end{pmatrix}$, and $Z_{GT_3(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 \\ & x_0 & \\ & & x_0 \end{pmatrix} \right\}$. This subgroup is commutative. Thus (A, B) is of type R_1, and there are $q(q-1)$ such branches. There are no further cases to see here. **Case:** $a_0 \neq c_0$. In Equation 3.2, choose x_2 so that $a_2' = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & a_1 & \\ & a_0 & \\ & & c_0 \end{pmatrix}$, and $Z_{GT_3(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & \\ & x_0 & \\ & & z_0 \end{pmatrix} \right\}$. This subgroup is commutative. Thus (A, B) is of type R_2, and there are $q^2(q-1) = q^3 - q^2$ such branches.

These are all the cases here. Thus, we have a total of $q^2 + q^3 - q^2 = q^3$ branches of type R. ☐

Proposition 3.4. *An upper triangular matrix of type A_2 has $q(q-1)$ branches of type A_2, and $q^2 - 1$ branches of type R_1, and $q(q-1)(q-2)$ branches of type R_2.*

Proof. Given $A = \begin{pmatrix} a & & 1 \\ & a & \\ & & a \end{pmatrix}$, the canonical form of a matrix of type A_2. The centralizer of A, $Z_{GT_3(\mathbf{F}_q)}(A)$ is $\left\{ \begin{pmatrix} x_0 & x_1 & x_2 \\ & y_0 & y_1 \\ & & x_0 \end{pmatrix} \mid x_0, y_0 \neq 0 \right\}$. Let $X = \begin{pmatrix} x_0 & x_1 & x_2 \\ & y_0 & y_1 \\ & & x_0 \end{pmatrix} \in Z_{GT_3(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} a_0 & a_1 & a_2 \\ & b_0 & b_1 \\ & & a_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & a_1' & a_2' \\ & b_0 & b_1' \\ & & a_0 \end{pmatrix} = XBX^{-1}$. Thus equating $XB = B'X$ gives us the following equations:

$$x_0a_1 + x_1b_0 = x_1a_0 + y_0a_1' \tag{3.3}$$

$$y_0b_1 + y_1a_0 = x_0b_1' + y_1b_0 \tag{3.4}$$

$$x_0a_2 + x_1b_1 = x_0a_2' + y_1a_1' \tag{3.5}$$

Using these we reduce B to the mentioned branches. □

Proposition 3.5. *An upper triangular matrix of type B_1 has $(q-1)^2$ branches of type B_1, and $(q-1)^2$ branches of type R_2, and $(q-1)^2(q-2)$ branches of type R_3.*

Proof. One of the canonical forms of an upper triangular matrix of type B_1 is $A = \begin{pmatrix} aI_2 & \\ & b \end{pmatrix}$, where $a \neq b \in \mathbf{F}_q^*$. Hence the centralizer of A is

$$Z_{GT_3(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} X & \\ & z_0 \end{pmatrix} \mid X \in GT_2(\mathbf{F}_q), z_0 \neq 0 \right\}.$$

Thus the branches of A are of the form $\begin{pmatrix} C & \\ & d \end{pmatrix}$, where $d \neq 0$, and C is a conjugacy class of $GT_2(\mathbf{F}_q)$. Hence, the result. □

Proposition 3.6. *For matrices of the* Regular *types:*

- *A matrix of type R_1 has $(q-1)q^2$ branches of type R_1.*
- *For type R_2, there are $(q-1)^2q$ branches of type R_2.*
- *For type R_3, there are $(q-1)^3$ branches of type R_3*

Proof. The result follows, as the centralizers of matrices of any of the Regular types are commutative. □

Proof of Theorem 3.1. From the data in Propositions 3.2 to 3.6, the branching rules are summarized to the branching table/matrix described in the statement of the theorem. □

4. Branching for $GT_4(q)$

In this section, we discuss the simultaneous conjugacy classes of tuples of commuting matrices of $GT_4(\mathbf{F}_q)$. The conjugacy classes of $GT_4(\mathbf{F}_q)$ is described in Appendix A. The branching rules are as follows (types written in the order listed in last column of Appendix A):

Theorem 4.1. *The branching matrix for $GT_4(\mathbf{F}_q)$ is of size 28 (22 types of $GT_4(\mathbf{F}_q)$ and 6 new types), which we write as $B_{GT_4(\mathbf{F}_q)} = (\mathcal{A} \mid \mathcal{B} \mid \mathcal{C})$ (split in three parts along the columns for convenience of writing) described in Table 1, 2 and 3.*

For the convenience, the branching of non-regular types are in part $\mathcal{A}$, those of regular types in part $\mathcal{B}$, and those of the new types in part $\mathcal{C}$. In each of the sub-tables, the regular branches are in blue, and the new types in red. The $0_{r,s}$ denotes the zero matrix of size $r \times s$. Rest of the section is devoted to proof of this.

TABLE 1. The matrix $\mathcal{A}$

C	A_1	A'_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	B_1	B_2	B_3	B_4	B_5	B_6
$q-1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$2q-2$	q^2-q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q-1$	0	q^2-q	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$2q-2$	0	0	q^2-q	0	0	0	0	0	0	0	0	0	0	0	0	0
$q-1$	0	0	0	q^2-q	0	0	0	0	0	0	0	0	0	0	0	0
$q-1$	0	0	0	0	q^2-q	0	0	0	0	0	0	0	0	0	0	0
$q-1$	0	q^2-q	q^2-q	q^2-q	0	q^3-q^2	$q(q-1)^2$	0	0	0	0	0	0	0	0	0
$q-1$	0	0	0	0	0	0	q^2-q	0	0	0	0	0	0	0	0	0
$2q-2$	q^2-q	0	q^2-q	0	0	0	0	q^3-q^2	0	0	0	0	0	0	0	0
$q-1$	q^2-q	0	0	$2q^2-2q$	0	0	0	0	q^3-q^2	0	0	0	0	0	0	0
$q-1$	0	0	0	q^2-q	0	0	0	0	0	q^3-q^2	0	0	0	0	0	0
$(3q-3).(q-2)$	0	0	0	0	0	0	0	0	0	0	$(q-1)^2$	0	0	0	0	0
$(4q-4).(q-2)$	0	0	0	0	0	0	0	0	0	0	0	$(q-1)^2$	0	0	0	0
$(8q-8).(q-2)$	$2(q^2-q).(q-2)$	$2(q^2-q).(q-2)$	$(q^2-q).(q-2)$	0	0	0	0	0	0	0	0	$(q-1)^2$	$q(q-1)^2$	0	0	0
$(4q-4).(q-2)$	0	0	$(q^2-q).(q-2)$	$2(q^2-q).(q-2)$	0	0	0	0	0	0	0	$(q-1)^2$	0	$q(q-1)^2$	0	0
$(6q-6).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	0	0	0	0	0	0	$2(q-1)^2$	0	0	0	$(q^2-q).(q-2)$	0
$(6q-6).(q-2).(q-3)$	0	0	0	0	0	0	0	0	0	0	$2q^3-8q^2+10q-4$	q^3-4q^2+5q-2	0	0	0	$(q-1)^3$
$q-1$	q^2-q	0	q^2-q	q^2-1	q^3-q^2	0	q^3-q^2	q^3-q^2	q^3-q	$q^4-q^3-q^2+q$	0	0	0	0	0	0
$(4q^{-}4).(q-2)$	$(2q-2).(q-2)$	$(2q-2).(q-2)$	$(3q-3).(q-2)$	$2q^3-4q^2-2q+4$	0	0	0	$(q^3-q^2).(q-2)$	$(q^3-q^2).(q-2)$	0	0	$(q-1)^2$	$q(q-1)^2$	q^3-q^2-q+1	0	0
$(3q-3).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	$(q^2-q).(q-2)$	$(q^3-q^2).(q-2)$	$(q^3-q^2).(q-2)$	$(q^3-q^2).(q-2)$	0	0	0	$(q-1)^2$	0	0	0	$q(q-1)^2$	0
$(6q-6).(q-2).(q-3)$	$q^4-6q^3+11q^2-6q$	$q^4-6q^3+11q^2-6q$	$q^4-6q^3+11q^2-6q$	$q^4-6q^3+11q^2-6q$	0	0	0	0	0	0	$2q^3-8q^2+10q-4$	q^3-4q^2+5q-2	$q^4-4q^3+5q^2-2q$	$q^4-4q^3+5q^2-2q$	$q^4-4q^3+5q^2-2q$	q^3-3q^2+3q-1
$4!\binom{q-1}{4}$	0	0	0	0	0	0	0	0	0	0	$q^4-6q^3+13q^2-12q+4$	$q^4-7q^3+17q^2-17q+6$	0	0	0	$q^4-5q^3+9q^2-7q+2$
0	q^2-q	0	0	0	$q(q-1)^2$	0	0	0	0	0	0	0	0	0	0	0
0	0	$2q^2-2q$	$q-1$	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	q^2-q	$2q-2$	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	$q-1$	0	0	q^3-q^2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	q^2-q	0	0	0	0	0	0	0	0	0	0	0
0	0	q^3+q^2-2q	q^2-1	0	0	q^4-q^2	q^3-q^2	0	0	0	0	0	0	0	0	0

Table 2. The matrix $\mathcal{B}$

$$\mathcal{B} = \left(\begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \\ & & & & \\ & & 0_{17\times 5} & & \\ & & & & \\ q^4-q^3 & 0 & 0 & 0 & 0 \\ 0 & q^4-2q^3+q^2 & 0 & 0 & 0 \\ 0 & 0 & q^4-2q^3+q^2 & 0 & 0 \\ 0 & 0 & 0 & q^4-3q^3+3q^2-q & 0 \\ 0 & 0 & 0 & 0 & (q-1)^4 \\ & & & & \\ & & 0_{6\times 5} & & \end{array}\right)$$

The first column of A corresponds to the central type C and the entries in the column are number of classes of each type in $GT_4(\mathbf{F}_q)$ which is the column two of table in Appendix A. For all the regular types R_1, R_2, R_3, R_4 and R_5, the only branch is that type itself, and the the number of branches is the size of its centralizer which is again listed in Appendix A. This fully describes the matrix $\mathcal{B}$. Thus, it only remain to explain the matrix $\mathcal{A}$ and $\mathcal{C}$.

4.1. **Branching rules for type A.** Let us deal with type A classes as in Section A.

Table 3. The matrix $\mathcal{C}$

$$\mathcal{C} = \left(\begin{array}{cccccc} tNT_1 & tNT_2 & tNT_3 & tNT_4 & tNT_5 & NR_1 \\ & & & & & \\ & & 0_{17\times 6} & & & \\ & & & & & \\ q^3-q^2 & 0 & q^3-q^2 & q^4-2q^3+q^2 & q^4-q^3-q^2+q & 0 \\ 0 & q^4-3q^3+2q^2 & q^4-3q^3+2q^2 & 0 & 0 & 0 \\ q^4-3q^3+2q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q^3-q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^3-q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^3-q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^3-q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^3-q & 0 \\ 0 & q^4-q^2 & q^3-q & q^4-q^3-q^2+q & 0 & q^5-q^4 \end{array}\right)$$

Proposition 4.2. *The branching rules of a matrix of type A_1 are:*

Branch	*No. Of Branches*	*Branch*	*No. of Branches*
A_1	$q(q-1)$	R_1	$(q-1)q$
A_7	$q(q-1)$	R_2	$2q(q-1)(q-2)$
A_8	$q(q-1)$	R_3	$q(q-1)(q-2)$
B_3	$2q(q-1)(q-2)$	R_4	$q(q-1)(q-2)(q-3)$
B_5	$q(q-1)(q-2)$	tNT_1	$q(q-1)$.

Here, a new type appears, called tNT_1, whose centralizer is $\left\{\begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix} \mid x_0z_0 \neq 0\right\}$.

Proof. A matrix of type A_1 has either of the canonical forms $\begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$, or $\begin{pmatrix} a & & & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$.

We may consider any one of them. WLOG, we take $A = \begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$. The centralizer $Z_{GT_4(\mathbf{F}_q)}(A)$ is $\left\{\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & d_0 \end{pmatrix} \mid a_0, c_0, d_0 \neq 0\right\}$. Let $B = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & d_0 \end{pmatrix}$, $B' = \begin{pmatrix} a_0 & a_1' & a_2' & a_3' \\ & a_0 & & \\ & & c_0 & c_1' \\ & & & d_0 \end{pmatrix} XBX^{-1}$, where $X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & w_0 \end{pmatrix}$. Equating $XB = B'X$ leads us to the following:

$$\begin{aligned} a_1' &= a_1 \\ \begin{pmatrix} z_0 & z_1 \\ & w_0 \end{pmatrix}\begin{pmatrix} c_0 & c_1 \\ & d_0 \end{pmatrix} &= \begin{pmatrix} c_0 & c_1' \\ & d_0 \end{pmatrix}\begin{pmatrix} z_0 & z_1 \\ & w_0 \end{pmatrix}. \end{aligned}$$

Let $C = \begin{pmatrix} c_0 & c_1 \\ & d_0 \end{pmatrix}$, and $Z = \begin{pmatrix} z_0 & z_1 \\ & w_0 \end{pmatrix}$. The second equation leads us to various conjugacy classes of $GT_2(\mathbf{F}_q)$. Hence, we take C to be some conjugacy class representative in $GT_2(\mathbf{F}_q)$, and $Z \in Z_{GT_2(\mathbf{F}_q)}(C)$. This leads us to the following equation:

$$x_0\begin{pmatrix} a_2 & a_3 \end{pmatrix} + \begin{pmatrix} x_2 & x_3 \end{pmatrix}.(C - a_0I_2) = \begin{pmatrix} a_2' & a_3' \end{pmatrix}.Z \tag{4.1}$$

When a_0 is an eigenvalue of C:

When $(a_2, a_3) = (0, 0)$: Equation 4.1 becomes $\begin{pmatrix} x_2 & x_3 \end{pmatrix}.(C - a_0I_2) = (0\ 0)$.

When $C = a_0 I_2$, Equation 4.1 is void, and we have B reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = Z_{GT_4(\mathbf{F}_q)}(A)$. Thus (A,B) is of type A_1, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$, Equation 4.1 becomes $\begin{pmatrix} x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} = (0\,0)$. Thus $x_2 = 0$. We have B reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix} \right\}$. This centralizer is not isomorphic to the centralizers of te known types. Thus (A,B) is of a new type, which we will call tNT_1 and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$ $(a_0 \neq b_0)$, Equation 4.1 becomes $\begin{pmatrix} x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} & \\ & (b_0 - a_0) \end{pmatrix} = (0\ 0)$. Thus $x_3 = 0$. We have B reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \\ & x_0 & & \\ & & z_0 & \\ & & & z_2 \end{pmatrix} \right\}$. Thus (A,B) is of type B_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & a_0 \end{pmatrix}$ $(a_0 \neq b_0)$, Equation 4.1 becomes $\begin{pmatrix} x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} (b_0 - a_0) & \\ & \end{pmatrix} = (0\ 0)$. Thus $x_2 = 0$. We have B reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & z_0 & \\ & & & z_2 \end{pmatrix} \right\}$. Thus (A,B) is of type B_3, and there are $q(q-1)(q-2)$ such branches.

When $(a_2, a_3) \neq (0,0)$:

When $C = a_0 I_2$, Equation 4.1 becomes: $\begin{pmatrix} a_2 & a_3 \end{pmatrix} = \begin{pmatrix} a_2' & a_3' \end{pmatrix} \begin{pmatrix} \frac{z_0}{x_0} & \frac{z_1}{x_0} \\ & \frac{z_2}{x_0} \end{pmatrix}$.

We have from this:

$$a_2 = \frac{z_0}{x_0}a_2' \tag{4.2}$$

$$a_3 = \frac{z_1}{z_0}a_2' + \frac{z_2}{x_0}a_3' \tag{4.3}$$

When $a_2 \neq 0$, choose x_0 such that $a_2' = 1$. In the equation below, choose z_1 so that $a_3' = 0$. So, B reduces to $\begin{pmatrix} a_0 & a_1 & 1 & \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & & \\ & & x_0 & \\ & & & z_2 \end{pmatrix} \right\}$. This (A, B) is of type A_7, and there are $q(q-1)$ such branches.

When $a_2 = 0$, $a_3 \neq 0$, choose z_2 such that $a_3' = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & a_1 & & 1 \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type A_8, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$, here $Z = \begin{pmatrix} x_0 & x_1 \\ & x_0 \end{pmatrix}$. Equation 4.1 becomes

$$\begin{pmatrix} a_2 & a_3 \end{pmatrix} + \begin{pmatrix} \frac{x_2}{x_0} & \frac{x_3}{x_0} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_2' & a_3' \end{pmatrix} \begin{pmatrix} \frac{z_0}{x_0} & \frac{z_1}{x_0} \\ & \frac{z_0}{x_0} \end{pmatrix}.$$

We have:

$$\begin{aligned} a_2 &= a_2' \frac{z_0}{x_0} \\ a_3 + \frac{x_2}{x_0} &= \frac{z_1}{z_0} a_2' + \frac{z_0}{x_0} a_3'. \end{aligned}$$

Choose x_2 such that $a_3' = 0$. As $(a_2, a_3) \neq (0, 0)$, and $a_3 = 0$, we have $a_2 \neq 0$. Choose z_0 such that $a_2' = 1$. So, B is reduced to $\begin{pmatrix} a_0 & a_1 & 1 & \\ & a_0 & & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & & \\ & & x_0 & x_2 \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type R_1, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$, where $a_0 \neq b_0$. Here $Z = \begin{pmatrix} z_0 & \\ & z_2 \end{pmatrix}$. Here, Equation 4.1 becomes:

$$\begin{pmatrix} \frac{x_2}{x_0} & \frac{x_3}{x_0} \end{pmatrix} \begin{pmatrix} 0 & \\ 0 & b_0 - a_0 \end{pmatrix} = \begin{pmatrix} a_2' & a_3' \end{pmatrix} \begin{pmatrix} \frac{z_0}{x_0} & \\ & \frac{z_2}{x_0} \end{pmatrix}.$$

We have:

$$\begin{aligned} a_2 &= a_2' \frac{z_0}{x_0} \\ a_3 + \frac{x_3}{x_0}(b_0 - a_0) &= \frac{z_0}{x_0} a_3'. \end{aligned}$$

As $b_0 - a_0 \neq 0$, choose x_3 so that $a_3' = 0$. So we are left with $a_2 \neq 0$. Choose z_0 such that $a_2' = 1$. So B is reduced to $\begin{pmatrix} a_0 & a_1 & 1 & \\ & a_0 & & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \\ & x_0 & & \\ & & x_0 & \\ & & & z_2 \end{pmatrix} \right\}$. Thus, (A, B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & a_0 \end{pmatrix}$, where $a_0 \neq b_0$. Here $Z = \begin{pmatrix} z_0 & \\ & z_2 \end{pmatrix}$. Here, Equation 4.1 becomes:

$$\begin{pmatrix} \frac{x_2}{x_0} & \frac{x_3}{x_0} \end{pmatrix} \begin{pmatrix} b_0 - a_0 & \\ & 0 \end{pmatrix} = \begin{pmatrix} a_2' & a_3' \end{pmatrix} \begin{pmatrix} \frac{z_0}{x_0} & \\ & \frac{z_2}{x_0} \end{pmatrix}.$$

We have:

$$\begin{aligned} a_2 + \frac{x_2}{x_0}(b_0 - a_0) &= a_2' \frac{z_0}{x_0} \\ a_3 &= \frac{z_0}{x_0} a_3'. \end{aligned}$$

As $b_0 - a_0 \neq 0$, choose x_2 so that $a_2' = 0$. So we are left with $a_3 \neq 0$. Choose z_2 such that $a_3' = 1$. So B is reduced to $\begin{pmatrix} a_0 & a_1 & & 1 \\ & a_0 & & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & z_0 & \\ & & & x_0 \end{pmatrix} \right\}$. Thus, (A, B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

Now, we come to the case of a_0 not being an eigenvalue of C. In Equation 4.1, the matrix $(C-a_0I_2)$ is invertible. So, we can choose x_2, x_3 such that both $a_2 = a_3 = 0$. Thus, on replacing a_2 and a_3 with 0 each in Equation 4.1, we get $\begin{pmatrix} x_2 & x_3 \end{pmatrix}(C-a_0I_2) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. So $x_2 = x_3 = 0$. Thus, we have:

When $C = b_0I_2$, $b_0 \neq a_0$, B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & x_1 & & \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_2 \end{pmatrix}\right\}$. (A,B) is of the type B_5, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & 1 \\ & b_0 \end{pmatrix}$, $b_0 \neq a_0$, B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & b_0 & 1 \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & x_1 & & \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix}\right\}$. (A,B) is of the type R_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & c_0 \end{pmatrix}$, $b_0 \neq a_0 \neq c_0 \neq b_0$, B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & b_0 & \\ & & & c_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & x_1 & & \\ & x_0 & & \\ & & z_0 & \\ & & & z_2 \end{pmatrix}\right\}$. (A,B) is of the type R_4, and there are $q(q-1)(q-2)(q-3)$ such branches.

We are left with no other cases.

□

Proposition 4.3. *The branching rules of a matrix of type A_1' are:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_1'	$q(q-1)$	R_2	$2q(q-1)(q-2)$
A_5	$q(q-1)$	R_3	$q(q-1)(q-2)$
B_3	$2q(q-1)(q-2)$	R_4	$q(q-1)(q-2)(q-3)$
B_5	$q(q-1)(q-2)$	NR_1	$q(q-1)(q+2)$
tNT_2	$2q(q-1)$		

Two new types NR_1 and tNT_2 appear here. The centralizers of these new types are $\left\{\begin{pmatrix} x_0I_2 & Y \\ & x_0I_2 \end{pmatrix} \mid Y \in M_2(\mathbf{F}_q), x_0 \neq 0\right\}$, and $\left\{\begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & y_0 \end{pmatrix} \mid x_0y_0 \neq 0\right\}$, respectively.

Proof. A matrix of type A_1' has the canonical form: $A = \begin{pmatrix} a & & & \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$. The centralizer $Z_{GT_4(\mathbf{F}_q)}(A)$, of A is $\left\{\begin{pmatrix} a_0 & & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & \\ & & & c_0 \end{pmatrix}\right\}$. Let $B = \begin{pmatrix} a_0 & & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & \\ & & & c_0 \end{pmatrix}$, $B' = \begin{pmatrix} a_0 & & a_2' & a_3' \\ & b_0 & b_1' & b_2' \\ & & b_0 & \\ & & & c_0 \end{pmatrix} = XBX^{-1}$, and where $X = \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & z_0 \end{pmatrix}$. Denote the submatrix $\begin{pmatrix} a_0 & a_3 \\ & c_0 \end{pmatrix}$ of B by C, and the submatrix $\begin{pmatrix} x_0 & x_3 \\ & z_0 \end{pmatrix}$ by Z. Then equating $XB = B'X$ leads us to $ZC = C'Z$. Thus, we can take C to be a canonical form in $GT_2(\mathbf{F}_q)$, and $Z \in Z_{GT_2(\mathbf{F}_q)}(C)$. Thus we have $b_1' = b_1$, and the following equaitons:

$$x_0a_2 + x_2b_0 = x_2a_0 + y_0a_2' \tag{4.4}$$

$$y_0b_2 + y_2c_0 = y_2b_0 + z_0b_2' \tag{4.5}$$

When C has b_0 as an eigenvalue:

When $(a_2, b_2) = (0, 0)$:

When $C = b_0I_2$, Equation 4.4 and 4.5 become void, and B becomes, $B = \begin{pmatrix} b_0 & & & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. (A, B) is of type A_1', and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} b_0 & 1 \\ & b_0 \end{pmatrix}$. Here too, Equations 4.4 and 4.5 are void. So, B reduces to $\begin{pmatrix} b_0 & & & 1 \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type A_5, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & c_0 \end{pmatrix}$, $b_0 \neq c_0$. Here Equation 4.4 stays void, but 4.5 becomes $y_2 c_0 = y_2 b_0$, thus $y_2 = 0$. So, B reduces to $\begin{pmatrix} b_0 & & & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & c_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type B_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$, $b_0 \neq a_0$. Here Equation 4.4 becomes $x_2 b_0 = x_2 a_0$, hence $x_2 = 0$, and Equation 4.5 stays void. So, B reduces to $\begin{pmatrix} c_0 & & & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type B_3, and there are $q^2(q-1) = q^3 - q^2$ such branches.

When $(a_2, b_2) \neq (0, 0)$:

When $C = b_0 I_2$, Equations 4.4 and 4.5 become

$$\begin{aligned} x_0 a_2 &= y_0 a_2' \\ y_0 b_2 &= z_0 b_2' \end{aligned}$$

When $a_2 \neq 0$, choose y_0 such that $a_2' = 1$. When $b_2 \neq 0$, choose z_0 such that $b_2' = 1$. So, B is reduced to $\begin{pmatrix} b_0 & & 1 & \\ & b_0 & b_1 & 1 \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} y_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \right\}$.

This centralizer is not of any known type in $GT_4(\mathbf{F}_q)$, and it is clearly a commutative one. We call this new type NR_1. There are $q(q-1)$ such branches.

When $b_2 = 0$, then Equation 4.5 is void, and B is reduced to $\begin{pmatrix} b_0 & & 1 & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and

$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} y_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. This centralizer too is not of any known type in $GT_4(\mathbf{F}_q)$, and definitely not of R_1, as this one is 6-dimensional. We call this new type tNT_2. There are q^2 such branches.

When $a_2 = 0$, and $b_2 \neq 0$, choose z_0 such that $b_2' = 1$, and B is reduced to $\begin{pmatrix} b_0 & & & \\ & b_0 & b_1 & 1 \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \right\}$. We have another q^2 branches of this new type tNT_2.

When $C = \begin{pmatrix} b_0 & 1 \\ & b_0 \end{pmatrix}$. Here $Z = \begin{pmatrix} x_0 & x_3 \\ & x_0 \end{pmatrix}$. Equation 4.4 becomes $x_0 a_2 = y_0 a_2'$, and Equation 4.5 becomes $y_0 b_2 = x_0 b_2'$.

When $a_2 \neq 0$, choose y_0 such that $a_2' = 1$. Now, on substituting a_2 by $a_2' = 1$ in the equation, we get $y_0 = x_0$, and thus $b_2' = b_2$. B is reduced to $\begin{pmatrix} b_0 & & 1 & 1 \\ & b_0 & b_1 & b_2 \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, with

$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} y_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \right\}$. (A,B) is of type NR_1, and there are $q^2(q-1)$ such branches.

When $a_2 = 0$, we look at $b_2' \neq 0$, Choose x_0 such that $b_2' = 1$. Then B is reduced to $\begin{pmatrix} b_0 & & & 1 \\ & b_0 & b_1 & 1 \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} y_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \right\}$. (A,B) is of type R_1, and there are another $q(q-1)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & c_0 \end{pmatrix}$, $b_0 \neq c_0$, then $Z = \begin{pmatrix} x_0 & \\ & z_0 \end{pmatrix}$. Equation 4.4 becomes $x_0 a_2 = y_0 a_2'$, and Equation 4.5 becomes $y_0 b_2 + y_2 c_0 = y_2 b_0 + z_0 b_2'$. As $b_0 \neq c_0$, choose y_2 such that $b_2' = 0$. So we have only one case here $a_2 \neq 0$. Thus, choose x_0 such that $a_2' = 1$. Thus B is reduced to $\begin{pmatrix} b_0 & & 1 & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & c_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} y_0 & & x_2 & \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$, $a_0 \neq b_0$, then $Z = \begin{pmatrix} x_0 & \\ & z_0 \end{pmatrix}$. Equation 4.4 becomes $x_0 a_2 + x_2 b_0 = y_0 a_2' + x_2 a_0$, and Equation 4.5 becomes $y_0 b_2 = z_0 b_2'$. As $b_0 \neq c_0$, choose x_2 such that $a_2' = 0$. So we have only one case here $b_2 \neq 0$. Thus, choose z_0 such that $b_2' = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & & & \\ & b_0 & b_1 & 1 \\ & & b_0 & \\ & & & b_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \right\}$. (A, B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

Now, the second main case of b_0 not being an eigenvalue of C, i.e., $b_0 \neq a_0$ and $b_0 \neq c_0$. Here in Equation 4.4, choose x_2 so that $a_2' = 0$, and in Equation 4.5 choose y_2 so that $b_2' = 0$.

When $C = a_0 I_2$, where $a_0 \neq b_0$, B is reduced to $\begin{pmatrix} a_0 & & & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & x_3 \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type B_5, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$, where $a_0 \neq b_0$, B is reduced to $\begin{pmatrix} a_0 & & & 1 \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & x_3 \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A, B) is of type R_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & c_0 \end{pmatrix}$, where $a_0, c_0 \neq b_0$, and $a_0 \neq c_0$, B is reduced to $\begin{pmatrix} a_0 & & & \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & c_0 \end{pmatrix}$,

and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type R, and there are $q(q-1)(q-2)(q-3)$ such branches.

With this there are no other cases left for us to analyse.

Adding up the branches of type NR_1, we have a total of $2q(q-1) + q^2(q-1) = q(q-1)(q+2)$ branches.

□

Proposition 4.4. *The branching rules of a matrix of type A_2 are given below:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_2	$q(q-1)$	R_2	$3(q^2-q)(q-2)$
A_5	$q(q-1)$	R_3	$(q^2-q)(q-2)$
A_7	$q(q-1)$	R_4	$q(q-1)(q-2)(q-3)$
B_3	$(q^2-q)(q-2)$	tNT_2	$q-1$
B_4	$(q^2-q)(q-2)$	tNT_3	$q(q-1)$
B_5	$(q^2-q)(q-2)$	NR_1	$q^2-1.$
R_1	$q(q-1)$		

A further new type tNT_3 appears here, whose centralizer is $\left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \mid x_0, y_0 \neq 0 \right\}$.

Proof. Matrices of this type have two non-similar canonical forms: $\begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$, and $\begin{pmatrix} a & & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$. Proving this for anyone of them is enough. We consider $A = \begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$.

The centralizer $Z_{GT_4(\mathbf{F}_q)}(A)$ of A is: $\left\{ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & a_0 & \\ & & & c_0 \end{pmatrix} \right\}$. Let $B = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & a_0 & \\ & & & c_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & a'_1 & a'_2 & a'_3 \\ & b_0 & b'_1 & b'_2 \\ & & a_0 & \\ & & & c_0 \end{pmatrix}$, be a conjugate of B by $X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & x_0 & \\ & & & z_0 \end{pmatrix}$. Denote $C = \begin{pmatrix} b_0 & b_2 \\ & c_0 \end{pmatrix}$, and $Z = \begin{pmatrix} y_0 & y_2 \\ & z_0 \end{pmatrix}$. Equating $XB = B'X$, we have first $ZC = C'Z$. Thus, we may take C to be a canonical form from $GT_2(\mathbf{F}_q)$, and $Z \in Z_{GT_2(\mathbf{F}_q)}(C)$. With these, we have the following equations:

$$x_0.\begin{pmatrix} a_1 & a_3 \end{pmatrix} + \begin{pmatrix} x_1 & x_3 \end{pmatrix}(C - a_0 I_2) = \begin{pmatrix} a'_1 & a'_3 \end{pmatrix}.Z \tag{4.6}$$

$$y_0 b_1 + y_1 a_0 = x_0 b'_1 + y_1 b_0 \tag{4.7}$$

$$x_0 a_2 + x_1 b_1 = x_0 a'_2 + y_1 a'_1 \tag{4.8}$$

We have two main cases, under each of which there are subcases:
When a_0 is an eigenvalue of C When $C = a_0 I_2$:, Equation 4.6 becomes:

$$\begin{pmatrix} x_0 a_1 & x_0 a_3 \end{pmatrix} = \begin{pmatrix} y_0 a'_1 & y_2 a'_1 + z_0 a'_3 \end{pmatrix} \tag{4.9}$$

Equation 4.7 becomes $y_0 b_1 = x_0 b'_1$.

When $a_1 = b_1 = 0$: From Equation 4.9 we have $x_0 a_3 = z_0 a'_3$, and from Equation 4.8 $a'_2 = a_2$

We have two subcases:

When $a_3 = 0$: B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. (A, B) is of type A_2, and there are $q(q-1)$ such branches.

When $a_3 \neq 0$, choose z_0 so that $a'_3 = 1$. B is reduced to $\begin{pmatrix} a_0 & & a_2 & 1 \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with

$Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. Now, this centralizer is not isomorphic to any known centralizer of a matrix in $GT_4(\mathbf{F}_q)$, and neither it is isomorphic to those of the three new types we encountered in the previous propositions. We have a new type tNT_3, and there are $q(q-1)$ such branches.

When $a_1 = 0$, and $b_1 \neq 0$. In Equation 4.7 choose y_0 such that $b'_1 = 1$. Then Equation 4.8 becomes $x_0 a_2 + x_1 = x_0 a'_2$. Choose x_1 such that $a'_2 = 0$.

Here, when $a_3 = 0$, B is reduced to $\begin{pmatrix} a_0 & & & \\ & a_0 & 1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type tNT_2, and there are $q-1$ such branches.

When $a_3 \neq 0$, choose z_0 such that $a_3' = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & & & 1 \\ & a_0 & 1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A, B) is of type NR_1, and there are $q-1$ such branches.

When $a_1 \neq 0$, in Equation 4.9, choose y_0 such that $a_1' = 1$. Thus, on replacing a_1 with $a_1' = 1$ in Equation 4.9, we get $y_0 = x_0$. In the same equation, choose y_2 such that $a_3' = 0$.

From Equation 4.7, we get $b_1' = b_1$. Equation 4.8 becomes $x_0a_2 + x_1b_1 = x_0a_2' + y_1$. Choose y_1 such that $a_2' = 0$. Here, B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & b_1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & b_1x_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. (A, B) is of type A_7, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$: Equation 4.6 becomes

$$(4.10) \qquad \begin{pmatrix} x_0a_1 & x_0a_3 \end{pmatrix} + \begin{pmatrix} 0 & x_1 \end{pmatrix} = \begin{pmatrix} y_0a_1' & y_2a_1' + y_0a_3' \end{pmatrix}.$$

Choose x_1 such that $a_3' = 0$. Hence, on replacing a_3 by $a_3' = 0$ in the above equation, we have $x_1 = a_1'y_2$.

Equation 4.7 becomes $x_0b_1' = y_0b_1$.

When $a_1 = b_1 = 0$, from Equation 4.8 $a_2' = a_2$, B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & a_0 & & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & x_0 & \\ & & & y_0 \end{pmatrix} \right\}$. (A,B) is of type A_5. There are $q(q-1)$ such branches.

When $a_1 = 0$, and $b_1 \neq 0$, we choose y_0 in Equation 4.7 so that $b_1' = 1$. So, Equation 4.8 becomes $x_0 a_2 = x_0 a_2'$, since $x_1 = y_2 a_1 = 0$. Hence $a_2' = a_2$. So B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & a_0 & 1 & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. So, we have another $q(q-1)$ branches of type NR_1 here.

When $a_1 \neq 0$, in Equation 4.6, choose y_0 so that $a_1' = 1$. So, $x_1 = y_2$, and on replacing a_1' with a_1 in the same equation, we have $y_0 = x_0$, and hence from Equation 4.7, $b_1' = b_1$.

With these, Equation 4.8 becomes $x_0 a_2 + x_1 b_1 = x_0 a_2' + y_1$. Choose y_1 so that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & b_1 & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & b_1 x_1 & x_1 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. By a routine check, one can see that this subgroup is commutative. Thus (A,B) is of type R_1, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$, $b_0 \neq a_0$: Here $Z = \begin{pmatrix} y_0 & \\ & z_0 \end{pmatrix}$. Equation 4.6 becomes

$$\begin{pmatrix} x_0 a_1 & x_0 a_3 \end{pmatrix} + \begin{pmatrix} 0 & (b_0 - a_0) x_3 \end{pmatrix} = \begin{pmatrix} y_0 a_1' & z_0 a_3' \end{pmatrix} \tag{4.11}$$

And, Equation 4.7 becomes $y_0 b_1 = x_0 b_1'$.

Choose x_3 such that $a_3' = 0$.

When $a_1 = b_1 = 0$, from Equation 4.8, we have $a_2' = a_2$, and B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & a_0 & & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \\ & y_0 & y_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. This (A,B) is of type B_4, and there are $q(q-1)(q-2)$ such branches.

When $a_1 = 0$, and $b_1 \neq 0$. In Equation 4.7, choose y_0 so that $b_1' = 1$. And, Equation 4.8 becomes $x_0 a_2 + x_1 = x_0 a_2'$. We choose x_1 so that $a_2' = 0$. Hence, B is reduced to

$\begin{pmatrix} a_0 & & & \\ & a_0 & 1 & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & \\ & x_0 & y_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. By a routine check, one can see that this subgroup is commutative. Thus (A,B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

When $a_1 \neq 0$, in Equation 4.6, choose y_0 so that $a_1' = 1$. Hence, on replacing a_1 by $a_1' = 1$ on both sides of Equation 4.11, we get $x_0 = y_0$. Hence, Equation 4.7 becomes $x_0 b_1 = x_0 b_1'$, thus leaving use with $b_1' = b_1$. Equation 4.8 becomes $x_0 a_2 + x_1 b_1 = y_1 + x_0 a_2'$. Thus, choose y_1 such that $a_2' = 0$. Hence B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & b_1 & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \\ & x_0 & b_1 x_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. By a routine check, one can see that this subgroup is commutative. Thus (A,B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & a_0 \end{pmatrix}$, $a_0 \neq b_0$: Here too $Z = \begin{pmatrix} y_0 & \\ & z_0 \end{pmatrix}$. In this case, Equation 4.6 is reduced to $\begin{pmatrix} x_0 a_1 & x_0 a_3 \end{pmatrix} + \begin{pmatrix} (b_0 - a_0) x_1 & 0 \end{pmatrix} = \begin{pmatrix} y_0 a_1' & z_0 a_3' \end{pmatrix}$. Choose x_1 so that $a_1' = 0$.

Equation 4.7 becomes $y_0 b_1 + y_1 a_0 = y_1 b_0 + x_0 b_1'$. As $a_0 \neq b_0$, choose y_1 such that $b_1' = 0$. With these, Equation 4.8 becomes $x_0 a_2 = x_0 a_2'$, thus leaving us with $a_2' = a_2$.

Now, we are left to deal with a_3.

When $a_3 = 0$, B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & b_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. This subgroup is isomorphic to the subgroup $\left\{ \begin{pmatrix} y_0 & & & \\ & x_0 & x_2 & x_3 \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type B_3. Hence, we have $q(q-1)(q-2)$ branches of type B_3.

When $a_3 \neq 0$, choose z_0 so that $a'_3 = 1$. Hence B is reduced to $\begin{pmatrix} a_0 & & a_2 & 1 \\ & b_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & & \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. By a routine check, one can see that this subgroup is commutative. Thus (A,B) is of type R_2, and there are $q(q-1)(q-2)$ such branches. With this we have looked at all the cases, when a_0 is an eigenvalue of C.

When a_0 is not an eigenvalue oc C: When a_0 is not an eigenvalue of C, $C - a_0 I_2 = \begin{pmatrix} b_0 - a_0 & b_2 \\ & c_0 - a_0 \end{pmatrix}$, with $b_0 - a_0 \neq 0$, and $c_0 - a_0 \neq 0$. Hence, in equation 4.6, we can choose $\begin{pmatrix} x_1 & x_3 \end{pmatrix}$ such that $a'_1 = 0$, and $a'_3 = 0$. In Equation 4.7 choose y_1 so that $b'_1 = 0$. Hence Equation 4.8 becomes $x_0 a_2 = x_0 a'_2$. Therefore $a'_2 = a_2$.

On replacing a_1 and a_3 by 0 in Equation 4.6, we get $x_1 = x_3 = 0$, and on replacing b_1 by 0 in Eqaution 4.7, we get $y_1 = 0$. Now, we can look at the various cases of C.

When $C = b_0 I_2, b_0 \neq a_0$**:** B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & b_0 & & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$ and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & \\ & y_0 & & y_2 \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. Thus (A,B) is of type B_5, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & 1 \\ & b_0 \end{pmatrix}, b_0 \neq a_0$**:** B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & b_0 & & 1 \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$ and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & x_2 & \\ & y_0 & & y_2 \\ & & x_0 & \\ & & & y_0 \end{pmatrix} \right\}$. Thus (A,B) is of type R_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & c_0 \end{pmatrix}, b_0 \neq c_0, a_0 \notin \{b_0, c_0\}$**:** B is reduced to $\begin{pmatrix} a_0 & & a_2 & \\ & b_0 & & \\ & & a_0 & \\ & & & c_0 \end{pmatrix}$ and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & \\ & y_0 & & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. Thus (A, B) is of type R_4, and there are $q(q-1)(q-2)(q-3)$ such branches.

Adding up the branches of type NR_1, we have a total of $q - 1 + q(q-1) = q^2 - 1$ branches of type R_1. □

Proposition 4.5. *For a matrix of type A_3, the branching rules are in the table below:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_3	$q(q-1)$	R_1	$q^2 - 1$
A_5	$q(q-1)$	R_2	$2(q^2-1)(q-2)$
A_8	$2q(q-1)$	R_3	$(q^2-q)(q-2)$
A_9	$q(q-1)$	R_4	$(q^2-q)(q-2)(q-3)$
B_4	$2(q^2-q)(q-1)$	tNT_3	$2(q-1)$
B_5	$(q^2-q)(q-1)$	tNT_4	$q-1.$

A new type tNT_4 appears here, whose centralizer is $\left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix} \mid x_0 \neq 0 \right\}$.

Proof. A matrix of type A_3 has the canonical form $A = \begin{pmatrix} a & & & 1 \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$. Then we have

$$Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & {}^t\overrightarrow{b} & a_1 \\ & C & \overrightarrow{d} \\ & & a_0 \end{pmatrix} \;\middle|\; \begin{array}{c} a_0 \neq 0, C \in T_2(\mathbf{F}_q) \\ {}^t\overrightarrow{b} = (b_1 \; b_2) \in M_{1\times 2}(\mathbf{F}_q) \\ \overrightarrow{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in M_{2\times 1}(\mathbf{F}_q) \end{array} \right\}.$$ Let $B = \begin{pmatrix} a_0 & {}^t\overrightarrow{b} & a_1 \\ & C & \overrightarrow{d} \\ & & a_0 \end{pmatrix}$,

and $B' = \begin{pmatrix} a_0 & {}^t\overrightarrow{b}' & a_1' \\ & C' & \overrightarrow{d}' \\ & & a_0 \end{pmatrix} = XBA^{-1}$, where $X = \begin{pmatrix} x_0 & {}^t\overrightarrow{y} & x_1 \\ & Z & \overrightarrow{w} \\ & & x_0 \end{pmatrix}$. $XB = B'X$ leads to firstly, $ZC = C'Z$, hence we shall take C to be a canonical conjugacy class representative in $GT_2(\mathbf{F}_q)$, and $Z \in Z_{GT_2(\mathbf{F}_q)}(C)$. Then we have the following set of equations:

$$x_0{}^t\overrightarrow{b} + {}^t\overrightarrow{y}.(C - a_0I_2) = {}^t\overrightarrow{b}'.Z \tag{4.12}$$
$$Z.\overrightarrow{d} + (a_0I_2 - C)\overrightarrow{w} = x_0\,\overrightarrow{d}' \tag{4.13}$$
$$x_0a_1 + y_1d_1 + y_2d_2 = x_0a_1' + b_1'w_1 + b_2'w_2 \tag{4.14}$$

When a_0 is an eigenvalue of C:

When $\overrightarrow{b} = \overrightarrow{d} = \overrightarrow{0}$:

Here, Equation 4.12 becomes ${}^t\overrightarrow{y}(C - a_0I_2) = {}^t\overrightarrow{0}$, Equation 4.13 becomes $(a_0I_2 - C)\overrightarrow{w} = \overrightarrow{0}$, and Equation 4.14 becomes $x_0a_1 = x_0a_1'$. Hence we have $a_1' = a_1$.

When $C = a_0I_2$: Equations 4.12 and 4.13 are void. Hence B is reduced to $\begin{pmatrix} a_0 & & & a_1 \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = Z_{GT_4(\mathbf{F}_q)}(A)$. (A,B) is of type A_2, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$: Here Equation 4.12 becomes: $\begin{pmatrix} 0 & y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and Equation 4.13 becomes $\begin{pmatrix} -w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $y_1 = 0$, and $w_2 = 0$. B is reduced to $\begin{pmatrix} a_0 & & & a_2 \\ & a_0 & 1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & y_2 & x_1 \\ & z_0 & z_1 & w_1 \\ & & z_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A,B) is of type A_5, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & c_0 \end{pmatrix}, a_0 \neq c_0$: Here Equation 4.12 becomes: $\begin{pmatrix} 0 & y_2(c_0 - a_0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and Equation 4.13 becomes $\begin{pmatrix} \\ (a_0 - c_0)w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $y_2 = 0$, and $w_2 = 0$. B is reduced to $\begin{pmatrix} a_0 & & & a_2 \\ & a_0 & & \\ & & c_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & & x_1 \\ & z_0 & & w_1 \\ & & z_2 & \\ & & & x_0 \end{pmatrix} \right\}$. (A,B) is of type B_4, and there are $q^(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & a_0 \end{pmatrix}, a_0 \neq b_0$: Here Equation 4.12 becomes: $\begin{pmatrix} (b_0 - a_0)y_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and Equation 4.13 becomes $\begin{pmatrix} (b_0 - a_0)w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $y_1 = 0$, and $w_1 = 0$.

B is reduced to $\begin{pmatrix} a_0 & & & a_2 \\ & b_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & y_2 & x_1 \\ & z_0 & & \\ & & z_2 & w_2 \\ & & & x_0 \end{pmatrix} \right\}$. This (A,B) too is of type B_4, and there are $q(q-1)(q-2)$ such branches.

When $(\overrightarrow{b}, \overrightarrow{d}) \neq (\overrightarrow{0}, \overrightarrow{0})$.

When $C = a_0 I_2$: Equation 4.12 becomes:

$$\begin{pmatrix} x_0 b_1 & x_0 b_2 \end{pmatrix} = \begin{pmatrix} z_0 b_1' & z_1 b_1' + z_2 b_2 \end{pmatrix}, \tag{4.15}$$

and Equation 4.13 becomes:

$$\begin{pmatrix} z_0 d_1 + z_1 d_2 \\ z_2 d_2 \end{pmatrix} = \begin{pmatrix} x_0 d_1' \\ x_0 d_2' \end{pmatrix}. \tag{4.16}$$

When $b_1 = 0$ and $b_2 \neq 0$. In Equation 4.15 choose z_2 so that $b_2' = 1$. Hence, on replacing b_2 by $b_2' = 1$ in Equation 4.15, we get $x_0 = z_2$. Hence in Equation 4.16, $x_0 d_2' = x_0 d_2$. Thus $d_2 = d_2'$.

Here, if $d_2 = 0$, in Equation 4.16, we have $x_0 d_1' = z_0 d_1$. When $d_1 = 0$; Equation 4.14 becomes $x_0 a_1 = x_0 a_1' + w_2$. Choose w_2 so that $a_1' = 0$. Hence B is reduced to $\begin{pmatrix} a_0 & & 1 & \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ & z_0 & z_1 & w_1 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A,B) is of type tNT_3, and there are $q-1$ such branches.

When $d_1 \neq 0$, choose z_0 so that $d_1' = 1$. Then Equation 4.14 becomes $x_0 a_1 + y_1 = x_0 a_1' w_2$. Choose w_2 such that $a_1' = 0$. With these, B is reduced to: $\begin{pmatrix} a_0 & & 1 & \\ & a_0 & & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ & x_0 & z_1 & w_1 \\ & & x_0 & y_1 \\ & & & x_0 \end{pmatrix} \right\}$.Now, this is a centralizer we have not seen so far. Thus we have a new type, tNT_4. There are $q-1$ such branches.

When $d_2 \neq 0$, in Equation 4.16, choose z_1 so that $d_1' = 0$. Equation 4.14 becomes $x_0 a_1 + y_2 d_2 = x_0 a_1' + w_2$. Choose w_2 such that $a_1' = 0$. So, B is reduced to $\begin{pmatrix} a_0 & & 1 & \\ & a_0 & & \\ & & a_0 & d_2 \\ & & & a_0 \end{pmatrix}$

and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ & z_0 & & w_1 \\ & & x_0 & d_2y_2 \\ & & & x_0 \end{pmatrix}\right\}$. (A,B) is of type A_8, and there are $(q-1)^2$ such branches.

When $b_1 = b_2 = d_2 = 0$. Here $d_1 \neq 0$. Choose z_0 so that $d_1' = 1$. Then, in Equation 4.14, we have $x_0a_1 + y_1 = x_0a_1'$. Choose y_1 so that $a_1' = 0$. Thus B is reduced to: $\begin{pmatrix} a_0 & & & \\ & a_0 & & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & & y_2 & x_1 \\ & x_0 & z_1 & w_1 \\ & & z_2 & w_2 \\ & & & x_0 \end{pmatrix}\right\}$. (A,B) is of type tNT_3. There are $q-1$ such branches.

When $b_1 = b_2 = 0$, and $d_2 \neq 0$. In Equation 4.16, choose z_2 such that $d_2' = 1$, and in the same equation, choose z_1 so that $d_1' = 0$. With these, Equation 4.14 becomes $x_0a_1 + y_2 = x_0a_1'$. Choose y_2 such that $a_1' = 0$. Thus B is reduced to: $\begin{pmatrix} a_0 & & & \\ & a_0 & & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & y_1 & & x_1 \\ & z_0 & & w_1 \\ & & x_0 & w_2 \\ & & & x_0 \end{pmatrix}\right\}$. This too is of type A_8. There are $q-1$ such branches.

When $b_1 \neq 0$. In Equation 4.15, choose z_0 so that $b_1' = 1$, and choose z_1 so that $b_2' = 0$. On replacing b_1 with $b_1' = 1$, and b_2 with $b_2' = 0$ in Equation 4.15, we get $z_0 = x_0$, and $z_1 = 0$. Putting these in equation 4.16 leaves us with $d_1' = d_1$ and $z_2d_2 = x_0d_2'$.

With these, Equation 4.14 is reduced to $x_0a_1 + d_1y_1 = x_0a_1' + w_1$. Choose w_1 so that $a_1' = 0$.

When $d_2 = 0$, B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & d_1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ & x_0 & & d_1y_1 \\ & & z_2 & w_2 \\ & & & x_0 \end{pmatrix}\right\}$. This too is of type A_8. There are $q(q-1)$ such branches.

When $d_2 \neq 0$, in Equation 4.16, choose z_2 so that $d_2' = 1$. With these Equation 4.14 becomes $x_0a_1 + y_1d_1 + y_2 = x_0a_1' + w_1$. Choose w_1 such that $a_1' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & d_1 \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ & x_0 & & y_2 + d_1 y_1 \\ & & x_0 & w_2 \\ & & & x_0 \end{pmatrix} \right\}$. This is of type A_9. There are $q(q-1)$ such branches.

So, with these, we are done with all the cases, when $C = a_0 I_2$.

When $C = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$**:** Here $Z = \begin{pmatrix} z_0 & z_1 \\ & z_0 \end{pmatrix}$. Equation 4.12 becomes:

$$\begin{pmatrix} x_0 b_1 & x_0 b_2 \end{pmatrix} + \begin{pmatrix} 0 & y_1 \end{pmatrix} = \begin{pmatrix} z_0 b_1' & z_1 b_1' + z_0 b_2' \end{pmatrix}$$

Choose y_1 such that $b_2' = 0$. On substituting b_2 with $b_2' = 0$ in the above equation, we have $y_1 = z_1 b_1'$.

Similarly, Equation 4.13 becomes

$$\begin{pmatrix} z_0 d_1 + z_1 d_2 \\ z_0 d_2 \end{pmatrix} + \begin{pmatrix} -w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 b_1' \\ x_0 b_2' \end{pmatrix}.$$

Choose w_2 such that $d_1' = 0$. On substituting d_1 with $d_1' = 0$ in the above equation, we have $w_2 = d_2 z_1$.

When $b_1 \neq 0$, choose z_0 so that $b_1' = 1$. Then, on substituting b_1 with $b_1' = 1$ in Equation 4.12, we get $z_0 = x_0$, and thus $d_2' = d_2$. With these, Equation 4.14 becomes $x_0 a_1 + y_2 d_2 = x_0 a_1' + w_1$. Choose w_1 such that $a_1' = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & 1 & \\ & & a_0 & d_2 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & z_1 & y_2 & x_1 \\ & x_0 & z_1 & d_2 y_2 \\ & & x_0 & d_2 z_1 \\ & & & x_0 \end{pmatrix} \right\}$. This (A,B) is of type R_1, and there are $q(q-1)$ such branches.

When $b_1 = 0$, and $d_2 \neq 0$ $y_1 = 0$. In Equation 4.13, choose z_0 so that $d_2' = 1$. With these, Equation 4.14 becomes $x_0 a_1 + y_2 = x_0 a_1'$. Choose y_2 so that $a_1' = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & & & \\ & a_0 & 1 & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & & x_1 \\ & x_0 & z_1 & w_1 \\ & & x_0 & z_1 \\ & & & x_0 \end{pmatrix} \right\}$. By a routine check, one can see that this is commutative. Thus (A,B) is of type R_1, and there are $q-1$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & b_0 \end{pmatrix}$, $b_0 \neq a_0$**:** Here $Z = \begin{pmatrix} z_0 & \\ & z_2 \end{pmatrix}$. Equation 4.12 becomes :

$$\begin{pmatrix} x_0 b_1 & x_0 b_2 \end{pmatrix} + \begin{pmatrix} 0 & (b_0 - a_0) y_2 \end{pmatrix} = \begin{pmatrix} z_0 b_1' & z_2 b_2' \end{pmatrix}$$

As $b_0 - a_0 \neq 0$, choose y_2 such that $b_2' = 0$. Hence, on replacing b_2 by $b_2' = 0$ in the above equation, we get $y_2 = 0$.

Similarly Equation 4.13 becomes:

$$\begin{pmatrix} z_0d_1 & z_2d_2 \end{pmatrix} + \begin{pmatrix} 0 \\ (a_0 - b_0)w_2 \end{pmatrix} = \begin{pmatrix} x_0d_1' & x_0d_2' \end{pmatrix}$$

Choose w_2 so that $d_2' = 0$. So, if we replace d_2 by $d_2' = 0$ in the above equation, we have $w_2 = 0$.

When $b_1 = 0$ and $d_1 \neq 0$, choose z_0 so that $d_1' = 1$. With these, Equation 4.14 becomes $x_0a_1 + y_1 = x_0a_1'$. Choose y_1 so that $a_1' = 0$. B is thus reduced to $\begin{pmatrix} a_0 & & & \\ & a_0 & & 1 \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & & x_1 \\ & x_0 & & w_1 \\ & & z_2 & \\ & & & x_0 \end{pmatrix} \right\}$. By a routine check, we can see that this centralizer is commutative. Thus (A,B) is of type R_2, and there are $(q-1)(q-2)$ such branches.

When $b_1 \neq 0$. in Equation 4.12 for this C, choose z_0 so that $b_1' = 1$. Thus on substituting b_1 with $b_1' = 1$ in the same, we get $z_0 = x_0$. Hence, from Equation 4.13 for this case, we have $d_1' = d_1$. With these Equation 4.14 becomes $x_0a_1 + d_1y_1 = x_0a_1' + w_1$. Choose w_1 so that $a_1' = 0$. Hence B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & d_1 \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$ and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & y_1 & & x_1 \\ & x_0 & & d_1y_1 \\ & & z_2 & \\ & & & x_0 \end{pmatrix} \right\}$. Easy to see that this centralizer too is commutative. Thus (A,B) is of type R_2, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & a_0 \end{pmatrix}, b_0 \neq a_0$**:** Here $Z = \begin{pmatrix} z_0 & \\ & z_2 \end{pmatrix}$. Equation 4.12 becomes :

$$\begin{pmatrix} x_0b_1 & x_0b_2 \end{pmatrix} + \begin{pmatrix} (b_0 - a_0)y_1 & 0 \end{pmatrix} = \begin{pmatrix} z_0b_1' & z_2b_2' \end{pmatrix}$$

As $b_0 - a_0 \neq 0$, choose y_1 such that $b_1' = 0$. Hence, on replacing b_1 by $b_1' = 0$ in the above equation, we get $y_1 = 0$.

Similarly Equation 4.13 becomes $\begin{pmatrix} z_0d_1 & z_2d_2 \end{pmatrix} + \begin{pmatrix} (a_0-b_0)w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0d_1' & x_0d_2' \end{pmatrix}$. Choose w_1 so that $d_1' = 0$. So, if we replace d_1 by $d_1' = 0$ in the above equation, we have $w_1 = 0$.

When $b_2 = 0$ and $d_2 \neq 0$, choose z_2 so that $d_2' = 1$. With these, Equation 4.14 becomes $x_0a_1 + y_2 = x_0a_1'$. Choose y_2 so that $a_1' = 0$. B is thus reduced to $\begin{pmatrix} a_0 & & & \\ & b_0 & & \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & & x_1 \\ & z_0 & & \\ & & x_0 & w_2 \\ & & & x_0 \end{pmatrix} \right\}$. By a routine check, we can see that this centralizer is commutative. This (A,B) is of type R_2, and there are $(q-1)(q-2)$ such branches.

When $b_2 \neq 0$, in Equation 4.12 for this C, choose z_2 so that $b_2' = 1$. Thus on substituting b_2 with $b_2' = 1$ in the same, we get $z_2 = x_0$. Hence, from Equation 4.13 for this case, we have $d_2' = d_2$. With these Equation 4.14 becomes $x_0a_1 + d_2y_2 = x_0a_1' + w_2$. Choose w_2 so that $a_1' = 0$. Hence B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & & d_1 \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$ and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & & y_2 & x_1 \\ & z_0 & & \\ & & x_0 & d_2y_2 \\ & & & x_0 \end{pmatrix} \right\}$. Easy to see that this centralizer is commutative. This (A,B) too is of type R, and there are $q(q-1)(q-2)$ such branches.

With these, we have covered all the subcases under the case of a_0 being an eigenvalue of C.

When a_0 is not an eigenvalue of C: In this case $C - a_0I_2$ is invertible. Hence, in Equation 4.12, choose y_1, y_2 so that $b_1' = b_2' = 0$. Similarly, in Equation 4.13, choose w_1, w_2 so that $d_1' = d_2' = 0$.

So, Equation 4.14 becomes $x_0a_1 = x_0a_1'$, thus $a_1' = a_1$.

When $C = b_0 I_2, b_0 \neq a_0$**:** B is reduced to $\begin{pmatrix} a_0 & & & a_1 \\ & b_0 & & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) =$ $\left\{ \begin{pmatrix} x_0 & & & x_1 \\ & z_0 & z_1 & \\ & & z_2 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type B_5. There are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & 1 \\ & b_0 \end{pmatrix}$, $b_0 \neq a_0$**:**In this case, B is reduced to $\begin{pmatrix} a_0 & & & a_1 \\ & b_0 & 1 & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & x_1 \\ & z_0 & z_1 & \\ & & z_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This one is a commutative centralizer. This (A, B) is of type R_3, and there are $q(q-1)(q-2)$ such branches.

When $C = \begin{pmatrix} b_0 & \\ & c_0 \end{pmatrix}$, $b_0, c_0 \neq a_0$**, and** $b_0 \neq c_0$**:**In this case, B is reduced to $\begin{pmatrix} a_0 & & & a_1 \\ & b_0 & & \\ & & c_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & x_1 \\ & z_0 & & \\ & & z_2 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type R, and there are $q(q-1)(q-2)(q-3)$ such branches.

So, those are all the cases available.

Adding up all the branches of type A_8, we have a total of $q - 1 + q(q-1) + (q-1)^2 = 2q(q-1)$ branches. □

Proposition 4.6. *A matrix of type A_4 has:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_4	$q(q-1)$	tNT_1	$q(q-1)^2$
R_1	$q^3 - q^2$	tNT_5	$q(q-1)$
R_3	$q^2(q-1)(q-2)$		

A new type tNT_5 appears with centralizer $\left\{ \begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ & a_0 & & b_0 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix} \mid a_0 \neq 0 \right\}$.

Proof. The canonical form of a matrix of this type is $A = \begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$. Then

$Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ & a_0 & & b_0 \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix} \mid a_0, c_0 \neq 0 \right\}$. Let $B = \begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ & a_0 & & b_0 \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix}$, and

$B' = \begin{pmatrix} a_0 & a_1 & b'_0 & b'_1 \\ & a_0 & & b'_0 \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix} = XBX^{-1}$, where $X = \begin{pmatrix} x_0 & x_1 & y_0 & y_1 \\ & x_0 & & y_0 \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix}$. $XB = B'X$ gives us the following:

$$x_0b_0 + y_0c_0 = z_0b'_0 + y_0a_0 \tag{4.17}$$

$$x_0b_1 + x_1b_0 + y_0c_1 + y_1c_0 = y_1a_0 + y_0a_1 + z_1b'_0 + z_0b'_1 \tag{4.18}$$

When $a_0 = c_0$: From Equation 4.17, we have $x_0b_0 = z_0b'_0$.

When $b_0 = 0$: Equation 4.18 becomes $x_0b_1 + y_0c_1 = z_0b'_1 + y_0a_1$. Here we first look at what happens when $a_1 = c_1$, and $b_1 = 0$. Here B reduces to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix}$, $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. (A, B) is of type A_4, and there are $q(q-1)$ such branches.

When $a_1 = c_1$, and $b_1 \neq 0$. We can choose x_0 such that $b'_1 = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & a_1 & & 1 \\ & a_0 & & \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & y_0 & y_1 \\ & x_0 & & y_0 \\ & & x_0 & z_1 \\ & & & x_0 \end{pmatrix} \right\}$. We see a centralizer, not isomorphic to the ones seen so far. Thus, we have a new type tNT_5. (A, B) is of type tNT_5, and there are $q(q-1)$ such branches.

When $a_1 \neq c_1$, in Equation 4.18, we can choose y_0, so that $b'_1 = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & & y_1 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix} \right\}$ (A, B) is of type tNT_1, and there are $q(q-1)^2$ such branches.

When $b_0 \neq 0$: In Equation 4.17, choose x_0 such that $b'_0 = 1$. Then, on replacing b_0 and b'_0 by 1 in the same equation, we have $x_0 = z_0$. Hence, Equation 4.18 becomes $x_0b_1 + x_1 + y_0c_1 = x_0b'_1 + z_1 + y_0a_1$. Hence, choose z_1 so that $b'_1 = 0$. Then, B is reduced to $\begin{pmatrix} a_0 & a_1 & 1 & \\ & a_0 & & 1 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & & y_1 \\ & x_0 & & \\ & & x_0 & \substack{x_1+\\ y_0(c_1-a_1)} \\ & & & x_0 \end{pmatrix} \right\}$ (A,B) is of type R_1, and there are $q^2(q-1)$ such branches.

When $a_0 \neq c_0$: In Equation 4.17, choose y_0 so that $b_0' = 0$. With this, Equation 4.18 becomes $x_0b_1 + y_1c_0 = z_0b_1' + y_1a_0$. Choose y_1 such that $b_1' = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} x_0 & x_1 & & \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix} \right\}$. (A,B) is of type R_3, and there are $q^2(q-1)(q-2) = q^4 - q^3$ such branches. Thus, there are no more cases left to deal with.

$\square$

Proposition 4.7. *An upper triangular matrix of type A_5 has $q^2(q-1)$ branches of type A_5, $q^2(q-1)(q-2)$ branches of type R_3, and $q^2(q^2-1)$ branches of type NR_1.*

Proof. A matrix of type A_5 has the canonical form: $A = \begin{pmatrix} a & & & 1 \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$. Thus its centralizer $Z_{GT_4(\mathbf{F}_q)}(A)$ is: $\left\{ \begin{pmatrix} a_0 & & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & \\ & & & a_0 \end{pmatrix} \right\}$. Let $B = \begin{pmatrix} a_0 & & a_2 & a_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & & a_2' & a_3' \\ & b_0' & b_1' & b_2 \\ & & b_0 & \\ & & & a_0 \end{pmatrix} = XBX^{-1}$, where $X = \begin{pmatrix} x_0 & & x_2 & x_3 \\ & y_0 & y_1 & y_2 \\ & & y_0 & \\ & & & x_0 \end{pmatrix}$. Thus, from $XB = B'X$, we have $a_3' = a_3$, $b_1' = b_1$, and the following equations:

$$x_0a_2 + x_2b_0 = y_0a_2' + x_2a_0 \tag{4.19}$$

$$y_0b_2 + a_0y_2 = x_0b_2' + y_2b_0 \tag{4.20}$$

Case $a_0 = b_0$. Equations 4.19 and 4.20 become $x_0a_2 = y_0a_2'$, and $y_0b_2 = x_0b_2'$ respectively.

If $a_2 = b_2 = 0$, the above equations are void, and we have B reduced to $\begin{pmatrix} a_0 & & & a_3 \\ & a_0 & b_1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. Thus (A, B) is a branch of type A_5, and there are $q^2(q-1)$ such branches.

If $a_2 \neq 0$, then choose x_0 so that $a_2' = 1$. Substituting a_2 with $a_2' = 1$ in the equation $x_0 a_2 = y_0 a_2'$, we get $x_0 = y_0$, thus leaving us with $b_2' = b_2$. Hence B is reduced to $\begin{pmatrix} a_0 & & 1 & a_3 \\ & a_0 & b_1 & b_2 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. Thus (A, B) is a branch of type NR_1, and there are $q^3(q-1)$ such branches.

If $a_2 = 0$ and $b_2 \neq 0$, then we choose y_0 such that $b_2' = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & & & a_3 \\ & a_0 & b_1 & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & x_2 & x_3 \\ & x_0 & y_1 & y_2 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This branch too is of type NR_1, and there are $q^2(q-1)$ such branches.

If $a_0 \neq b_0$. Then, in Equation 4.19, choose x_2 such that $a_2' = 0$. Similarly in Equation 4.20, choose y_2 such that $b_2' = 0$. Thus B boils down to $\begin{pmatrix} a_0 & & & a_3 \\ & b_0 & b_1 & \\ & & b_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & & & x_3 \\ & y_0 & y_1 & \\ & & y_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type R_3, and there are $q2^2(q-1)(q-2)$ such branches.

Adding up the branches of type NR_1, we have a total of $q^3(q-1)+q^2(q-1) = q^2(q^2-1)$ branches of type NR_1. $\square$

Proposition 4.8. *For a matrix of type A_6, the branchings are:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_6	$q(q-1)$	R_3	$q^2(q-1)(q-2)$
A_5	$q(q-1)^2$	tNT_4	$q^2(q-1)$
R_1	$q^2(q-1)$	NR_1	$q^2(q-1)$.

Proof. A matrix of type A_6 has the canonical form $\begin{pmatrix} a & & 1 & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$. The centralizer subgroup $Z_{GT_4(\mathbf{F}_q)}(A)$ is $\left\{ \begin{pmatrix} C & D \\ & C \end{pmatrix} \mid C \in GT_2(\mathbf{F}_q) \right\}$, where $D = \begin{pmatrix} d_0 & d_1 \\ d_2 & d_3 \end{pmatrix}$, and $W = \begin{pmatrix} w_0 & w_1 \\ w_2 & w_3 \end{pmatrix}$.

Let $B = \begin{pmatrix} C & D \\ & C \end{pmatrix}$, and $B' = \begin{pmatrix} C' & D' \\ & C' \end{pmatrix} = XBX^{-1}$, where $X = \begin{pmatrix} Z & W \\ & Z \end{pmatrix}$. So $XB = B'X$ leads to $ZC = C'Z$. Hence, we can take C to be a representative of a conjugacy class in $GT_2(\mathbf{F}_q)$, and $Z = Z_{GT_2(\mathbf{F}_q)}(C)$. We have the following equation

$$ZD + WC = CW + D'Z \tag{4.21}$$

So the cases to deal with here are the three conjugacy class types in $GT_2(\mathbf{F}_q)$.

Case $C = \left(\begin{smallmatrix} a_0 & 1 \\ & a_0 \end{smallmatrix}\right)$**:** here $Z = \begin{pmatrix} x_0 & x_1 \\ & x_0 \end{pmatrix}$, and Equation 4.21 becomes:

$$\begin{pmatrix} x_0d_0 + x_1d_2 & x_0d_1 + x_1d_3 + w_0 \\ x_0d_2 & x_0d_3 + w_2 \end{pmatrix} = \begin{pmatrix} w_2 + x_0d_0' & w_3 + x_1d_0' + x_0d_1 \\ x_0d_2' & x_1d_2' + x_0d_3 \end{pmatrix}$$

Choose w_2 so that $d_0' = 0$. Thus, on replacing d_0 by 0, we get $w_2 = x_1d_2$, and hence $d_3' = d_3$.

We can choose w_0 such that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & d_2 & d_3 \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and

$Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & w_0 & w_1 \\ & x_0 & x_1d_2 & w_0 + x_1d_3 \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type R_1, and there are $q^2(q-1)$ such branches.

Case $C = \left(\begin{smallmatrix} a_0 & \\ & b_0 \end{smallmatrix}\right), a_0 \neq b_0$**:** here $Z = \begin{pmatrix} x_0 & \\ & x_3 \end{pmatrix}$, and Equation 4.21 becomes:

$$\begin{pmatrix} x_0d_0 + a_0w_0 & x_0d_1 + w_1b_0 \\ x_3d_2 + a_0w_2 & x_3d_3 + b_0w_3 \end{pmatrix} = \begin{pmatrix} a_0w_0 + x_0d_0' & w_1a_0 + x_3d_1 \\ b_0w_2 + x_0d_2 & b_0w_3 + x_3d_3 \end{pmatrix}.$$

We have $d_0' = d_0$ and $d_3' = d_3$. As $a_0 \neq b_0$, choose w_2 such that $d_2' = 0$, and w_1 so that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & & d_0 & \\ & b_0 & & d_3 \\ & & a_0 & \\ & & & d_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & & w_0 & \\ & x_3 & & w_3 \\ & & x_0 & \\ & & & x_3 \end{pmatrix}\right\}$. This (A,B) is of type R_3, and there are $q^2(q-1)(q-2)$ such branches.

Case $C = a_0 I_2$**:** Here Equation 4.21 becomes: $ZD = D'Z$, where $Z \in T_2(\mathbf{F}_q)$. With $Z = \begin{pmatrix} x_0 & x_1 \\ & x_2 \end{pmatrix}$, we see that:

$$\begin{pmatrix} x_0 d_0 + x_1 d_2 & x_0 d_1 + x_1 d_3 \\ x_3 d_2 & x_3 d_3 \end{pmatrix} = \begin{pmatrix} x_0 d_0' & x_1 d_0' + x_3 d_1' \\ x_0 d_2' & x_1 d_2' + x_3 d_3' \end{pmatrix}. \tag{4.22}$$

We see that $x_0 d_2' = x_3 d_2$. We have two main cases here:

Case $d_2 = 0$**.** In this case, from Equation 4.22 we have $d_0' = d_0$, and $d_3' = d_3$, and we have $x_0 d_1 + (d_3 - d_0)x_1 = x_3 d_1'$.

When $d_0 = d_3$, we have $x_0 d_1 = x_3' d_1$. Now, if $d_1 = 0$. we have $B = \begin{pmatrix} a_0 I_2 & d_0 I_2 \\ & a_0 I_2 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = Z_{GT_4(\mathbf{F}_q)}(A)$. Thus, (A,B) is of type A_6, and there are $q(q-1)$ such branches.

If $d_1 \neq 0$, choose x_0 so that $d_1' = 1$. Thus B is reduced to $\begin{pmatrix} a_0 & & d_0 & 1 \\ & a_0 & & d_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and

$$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & x_1 & w_1 & w_2 \\ & x_0 & w_2 & w_3 \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix}\right\}.$$

(A,B) is therefore of type tNT_4, and there are $q^2(q-1)$ such branches.

When $d_0 \neq d_3$, in the (1,2)th entry of Equation 4.22, we choose x_1 so that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & & d_0 & \\ & a_0 & & d_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{\begin{pmatrix} x_0 & & w_1 & w_2 \\ & x_3 & w_2 & w_3 \\ & & x_0 & \\ & & & x_3 \end{pmatrix}\right\}$.

This is isomorphic to the centralizer of a matrix of type A_5. Thus (A, B) is a branch of type A_5, and there are $q^2(q-1)$ such branches.

Case $d_2 \neq 0$. First, we choose x_0 such that $d_2' = 1$. On replacing d_2 with $d_2' = 1$ in Equation 4.22, and equating, we get $x_0 = x_3$.

In the same equation, we can choose x_1 such that $d_0' = 0$. On replacing d_0 with $d_0' = 0$ and equating, we get $x_1 = 0$. Thus, $d_3' = d_3$. Lastly, we have $x_0 d_1 = x_0 d_1'$, hence $d_1' = d_1$.

Thus B is reduced to $\begin{pmatrix} a_0 & & & d_1 \\ & a_0 & 1 & d_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 I_2 & W \\ & x_0 I_2 \end{pmatrix} \mid W \in M_2(\mathbf{F}_q) \right\}$. (A, B) is a branch of type NR_1, and there are $q^2(q-1)$ such branches.

There are no other cases. $\square$

Proposition 4.9. *The branching rules of remaining A types are as follows.*

(1) For a matrix of type A_7, there are $q^2(q-1)$ branches of type A_7, $q^2(q-1)$ branches of type R_1, and $q^2(q-1)(q-2)$ branches of type R_2.
(2) The type A_8 has $q^2(q-1)$ branches of type A_8, $q^3 - q$ branches of type R_1, and $q^2(q-1)(q-2)$ branches of type R_2.
(3) The type A_9 has $q^2(q-1)$ branches of type A_9, $(q^2 - q)(q^2 - 1)$ branches of type R_1.

Proof. (1) A matrix of type A_7 has two non-similar canonical forms, $\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$, and $\begin{pmatrix} a & & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{pmatrix}$. As their centralizer subgroups in $T_4(\mathbf{F}_q)$ are conjugate in $GL_4(\mathbf{F}_q)$, we may prove the branching for any one. Let $A = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$.

Then $Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & a_1 & \\ & & a_0 & \\ & & & d_0 \end{pmatrix} \mid a_0, d_0 \neq 0 \right\}$.

Let $B = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & a_1 & \\ & & a_0 & \\ & & & d_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & a_1' & a_2' & a_3' \\ & a_0 & a_1' & \\ & & a_0 & \\ & & & d_0 \end{pmatrix} = XBX^{-1}$, where $X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & x_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix}$. From $XB = B'X$ we have $a_1' = a_1$, $a_2' = a_2$, and this equation:

$$x_0a_3 + x_3d_0 = z_0a_3' + x_3a_0 \tag{4.23}$$

.

If $a_0 = d_0$, then Equation 4.23 becomes $x_0a_3 = z_0z_3'$.

Here, if $a_3 = 0$, then B is reduced to $\begin{pmatrix} a_0 & a_1 & a_2 & \\ & a_0 & a_1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. Thus (A, B) is of type A_7, and there are $q^2(q-1)$ such branches,.

If $a_3 \neq 0$, then choose z_0 so that $a_3' = 1$. Thus, B is reduced to $\begin{pmatrix} a_0 & a_1 & a_2 & 1 \\ & a_0 & a_1 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & x_1 & \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) is of type R_1, and there are $q^2(q-1)$ such branches.

When $a_0 \neq d_0$, then, in Equation 4.23, choose x_3 so that $a_3' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & a_1 & a_2 & \\ & a_0 & a_1 & \\ & & a_0 & \\ & & & d_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \\ & x_0 & x_1 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. This too is commutative (by a routine check). (A, B) is of type R, and there are $q^2(q-1)(q-2)$ such branches. There are no other cases left to analyze, so these are all the branches.

(2) Matrices of type A_8 have either of the two non-similar canonical forms: $\begin{pmatrix} a & 1 & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$,

and $\begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$. As their centralizers are conjugate in $GL_4(\mathbf{F}_q)$, it is enough

to prove for any one of the canonical forms. Let $A = \begin{pmatrix} a & 1 & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$. Then the

centralizer of A is $Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & a_1 & b & a_2 \\ & a_0 & & a_1 \\ & & d & c \\ & & & a_0 \end{pmatrix} \right\}$. let $B \in Z_{GT_4(\mathbf{F}_q)}(A)$

be the matrix $\begin{pmatrix} a_0 & a_1 & b & a_2 \\ & a_0 & & a_1 \\ & & d & c \\ & & & a_0 \end{pmatrix}$, and let $B' = \begin{pmatrix} a_0 & a_1' & b' & a_2' \\ & a_0 & & a_1' \\ & & d & c' \\ & & & a_0 \end{pmatrix} = XBX^{-1}$,

where $X = \begin{pmatrix} x_0 & x_1 & y & x_2 \\ & x_0 & & x_1 \\ & & z & w \\ & & & x_0 \end{pmatrix}$. Now $XB = XB'X$ leads us to $a_1' = a_1$, and the

following equations:

$$x_0 b + yd = zb' + ya_0 \tag{4.24}$$

$$zc + wa_0 = x_0 c' + wd \tag{4.25}$$

$$x_0 a_2 + yc = wb' + x_0 a_2' \tag{4.26}$$

When $a_0 = d$: Here, Equations 4.24 and 4.25 become $x_0 b = zb'$, and $zc = x_0 c'$ respectively.

When $b = c = 0$, Equation 4.26 becomes $x_0 a_2 = x_0 a_2'$, hence $a_2' = a_2$. B is

reduced to $\begin{pmatrix} a_0 & a_1 & & a_2 \\ & a_0 & & a_1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. (A, B) is of

type A_8, and there are $q^2(q-1)$ such branches.

When $b \neq 0$, choose z such that $b' = 1$. Then, on substituting b with $b' = 1$ in Equation 4.24, we get $z = x_0$. Thus, we have $c = c'$. And, in Equation 4.26, choose w so that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} a_0 & a_1 & 1 & \\ & a_0 & & a_1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & y & x_2 \\ & x_0 & & x_1 \\ & & x_0 & cy \\ & & & x_0 \end{pmatrix} \right\}$. (A, B) is of type R_1, and there are $q^2(q-1)$ such branches.

When $b = 0$ and $c \neq 0$, in Equation 4.25, choose x_0 such that $c' = 1$. Then Equation 4.26 becomes $x_0 a_2 + y = x_0 a_2'$. Thus, choose y so that $a_2' = 0$. Hence B is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & a_1 \\ & & a_0 & 1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_2 \\ & x_0 & & x_1 \\ & & x_0 & w \\ & & & x_0 \end{pmatrix} \right\}$. (A, B) is of type R_1, and there are $q(q-1)$ such branches.

There are no further cases for us to look at here. We now look at the case of $a_0 \neq d$.

When $a_0 \neq d$: In Equation 4.24, choose y such that $b' = 0$, and in Equation 4.25, choose w such that $c' = 0$. Then Equation 4.26 becomes $x_0 a_2 = a_0 a_2'$, implying $a_2' = a_2$. B reduces to $\begin{pmatrix} a_0 & a_1 & & a_2 \\ & a_0 & & a_1 \\ & & d & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_2 \\ & x_0 & & x_1 \\ & & z & \\ & & & x_0 \end{pmatrix} \right\}$. This too is a commutative centralizer. (A, B) is of type R_2, and there are $q^2(q-1)(q-2)$ such branches. Now, there are no more cases to look at. Adding up all the branches of type R_1, we have a total of $q^2(q-1) + q(q-1) = q^3 - q$ branches of type R_1.

(3) A matrix of type A_9 has the following canonical form: $A = \begin{pmatrix} a & 1 & 1 & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$. Then we have $Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} a_0 & a_1 & b & a_2 \\ & a_0 & & c \\ & & a_0 & b-c \\ & & & a_0 \end{pmatrix} \right\}$. Let $B = \begin{pmatrix} a_0 & a_1 & b & a_2 \\ & a_0 & & c \\ & & a_0 & b-c \\ & & & a_0 \end{pmatrix}$, and $B' = \begin{pmatrix} a_0 & a_1' & b'' & a_2 \\ & a_0 & & c' \\ & & a_0 & b'-c' \\ & & & a_0 \end{pmatrix} = XBX^{-1}$, where $X = \begin{pmatrix} x_0 & x_1 & y & x_2 \\ & x_0 & & w \\ & & x_0 & y-w \\ & & & x_0 \end{pmatrix}$, with $x_0 \neq 0$. So, $XB = B'X$ leaves us with $a_1' = a_1$, $b' = b$, and $c' = c$, and the following equation:

$$x_0a_2 + (x_1 - x_2)c = x_0a_2' + (a_1 - b)w \tag{4.27}$$

When $a_1 = b$ **and** $c = 0$ Here Equation 4.27 ends up as $a_2' = a_2$. B is thus reduced to $\begin{pmatrix} a_0 & a_1 & a_1 & a_2 \\ & a_0 & & \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B) = Z_{GT_4(\mathbf{F}_q)}(A)$. Thus (A, B) is of type A_9, and there are $q^2(q-1)$ such branches.

When $a_1 \neq b$: Here, in Equation 4.27, we choose w such that $a_2' = 0$. B is thus reduced to $\begin{pmatrix} a_0 & a_1 & b & \\ & a_0 & & c \\ & & a_0 & b-c \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & y & x_2 \\ & x_0 & & \frac{(x_1-y)}{a_1-b}c \\ & & x_0 & y - \frac{(x_1-y)}{a_1-b}c \\ & & & x_0 \end{pmatrix} \right\}$. (A, B) is therefore of type R_1, and there are $q^2(q-1)^2$ such branches.

When $a_1 = b$**, and** $c \neq 0$**:** In Equation 4.27, choose x_1 or y such that $a_2' = 0$. Thus, B is reduced to $\begin{pmatrix} a_0 & a_1 & a_1 & \\ & a_0 & & c \\ & & a_0 & a_1 - c \\ & & & a_0 \end{pmatrix}$. So $Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & x_1 & x_2 \\ & x_0 & & w \\ & & x_0 & x_1 - w \\ & & & x_0 \end{pmatrix} \right\}$. This (A, B) too is of type R_1, and there are $q(q-1)^2$ such branches.

With this, we have no other cases to look at. Thus, we have q^3 branches of type A_9, and $q(q-1)^2 + q^2(q-1)^2 = (q^2 - q)(q^2 - 1)$ branches of type R_1.

□

4.2. **Branching rules for type** B**.** Matrices of types $B1, B2, B3, B4, B5$ are in block form of the kind $A = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix}$, where $C_1 \in GT_{m_1}(\mathbf{F}_q)$, and $C_2 \in GT_{m_2}(\mathbf{F}_q)$, where $m_1 + m_2 = 4$. Thus, $Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} \right\}$ where $X_1 \in Z_{GT_{m_1}}(C_1)$ and $X_2 \in Z_{GT_{m_2}}(C_2)$. Thus, the branches of A are of the form $\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}$, where D_1 is a branch of C_1, and D_2 is a branch of C_2. With this argument, we can prove the following proposition.

Proposition 4.10. *The branching rules are as follows:*

(1) *For a matrix of type B_1, there are:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
B_1	$(q-1)^2$	R_3	$(q-1)^2$
B_5	$2(q-1)^2$	R_4	$2(q-1)^2(q-2)$
B_6	$2(q-1)^2(q-2)$	R_5	$(q-1)^2(q-2)^2$

(2) *For a matrix of type B_2, there are:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
B_2	$(q-1)^2$	R_2	$(q-1)^2$
B_3	$(q-1)^2$	R_4	$(q-1)^2(q-2)$
B_4	$(q-1)^2$	R_5	$(q-1)^2(q-2)(q-3)$
B_6	$(q-1)^2(q-2)$		

(3) *For a matrix of type B_3, there are $q(q-1)^2$ branches of type B_3, $q(q-1)^2$ branches of type R_2, and $q(q-1)^2(q-2)$ branches of type R_4.*

(4) *For a matrix of type B_4, there are, $q(q-1)^2$ branches of type B_4, $(q^2-1)(q-1)$ branches of type R_2, and $q(q-1)^2(q-2)$ branches of type R_4.*

(5) *For a matrix of type B_5, there are $q(q-1)^2$ branches of type B_5, $q(q-1)^2$ branches of type R_3, and $q(q-1)^2(q-2)$ branches of type R_4.*

Finally,

Proposition 4.11. *For a matrix of type B_6, there are, $(q-1)^3$ branches of type B_6, $(q-1)^3$ branches of type R_4, and $(q-1)^3(q-2)$ branches of type R_5.*

Proof. A matrix of type B_6 has the canonical form: $A = \begin{pmatrix} a & & & \\ & a & & \\ & & b & \\ & & & c \end{pmatrix}$. Here, $Z_{GT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} C & & \\ & c_0 & \\ & & d_0 \end{pmatrix} \mid C \in GT_2(\mathbf{F}_q), c_0, d_0 \neq 0 \right\}$. Enumerating the conjugacy classes of $GT_2(\mathbf{F}_q)$ gives us the branches mentioned. $\square$

4.3. **Branching Rules of the New Types.** While determining the branching rules of the existing types of conjugacy classes of $GT_4(\mathbf{F}_q)$, we came across six new types of simultaneous conjugacy classes of pairs of commuting matrices. We called them tNT_1, tNT_2, tNT_3, tNT_4,tNT_5, and NR_1. In this subsection, we shall focus on the branching rules of these new types.

Proposition 4.12. *A commuting tuple of type tNT_1 has $q^2(q-1)$ branches of type tNT_1, $q^2(q-1)$ branches of type R_1, and $q^2(q-1)(q-2)$ branches of type R_3.*

Proof. For a commuting pair (A,B) of matrices of type tNT_1, the centralizer is $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} a_0 & a_1 & & a_3 \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix} \mid a_0, c_0 \neq 0 \right\}$. Let $C = \begin{pmatrix} a_0 & a_1 & & a_3 \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix}$, and $C' = \begin{pmatrix} a_0 & a'_1 & & a'_3 \\ & a_0 & & \\ & & c_0 & c'_1 \\ & & & c_0 \end{pmatrix} =$ XCX^{-1} by $X = \begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix}$. $XC = C'X$ leads us to $a'_1 = a_1$, $c'_1 = c_1$, and just one equation:

$$x_0a_3 + x_3b_0 = z_0a'_3 + x_3a_0. \tag{4.28}$$

When $a_0 = c_0$: Here Equation 4.28 becomes $x_0a_3 = z_0a'_3$.

So, we have two cases over here: $a_3 = 0$, and $a_3 \neq 0$.

When $a_3 = 0$, C is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = Z_{GT_4(\mathbf{F}_q)}(A,B)$. (A,B,C) is of type tNT_1, and there are $q^2(q-1)$ such branches.

When $a_3 \neq 0$, we choose z_0 such that $a'_3 = 1$. Here, C is reduced to $\begin{pmatrix} a_0 & a_1 & & 1 \\ & a_0 & & \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & x_1 & & x_3 \\ & x_0 & & \\ & & x_0 & z_1 \\ & & & x_0 \end{pmatrix} \right\}$. This (A,B,C) is of type R_1, and there are $q^2(q-1)$ such branches.

So now, with $a_0 = c_0$, we have no other cases left to analyse. We move on to the case of $a_0 \neq c_0$.

When $a_0 \neq c_0$: Here, in Equation 4.28, we can choose x_3 so that $a_3' = 0$. So C is reduced to $\begin{pmatrix} a_0 & a_1 & & \\ & a_0 & & \\ & & c_0 & c_1 \\ & & & c_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & x_1 & & \\ & x_0 & & \\ & & z_0 & z_1 \\ & & & z_0 \end{pmatrix} \right\}$. This (A,B,C) is of type R_3, and there are $q^2(q-1)(q-2)$ such branches.

So, with this, we have no other cases to look at. $\square$

Proposition 4.13. *The new type tNT_2 has $q^2(q-1)$ branches of type tNT_2, $q^2(q-1)(q-2)$ branches of type R_2, and $q^2(q^2-1)$ branches of type NR_1.*

Proof. For a commuting pair (A,B) of type tNT_2, the centralizer is

$$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} a_0 & & b_0 & b_1 \\ & a_0 & b_2 & b_3 \\ & & a_0 & \\ & & & c_0 \end{pmatrix} \mid \begin{matrix} a_0, b_0, b_1 \\ b_2, b_3, c_0 \in \mathbf{F}_q \end{matrix} \right\}.$$ Let $C = \begin{pmatrix} a_0 & & b_0 & b_1 \\ & a_0 & b_2 & b_3 \\ & & a_0 & \\ & & & c_0 \end{pmatrix}$, and $C' = \begin{pmatrix} a_0 & & b_0' & b_1' \\ & a_0 & b_2' & b_3' \\ & & a_0 & \\ & & & c_0 \end{pmatrix} = XCX^{-1}$ for some $X = \begin{pmatrix} x_0 & & y_0 & y_1 \\ & x_0 & y_2 & y_3 \\ & & x_0 & \\ & & & z_0 \end{pmatrix}$. So, equating $XC = C'X$ leads us to $b_0' = b_0$, $b_2' = b_2$, and the following equations:

$$x_0 b_1 + y_1 c_0 = z_0 b_1' + y_1 a_0 \tag{4.29}$$

$$x_0 b_3 + y_3 c_0 = z_0 b_3' + y_3 a_0 \tag{4.30}$$

We have two main cases: $a_0 = c_0$, and $a_0 \neq c_0$:

When $a_0 = c_0$: Here, Equation 4.29 becomes $x_0 b_1 = z_0 b_1'$, and Equation 4.30 becomes $x_0 b_3 = z_0 b_3'$.

When $b_1 = b_3 = 0$, C is reduced to $\begin{pmatrix} a_0 & & b_0 & \\ & a_0 & b_2 & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = Z_{GT_4(\mathbf{F}_q)}(A,B)$. Thus (A,B,C) is of type tNT_2, and there are $q^2(q-1)$ such branches.

When $b_1 \neq 0$. In Equation 4.29, choose z_0 such that $b_1' = 1$. Then, on replacing b_1 and b_1' by 1 in the same equation, we get $z_0 = x_0$. Hence, Equation 4.30 becomes $x_0 b_3 =$

x_0b_3', hence $b_3' = b_3$. C is reduced to $\begin{pmatrix} a_0 & & b_0 & 1 \\ & a_0 & b_2 & b_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & & y_0 & y_1 \\ & x_0 & y_2 & y_3 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A,B,C) is of type NR_1. There are $q^3(q-1)$ such branches.

When $b_1 = 0$, and $b_3 \neq 0$. In Equation 4.30, choose z_0 so that $b_3' = 1$. Thus C is reduced to $\begin{pmatrix} a_0 & & b_0 & \\ & a_0 & b_2 & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & & y_0 & y_1 \\ & x_0 & y_2 & y_3 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A,B,C) is of type NR_1. There are $q^2(q-1)$ such branches. We have exhausted all the cases under $a_0 = c_0$.

When $a_0 \neq c_0$: Here, in Equation 4.29, choose y_1 so that $b_1' = 0$, and in Equation 4.30, choose y_3 so that $b_3' = 0$. C is thus reduced to $\begin{pmatrix} a_0 & & b_0 & \\ & a_0 & b_2 & \\ & & a_0 & \\ & & & b_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & & y_0 & \\ & x_0 & y_2 & \\ & & x_0 & \\ & & & z_0 \end{pmatrix} \right\}$. This (A,B,C) is of type R_2, and there are $q^2(q-1)(q-2)$ such branches.

This leaves us with no further cases to analyse. Adding up the branches of type NR_1, we have a total of $q^2(q-1) + q^3(q-1) = q^2(q^2-1)$ branches of type NR_1. $\square$

Proposition 4.14. *A commuting pair of type tNT_3 has $q^2(q-1)$ branches of type tNT_3, $q^2(q-1)$ branches of type R_1, $q^2(q-1)(q-2)$ branches of type R_2, and $q(q^2-1)$ branches of type NR_1.*

Proof. Let (A,B) be a pair of commuting matrices of type tNT_3. Their common centralizer is $Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} D & E \\ & D_{11}I_2 \end{pmatrix} \mid D \in T_2(\mathbf{F}_q), E \in M_2(\mathbf{F}_q) \right\}$. Let $C = \begin{pmatrix} D & E \\ & a_0I_2 \end{pmatrix}$, where $D = \begin{pmatrix} a_0 & a_1 \\ & b_0 \end{pmatrix}$ and $E = \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix}$. Let $C' = \begin{pmatrix} D' & E' \\ & a_0I_2 \end{pmatrix} = XCX^{-1}$, where $X = \begin{pmatrix} Z & Y \\ & x_0I_2 \end{pmatrix} \in Z_{GT_4(\mathbf{F}_q)}(A,B)$, where $Z = \begin{pmatrix} x_0 & x_1 \\ & z_0 \end{pmatrix} \in GT_2(\mathbf{F}_q)$, and $Y = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}$.

So $XC = C'X$ leaves us with the following $ZD = D'Z$. Thus D can be taken to be a representative of a conjugacy class in $GT_2(\mathbf{F}_q)$, and $Z \in Z_{GT_2(\mathbf{F}_q)}(D)$. We are therefore left with the following equation:

$$ZE + a_0 Y = DY + x_0 E'$$

Exapanding this, we have:

$$\begin{pmatrix} x_0b_0 + x_1b_2 & x_0b_1 + x_1b_3 \\ z_0b_2 & z_0b_3 \end{pmatrix} + \begin{pmatrix} -a_1y_2 & -a_1y_3 \\ (a_0 - b_0)y_2 & (a_0 - b_0)y_3 \end{pmatrix} = \begin{pmatrix} x_0b'_0 & x_0b'_1 \\ x_0b'_2 & x_0b'_3 \end{pmatrix} \tag{4.31}$$

When $D = a_0I_2$: Here Equation 4.31 becomes:

$$\begin{pmatrix} x_0b_0 + x_1b_2 & x_0b_1 + x_1b_2 \\ z_0b_2 & z_0b_3 \end{pmatrix} + = \begin{pmatrix} x_0b'_0 & x_0b'_1 \\ x_0b'_2 & x_0b'_3 \end{pmatrix}$$

When $b_2 = b_3 = 0$, we have $b'_0 = b_0$, and $b'_1 = b_1$. Thus, C is reduced to $\begin{pmatrix} a_0 & & b_0 & b_1 \\ & a_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $Z_{GT_4(\mathbf{F}_q)}(A, B, C) = Z_{GT_4(\mathbf{F}_q)}(A, B)$. (A, B, C) is of type tNT_3, and there are $q^2(q-1)$ such branches.

When $b_2 \neq 0$, choose z_0 such that $b'_2 = 1$. Thus, on replacing b_0 by $b'_0 = 1$ in Equation 4.31, we get $z_0 = x_0$. Hence $b'_3 = b_3$. With these, Eqaution 4.31 becomes

$$\begin{pmatrix} x_0b_0 + x_1 & x_0b_1 + x_1b_2 \\ 1 & b_3 \end{pmatrix} + = \begin{pmatrix} x_0b'_0 & x_0b'_1 \\ 1 & b'_3 \end{pmatrix}$$

Choose x_1 so that $b'_0 = 0$. On replacing b_0 by $b'_0 = 0$ in the above equation, we have $x_1 = 0$. Thus $b'_1 = b_1$. So C is reduced to $\begin{pmatrix} a_0 & & & b_1 \\ & a_0 & 1 & b_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$ with $Z_{GT_4(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} x_0 & & y_0 & y_1 \\ & x_0 & y_2 & y_3 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. (A, B, C) is of type NR_1, and there are $q^2(q-1)$ such branches.

When $b_2 = 0$ and $b_3 \neq 0$. Choose z_0 so that $b'_3 = 1$. Equation 4.31 becomes

$$\begin{pmatrix} x_0b_0 & x_0b_1 + x_1 \\ 0 & 1 \end{pmatrix} + = \begin{pmatrix} x_0b'_0 & x_0b'_1 \\ 0 & 1 \end{pmatrix}$$

Hence, $b_0' = b_0$, and choose x_1 so that $b_1' = 0$. C is reduced to $\begin{pmatrix} a_0 & & b_0 & \\ & a_0 & & 1 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$,

with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & & y_0 & y_1 \\ & x_0 & y_2 & y_3 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A,B,C) too is of type NR_1, and there are $q(q-1)$ such branches.

With this, we have no other cases to analyse when $D = a_0 I_2$.

When $D = \begin{pmatrix} a_0 & 1 \\ & a_0 \end{pmatrix}$: Here $Z = \begin{pmatrix} x_0 & x_1 \\ & x_0 \end{pmatrix}$. Equation 4.31 becomes:

$$\begin{pmatrix} x_0b_0 + x_1b_2 & x_0b_1 + x_1b_3 \\ x_0b_2 & x_0b_3 \end{pmatrix} + \begin{pmatrix} -y_2 & -y_3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_0b_0' & x_0b_1' \\ x_0b_2' & x_0b_3' \end{pmatrix}$$

We have from this $b_2' = b_2$, $b_3' = b_3$, and we can choose y_2 so that $b_0' = 0$ and y_3 such that $b_1' = 0$. Hence C is reduced to $\begin{pmatrix} a_0 & 1 & & \\ & a_0 & b_2 & b_3 \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ & x_0 & b_2x_1 & b_3x_1 \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. This (A,B,C) is of type R_1, and there are $q^2(q-1)$ such branches.

When $C = \begin{pmatrix} a_0 & \\ & c_0 \end{pmatrix}$**,** $c_0 \neq a_0$: Here $Z = \begin{pmatrix} x_0 & \\ & z_0 \end{pmatrix}$. Equation 4.31 becomes:

$$\begin{pmatrix} x_0b_0 & x_0b_1 \\ z_0b_2 & z_0b_3 \end{pmatrix} + \begin{pmatrix} & \\ (a_0-c_0)y_2 & (a_0-c_0)y_3 \end{pmatrix} = \begin{pmatrix} x_0b_0' & x_0b_1' \\ x_0b_2' & x_0b_3' \end{pmatrix}$$

We have $b_0' = b_0$ and $b_1' = b_1$. Choose y_2 and y_3 such that $b_2' = b_3' = 0$. C is reduced to $\begin{pmatrix} a_0 & & b_0 & b_1 \\ & c_0 & & \\ & & a_0 & \\ & & & a_0 \end{pmatrix}$, and $\left\{ \begin{pmatrix} x_0 & & y_0 & y_1 \\ & z_0 & & \\ & & x_0 & \\ & & & x_0 \end{pmatrix} \right\}$. Here (A,B,C) is of type R_2, and there are $q^2(q-1)(q-2)$ such branches.

With this, we have no other cases to deal with.

Adding up the branches of type NR_1, we have a total of $q(q-1)+q^2(q-1) = q(q^2-1)$ branches of this type. $\square$

Proposition 4.15. *For a pair of commuting matrices of type tNT_4, there are $q^2(q-1)$ branches of type tNT_4, $q^2(q-1)^2$ branches of type R_1, and $q(q^2-1)(q-1)$ branches of type NR_1.*

Proof. The centralizer of a commuting pair (A,B) of this type is

$$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} \begin{smallmatrix} a_0 & a_1 \\ & a_0 \end{smallmatrix} & B_1 \\ & \begin{smallmatrix} a_0 & a_1 \\ & a_0 \end{smallmatrix} \end{pmatrix} \mid a_0 \neq 0, B_1 \in M_2(Fq) \right\}.$$

This was seen, and proved in [Sh1, , Lemma 5.14] as the new type NT_1. $\square$

Proposition 4.16. *For a commuting pair of type tNT_5, there are $q^2(q-1)$ branches of type tNT_5, and $q(q^2-1)(q-1)$ branches of type R_1.*

Proof. The centralizer of a commuting pair (A,B) of type tNT_5 is:

$$Z_{GT_4(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ & a_0 & & b_0 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix} \mid a_0 \neq 0 \right\}.$$

Let $C = \begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ & a_0 & & b_0 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$, and $C' = \begin{pmatrix} a_0 & a_1 & b'_0 & b'_1 \\ & a_0 & & b'_0 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix} = XCX^{-1}$, for some $X = \begin{pmatrix} x_0 & x_1 & y_0 & y_1 \\ & x_0 & & y_0 \\ & & x_0 & z_1 \\ & & & x_0 \end{pmatrix}$. So $XC = C'X$ leads us to $b'_0 = b_0$, and the equation:

$$x_0b_1 + x_1b_0 + y_0c_1 = x_0b'_1 + z_1b_0 + y_0a_1. \tag{4.32}$$

We have two main cases: $a_1 = c_1$ and $a_1 \neq c_1$.
When $a_1 = c_1$**:** Equation 4.32 becomes $x_0b_1 + x_1b_0 = x_0b'_1 + z_1b_0$.

When $b_0 = 0$, we have $b'_1 = b_1$. C is reduced to $\begin{pmatrix} a_0 & a_1 & & b_1 \\ & a_0 & & \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix}$, with $Z_{GT_4(\mathbf{F}_q)}(A,B,C) = Z_{GT_4(\mathbf{F}_q)}(A,B)$. (A,B,C) is thus of type tNT_5, and there are $q^2(q-1)$ such branches.

When $b_0 \neq 0$, choose z_1 such that $b_1' = 0$. C is reduced to $\begin{pmatrix} a_0 & a_1 & b_0 & \\ & a_0 & & b_0 \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix}$, with

$Z_{GT_4(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} x_0 & x_1 & y_0 & y_1 \\ & x_0 & & y_0 \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix} \right\}$. (A, B, C) is of type R_1, and there are $q(q-1)^2$ such branches.

So, we have no other cases to look at for $a_1 = c_1$.

$a_1 \neq c_1$**:** In Equation 4.32, choose y_0 so that $b_1' = 0$. Thus, C is reduced to $\begin{pmatrix} a_0 & a_1 & b_0 & \\ & a_0 & & b_0 \\ & & a_0 & c_1 \\ & & & a_0 \end{pmatrix}$,

with $Z_{GT_4(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} x_0 & x_1 & \frac{b_0(z_1-x_1)}{c_1-a_1} & \\ & x_0 & & \frac{b_0(z_1-x_1)}{c_1-a_1} \\ & & x_0 & z_1 \\ & & & x_0 \end{pmatrix} \right\}$. Here (A, B, C) is of type R_1, and there are $q^2(q-1)^2$ such branches.

With this, we have no other case to look at. So, adding up the branches of type R, we have a total of $q(q-1)^2 + q^2(q-1)^2 = q(q^2-1)(q-1)$ branches of type R_1.

□

Proposition 4.17. *For a commuting pair of type NR_1, there are $q^4(q-1)$ branches of type NR_1.*

Proof. The centralizer of a commuting pair (A, B) of type NR_1 is

$$Z_{GT_4(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} a_0 I_2 & D \\ & a_0 I_2 \end{pmatrix} \mid a_0 \neq 0, D \in M_2(\mathbf{F}_q) \right\}.$$

The result follows, as this is a commutative subgroup. □

5. Branching in $UT_3(q)$

For the unitriangular group $UT_3(\mathbf{F}_q)$, the conjugacy classes are as follows:

Canonical Form	No. of Classes	Centralizer	Name of Type
$\left(\begin{smallmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a \in \mathbf{F}_q$	q	$UT_3(\mathbf{F}_q)$	C
$\left(\begin{smallmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a \in \mathbf{F}_q^*$.	$(q-1)$	$\left\{\left(\begin{smallmatrix} 1 & x_0 & x_1 \\ & 1 & \\ & & 1 \end{smallmatrix}\right) \mid x_0, x_1 \in \mathbf{F}_q\right\}$	R_1
$\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a \in \mathbf{F}_q^*$.	$(q-1)$	$\left\{\left(\begin{smallmatrix} 1 & & x_1 \\ & 1 & x_0 \\ & & 1 \end{smallmatrix}\right) \mid x_1, x_0 \in \mathbf{F}_q\right\}$	R_1
$\left(\begin{smallmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a, b \in \mathbf{F}_q^*$.	$(q-1)^2$	$\left\{\left(\begin{smallmatrix} 1 & x_0 & x_1 \\ & 1 & \frac{b}{a}x_0 \\ & & 1 \end{smallmatrix}\right) \mid x_0, x_1 \in \mathbf{F}_q\right\}$	R_2

We see that there are two types here: central C and regular R. Note that the centralizers of both regulars R_1 and R_2 are isomorphic (not conjugate). For the type C, the centralizer is full group $UT_3(\mathbf{F}_q)$, thus all types appear in the first column. For the regular type, it has q^2 branches of the same R type, as the centralizer is commutative, of size q^2, hence the number of branches is q^2.

Theorem 5.1. *The branching matrix (with the order of type $C, R1$):*

$$B_{UT_3(\mathbf{F}_q)} = \begin{pmatrix} q & 0 \\ q^2 - 1 & q^2 \end{pmatrix}.$$

We prove the branching rules below.

Proposition 5.2. *An upper unitriangular matrix of type C has q branches of type C, and $q^2 - 1$ branches of the type R.*

Proof. The result follows as matrices of this type are central. □

Proposition 5.3. *A matrix of of any of the R types has q^2 branches of the same R type.*

Proof. A matrix of any of the R types is a Regular type, hence its centralizer in $UT_3(\mathbf{F}_q)$ is commutative, of size q^2, hence the number of branches is q^2. □

6. Branching in $UT_4(q)$

We shift our focus to commuting tuples of matrices in $UT_4(\mathbf{F}_q)$. The conjugacy classes according to the types of this group are listed in Appendix B.

Theorem 6.1. *The branching rules for the upper unitriangular group is given by the following matrix (with order $C, A_1, A_2, A_3, R_1, R_2$):*

$$B_{UT_4(\mathbf{F}_q)} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 2(q-1) & q^2 & 0 & 0 & 0 & 0 \\ (q-1)^2 & 0 & q^2 & 0 & 0 & 0 \\ q(q^2-1) & 0 & 0 & q^2 & 0 & 0 \\ q(q-1) & q(q^2-1) & q^2(q-1) & q(q^2-1) & q^4 & 0 \\ (q^2-1)(q-1) & q^2(q-1) & q(q^2-1) & 0 & 0 & q^3 \end{pmatrix}.$$

The first column corresponds to type C, thus all types of $UT_4(\mathbf{F}_q)$ appears. The last two columns are the regular types. There are no new types here. The proof for other columns is listed below in propositions.

Proposition 6.2. *An upper unitriangular matrix of type A_1 has q^2 branches of type A_1, and $q(q^2-1)$ branches of type R_1, and $q^2(q-1)$ branches of type R_2.*

Proof. Let $A = \begin{pmatrix} 1 & & & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix}$, a matrix of type A_1. The centralizer $Z_{UT_4}(A)$ of A is: $\left\{ \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, y_i, z_0 \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$, be an element of $Z_{UT_4}(A)$. Let $B = \begin{pmatrix} 1 & & b_1 & b_2 \\ & 1 & c_0 & c_1 \\ & & 1 & d_0 \\ & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & & b'_1 & b'_2 \\ & 1 & c'_0 & c'_1 \\ & & 1 & d'_0 \\ & & & 1 \end{pmatrix}$ be the conjugate of B by X, i.e., $B' = XBX^{-1}$. Thus equating $XB = B'X$ leads us to $b'_0 = b_0$, $c'_0 = c_0$, $c'_1 = c_1$, and the following equations:

$$\begin{aligned} x_0c_0 + b_1 &= y_0b'_0 + b'_1 \\ x_0c_1 + b_2 &= y_1b'_0 + b'_2 \end{aligned}$$

We use these to simplify B to the branches mentioned in the statement of the proposition. $\square$

Proposition 6.3. *An upper unitriangular matrix of type A_2 has q^2 branches of type A_2, and $q^2(q-1)$ branches of type R_1, and $q(q^2-1)$ branches of R_2.*

Proof. Given $A = \begin{pmatrix} 1 & & a & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{pmatrix}$, where $a, b \in \mathbf{F}_q^*$. the canonical form of a matrix of type A_2. The centralizer of A, $Z_{UT_4}(A)$ is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & \lambda x_0 \\ & & & 1 \end{pmatrix} \mid \lambda = \frac{b}{a}, x_i, y_i, z_0 \in \mathbf{F}_q \right\}$.

Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & \lambda x_0 \\ & & & 1 \end{pmatrix}$ be an element of $Z_{UT_4}(A)$. Let $B = \begin{pmatrix} 1 & b_0 & b_1 & b_2 \\ & 1 & c_0 & c_1 \\ & & 1 & \lambda b_0 \\ & & & 1 \end{pmatrix}$,
and $B' = \begin{pmatrix} 1 & b'_0 & b'_1 & b'_2 \\ & 1 & c'_0 & c'_1 \\ & & 1 & \lambda b'_0 \\ & & & 1 \end{pmatrix}$ be the conjugate of B by X. Thus equating $XB = B'X$ gives us the following equations:

$$\begin{gathered} b_0 = b'_0 \\ c_0 = c'_0 \\ x_0 c_0 + b_1 = y_0 b'_0 + b'_1 \\ \lambda b_0 y_0 + c_1 = \lambda x_0 c'_0 + c'_1 \\ x_0 c_1 + \lambda b_0 x_1 + b_2 = y_1 b'_0 + \lambda b'_1 x_0 b'_2 \end{gathered}$$

Using these we reduce B to the mentioned branches. □

Proposition 6.4. *An upper triangular matrix of type A_3 has q^2 branches of type A_3, and $q(q^2-1)$ branches of type R_1.*

Proof. One of the canonical forms of an upper triangular matrix of type A_3 is $A = \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, where $a \in \mathbf{F}_q^*$. Here $Z_{UT_4(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, z_0 \in \mathbf{F}_q \right\}$.

Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$, be an element of $Z_{UT_4}(A)$. Let $B = \begin{pmatrix} 1 & b_0 & b_1 & b_2 \\ & 1 & & \\ & & 1 & d_0 \\ & & & 1 \end{pmatrix}$, and

$B' = \begin{pmatrix} 1 & b_0' & b_1' & b_2' \\ & 1 & & \\ & & 1 & d_0' \\ & & & 1 \end{pmatrix}$ be the conjugate of B by X, i.e., $B' = XBX^{-1}$. Thus equating $XB = B'X$ leads us to the $b_0' = b_0$, $b_1' = b_1$, $d_0' = d_0$, and the following equation:

$$x_1 d_0 + b_2 = z_0 b_1' + b_2'$$

We use these to simplify B to the branches mentioned in the statement of the proposition. □

Proposition 6.5. *A matrix of the R_1 type has q^4 branches of type R_1 and A matrix of the R_2 type has q^3 branches of type R_2.*

Proof. The type R_1 and R_2 are Regular types, hence the centralizer of matrices of such a type is a commutative. □

Proof of Theorem 6.1. From the data in Propositions 6.2 to 6.5, we summarize the branching rules for UT_4, as in the table described in the theorem. □

Here are some isomorphisms between centralizers of matrices of the same z-class for some z-classes in $UT_4(\mathbf{F}_q)$.

Proposition 6.6. *The centralizer of conjugacy classes with representative* $\begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & & & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix}$, *for $a \in \mathbf{F}_q^*$ are isomorphic.*

Proof. The centralizer of conjugacy class with representative $\begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ is

$$\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid x_i, y_i \in \mathbf{F}_q \right\}.$$

The centralizer of conjugacy class with representative $\begin{pmatrix} 1 & & & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix}$ is

$$\left\{ \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, y_i, z_0 \in \mathbf{F}_q \right\}.$$

The following map gives isomorphism between these two centralizers.

$$\begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & y_0 & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -z_0 & y_1 - z_0 y_0 & x_2 - x_1 z_0 \\ & 1 & y_0 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ □

Proposition 6.7. *The centralizers of all conjugacy classes of type A_3 are isomorphic.*

Proof. There are six conjugacy classes of type A_3. In the following table, we give the centralizer of these conjugacy classes. We also set a notation for these conjugcay classes which will be used later in this proof.

Class Representative	Centralizer in $UT_4(\mathbf{F}_q)$	Name of Conjugacy class
$\begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, a \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, z_0 \in \mathbf{F}_q \right\}$	A_{3_1}
$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & a \\ & & & 1 \end{pmatrix}, a \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, y_1, z_0 \in \mathbf{F}_q \right\}$	A_{3_2}
$\begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & b \\ & & & 1 \end{pmatrix}, a, b \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & \frac{a}{b} y_1 & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, y_1, z_0 \in \mathbf{F}_q \right\}$	A_{3_3}
$\begin{pmatrix} 1 & a & & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{pmatrix}, a, b \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & \frac{b}{a} x_0 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mid x_i, z_0 \in \mathbf{F}_q \right\}$	A_{3_4}
$\begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & b \\ & & & 1 \end{pmatrix}, a, b \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & y_1 \\ & & 1 & \frac{b}{a} x_1 \\ & & & 1 \end{pmatrix} \mid x_i, y_1 \in \mathbf{F}_q \right\}$	A_{3_5}
$\begin{pmatrix} 1 & a & b & \\ & 1 & & \\ & & 1 & c \\ & & & 1 \end{pmatrix}, a, b, c \in \mathbf{F}_q^*$	$\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & y_1 \\ & & 1 & \frac{c}{b} x_1 - \frac{a}{b} y_1 \\ & & & 1 \end{pmatrix} \mid x_i, y_1 \in \mathbf{F}_q \right\}$	A_{3_6}

(1) The following map gives isomorphism between centralizers of representative of conjugacy classes A_{3_1} and A_{3_2}.

$$\begin{pmatrix} 1 & x_0 & & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z_0 & y_1 & x_2 - y_1 x_0 \\ & 1 & & \\ & & 1 & -x_0 \\ & & & 1 \end{pmatrix}$$

(2) The following map gives isomorphism between centralizers of representative of conjugacy classes A_{3_1} and A_{3_4}.

$$\begin{pmatrix} 1 & x_0 & x_1 & x_2 \\ & 1 & & \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x_0 & x_1 & x_2 - \left(\frac{x_0(x_0-1)}{2}\right)\lambda \\ & 1 & & \lambda x_0 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$$

(3) The following map gives isomorphism between centralizers of representative of conjugacy classes A_{3_2} and A_{3_5}.

$$\begin{pmatrix} 1 & x_0 & & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x_0 & \lambda z_0 & x_2 + \left(\frac{z_0(z_0-1)}{2}\right)\lambda \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$$

(4) The following map gives isomorphism between centralizers of representative of conjugacy classes A_{3_2} and A_{3_3}.

$$\begin{pmatrix} 1 & x_0 & & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x_0 + \lambda z_0 & \lambda y_1 & x_2 + \lambda y_1 z_0 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$$

(5) The following map gives isomorphism between centralizers of representative of conjugacy classes A_{3_2} and A_{3_6}.

$$\begin{pmatrix} 1 & x_0 & & x_2 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x_0 + \lambda_2 z_0 & \lambda_1 z_0 + \lambda_2 y_1 & x_2 + \lambda_2 y_1 z_0 + \left(\frac{z_0(z_0-1)}{2}\right)\lambda_1 \\ & 1 & & y_1 \\ & & 1 & z_0 \\ & & & 1 \end{pmatrix}$$

□

7. Branching rules for $UT_5(\mathbf{F}_q)$

In this section, we will discuss the simultaneous conjugacy classes of tuples of commuting matrices of $UT_5(\mathbf{F}_q)$. The types are listed in Section B. The branching matrix is as follows:

Theorem 7.1. *The branching rule of $UT_5(\mathbf{F}_q)$ has 3 new types. The branching matrix $B_{UT_5(\mathbf{F}_q)}$ is in table 4 which is a 20×20 matrix.*

Once again it's easy to see the branches for central and regular types.

Table 4. Branching matrix of $UT_5(\mathbf{F}_q)$

C	A_1	A_2	A_3	A_4	A_5	B_1	B_2	B_3	B_4	B_5	B_6	D_1	D_2	R_1	R_2	R_3	UNT_1	UNT_2	UNT_3
q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$2(q-1)$	q^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q^2-q	$q(q^2-1)$	q^4	0	0	0	$3q^2-3q$	0	0	0	0	0	0	0	0	0	0	0	0	0
$2q^2-2q$	0	0	q^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$2q^2-2q$	$2q^2(q-1)$	0	$q(q^2-1)$	q^4	0	0	q^3-q	0	0	0	0	0	0	0	0	0	0	0	0
$(q^2-1).(2q-1)$	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(q-1)^2$	0	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0	0	0	0	0
$2q^2-2q$	0	0	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0	0	0	0
$2(q-1)^2$	$q^2(q-1)$	0	$q^2(q-1)$	0	0	0	0	q^3	0	0	0	0	0	0	0	0	0	0	0
$(2q^2+4).(q-1)^2$	$q(q-1).(q^3+q^2-1)$	0	0	0	$2q(q-1)$	0	q^2-q	0	q^3	0	0	$2q(q-1)$	0	0	0	0	0	0	0
$q(q-1)^2$	0	0	$q^3(q-1)$	0	0	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0
$2q(q-1)^2$	0	0	0	0	$q^2(q-1)$	$q^2(q-1)$	q^4-q^3	0	0	$(q^3+q).(q^2-1)$	q^3	0	0	0	0	0	0	0	0
$(q-1)^3$	0	0	0	0	0	0	0	0	0	0	0	q^2	0	0	0	0	0	0	0
$(2q+1).(q-1)^3$	0	0	0	0	0	0	0	0	0	0	0	0	q^3	0	0	0	0	0	0
$2(q-1)^2$	$2q^3-2q^2$	$2q^4-2q^2$	$q(q-1).(q^2+q-1)$	0	0	$2q^3-4q+2$	$(q^2-q).(q^2+q-1)$	$q^2(q-1).(q^2+q-1)$	0	0	0	0	0	q^6	0	0	0	q^5-q^2	0
$q(q-1)^2$	$q(q-1)^2.(q+1)$	$q(q^2-1)^2$	$q(q^2-1).(q-1)$	$q^4(q-1)$	$q^2(q-1)$	$q(q-1)^2.(q+2)$	$(q-1).(q^3-q)$	0	q^4-q^2	0	0	0	0	0	q^5	0	q^4-q^2	q^4-q^3	q^4-q^2
$(q^2-1).(q-1)^2$	$q^2(q-1)^2$	0	$q^2(q-1)^2$	$q^3(q^2-1)$	$q(q-1).(q^2-1)$	$(q-1)^2.(q^2+q+1)$	$q^2(q-1)^2$	$q^3(q-1)$	0	q^4-q^3	$q^2(q^2-1)$	$q^2(q^2-1)$	$q^2(q^2-1)$	0	0	q^4	q^4-q^3	q^4-q^3	q^4-q^3
0	0	0	$q^2(q-1)$	0	$q(q-1)^2$	$(q-1)^2$	$q(q-1)^2$	0	0	0	0	0	0	0	0	0	q^3	0	0
0	0	0	0	0	0	$2q(q-1)$	q^3-q^2	0	0	0	0	0	0	0	0	0	0	q^3	0
0	0	0	0	0	0	$(q-1)^3$	0	0	0	0	0	$q(q-1)^2$	0	0	0	0	0	0	q^3

7.1. **Branching of type A.**

Proposition 7.2. *An upper unitriangular matrix of type A_1 has the following branches:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_1	q^2	B_4	$q(q-1)(q^3+q^2-1)$
A_2	$q(q^2-1)$	R_1	$2q^2(q-1)$
A_4	$2q^2(q-1)$	R_2	$q(q-1)^2(q+1)$
B_3	$q^2(q-1)$	R_3	$q^2(q-1)^2$

Proof. For a matrix of type A_1, there are two canonical forms: I_5+aE_{14}, and I_5+aE_{25}, where $a\neq 0$. We will take our matrix A of type A_1, to be the canonical form I_5+aE_{14}, $a\neq 0$. So the centralizer of A is $Z_{UT_5(\mathbf{F}_q)}(A)=\left\{\begin{pmatrix}1 & a_1 & b_1 & b_2 & a_2\\ & 1 & c_1 & c_2 & d_1\\ & & 1 & c_3 & d_2\\ & & & 1 & \\ & & & & 1\end{pmatrix}\right\}$. Let $B=\begin{pmatrix}1 & a_1 & b_1 & b_2 & a_2\\ & 1 & c_1 & c_2 & d_1\\ & & 1 & c_3 & d_2\\ & & & 1 & \\ & & & & 1\end{pmatrix}$, $B'=\begin{pmatrix}1 & a_1' & b_1 & b_2' & a_2'\\ & 1 & c_1' & c_2' & d_1'\\ & & 1 & c_3' & d_2'\\ & & & 1 & \\ & & & & 1\end{pmatrix}$, and $X=\begin{pmatrix}1 & x_1 & y_1 & y_2 & x_2\\ & 1 & z_1 & z_2 & w_1\\ & & 1 & z_3 & w_2\\ & & & 1 & \\ & & & & 1\end{pmatrix}$, be such that $XB=B'X$. From $XB=B'X$, we get that $a_1'=a_1$. Let C denote the middle 3×3 unitriangular block $\begin{pmatrix}1 & c_1 & c_2\\ & 1 & c_3\\ & & 1\end{pmatrix}$ in the matrix B, and let Z denote the middle block, $\begin{pmatrix}1 & z_1 & z_2\\ & 1 & z_3\\ & & 1\end{pmatrix}$, from X. Likewise, we have C'. We see that from $XB=B'X$, we have $ZC=C'Z$. Thus we take C to be a conjugacy class representative from $UT_3(\mathbf{F}_q)$, and Z to be its centralizer element in $UT_3(\mathbf{F}_q)$. Now, with this, we have the following set of equations:

$$\begin{pmatrix}x_1 & y_1 & y_2\end{pmatrix}C+\begin{pmatrix}a_1 & b_1 & b_2\end{pmatrix} = \begin{pmatrix}a_1 & b_1' & b_2'\end{pmatrix}Z+\begin{pmatrix}x_1 & y_1 & y_2\end{pmatrix} \tag{7.1}$$

$$Z\begin{pmatrix}d_1\\ d_2\\ 0\end{pmatrix}+\begin{pmatrix}w_1\\ w_2\\ 0\end{pmatrix} = C\begin{pmatrix}w_1\\ w_2\\ 0\end{pmatrix}+\begin{pmatrix}d_1'\\ d_2'\\ 0\end{pmatrix} \tag{7.2}$$

$$x_1d_1+y_1d_2+a_2 = a_1w_1+b_1'w_2+a_2' \tag{7.3}$$

We look at two main cases, $a_1=0$, and $a_1\neq 0$.

Case $a_1 = 0$: Here Equation 7.3 is reduced to $x_1 d_1 + y_1 d_2 + a_2 = b_1' w_2 + a_2'$. Here we look at subcases:

When $(b_1, b_2) = (d_1, d_2) = (0, 0)$: Thus Equations 7.1 and 7.2 become:

$$\begin{aligned}\begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} C &= \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} \\ \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix} &= C \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix}\end{aligned}$$

and $a_2' = a_2$.

When $C = I_3$: Equations 7.1 and 7.2 are void, and B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$.

Thus $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type A_1, and there are q branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$, $c \neq 0$: Equation 7.2 remains void, but from Equation 7.1, we get $cx_1 + y_2 = y_2$, which leaves us with $x_1 = 0$, as $c \neq 0$. Thus the branch is $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & & c & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of one of the canonical forms of type A_2. So (A, B) is a branch of type A_2, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}$, $c \neq 0$: Here we have $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. From Equations 7.1 and 7.2, we have $cx_1 + y_1 = y_1$ and $w_1 + cw_2 = w_1$, thus we have $x_1 = w_2 = 0$. So we have $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & c & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, and by a routine check, we see that $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative, of size q^6. (A, B) is of the regular type R_1, and there are $(q-1)q$ branches of this type.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}$, $c \neq 0$: We have $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. In this case Equation 7.2 becomes void, and from Equation 7.1, we have $cy_1 + y_2 = y_2$, thus leading to $y_1 = 0$.

Hence, $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. We have $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a unitriangular matrix of type A_4. So (A,B) is a branch of type A_4, and there are $q(q-1)$ branches.

When $C = \left(\begin{smallmatrix} 1 & c_1 & \\ & 1 & c_2 \\ & & 1 \end{smallmatrix}\right)$, $c_1, c_2 \neq 0$**:** We have $Z = \left(\begin{smallmatrix} 1 & z_1 & z_2 \\ & 1 & \lambda z_1 \\ & & 1 \end{smallmatrix}\right)$, where $\lambda = c_2/c_1$. From Equation 7.1, we have $c_1x_1 + y_1 = y_1$, which leaves us with $x_1 = 0$, and then we have $c_2y_1 + y_2 = y_2$, which leaves us with $y_1 = 0$. Then, from Equation 7.2, we have $w_1 + c_1w_2 = w_1$, leaving us with with $w_2 = 0$. So, we have $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & c_1 & & \\ & & 1 & c_2 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & \lambda z_1 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. This is of size q^5, and by a routine check, it can be seen that $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is commutative. Thus (A,B) is of the regular type R_2, and there are $q(q-1)^2$ branches.

When $((b_1,b_2),(d_1,d_2)) \neq ((0,0),(0,0))$**:** We shall start with $C = I_3$.

When $C = I_3$**:** Here Z is any aribtrary matrix in $UT_3(\mathbf{F}_q)$, and Equations 7.1 and 7.2 become:

$$\begin{pmatrix} 0 & b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & b_1' & b_2' \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$$

From the above equation, we have $b_1' = b_1$, and $b_2 = b_2' + b_1z_3$, and we have $d_1 + z_1d_2 = d_1'$, and $d_2' = d_2$.

Firstly, if both $b_1 \neq 0$ and $d_2 \neq 0$. Then we can choose a z_3 such that $b_2' = 0$, and similarly we can choose z_1 such that $d_1' = 0$. Hence, with this Equation 7.3 is reduced to $y_1d_2 + a_2 = b_1w_2 + a_2'$. We may choose a w_2 such that $a_2' = 0$. Thus, we have reduced B

to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & & \lambda y_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = d_2/b_1 \right\}$. This is the centralizer of a matrix of type A_4. Thus, we have $(q-1)^2$ branches of this type.

When $b_1 \neq 0$ and $d_2 = 0$, we again pick a z_3 such that $b_2' = 0$, and Equation 7.3 is reduced to $x_1 d_1 + a_2 = b_1 w_2 + a_2'$. Again, choose w_2 so that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \lambda x_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = d_1/b_1 \right\}$, which is isomorphic (conjugation by the matrix that swaps the 4th and 5th rows and columns) to centralizer of a matrix of type B_3. Thus there are $q(q-1)$ branches of this type.

When $b_1 = 0$, we have $b'2_2 = b_2$. We consider $d_2 \neq 0$, and choose a suitable z_1 so that $d_1' = 0$. Equation 7.3 is reduced to $y_1 d_2 + a_2 = a_2'$. Thus, we choose an apporpriate y_1 so that $a_2' = 0$. B is thus reduced to $\begin{pmatrix} 1 & & b_2 & & \\ & 1 & & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = d_1/b_1 \right\}$, which is the centralizer of one of the canonical forms of type A_4. There are $q(q-1)$ such branches.

When $b_2 \neq 0$, $d_2 = 0$, we have $d_1' = d_1$. We first take $d_1 = 0$. Then Equation 7.3 is reduced to $a_2' = a_2$. We thus have B reduced to $\begin{pmatrix} 1 & & & b_2 & a_2 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and thus $Z_{UT_5(\mathbf{F}_q)}(A,B) = Z_{UT_5(\mathbf{F}_q)}(A)$. Hence, (A,B) is a branch of type A_1, and there are $q(q-1)$ branches.

When $b_1 = 0$, with $d_2 = 0$, and $d_1 \neq 0$. Equation 7.3 is reduced to $x_1 d_1 + a_2 = a_2'$. With a suitable x_1, we can get rid of d_1. Hence B is reduced to $\begin{pmatrix} 1 & & & b_2 & \\ & 1 & & & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A, B) is of type A_2, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$: Equation 7.1 is reduced to $\begin{pmatrix} 0 & b_1 & b_2 + cx_1 \end{pmatrix} = \begin{pmatrix} 0 & b_1' & b_1' z_3 + b_2' \end{pmatrix}$. Thus, we have $b_1' = b_1$, and we can choose x_1 such that $b_2' = 0$. Now, here, on replacing b_2' and b_2 by 0 in the above equation, we get that $x_1 = \frac{b_1}{c} z_3$. From Equation 7.2, we have $d_2' = d_2$, and $d_1' = d_1 + z_1 d_2$. Equation 7.3 becomes $\frac{b_1}{c} z_3 d_1 + y_1 d_2 + a_2 = w_2 b_1 + a_2'$.

We now look at the case when $b_1 \neq 0$, and $d_2' \neq 0$. We choose z_1 so that $d_1' = 0$, and w_2 such that $a_2' = 0$. Hence, we reduce B to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & c & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and we have

$$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & \lambda z_3 & y_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & \mu y_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b_1}{c}, \mu = \frac{d_2}{b_1} \right\},$$

which is isomorphic to the centralizer of some canonical matrix of type B_4. There are $(q-1)^3$ such branches.

When $b_1 \neq 0$, and $d_2 = 0$, then $d_1' = d_1$. Equation 7.3 becomes $\frac{b_1}{c} z_3 d_1 + a_2 = w_2 b_1 + a_2'$. Choose a suitable w_2, to make $a_2' = 0$. Then B is reduced to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & c & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & \lambda z_3 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & \mu z_3 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b_1}{c}, \mu = \frac{d_1}{c} \right\}$. If we write z_3 in terms of z_1, then $Z_{UT_5(\mathbf{F}_q)}$ will be this: $\left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & \lambda x_1 & \mu x_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{c}{b_1}, \mu = \frac{d_1}{b_1} \right\}$. If we conjugate this centralizer by the matrix $I + \frac{\mu}{\lambda} E_{45}$, we get the centralizer of a canonical unitriangular matrix of type B_3. Thus (A, B) is a branch of type B_3, and there are $q(q-1)^2$ such branches.

Now, when $b_1 = 0$, and $(d_1, d_2) \neq (0, 0)$. We have $x_1 = \frac{b_1}{c} z_3 = 0$,a nd Equation 7.3 becomes $y_1 d_2 + a_2 = a_2'$. First, when $d_2 \neq 0$, then we choose z_1 so that $d_1' = 0$, and choose y_1 so that $a_2' = 0$. So, B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & & c & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is commutative of size q^6, (A, B) is of regular type R_1, and there are $q(q-1)$ such branches.

When $b_1 = d_2 = 0$, $d_1 \neq 0$. We have $d_1' = d_1$, and Equation 7.3 reduces to $a_2' = a_2$. Thus, B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & & c & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & & c & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type A_2. Thus (A, B) is of type A_2, and there are $q(q-1)^2$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}$, $c \neq 0$**:** Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$, and Equation 7.1 becomes:

$$\begin{pmatrix} cx_1 + b_1 & b_2 \end{pmatrix} = \begin{pmatrix} b_1' & b_2' \end{pmatrix}.$$

Using a nice x_1, we can make $b_1' = 0$, and $b_2' = b_2$. So, if we replace b_1 by $b_1' = 0$ in the above equation, we have $x_1 = 0$. Next, Equation 7.2 becomes:

$$\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} cw_2 + d_1' \\ d_2' \\ 0 \end{pmatrix}.$$

As $c \neq 0$, we choose a w_2 so that $d_1' = 0$. We have $d_2' = d_2$. With these, Equation 7.3 becomes

$$y_1 d_2 + a_2 = a_2' \tag{7.4}$$

When $d_2 \neq 0$, choose y_1 such that $a_2' = 0$. B is reduced to $\begin{pmatrix} 1 & & & b_2 & \\ & 1 & c & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \lambda z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{d_2}{c} \right\}$. Thus (A, B) is of regular type R_2, and there are $q(q-1)^2$ branches of this type.

When $d_2 = 0$, then we are left with $b_2 \neq 0$. Hence Equation 7.3 is reduced to $a_2' = a_2$. Hence B is reduced to $\begin{pmatrix} 1 & & & b_2 & a_2 \\ & 1 & c & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_1. (A, B) is a branch of type R_1, and there are $(q-1)^2 q$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}$, $c \neq 0$**:** Here $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. With these, Equation 7.1 becomes:

$$\begin{pmatrix} b_1 & b_2 + cy_1 \end{pmatrix} = \begin{pmatrix} b_1' & b_2' + b_1' z_3 \end{pmatrix}$$

So, we have $b_1' = b_1$, and we can choose y_1 so that $b_2' = 0$. Thus, on equating the above equation, with b_2 replaced by 0, we get that $y_1 = \frac{b_1}{c} z_3$; and from Equation 7.2, we have $d_1' = d_1$, and $d_2' = d_2$, and thus Equation 7.3 boils down to $x_1 d_1 + \frac{b_1}{c} z_3.d_1 = b_1 w_2 + a_2'$. We first look at the case, when $b_1 \neq 0$. Then choose w_2 so that $a_2' = 0$. So B reduces to $\begin{pmatrix} 1 & b_1 & & & \\ & 1 & & & d_1 \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & \lambda_1 z_3 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & \lambda_2 z_3 + \mu x_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid {\scriptstyle \lambda_1 = \frac{b_1}{c}, \lambda_2 = \frac{d_2}{c}, \atop \mu = \frac{d_1}{b_1}} \right\}$. This is isomorphic to a centralizer of canonical form of type A_4. So (A,B) is a branch of type A_4, and there are $q^2(q-1)^2$ such branches.

When $b_1 = 0$. Then we have $y_1 = 0$. Hence Equation 7.3 becomes $x_1 d_1 + a_2 = a_2'$. When $d_1 \neq 0$, choose x_1 so that $a_2' = 0$. B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & & & d_1 \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_1. Thus (A,B) is of type R_1, and thus there are $q(q-1)^2$ branches of this type.

When $d_1 = 0$, and $d_2 \neq 0$. Equation 7.3 ends up becoming $a_2' = a_2$, and B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & & & \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, hence $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A,B) is a branch of type A_4, and there are $q(q-1)^2$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c_1 & \\ & 1 & c_2 \\ & & 1 \end{smallmatrix}\right)$, $c \neq 0$: Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \lambda_0 z_1 \\ & & 1 \end{pmatrix}$, where $\lambda_0 = \frac{c_2}{c_1}$. Thus, from Equation 7.1, we have: $\begin{pmatrix} c_1 x_1 + b_1 & c_2 y_1 + b_2 \end{pmatrix} = \begin{pmatrix} b_1' & \lambda_0 z_1 b_1' + b_2' \end{pmatrix}$. So, we choose x_1 so that $b_1' = 0$. Similarly, we choose y_1 such that $b_2' = 0$. Thus, on replacing b_1, and b_2 by 0 in the above equation, we get that $x_1 = 0$, and $y_1 = 0$.

Equation 7.2 becomes $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + c_1 w_2 \\ d_2' \\ 0 \end{pmatrix}$. Thus $d_2' = d_2$, and we can choose w_2 so that $d_1' = 0$. So we are left with $d_2 \neq 0$. With $x_1 = y_1 = b_1 = 0$, Equation 7.3 becomes $a_2' = a_2$. Hence B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & c_1 & & \\ & & 1 & c_2 & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & \lambda_0 z_1 & \lambda_1 z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda_0 = \frac{c_2}{c_1}, \lambda_1 = \frac{d_2}{c_1} \right\}$, which is a centralizer of type R_2. (A,B) is a branch of type R_2, and there are $q(q-1)^3$ branches of this type.

Case $a_1 \neq 0$: We look at the various types of C as our subcases.

When $C = I_3$**:** Here Equation 7.1 becomes:

$$\begin{pmatrix} a_1 & b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1' + a_1 z_1 & b_2' + b_1' z_3 + z_2 a_1 \end{pmatrix}.$$

Using a suitable z_1, we can make $b_1' = 0$, and using a suitable z_2, we can make $b_2' = 0$. Thus, on replacing b_1 and b_2' by 0 in the above equation, we have $z_1 = z_2 = 0$. Hence with this, Equation 7.2 becomes $\begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$. Equation 7.3 is reduced to $a_2 + x_1 d_1 + y_1 d_2 = a_2' + a_1 w_1$. So we choose w_1 such that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & & & d_1 \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & & \lambda x_1 + \mu y_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{d_1}{a_1}, \mu = \frac{d_2}{a_1} \right\}$, which is the centralizer of type B_4. (A,B) is thus a branch of type B_4, and there are $q^2(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$**:** Equation 7.1 becomes:

$$\begin{pmatrix} a_1 & b_1 & b_2 + cx_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_1' z_3 + b_2' \end{pmatrix}.$$

Choose z_1 and z_2 such that $b_1' = b_2' = 0$. Again, like in the previous case on replacing b_1 and b_2 by 0 in the above equation, we have $z_1 = 04$ and $z_2 = \frac{c}{a_1} x_1$. From Equation 7.2, we get $d_1' = d_1$ amd $d_2' = d_2$. Equation 7.3 is reduced to $x_1 d_1 + y_1 d_2 + a_2 = w_1 a_1 + a_2'$. We choose w_1 such that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & & c & d_1 \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & \lambda_1 x_1 & \lambda_2 x_1 + \mu y_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda_1 = \tfrac{c}{a_1}, \lambda_2 = \tfrac{d_1}{a_1}, \mu = \tfrac{d_2}{a_1} \right\}.$$ This

is of type B_4. Hence (A, B) is a branch of type B_4, and there are $(q-1)^2 q^2$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$**:** $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. Equation 7.1 becomes

$$\begin{pmatrix} a_1 & cy_1 + b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_2' \end{pmatrix}$$

Choose z_1 such that $b_1' - 0$, and choose z_2 such that $b_2' = 0$. So, on substituting b_1 and b_2 with 0 in the above, we have $z_1 = \frac{c}{a_1} y_1$, and $z_2 = 0$. Thus Equation 7.2 is reduced to $\begin{pmatrix} d_1 + \frac{c}{a_1} y_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} cw_2 + d_1' \\ d_2' \\ 0 \end{pmatrix}$. We have $d_2' = d_2$. Choose w_2 so that $d_1' = 0$. Equation 7.3 is reduced to $y_1 d_2 + a_2 = a_1 w_1 + a_2'$. Choose w_1 such that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & c & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \frac{c}{a_1} y_1 & & \frac{d_2}{a_1} y_1 \\ & & 1 & & \frac{d_2}{a_1} y_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is a centralizer of a matrix of type R_3. So (A, B) is a branch of type R_3, and there are $(q-1)^2 q$ branches of this type.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0$**:** Here $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. Equation 7.1 becomes

$$\begin{pmatrix} a_1 & b_1 & b_2 + cy_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1' & a_1 z_2 + b_1' z_3 + b_2' \end{pmatrix}.$$

We have $b_1' = b_1$. we can choose y_2 such that $b_2' = 0$. Thus, on replacing b_2 by 0 in the above equation, we have $y_2 = \frac{a_1}{c} z_2 + \frac{b_1}{c} z_3$. And Equation 7.2 ends up giving us $d_1' = d_1$, and $d_2' = d_2$. Thus Equation 7.3 stays as it is. Since $a_1 \neq 0$, we choose w_1 so that $a_2' = 0$. B is therefore reduced to $\begin{pmatrix} 1 & a_1 & b_1 & & \\ & 1 & & & d_1 \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & \frac{a_1}{c} z_2 + \frac{b_1}{c} z_3 & x_2 \\ & 1 & & z_2 & \frac{d_1}{a_1} x_1 + \frac{d_2}{a_1} y_1 - \frac{b_1}{a_1} w_2 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is that of type B_4. (A, B) is of type B_4, and the number of branches is $q^3(q-1)^2$.

When $C = \begin{pmatrix} 1 & c_1 & \\ & 1 & c_2 \\ & & 1 \end{pmatrix}, c_1, c_2 \neq 0$**:** Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \frac{c_2}{c_1} z_1 \\ & & 1 \end{pmatrix}$. Equation 7.1 becomes:

$$\begin{pmatrix} a_1 & b_1 + c_1 x_1 & b_2 + c_2 y_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_1' \frac{c_2}{c_1} z_1 + b_2' \end{pmatrix}.$$

Choose z_1 such that $b_1' = 0$, and choose z_2 such that $b_2 = 0$. On replacing b_1 and b_2 by 0 in the above equation, we see that $z_1 = \frac{c_1}{a_1} x_1$, and $z_2 = \frac{c_2}{a_1} y_1$. From Equation 7.2, we have $d_1 + \frac{c_1}{a_1} x_1 d_2 = c_1 w_2 + d_1'$, and $d_2' = d_2$. So we choose w_2 such that $d_1' = 0$. Equation 7.3 becomes: $y_1 d_2 + a_2 = w_1 a_1 + a_2'$. Choose w_1 such that $a_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & c_1 & & \\ & & 1 & c_2 & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \frac{c_1}{a_1} x_1 & \frac{c_2}{a_1} y_1 & \frac{d_2}{a_1} y_1 \\ & & 1 & \frac{c_2}{a_1} x_1 & \frac{d_2}{a_1} x_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_3. Thus (A, B) is branch of type R_3, and there are $q(q-1)^3$ such branches. Hence, adding up the branches of each type, we get the numbers as mentioned in the statement of this proposition. $\square$

Proposition 7.3. *An upper unitriangular matrix of type* A_2 *has* q^4 *branches of type* A_2, $2q^2(q^2-1)$ *branches of regular type* R_1, *and* $q(q^2-1)^2$ *branches of regular type* R_2.

Proof. Let $A = \begin{pmatrix} 1 & & & & \\ & 1 & & a & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a \neq 0$ a matrix of type A_2. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid, x_i, y_i, z_i \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & & a_1 & a_2 & a_3 \\ & 1 & b_0 & b_1 & b_2 \\ & & 1 & c_0 & c_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & & a'_1 & a'_2 & a'_3 \\ & 1 & b'_0 & b'_1 & b'_2 \\ & & 1 & c'_0 & c'_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a'_1 = a_1$, $b'_0 = b_0$, $c'_0 = c_0$, $c'_1 = c_1$, and the following equations:

$$\begin{aligned} x_1c_0 + a_2 &= a_1z_0 + a'_2 \\ x_1c_1 + a_3 &= a_1z_1 + a'_3 \\ y_0c_0 + b_1 &= b'_1 + b'_0z_0 \\ y_0c_1 + b_2 &= b'_2 + b'_0z_1 \end{aligned}$$

We consider two cases when $(a_1, b_0, c_0, c_1) = \mathbf{0}$ and when $(a_1, b_0, c_0, c_1) \neq \mathbf{0}$.

Case: $(a_1, b_0, c_0, c_1) = \mathbf{0}$. In this case, we get $a_2 = a'_2,\ a_3 = a'_3,\ b_1 = b'_1$ and $b_2 = b'_2$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type A_2, and there are q^4 branches.

Case: $(a_1, b_0, c_0, c_1) \neq \mathbf{0}$. First we consider that $c_1 \neq 0$. We choose x_1 and y_0 in such a way that we get $a_3 = b_2 = 0$. Now if $(a_1, b_0) = (0, 0)$, then by simple calculations, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^3(q-1)$ branches of this type. If we consider that case when at least one of a_1 and b_0 is non-zero, then we can choose z_0 suitably so that we get one of a_2 or b_1 equal to zero. By routine check, we get that $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^5. Thus (A, B) is of regular type R_2, and there are $(q^3 - q^2)(q^2 - 1)$ branches of this type.

Now we consider that $c_1 = 0$ and $c_0 \neq 0$. We choose x_1 and y_0 in such a way that we get $a_2 = b_1 = 0$. Now if $(a_1, b_0) = 0$, then by simple calculations, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^2(q-1)$ branches of this type. If we consider that case when at least one of a_1 and b_0 is non-zero, then we can choose z_1 suitably so that we get one of a_3 or b_2 equal to zero. By routine check, we get that $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of size q^5. Thus (A,B) is of regular type R_2, and there are $(q^2-q)(q^2-1)$ branches of this type.

Next we consider when $c_1 = c_0 = 0$ and $b_0 \neq 0$. We choose z_0 and z_1 in such a way that we get $b_1 = b_2 = 0$. Now by simple calculations, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of size q^6. Thus (A,B) is of regular type R_1, and there are $q^3(q-1)$ branches of this type.

Finaly we consider when $c_1 = c_0 = b_0 0$ and $a_1 \neq 0$. We choose z_0 and z_1 in such a way that we get $a_2 = a_3 = 0$. Now by simple calculations, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of size q^6. Thus (A,B) is of regular type R_1, and there are $q^2(q-1)$ branches of this type.

Therefore a matrix of type A_2 has q^4 branches of type A_2, $2q^2(q^2-1)$ branches of regular type R_1, and $q(q^2-1)^2$ braches of regular type R_2.

$\square$

Proposition 7.4. *An upper unitriangular matrix of type A_3 has*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
A_3	q^2	R_1	$q(q^2+q-1)(q-1)$
A_4	$q(q^2-1)$	R_2	$q(q^2-1)(q-1)$
B_3	$q^2(q-1)$	R_3	$q^2(q-1)^2$
B_5	$q^3(q-1)$	UNT_1	$q^2(q-1)$.

It has a new type branch, named UNT_1, with common centralizer $\left\{\begin{pmatrix} 1 & x_0 & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix}\right\}$.

Proof. Let $A = \begin{pmatrix} 1 & & a & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a \neq 0$ a matrix of type A_3. The centralizer $Z_{UT_5}(A)$ of A is $\left\{\begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \mid, x_i, y_i, w_0 \in \mathbf{F}_q\right\}$. Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & & \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & b_0 & b_1 & b_2 \\ & & 1 & & \\ & & & 1 & d_0 \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a'_0 & a'_1 & a'_2 & a'_3 \\ & 1 & b'_0 & b'_1 & b'_2 \\ & & 1 & & \\ & & & 1 & d'_0 \\ & & & & 1 \end{pmatrix}$

be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_0' = a_0$, $b_0' = b_0$, $b_1' = b_1$, $d_0' = d_0$, and the following equations:

$$\begin{aligned} a_1 + x_0 b_0 &= a_0 y_0 + a_1' \\ a_2 + x_0 b_1 &= a_0 y_1 + a_2' \\ a_3 + x_0 b_2 + x_2 d_0 &= a_0 y_2 + a_2' z_0 + a_3' \\ b_2 + y_1 d_0 &= z_0 b_1 + b_2' \end{aligned}$$

We consider two cases when $(a_0, b_0, b_1, d_0) = \mathbf{0}$ and when $(a_0, b_0, b_1, d_0) \neq \mathbf{0}$.

Case: $(a_0, b_0, b_1, d_0) = \mathbf{0}$. In this case, we get $a_1' = a_1$, $a_2' = a_2$, $b_2' = b_2$, and $a_3 + x_0 b_2 = a_2 z_0 + a_3'$.

If $(a_2, b_2) = \mathbf{0}$, then we get $b_3 = b_3'$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type A_3, and there are q^2 branches. Now we consider that $a_2 \neq 0$. In this case, we can choose w_0 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a group of order q^7 and (A, B) is the type B_3, and there are $q^2(q-1)$ branches.

If we consider $a_2 = 0$ and $b_2 \neq 0$, choose x_0 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a group of order q^7 and (A, B) is a branch of type A_4, and there are $q(q-1)$ branches.

Case: $(a_0, b_0, b_1, d_0) \neq \mathbf{0}$. First we consider that $a_0 \neq 0$. In this case, we can choose y_0, y_1 and y_2 in such a way that we get $a_1 = a_2 = a_3 = 0$ and $b_2 + \frac{d_0 b_1}{b_0} x_0 = z_0 b_1 + b_2'$. Now if $b_1 = 0$, then we get $b_2 = b_2'$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a group of order q^5 and (A, B) is a branch of type B_5, and there are $q^3(q-1)$ branches. On the other hand if $b_1 \neq 0$, then we choose z_0 in such a way that we get $b_2 = 0$ By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of order q^4 and (A, B) is a branch of regular type R_3, and there are $q^2(q-1)^2$ branches.

Now we consider that $a_0 = 0$ and $b_0 \neq 0$. In this case, we can choose x_0 in such a way that we get $a_1 = 0$ and this implies $x_0 = 0$. Thus we get $b_2 = b_2'$ and the following equalities: $\begin{aligned} a_3 + x_2 d_0 &= a_2 z_0 + a_3' \\ b_2 + d_0 y_1 &= z_0 b_1 + b_2' \end{aligned}$ Now if $(d_0, a_2, b_1) = \mathbf{0}$, then we get $a_3 = a_3'$ and $b_2 = b_2'$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a group of order q^7 and (A, B) is a branch of type A_4, and there are $q^2(q-1)$ branches. If $d_0 \neq 0$, then we choose x_2 and y_1 in such a way that we get $a_3 = b_2 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of order q^5 and (A, B) is a branch of regular type R_2, and there are $q^2(q-1)^2$ branches.

If $d_0 = 0$ and $a_2 \neq 0$, then we choose w_0 in such a way that we get $a_3 = 0$ and this implies $w_0 = 0$. Thus we get $b_2 = b_2'$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of order q^6 and (A, B) is a branch of regular type R_1, and there are $q^2(q-1)^2$ branches.

If $d_0 = a_2 = 0$ and $b_1 \neq 0$, then we get $a_3 = a'_3$ and we choose w_0 in such a way that we get $b_2 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of order q^6 and (A,B) is a branch of regular type R_1, and there are $q(q-1)^2$ branches.

Now we consider that $a_0 = b_0 = 0$ and $b_1 \neq 0$. In this case, we can choose x_0 and z_0 in such a way that we get $a_2 = b_2 = 0$. In addition to this, if $d_0 = 0$, then we get $a_3 = a'_3$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of order q^6 and (A,B) is a branch of regular type R_3, and there are $q^2(q-1)$ branches. Now if we consider $d_0 \neq 0$, then we can choose x_2 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a commutative group of order q^5 and (A,B) is a branch of regular type R_2, and there are $q(q-1)^2$ branches.

Finally we consider the case when $a_0 = b_0 = b_1 = 0$ and $d_0 \neq 0$, then we get $a_2 = a'_2, a_1 = a'_1$ and we can choose y_1 and x_2 in such a way that we get $a_3 = b_2 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is a group of order q^6, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \right\}$. As we have not seen this centralizer before, and This (A,B) is a branch of new type, which we call UNT_1 and there are $q^2(q-1)$ branches. $\square$

Proposition 7.5. *An upper unitriangular matrix of type A_4 has q^4 branches of type A_4, $q^3(q^2-1)$ branches of regular type R_1, and $q^4(q-1)$ breaches of regular type R_2.*

Proof. Let $A = \begin{pmatrix} 1 & & & & \\ & 1 & a & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a \neq 0$ a matrix of type A_4. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & & \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix} \mid , x_i, y_i, w_0 \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & & \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & & a_1 & a_2 & a_3 \\ & 1 & b_0 & b_1 & b_2 \\ & & 1 & & \\ & & & 1 & d_0 \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & & a'_1 & a'_2 & a'_3 \\ & 1 & b'_0 & b'_1 & b'_2 \\ & & 1 & & \\ & & & 1 & d'_0 \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_1 = a'_1$ $a_{,2} = a'_2$, $b_0 = b'_0$, $b_1 = b'_1$, $d_0 = d'_0$ and the following equations:

$$a_3 + x_2 d_0 = a'_2 w_0 + a'_3$$
$$b_2 + d_0 y_1 = w_0 b'_1 + b'_2$$

We consider two cases when $(a_2, b_1, d_0) = \mathbf{0}$ and when $(a_2, b_1, d_0) \neq \mathbf{0}$.

Case: $(a_2, b_1, d_0) = \mathbf{0}$. In this case, we get $a_3 = a_3'$ and $b_2 = b_2'$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type A_4, and there are q^4 branches.

Case: $(a_2, b_1, d_0) \neq \mathbf{0}$. First we consider that $d_0 \neq 0$. Now we can choose x_2 and y_1 in such a way that we get $a_3 = b_2 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^5. Thus (A, B) is of regular type R_2, and there are $q^4(q-1)$ branches of this type.

Now we consider that $d_0 = 0$ and $a_2 \neq 0$. In this case, we can choose w_0 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^4(q-1)$ branches of this type.

Finaly we consider when $d_0 = a_2 = 0$ and $b_1 \neq 0.$, now we can choose w_0 in such a way that we get $b_2 = 0$. Again, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^3(q-1)$ branches of this type.

Therefore we get that a matrix of type A_4 has q^4 branches of type A_4, $q^3(q^2-1)$ braches of regular type R_1, and $q^4(q-1)$ braches of regular type R_2. $\square$

Proposition 7.6. *An upper unitriangular matrix of type A_5 has:*

Branch Type	*No. of Branches*	*Branch Type*	*No. of Branches*
A_5	q^2	R_2	$q^2(q-1)$
B_4	$2q(q-1)$	R_3	$q(q-1)(q^2-1)$
B_6	$q^2(q^2-1)$	UNT_1	$q(q-1)^2$.

It has the new branch UNT_1 already seen in previous case.

Proof. There are several canonical forms for a matrix in $UT_5(\mathbf{F}_q)$, of type A_5. We prove this proposition for the canonical form $A = \begin{pmatrix} 1 & a & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, where $a \neq 0$. We have:

$Z_{U_5(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & & & \\ & & 1 & b_0 & b_1 \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix} \right\}$. We can rewrite this centralizer subgroup as $\left\{ \begin{pmatrix} 1 & a_0 & {}^t\overrightarrow{b} \\ & 1 & \\ & & C \end{pmatrix} \mid \begin{matrix} C \in UT_3(\mathbf{F}_q) \\ {}^t\overrightarrow{b} = (b_1\ b_2\ b_3) \end{matrix} \right\}$. Let $B = \begin{pmatrix} 1 & a_0 & {}^t\overrightarrow{b} \\ & 1 & \\ & & C \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a_0' & {}^t\overrightarrow{b} \\ & 1 & \\ & & C' \end{pmatrix}$

be a conjugate in UT_5 of B. $B' = XBX^{-1}$, where $X = \begin{pmatrix} 1 & x_0 & {}^t\overrightarrow{y} \\ & 1 & \\ & & Z \end{pmatrix}$. So, equating $XB = B'X$ gives us $a'_0 = a_0$, $ZC = C'Z$. So, we may take C to be the representative of a conjugacy class in $UT_3(\mathbf{F}_q)$, and we have the equation:

$$ {}^t\overrightarrow{y}.C + {}^t\overrightarrow{b} = {}^t\overrightarrow{b'}Z + {}^t\overrightarrow{y} $$

We rewrite this equation slightly to get:

$$ \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} (C - I_3) + \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} b'_1 & b'_2 & b'_3 \end{pmatrix} Z \tag{7.5} $$

The cases:

When $C = I_3$. Here Equation 7.5 becomes: $\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} b'_1 & b_2 & b'_3 \end{pmatrix} \begin{pmatrix} 1 & z_0 & z_1 \\ & 1 & z_2 \\ & & 1 \end{pmatrix}$, which gives us $b'_1 = b_1$, and the following equation:

$$ b_2 = b'_2 + z_0 b_1 \tag{7.6} $$

$$ b_3 = b'_3 + z_1 b_1 + z_2 b'_2 \tag{7.7} $$

We have two subcases here: When $b_1 = 0$ and when $b_1 \neq 0$.

When $b_1 = 0$ Equation 7.6 becomes $b'_2 = b_2$, and Equation 7.7 becomes $b_3 = b'_3 + z_2 b_2$.

When $b_2 = 0$, we have $b'_3 = b_3$. So B is reduced to $\begin{pmatrix} 1 & a_0 & & & b_3 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A, B) = Z_{U_5(\mathbf{F}_q)}(A)$. Thus (A, B) is of type A_5, and there are q^2 such branches.

When $b_2 \neq 0$, in Equation 7.6, choose z_2 so that $b'_3 = 0$. So, We have B reduced to $\begin{pmatrix} 1 & a_0 & & b_2 & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_0 & y_1 & y_2 & y_3 \\ & 1 & & & \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type B_4 and there are $q(q-1)$ such branches.

When $b_1 \neq 0$: In Equation 7.6, choose z_0 such that $b'_2 = 0$, and in Equation 7.6, choose z_1 such that $b'_3 = 0$. So B is reduced to $\begin{pmatrix} 1 & a_0 & b_1 & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A, B) =$

$\left\{\begin{pmatrix}1 & x_0 & y_1 & y_2 & y_3\\ & 1 & & & \\ & & 1 & & \\ & & & 1 & z_2\\ & & & & 1\end{pmatrix}\right\}$. (A, B) is a branch of type B_6, and there are $q(q-1)$ such branches.

When $C = \left(\begin{smallmatrix}1 & & c\\ & 1 & \\ & & 1\end{smallmatrix}\right), c \neq 0$**:** From Equation 7.5, we have $b_1' = b_1$, and the following following equations:

$$b_2 = b_2' + z_0 b_1 \tag{7.8}$$

$$b_3' + cy_1 = b_3' + z_1 b_1' + z_2 b_2'. \tag{7.9}$$

As $c \neq 0$, choose y_1 so that $b_3' = 0$.

Case: $b_1 = 0$

We have $b_2' = b_2$. When $b_2 = 0$, B is reduced to $\begin{pmatrix}1 & a_0 & & & \\ & 1 & & & \\ & & 1 & & c\\ & & & 1 & \\ & & & & 1\end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A, B) =$ $\left\{\begin{pmatrix}1 & x_0 & & y_2 & y_3\\ & 1 & & & \\ & & 1 & z_0 & z_1\\ & & & 1 & z_2\\ & & & & 1\end{pmatrix}\right\}$. (A, B) is of type B_4, and there are $q(q-1)$ such branches.

When $b_2 \neq 0$, we have B reduced to $\begin{pmatrix}1 & a_0 & & a_2 & \\ & 1 & & & \\ & & 1 & & c\\ & & & 1 & \\ & & & & 1\end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A, B) =$ $\left\{\begin{pmatrix}1 & x_0 & \frac{b_2}{c}z_2 & y_2 & y_3\\ & 1 & & & \\ & & 1 & z_0 & z_1\\ & & & 1 & z_2\\ & & & & 1\end{pmatrix}\right\}$. This centralizer is isomorphic to that of a new type, UNT_1, which we had come across earlier. There are $q(q-1)^2$ such branches.

When $b_1 \neq 0$. We can choose z_0 so that $b_2' = 0$. Here B is reduced to $\begin{pmatrix} 1 & a_0 & a_1 & & \\ & 1 & & & \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{U_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & \frac{b_1}{c_1} z_1 & y_2 & y_3 \\ & 1 & & & \\ & & 1 & & z_1 \\ & & & 1 & z_2 \\ & & & & 1 \end{pmatrix} \right\}$. Hence (A,B) is of type B_6, and there are $q(q-1)^2$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}$, $c \neq 0$: Here $Z = \begin{pmatrix} 1 & z_0 & z_1 \\ & 1 & \\ & & 1 \end{pmatrix}$. With this, Equation 7.5 becomes: $\begin{pmatrix} b_1 & b_2 + cy_1 & b_3 \end{pmatrix} = \begin{pmatrix} b_1' & z_0 b_1' + b_2' & z_1 b_1' + b_3' \end{pmatrix}$. Now, as $c \neq 0$, choose y_1 so that $b_2' = 0$.

When $b_1 = 0$, we have $b_3' = b_3$. Thus, B is reduced to $\begin{pmatrix} 1 & a_0 & & & b_3 \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{U_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & & y_2 & y_3 \\ & 1 & & & \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. By a routine check, we can see that this centralizer is commutative, and of size q^5. (A,B) is of type R_2, and there are $q^2(q-1)$ such branches.

When $b_1 \neq 0$, choose z_1 such that $b_3' = 0$. Thus, B is reduced to $\begin{pmatrix} 1 & a_0 & a_1 & & \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{U_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & \frac{b_1}{c}z_0 & y_2 & y_3 \\ & 1 & & & \\ & & 1 & z_0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. This centralizer is of size q^4, and is commutative. Thus (A,B) is of type R_3, and there are $(q-1)^2 q$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0$: Here $Z = \begin{pmatrix} 1 & & z_1 \\ & 1 & z_2 \\ & & 1 \end{pmatrix}$. Here Equation 7.5 becomes: $\begin{pmatrix} b_1 & b_2 & b_3 + cy_2 \end{pmatrix} = \begin{pmatrix} b_1' & b_2' & b_3' + z_1 b_1' + z_2 b_2' \end{pmatrix}$.

We have $b_1' = b_1$ and $b_2' = b_2$, and choose y_2 so that $b_3' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & a_0 & b_1 & b_2 & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & c \\ & & & & 1 \end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & y_1 & \frac{b_1}{c}z_1 + \frac{b_2}{c}z_2 & y_3 \\ & 1 & & & \\ & & 1 & & z_1 \\ & & & 1 & z_2 \\ & & & & 1 \end{pmatrix} \right\}$. This too is of type B_6, and there are $q^3(q-1)$ such branches.

And now we have the last case: **When** $C = \begin{pmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{pmatrix}, c, d \neq 0$: Here $Z = \begin{pmatrix} 1 & z_0 & z_1 \\ & 1 & \lambda z_0 \\ & & 1 \end{pmatrix}$,

where $\lambda = \frac{d}{c}$. Equation 7.5 becomes: $\begin{pmatrix} b_1 & b_2 + cy_1 & b_3 + dy_2 \end{pmatrix} = \begin{pmatrix} b_1' & b_2' + z_0 b_1' & b_3' + z_1 b_1' + \lambda z_0 b_2' \end{pmatrix}$.
We have $b_1' = b_1$, and choose y_1 so that $b_2' = 0$, and y_2 so that $b_3' = 0$. Hence B is reduce to $\begin{pmatrix} 1 & a_0 & b_1 & & \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & d \\ & & & & 1 \end{pmatrix}$, and $Z_{U_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_0 & \frac{b_1}{c}z_0 & \frac{b_1}{d}z_1 & y_3 \\ & 1 & & & \\ & & 1 & z_0 & z_1 \\ & & & 1 & \lambda z_0 \\ & & & & 1 \end{pmatrix} \right\}$. This centralizer is 4 dimensional, and commutative. Thus (A,B) is of type R_3, and there are $(q-1)^2 q^2$ such branches.

With this, we have no other cases to analyse. So from the calculations, we have:

- q^2 branches of type A_5.
- $q(q-1) + q(q-1) = 2q(q-1)$ branches of type B_4.
- $q(q-1) + q(q-1)^2 + q^3(q-1) = q^4 - q^2$ branches of type B_6.

- $q^2(q-1)$ branches of type R_2.
- $q(q-1)^2 + q^2(q-1)^2 = q(q-1)(q^2-1)$ branches of type R_3, and
- $q(q-1)^2$ branches of the new type UNT_1.

$\square$

7.2. **Branching of type B.** Now we look at the B types and decide its branching.

Proposition 7.7. *An upper unitriangular matrix of type B_1 has the following branches:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
B_1	q^2	R_3	$(q-1)^2(q^2+q+1)$
A_2	$3q^2-3q$	UNT_1	$(q-1)^2$
R_1	$2q^3-4q+2$	UNT_2	$2q^2-2q$
R_2	$q(q-1)^2(q+2)$	UNT_3	$(q-1)^3.$
B_6	$q^2(q-1)$		

We have seen UNT_1 earlier. There are two more new types here UNT_2 with centralizer $\left\{\begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \lambda x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}\right\}$ *and UNT_3 with centralizer* $\left\{\begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \lambda_1 x_1 & z_2 & w_1 \\ & & 1 & \lambda_2 x_1 & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}\right\}$.

Proof. A matrix of type B_1 has the canonical form: $\begin{pmatrix} 1 & & & a & \\ & 1 & & & b \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. We may take $a = b = 1$. Then $Z_{UT_5(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} 1 & {}^t\overrightarrow{b} & a_2 \\ & C & \overrightarrow{d} \\ & & 1 \end{pmatrix} \mid C \in UT_3(\mathbf{F}_q), \begin{array}{l} {}^t\overrightarrow{b} = (a_1 \ b_1 \ b_2) \\ \overrightarrow{d} = \begin{pmatrix} d_1 \\ d_2 \\ a_1 \end{pmatrix} \end{array} \right\}$. Let $B = \begin{pmatrix} 1 & {}^t\overrightarrow{b} & a_2 \\ & C & \overrightarrow{d} \\ & & 1 \end{pmatrix}$, $X = \begin{pmatrix} 1 & {}^t\overrightarrow{y} & x_2 \\ & Z & \overrightarrow{w} \\ & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & {}^t\overrightarrow{b'} & a_2' \\ & C' & \overrightarrow{d'} \\ & & 1 \end{pmatrix} = XBX^{-1}$. Then $XB = B'X$ leads to firstly $ZC = C'Z$, so we might as well take C to be a conjugacy class representative in $UT_3(\mathbf{F}_q)$, and Z, a centralizer matrix of C. We also get in ${}^t\overrightarrow{b}$, and $\overrightarrow{d}$, $a_1' = a_1$, and the following equations:

$$\begin{pmatrix} a_1 & b_1 & b_2 \end{pmatrix} + \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} C = \begin{pmatrix} a_1' & b_1' & b_2' \end{pmatrix} Z + \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} \tag{7.10}$$

$$Z \begin{pmatrix} d_1 \\ d_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ a_1 \end{pmatrix} + C \begin{pmatrix} w_1 \\ w_2 \\ x_1 \end{pmatrix} \tag{7.11}$$

$$a_2 + x_1 d_1 + y_1 d_2 + y_2 d_1 = a_2' + a_1 w_1 + b_1' w_2 + b_2' x_1 \tag{7.12}$$

We look at two main cases: $a_1 = 0$, and $a_1 \neq 0$.

Case $a_1 = 0$**:** First we look at $b_1 = b_2 = d_1 = d_2 = 0$. Here Equation 7.10 reduces to $\begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} C = \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix}$, Equation 7.11 reduces to $C \begin{pmatrix} w_1 & w_2 & x_1 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & x_1 \end{pmatrix}$, and from Equation 7.12, we have $a_2' = a_2$.

When $C = I_3$, Equations 7.10 and 7.11 are void, and we have $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. Thus (A, B) is of type B_1, and there are q such branches.

When $C = \left(\begin{smallmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{smallmatrix}\right), c \neq 0$, we have from Equation 7.10: $cx_1 = 0$. Hence $x_1 = 0$. With this Equation 7.11 becomes void. So, we have $B = \begin{pmatrix} 1 & & & & a_2 \\ & 1 & & c & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a canonical form of type A_2. (A, B) is a branch of type A_2, and there are $q(q-1)$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), c \neq 0$, we have $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. From equation 7.10, with this C, we get: $\begin{pmatrix} x_1 & y_1 + cx_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix}$, which leaves us with $x_1 = 0$. Equation 7.11 becomes: $\begin{pmatrix} w_1 \\ w_2 \\ \\ \end{pmatrix} = \begin{pmatrix} w_1 + cw_2 \\ w_2 \\ \\ \end{pmatrix}$, thus we have $w_2 = 0$. So we have $B = I_5 + cE_{24} + a_2E_{15}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_1. Thus (A, B) is of type R_1, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0, Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. Equation 7.10 becomes: $\begin{pmatrix} x_1 & y_1 & cy_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix}$, which leaves us with $y_1 = 0$. Equation 7.11 becomes $\begin{pmatrix} w_1 \\ w_2 + cx_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ x_1 \end{pmatrix}$, which leads to $x_1 = 0$. So $B = I_5 + a_2E_{15} + cE_{34}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_1. (A, B) is a branch of type R_1, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{pmatrix}, c_1, c_2 \neq 0,\ Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \frac{d}{c}z_1 \\ & & 1 \end{pmatrix}$. Equation 7.10 becomes $\begin{pmatrix} x_1 & cx_1 + y_1 & dy_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_2 & y_2 \end{pmatrix}$, which leaves us with $x_1 = y_1 = 0$. Equation 7.11 becomes $\begin{pmatrix} w_1 + cw_2 \\ w_2 \\ \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \end{pmatrix}$, which leaves us with $w_2 = 0$. Hence $B = I_5 + a_2E_{15} + cE_{23} + dE_{34}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & \frac{d}{c}z_1 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type R_2. (A, B) is a branch of type R_2, and there are $q(q-1)^2$ branches.

When $((b_1, b_2), (d_1, d_2)) \neq (\overrightarrow{0}, \overrightarrow{0})$**:**

We start with $C = I_3$**:** Thus Equation 7.10 becomes $\begin{pmatrix} 0 & b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & b_1' & b_1'z_3 + b_2' \end{pmatrix}$. We have $b_1' = b_1$, and thus $b_2 = b_2' + b_1z_3$. First, when $b_1 \neq 0$, we choose z_3 so that $b_2' = 0$. Thus, on replacing b_2 by $b_2' = 0$ in the equation above, we have $z_3 = 0$. So Equation 7.11 boils down to $\begin{pmatrix} d_1 + z_1d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$. So we have $d_2' = d_2$. So, again, over here, when $d_2 \neq 0$, choose z_1 such that $d_1' = 0$. With these, Equation 7.12 becomes $y_1d_2 + a_2 = a_2' + w_2b_1$. So, choose w_2 such that $a_2' = 0$. So, B is reduced to

$I_5 + b_1E_{13} + d_2E_{35}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & & \frac{d_2}{b_1} y_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This centralizer is isomorphic to that of the new type UNT_1 (as seen in Proposition 7.4), via the isomorphism that maps generators to generators, and extended product-wise.

When $b_1 \neq 0$, and $d_2 = 0$, we have $d_1' = d_1$, and Equation 7.12 becomes $x_1 d_1 + = a_2' + b_1 w_2$. So, we choose w_2 such that $a_2' = 0$. So B is reduced to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \frac{d_1}{b_1} x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A, B) is of a new type, which we call UNT_2, and there are $q(q-1)$ such branches.

When $b_1 = 0$, $b_2' = b_2$. First, when $d_2 \neq 0$, we choose z_1 so that $d_1' = 0$, and hence Equation 7.12 becomes $a_2 + y_1 d_2 = a_2' z + b_2 x_1$. As $d_2 \neq 0$, choose y_1 so that $a_2' = 0$. Hence B is reduced to $\begin{pmatrix} 1 & & & b_2 & \\ & 1 & & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & \frac{b_2}{d_2} x_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This too is of type UNT_2, and there are $q(q-1)$ such branches.

When $d_2 = 0$, we have $d_1' = d_1$. So Equation 7.12 looks like: $a_2 + x_1 d_1 = a_2' + x_1 b_2$. When $b_2 = d_1 \neq 0$, we have $a_2' = a_2$, and B is reduced to $\begin{pmatrix} 1 & & & b_2 & a_2 \\ & 1 & & & b_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. Thus (A, B) is of type B_1, and there are $q(q-1)$ such branches.

When $b_2 \neq d_1$, choose an x_1 such that $a'_2 = 0$. So, B is boiled down to $\begin{pmatrix} 1 & & & b_1 & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type A_2, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$, $c \neq 0$**:** Equation 7.10 becomes: $\begin{pmatrix} 0 & b_1 & b_2 + cx_1 \end{pmatrix} = \begin{pmatrix} 0 & b'_1 & b'_1 z_3 + b'_2 \end{pmatrix}$. We have $b'_1 = b_1$, and as $c \neq 0$, we can choose x_1 so that $b'_2 = 0$. So, on replacing b_2 by 0 in the above equation, we have $x_1 = \frac{b_1}{c} z_3$.

Then from Equation 7.11, we have $\begin{pmatrix} d_1 + d_2 z_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d'_1 + b_1 z_3 \\ d'_2 \\ 0 \end{pmatrix}$. First, when $b_1 \neq 0$, we choose a z_3 so that $d'_1 = 0$. With these, Equation 7.12 becomes: $a_2 + y_1 d_2 = a'_2 + b_1 w_2$. As $b_1 \neq 0$, choose w_2 so that $a'_2 = 0$. So B is reduced to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & c & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. When $d_2 \neq 0$ $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & \frac{d_2}{c} z_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & \frac{d_2}{b_1} z_1 & \frac{d_2}{b_1} y_1 \\ & & & 1 & \frac{d_2}{c} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. This isn't isomorphic to the centralizer of any matrix in $UT_5(\mathbf{F}_q)$, hence (A,B) is of a new type UNT_3, and there are $(q-1)^3$ such branches. When $d_2 = 0$, $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. Hence (A,B) is of type R_1, and there are $(q-1)^2$ such branches.

When $b_1 = 0$, we have $x_1 = \frac{b_1}{c} z_3 = 0$, we have $d'_1 = d_1 + z_1 d_2$. When $d_2 \neq 0$, choose z_1 such that $d'_1 = 0$. Equation 7.12 becomes $a_2 + y_1 d_2 = a'_2$. Choose y_1 so that $a'_2 = 0$.

So, B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & & c & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$.
(A,B) is of type R_1, and there are $(q-1)^2$ such branches.

When $b_1 = 0$, and $d_2 = 0$. Then $d_1' = d_1$, which is $\neq 0$ Then from Equation 7.12, we simply have $a_2' = a_2$, and B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & & c & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. $Z_{UT_5(\mathbf{F}_q)}(A,B) =$
$\left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type A_2, and there are $q(q-1)$ such branches.

When $C\begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$**:** $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. Equation 7.10 becomes $\begin{pmatrix} 0 & b_1 + cx_1 & b-2 \end{pmatrix} =$ $\begin{pmatrix} 0 & b_1' & b_2' \end{pmatrix}$. We get that $b_2' = b_2$. We choose x_1 so that $b_1' = 0$. Thus, on replacing b_1 by 0, and equating the above equation, we have $x_1 = 0$. Equation 7.11 becomes $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} cw_2 + d_1' \\ d_2' \\ 0 \end{pmatrix}$. Again, over here, we choose w_2 such that $d_1' = 0$. Now, on substituting b_1 with 0, we have $w_2 = \frac{d_1}{c} z_1$. So, Equation 7.12 becomes $a_2 + y_1 d_2 = a_2'$.

When $d_2 \neq 0$, choose y_1 such that $a_2' =$. So B is reduced to $\begin{pmatrix} 1 & & & b_2 & \\ & 1 & c & & \\ & & 1 & & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \frac{d_2}{c} z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is therefore of type R_2, and there are $q(q-1)^2$ such branches.

When $d_2 = 0$, and $b_2 \neq 0$. Then Equation 7.12 becomes $a_2' = a_2$. So, B is reduced to $\begin{pmatrix} 1 & & & b_2 & a_2 \\ & 1 & c & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. So (A, B) is of type R_1, and there are $q(q-1)^2$ branches of this type.

When $C \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0$**:** $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. Equation 7.10 becomes $\begin{pmatrix} 0 & b_1 & b_2 + cy_1 \end{pmatrix} = \begin{pmatrix} 0 & b_1' & b_1' z_3 + b_2' \end{pmatrix}$. $b_1' = b_1$. Choose y_1 such that $b_2' = 0$. So, on substituting b_2 with 0, we have $y_1 = \frac{b_1}{c} z_3$. Equation 7.11 becomes $\begin{pmatrix} d_1 + d_2 z_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' + cx_1 \\ 0 \end{pmatrix}$. Choose x_1 such that $d_2' = 0$. So $d_1' = d_1$. Hence, on replacing d_2 by 0, we get $x_1 = 0$. Hence, Equation 7.12 becomes $a_2 = a_2' + b_1 w_2$. When $b_1 \neq 0$, choose w_2 such that $a_2' = 0$. Thus, B is reduced to $\begin{pmatrix} 1 & & b_1 & & \\ & 1 & & & \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & \frac{b_1}{c} z_3 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. So (A, B) is of type R_2, and there are $q(q-1)^2$ such branches.

When $b_1 = 0$, and $d_1' = d_1$, we get from Equation 7.12, we get $a_2' = a_2$. Hence B is reduced to $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & & & \\ & & 1 & c & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & z_3 & w_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type R_1, and there are $q(q-1)^2$ such branches.

When $C \begin{pmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{pmatrix}, c, d \neq 0$**:** Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \frac{d}{c} z_1 \\ & & 1 \end{pmatrix}$. Equation 7.10 becomes $\begin{pmatrix} 0 & b_1 + cx_1 & b_2 + dy_1 \end{pmatrix} = \begin{pmatrix} 0 & b_1' & b_2' + \frac{d}{c} z_1 b_1 \end{pmatrix}$. Choose x_1 such that $b_1' = 0$, and then choose y_1 such that $b_2' = 0$. So, on substituting b_1 with 0, we get $x_1 = 0$. Then, on substituting b_2 with 0, we get $y_1 = 0$. Equation 7.11 becomes $\begin{pmatrix} d_1 + d_2 z_1 \\ d_2 \\ 0 \end{pmatrix} =$

$\begin{pmatrix} d_1' + cw_2 \\ d_2' \\ 0 \end{pmatrix}$. We have $d_2' = d_2 \neq 0$, choose w_2 such that $d_1' = 0$. Thus, with these Equation 7.12 becomes $a_2' = a_2$. So B becomes $\begin{pmatrix} 1 & & & & a_2 \\ & 1 & c & & \\ & & 1 & d & d_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. $Z_{UT_5(\mathbf{F}_q)}(A, B) =$

$\left\{ \begin{pmatrix} 1 & & & y_2 & x_2 \\ & 1 & \frac{d}{c}z_1 & z_2 & w_1 \\ & & 1 & z_1 & \frac{d_2}{c}z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type R_2. There are $q(q-1)^3$ such branches.

When $a_1 \neq 0$: We now look at the branches, where the entry $a_1 \neq 0$.

When $C = I_3$: Equation 7.10 becomes $\begin{pmatrix} a_1 & b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_1' z_3 + b_2' \end{pmatrix}$. As $a_1 \neq 0$, choose z_1 such that $b_1' = 0$. Then choose z_2 such that $b_2' = 0$. Now, when we replace b_1 and b_2 by 0 in the above equation, we get $z_1 = z_2 = 0$. Then Equation 7.11 becomes $\begin{pmatrix} d_1 \\ d_2 + a_1 z_3 \\ a_1 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ a_1 \end{pmatrix}$. Choose z_3 such that $d_2' = 0$. Equation 7.12 becomes $a_2 + x_1 d_1 + a_1 y_2 = a_2' + a_1 w_1$. Choose w_1 such that $a_2' = 0$. So, B is boiled down to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & & & d_1 \\ & & 1 & & \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & & y_2 + \frac{d_1}{a_1}x_1 \\ & & 1 & & w_2 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. Now, we see that this centralizer is of size q^5, hence we expect it to be a commutative one. But it isn't. We also know that no matrix in $UT_5(\mathbf{F}_q)$ has a non-commutative centralizer of size q^5, and it is isomorphic to that of the type B_6. Thus, (A, B) is of type B_6, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$: Equation 7.10 in this case is

$$\begin{pmatrix} a_1 & b_1 & b_2 + cx_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_1' z_3 + b_2' \end{pmatrix}.$$

Choose z_1 so that $b_1' = 0$. Then, we choose z_2 so that $b_2' = 0$. Thus, on substituting b_1 and b_2 with 0, we get $z_1 = 0$, and $z_2 = \frac{c}{a_1}x_1$. Then Equation 7.11 becomes $\begin{pmatrix} d_1 \\ d_2 + z_3 a_1 \\ a_1 \end{pmatrix} =$

$\begin{pmatrix} d_1' \\ d_2' \\ a_1 \end{pmatrix}$. Choose z_3 such that $d_2' = 0$. Equation 7.12 becomes $a_2 + x_1 d_1 + a_1 y_2 = a_2' + a_1 w_1$. Choose w_1 such that $a_2' = 0$. So B boils down to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & & c & d_1 \\ & & 1 & & \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & & \frac{c}{a_1} x_1 & \frac{d_1}{a_1} x_1 + y_2 \\ & & 1 & & w_2 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This too is a branch of type B_6, and there are $q(q-1)^2$ such branches.

When $C = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$: Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. Equation 7.10 in this case is

$$\begin{pmatrix} a_1 & b_1 + c x_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b_1' & a_1 z_2 + b_2' \end{pmatrix}.$$

Choose z_1 so that $b_1' = 0$, and choose z_2 so that $b_2' = 0$. Thus, on substituting b_1 and b_2 with 0, we get $z_1 = \frac{c}{a_1} x_1$, and $z_2 = 0$. Then Equation 7.11 becomes $\begin{pmatrix} d_1 + \frac{c}{a_1} x_1 d_2 \\ d_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} d_1' + c w_2 \\ d_2' \\ a_1 \end{pmatrix}$. So $d_2' = d_2$, and we choose w_2 such that $d_1' = 0$. Equation 7.12 becomes $a_2 + y_1 d_2 + a_1 y_2 = a_2' + a_1 w_1$. Choose w_1 such that $a_2' = 0$. So B boils down to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & c & & \\ & & 1 & & d_2 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \frac{c}{a_1} x_1 & & \frac{d_2}{a_1} y_1 + y_2 \\ & & 1 & & \frac{d_2}{a_1} x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This is a branch of type R_3, and there are $q(q-1)^2$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0$: Here $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. Equation 7.10 in this case is

$$\begin{pmatrix} a_1 & b_1 & b_2 + c y_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1' & a_1 z_2 + b_1' z_3 + b_2' \end{pmatrix}.$$

We get $b'_1 = b_1$. Choose y_1 so that $b'_2 = 0$. Thus, on substituting b_2 with 0, we get $y_1 = \frac{a_1}{c}z_2 + \frac{b_1}{c}z_3$. Then Equation 7.11 becomes $\begin{pmatrix} d_1 + a_1 z_2 \\ d_2 + a_1 z_3 \\ a_1 \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 + cx_1 \\ a_1 \end{pmatrix}$. Choose z_2 such that $d'_1 = 0$, and z_3 such that $d'_2 = 0$. Equation 7.12 becomes $a_2 + a_1 y_2 = a'_2 + b_1 w_2 + a_1 w_1$. Choose w_1 such that $a'_2 = 0$. So B boils down to $\begin{pmatrix} 1 & a_1 & b_1 & & \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & x_1 & \frac{b_1}{a_1}x_1 & y_2 & x_2 \\ & 1 & & & y_2 - \frac{b_1}{a_1}w_2 \\ & & 1 & \frac{c}{a_1}x_1 & w_2 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This is a branch of type R_3, and there are $q(q-1)^2$ such branches.

And, lastly:

When $C = \begin{pmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{pmatrix}, c, d \neq 0$**:** Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \frac{d}{c}z_1 \\ & & 1 \end{pmatrix}$. Equation 7.10 in this case is

$$\begin{pmatrix} a_1 & b_1 + cx_1 & b_2 + dy_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 z_1 + b'_1 & a_1 z_2 + \frac{d}{c}b'_1 z_1 + b'_2 \end{pmatrix}.$$

Choose x_1 so that $b'_1 = 0$, and choose y_1 so that $b'_2 = 0$. Thus, on substituting b_1 and b_2 with 0, we get $x_1 = \frac{a_1}{c}z_1$, and $y_1 = \frac{a_1}{d}z_2$. Then Equation 7.11 becomes $\begin{pmatrix} d_1 + z_1 d_2 + z_2 a_1 \\ d_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} d'_1 + cw_2 \\ d'_2 \\ a_1 \end{pmatrix}$. So $d'_2 = d_2$, and we choose w_2 such that $d'_1 = 0$. Equation 7.12 becomes $a_2 + \frac{a_1}{d}z_2 d_2 + a_1 y_2 = a'_2 + a_1 w_1$. Choose w_1 such that $a'_2 = 0$. So B boils down to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & c & & \\ & & 1 & d & d_2 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & \frac{a_1}{c}z_1 & \frac{a_1}{d}z_2 & y_2 & x_2 \\ & 1 & z_1 & z_2 & \frac{d_2}{d}z_2 + y_2 \\ & & 1 & & \frac{d_2}{c}z_1 + \frac{a_1}{c}z_2 \\ & & & 1 & \frac{a_1}{c}x_1 \\ & & & & 1 \end{pmatrix} \right\}$. This is a branch of type R_3, and there are $q(q-1)^3$ such branches.

$\square$

Proposition 7.8. *An upper unitriangular matrix of type B_2 has*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
B_2	q^2	R_3	$(q-1)^2q^2$
A_4	q^3-q	UNT_1	$q(q-1)^2$
B_4	q^2-q	UNT_2	q^3-q^2
R_1	$(q^2-q)(q^2+q-1)$	B_6	$q^4-q^3.$
R_2	$(q-1)(q^3-q)$		

Proof. We may take $A = \begin{pmatrix} 1 & & 1 & & \\ & 1 & & & 1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. The first of the two canonical forms mentioned for a matrix of type B_2. For this A, we have $Z_{UT_5(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} 1 & c_1 & c_2 & b_1 & d_1 \\ & 1 & c_3 & b_2 & d_2 \\ & & 1 & & c_1 \\ & & & 1 & a \\ & & & & 1 \end{pmatrix} \right\}$.

We rewrite such a matrix as $\begin{pmatrix} C & \overrightarrow{b} & \overrightarrow{d} \\ & 0 & c_1 \\ & 1 & a \\ & & 1 \end{pmatrix}$, where $C \in UT_3(\mathbf{F}_q)$, and c_1 is the $(1,2)^{th}$ entry of C, and $\overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and $\overrightarrow{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

Let $B = \begin{pmatrix} C & \overrightarrow{b} & \overrightarrow{d} \\ & 0 & c_1 \\ & 1 & a \\ & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} C' & \overrightarrow{b'} & \overrightarrow{d'} \\ & 0 & c'_1 \\ & 1 & a' \\ & & 1 \end{pmatrix}$ be a conjugate of B by $X = \begin{pmatrix} Z & \overrightarrow{y} & \overrightarrow{w} \\ & 0 & z_1 \\ & 1 & x \\ & & 1 \end{pmatrix}$ Then from $XB = B'X$, we have $ZC = C'Z$. Thus, we can C to be a conjugacy class representative in $UT_3(\mathbf{F}_q)$, and $Z \in Z_{UT_3(\mathbf{F}_q)}(C)$, and we also have $a' = a$. With this, we have the following equations

$$Z \begin{pmatrix} \overrightarrow{b} \\ 0 \end{pmatrix} = (C - I_3) \begin{pmatrix} \overrightarrow{y} \\ 0 \end{pmatrix} + \begin{pmatrix} \overrightarrow{b'} \\ 0 \end{pmatrix} \tag{7.13}$$

$$Z \begin{pmatrix} \overrightarrow{d} \\ c_1 \end{pmatrix} + \begin{pmatrix} a\overrightarrow{y} \\ 0 \end{pmatrix} = (C - I_3) \begin{pmatrix} \overrightarrow{w} \\ z_1 \end{pmatrix} + \begin{pmatrix} x\overrightarrow{b} \\ 0 \end{pmatrix} + \begin{pmatrix} \overrightarrow{d'} \\ c_1 \end{pmatrix} \tag{7.14}$$

We first look at the case $\overrightarrow{b} = \overrightarrow{0}$:

When $a = 0$: Here, Equation 7.13 becomes $(C - I_3)\begin{pmatrix} \overrightarrow{y} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

When $C = I_3$: Here Equation 7.13 becomes void, and Equation 7.14 becomes $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$.

When $d_2 = 0$, we have $d_1' = d_1$. Thus $B = \begin{pmatrix} 1 & & & & d_1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type B_2, and there are q such branches.

When $d_2 \neq 0$, choose z_1 so that $d_1' = 0$. Then B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$.

Hence $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$, which is the centralizer of a matrix of type A_4. So (A, B) is of type A_4, and there are $(q - 1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$: Here too, Equation 7.13 stays void. So, we directly look at Equation 7.14, which boils down to: $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + c z_1 \\ d_2' \\ 0 \end{pmatrix}$. We have $d_2' = d_2$.

We look at two cases here:

When $d_2 = c$, we get $d_1' = d_1$, and thus B is reduced to $\begin{pmatrix} 1 & & c & & d_1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. (A, B) is of type B_2, and there are $q(q-1)$ such branches.

When $d_2 \neq c$, choose z_1 such that $d_1' = 0$. Thus B boils down to $\begin{pmatrix} 1 & & c & & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type A_4, and there are $(q-1)^2$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), c \neq 0$: Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. From Equation 7.13 we have $\begin{pmatrix} 0 & c & 0 \\ & & \\ & & \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \end{pmatrix} = \overrightarrow{0}$. We have $cy_2 = 0$, thus $y_2 = 0$. Equation 7.14 becomes: $\begin{pmatrix} d_1 + z_1 d_2 + z_2 c \\ d_2 \\ c \end{pmatrix} = \begin{pmatrix} d_1' + cw_2 \\ d_2' \\ c \end{pmatrix}$. Choose w_2 such that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & c & & & \\ & 1 & & & d_2 \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & & & z_2 + \frac{d_2}{c} z_1 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type B_6, and there are $(q-1)q$ such branches.

When $C = \left(\begin{smallmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{smallmatrix}\right), c \neq 0$: Here $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. In this case, Equation 7.13 stays void. So we directly jump to Equation 7.14. We have $\begin{pmatrix} d_1 \\ d_2 \\ \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ \end{pmatrix}$. So we have

$$B = \begin{pmatrix} 1 & & & & d_1 \\ & 1 & c & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \text{ and } Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}. \text{ Hence } (A,B)$$

is of type A_4, and there are $q^2(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c, d \neq 0$**:** Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \frac{d}{c} z_1 \\ & & 1 \end{pmatrix}$. Equation 7.13 becomes $\begin{pmatrix} cy_2 \\ 0 \\ 0 \end{pmatrix} = \overrightarrow{0}$. Thus $y_2 = 0$. From Equation 7.14 we have $\begin{pmatrix} d_1 + z_1 d_2 + cz_2 \\ d_2 + z_1 d \\ c \end{pmatrix} = \begin{pmatrix} d_1' + cw_2 \\ d_2' + dz_1 \end{pmatrix}$. Hence $d_2' = d_2$, and choose w_2 such that $d_1' = 0$. So B boils down to $\begin{pmatrix} 1 & c & & & \\ & 1 & d & & d_2 \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{d}{c} z_1 & & z_2 + \frac{d_2}{c} z_1 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type B_6 and there are $(q-1)^2 q$ such branches.

bfseries $a \neq 0$: We are still dealing with $\overrightarrow{b} = \overrightarrow{0}$ here. So Equation 7.13 becomes $(C - I_3) \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. And Equation 7.14 becomes: $Z \begin{pmatrix} d_1 \\ d_2 \\ c_1 \end{pmatrix} + a \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ c_1 \end{pmatrix} + (C - I_3) \begin{pmatrix} w_1 \\ w_2 \\ z_1 \end{pmatrix}$

When $C = I_3$**:** Equation 7.13 becomes void, and from Equation 7.14, we have $\begin{pmatrix} d_1 + z_1 d_2 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$. Choose y_2 and y_1 such that, d_2' and d_1' become 0. Hence, $B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(a, b) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & & w_1 \\ & 1 & z_3 & & w_2 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type B_4, and there are $q - 1$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$**:** Here also, Equation 7.13 remains void. Equation 7.14 becomes: $\begin{pmatrix} d_1 + z_2 d_2 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + cz_1 \\ d_2' \\ 0 \end{pmatrix}$. Choose y_2 and y_1 so that $d_2' = d_1' = 0$. Thus,

B is reduced to $\begin{pmatrix} 1 & & c & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & \frac{c}{a}z_1 & w_1 \\ & 1 & z_3 & & w_2 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$.
(A,B) is of type B_4, and there are $(q-1)^2$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), c \neq 0$**:** From Equation 7.13, like we saw before, $y_2 = 0$. Thus Equation 7.14 boils down to $\begin{pmatrix} d_1 + z_1 d_2 + z_1 c + a y_1 \\ d_2 \\ c \end{pmatrix} = \begin{pmatrix} d_1' + c w_2 \\ d_2 \\ c \end{pmatrix}$. We see that $d_2' = d_2$. Choose w_2 such that $d_1' = 0$. So B is reduced to $\begin{pmatrix} 1 & c & & & \\ & 1 & & & d_2 \\ & & 1 & & c \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & & & z_2 + \frac{d_2}{c}z_1 + \frac{a}{c}y_1 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type B_6, and there are $(q-1)^2 q$ such branches.

When $C = \left(\begin{smallmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{smallmatrix}\right), c \neq 0$**:** Here, Equation 7.13 stays void, and Equation 7.14 becomes $\begin{pmatrix} d_1 + a y_1 \\ d_2 + a y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \\ 0 \end{pmatrix}$. Choose y_1, y_2 such that $d_1' = d_2' = 0$. SO B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & c & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & z_2 & & w_1 \\ & 1 & z_3 & & w_2 \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is therefore of type R_2, and there are $(q-1)^2$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{smallmatrix}\right), c, d \neq 0$**:** Here, like earlier, from Equation 7.13, we get $y_2 = 0$. Hence Equation 7.14 boils down to $\begin{pmatrix} d_1 + z_1 d_2 + z_2 c + a y_1 \\ d_2 + z_1 d \\ c \end{pmatrix} = \begin{pmatrix} d_1' + c w_2 \\ d_2' + d z_1 \\ c \end{pmatrix}$. This leaves us with $d_2' = d_2$, and choose y_1 such that $d_1' = 0$. So we have B boiling down to

$\begin{pmatrix} 1 & c & & & \\ & 1 & d & & d_2 \\ & & 1 & & c \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$ and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & \frac{c}{a}w_2 - \frac{c}{a}z_2 - \frac{d_2}{a}z_1 & w_1 \\ & 1 & \frac{d}{c}z_1 & & w_2 \\ & & 1 & & z_1 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$.
(A,B) is thus, of type B_6, and there are $q(q-1)^3$ such branches.
Now we look at what happens, when $\overrightarrow{b} \neq \overrightarrow{0}$.

When $C = I_3$: Subcase $a = 0$: From Equation 7.13 we have $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1' \\ b_2' \\ 0 \end{pmatrix}$.
When $b_2 \neq 0$, we choose z_1 such that $b_1' = 0$. Thus replacing b_1 by 0 in the above equation, we obtain $z_1 = 0$. Hence, Equation 7.14 boils down to $\begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' + x b_2 \\ 0 \end{pmatrix}$.

We have $d_1' = d_1$. Choose x such that $d_2' = 0$. So B boils down to $\begin{pmatrix} 1 & & & & d_1 \\ & & & b_2 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type R_1, and there are $q(q-1)$ such branches.

When $b_2 = 0$, we have to look at $b_1 \neq 0$, and we have $b_1' = b_1$. Equation 7.14 becomes $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + x b_1 \\ d_2' \\ 0 \end{pmatrix}$. So $d_2' = d_2$, and choose x such that $d_1' = 0$. Hence $B = \begin{pmatrix} 1 & & & b_1 & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{d_2}{b_1}z_1 \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A,B) is of the new type UNT_2, and there are $(q-1)q$ such branches.

Subcase $a \neq 0$, and $b_2 \neq 0$: In Equation 7.13, we choose z_1 to get rid of b_1', and like before $z_1 = 0$. Equation 7.14 becomes: $\begin{pmatrix} d_1 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' + xb_2 \\ 0 \end{pmatrix}$. Choose y_1 such that $d_1' = 0$, and x such that $d_2' = 0$. So $B = \begin{pmatrix} 1 & & & & \\ & 1 & & b_2 & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \frac{a}{b_2} y_2 \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A, B) is of type R_2, and there are $(q-1)^2$ such branches.

Subcase $a \neq 0$ and $b_2 = 0$. Here we have $b_1' = b_1 \neq 0$. From Equation 7.14 we have $\begin{pmatrix} d_1 + z_1 d_2 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + b_1 x \\ d_2' \\ 0 \end{pmatrix}$. Choose y_2 such that $d_2' = 0$, and x such that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & & & b_1 & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & z_3 & & w_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{a}{b_1} y_1 \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of new type UNT_1, and there are $(q-1)^2$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$: Here Equation 7.13 stays as it was in the previous case, i.e., $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1' \\ b_2' \\ 0 \end{pmatrix}$.

When $b_2 \neq 0$, choose z_1 so that $b_1' = 0$, and on replacing b_1 with $b_1' = 0$ in the above equation, we get $z_1 = 0$. Hence, Equation 7.14 becomes: $\begin{pmatrix} d_1 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' + xb_2 \\ 0 \end{pmatrix}$. We can choose x such that $d_2' = 0$.

Subcase $a = 0$. We have in this $d_1' = d_1$. B reduces to $\begin{pmatrix} 1 & & c & & d_1 \\ & 1 & & b_2 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. So, (A, B) is of type R_1, and there are $q(q-1)^2$ such branches.

Subcase $a \neq 0$. Here, in addition to getting rid of d_2', we choose y_1 such that $d_1' = 0$. So, B reduces to $\begin{pmatrix} 1 & & c & & \\ & 1 & & b_2 & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \frac{a}{b_2} y_2 \\ & & & & 1 \end{pmatrix} \right\}$. So (A, B) is of type R_2, and there are $(q-1)^3$ such branches.

When $b_2 = 0$, here $b_1' = b_1 \neq 0$. Equation 7.14 becomes $\begin{pmatrix} d_1 + z_1 d_2 + ay_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + b_1 x + cz_1 \\ d_2' \\ 0 \end{pmatrix}$. Choose x so that $d_1' = 0$.

Subcase $a = 0$. Here $d_2' = d_2$, and B thus reduces to $\begin{pmatrix} 1 & & c & b_1 & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with

$$Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{d_2 - c}{b_1} z_1 \\ & & & & 1 \end{pmatrix} \right\}.$$ Hence (A,B) is of the new type UNT_2, and there are $(q-1)^2 q$ such branches.

Subcase $a \neq 0$. Here, choose y_2 such that $d_2' = 0$. Hence B is reduced to $\begin{pmatrix} 1 & & c & b_1 & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$,

with $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & z_3 & & w_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{a}{b_1} y_1 - \frac{c}{b_1} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. Hence (A,B) is of type UNT_1, and there are $(q-1)^3$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{smallmatrix}\right)$, $c \neq 0$ **:** Here Equation 7.13 becomes $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1' + c y_2 \\ b_2' \\ 0 \end{pmatrix}$. Choose y_2 such that $b_1' = 0$. We have $b_2' = b_2 \neq 0$. On replacing b_1 with 0 in the above equation, we get $y_2 = \frac{b_2}{c} z_1$. Equation 7.14 thus becomes $\begin{pmatrix} d_1 + z_1 d_2 + c z_2 + a y_1 \\ d_2 + \frac{a b_2}{c} z_1 \\ c \end{pmatrix} = \begin{pmatrix} d_1' + c w_2 \\ d_2' + x b_2 \\ c \end{pmatrix}$. Choose w_2 such that $d_1' = 0$, and x such that $d_2' = 0$. Hence B is reduced to $\begin{pmatrix} 1 & c & & & \\ & 1 & & b_2 & \\ & & 1 & & c \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & & \frac{b_2}{c} z_1 & \frac{a}{c} y_1 + z_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{a}{c} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. (A,B) is of type R_3, and there are $(q-1)^2 q$ such branches.

When $C = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, c \neq 0$ **:** Here Equation 7.13 becomes $\begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ 0 \end{pmatrix}$. We have $b'_1 = b_1$, and $b'_2 = b_2$ Equation 7.14 thus becomes $\begin{pmatrix} d_1 + ay_1 \\ d_2 + ay_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d'_1 + xb_1 \\ d'_2 + xb_2 \\ 0 \end{pmatrix}$.

Subcase $a = 0$: When $b_1 \neq 0$, choose x such that $d'_1 = 0$. Thus, on replacing d_1 with $d'_1 = 0$, we get $x = 0$, and thus $d'_2 = d_2$. Hence B is reduced to $\begin{pmatrix} 1 & & & b_1 & \\ & 1 & c & b_2 & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A, B) is of type R_1, and there are $(q-1)^2 q^2$ such branches.

When $b_1 = 0$, we work with $b_2 \neq 0$. Choose x such that $d'_2 = 0$, and with this on replacing d_2 with $d'_2 = 0$, we have $x = 0$, which leaves us with $d'_1 = d_1$. B is reduced to $\begin{pmatrix} 1 & & & & d_1 \\ & 1 & c & b_2 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. Hence we have another branch of type R_1, and these are $(q-1)^2 q$ in number.

Subcase $a \neq 0$. We just choose y_1, y_2 such that $d'_1 = d'_2 = 0$. Here $(b_1, b_2) \neq (0, 0)$. So, $B = \begin{pmatrix} 1 & & & b_1 & \\ & 1 & c & b_2 & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & \frac{b_1}{a}x & w_1 \\ & 1 & z_3 & \frac{b_2}{a}x & w_2 \\ & & 1 & & \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A, B) is of type R_2, and there are $(q-1)^2(q^2-1)$ such branches (as $(b_1, b_2) \neq (0, 0)$).

When $C = \begin{pmatrix} 1 & c & \\ & 1 & d \\ & & 1 \end{pmatrix}, c, d \neq 0$ **:** Here Equation 7.13 becomes $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b'_1 + cy_2 \\ b'_2 \\ 0 \end{pmatrix}$. Choose y_2 such that $b'_1 = 0$. We have $b'_2 = b_2 \neq 0$. On replacing b_1 with 0 in the above

equation, we get $y_2 = \frac{b_2}{c} z_1$. Equation 7.14 thus becomes $\begin{pmatrix} d_1 + z_1 d_2 + c z_2 + a y_1 \\ d_2 + \frac{ab_2}{c} z_1 + d z_1 \\ c \end{pmatrix} = \begin{pmatrix} d_1' + c w_2 \\ d_2' + x b_2 + d z_1 \\ c \end{pmatrix}$. Choose w_2 such that $d_1' = 0$, and x such that $d_2' = 0$. Hence B is reduced to $\begin{pmatrix} 1 & c & & & \\ & 1 & d & b_2 & \\ & & 1 & & c \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{d}{c} z_1 & \frac{b_2}{c} z_1 & \frac{a}{c} y_1 + z_2 \\ & & 1 & & z_1 \\ & & & 1 & \frac{a}{c} z_1 \\ & & & & 1 \end{pmatrix} \right\}$.
(A, B) is of type R_3, and there are $(q-1)^3 q$ such branches. □

Proposition 7.9. *An upper unitriangular matrix of type B_3 has q^3 branches of type B_3, $q^2(q^2+q+1)(q-1)$ branches of regular type R_1, and $q^3(q-1)$ branches of regular type R_3.*

Proof. Let $A = \begin{pmatrix} 1 & & a & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a, b \neq 0$ a matrix of type B_3. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & \lambda x_0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b}{a}, x_i, y_i \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & y_0 & y_1 & y_2 \\ & & 1 & \lambda x_0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & b_0 & b_1 & b_2 \\ & & 1 & \lambda a_0 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a_0' & a_1' & a_2' & a_3' \\ & 1 & b_0' & b_1' & b_2' \\ & & 1 & \lambda a_0' & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_0 = a_0'$, $b_0 = b_0'$, $b_2 = b_2'$, and the following equations:

$$\begin{aligned} a_1 + b_0 x_0 &= a_0' y_0 + a_1' \\ a_2 + x_0 b_1 + \lambda x_1 a_0 &= a_0' y_1 + \lambda x_0 a_1' + a_2' \\ a_3 + x_0 b_2 &= a_3' + a_0' y_2 \\ b_1 + \lambda a_0 y_0 &= \lambda x_0 b_0' + b_1' \end{aligned}$$

We look at three cases, the first case is when $\lambda a_1 = b_1$ and $(a_0, b_0, b_2) = \mathbf{0}$. The second case is when $\lambda a_1 \neq b_1$ and $(a_0, b_0, b_2) = \mathbf{0}$. The third case is when $(a_0, b_0, b_2) \neq \mathbf{0}$.

Case: $\lambda a_1 = b_1$ **and** $(a_0, b_0, b_2) = \mathbf{0}$. In this case, we get $a_2 = a_2'$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type B_3, and there are q^3 branches.

Case: $\lambda a_1 \neq b_1$ **and** $(a_0, b_0, b_2) = \mathbf{0}$. In this case, we can choose x_0 in such a way that we get $a_2 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^2(q-1)$ branches of this type.

Case: $(a_0, b_0, b_2) \neq \mathbf{0}$. We first consider that $a_0 \neq 0$, then we can choose y_0, y_1 and y_2 in such a way that we get $a_1 = a_2 = a_3 = 0$ and $b_1 = b_1'$. By simple calculations, we get that $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is a commutative group of size q^4. Thus (A, B) is of regular type R_3, and there are $q^3(q-1)$ branches of this type.

Next we consider the case when $a_0 = 0$ and $b_0 \neq 0$. Here we can choose x_0 in such a way that we get $a_1 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^4(q-1)$ branches of this type.

Finaly we consider the case when $a_0 = b_0 = 0$ and $b_2 \neq 0$., now we can choose x_0 in such a way that we get $a_3 = 0$. Again, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^6. Thus (A, B) is of regular type R_1, and there are $q^3(q-1)$ branches of this type.

Therefore we get that a matrix of type B_3 has q^3 branches of type B_3, $q^2(q^2+q+1)(q-1)$ braches of regular type R_1, and $q^3(q-1)$ braches of regular type R_3.

□

Proposition 7.10. *An upper unitriangular matrix of type B_4 has q^3 branches of type B_4, $q^2(q^2-1)$ branches of regular type R_2, and $q^3(q-1)$ branches of regular type R_3.*

Proof. Let $A = \begin{pmatrix} 1 & a & & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a, b \neq 0$ a matrix of type B_4. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & & \lambda x_0 & \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b}{a}, x_i, z_i \in \mathbf{F}_q \right\}$.

Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & & \lambda x_0 & \\ & & 1 & z_0 & z_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & & \lambda a_0 & \\ & & 1 & c_0 & c_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a'_0 & a'_1 & a'_2 & a'_3 \\ & 1 & & \lambda a'_0 & \\ & & 1 & c'_0 & c'_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_0 = a'_0$, $a_1 = a'_1$, $c_0 = c'_0$, $c_1 = c'_1$, and the following equations:

$$\begin{aligned} x_1c_1 + a_3 &= z_1a_1 + a'_3 \\ x_1c_0 + a_2 &= z_0a_1 + a'_2 \end{aligned}$$

We look at two cases, when $(a_1, c_0, c_1) = \mathbf{0}$ and $(a_1, c_0, c_1) \neq bf0$.

Case: $(a_1, c_0, c_1) = \mathbf{0}$**:** In this case, we get $a_2 = a'_2$ and $a_3 = a'_3$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type B_4, and there are q^3 branches.

Case: $(a_1, c_0, c_1) \neq \mathbf{0}$**:** When $a_1 \neq 0$, then we choose z_0 and z_1 in such a way that we get $a_2 = a_3 = 0$. By routine check, we get that $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^4. Thus (A, B) is of the regular type R_3, and there are $q^3(q-1)$ branches of this type.

When $a_1 = 0$ and one of c_0 and c_1 is non-zero. We can choose x_1 in such a way that we get either $a_2 = 0$ or $a_3 = 0$. Again by simple calculations, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^5. Thus (A, B) is of the regular type R_2, and there are $q^2(q^2-1)$ branches of this type. □

Proposition 7.11. *An upper unitriangular matrix of type B_5 has q^2 branches of type B_5, $(q^5 - q)$ branches of regular type B_6.*

Proof. Let $A = \begin{pmatrix} 1 & a & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & b \\ & & & & 1 \end{pmatrix}$, $a, b \neq 0$ a matrix of type B_5. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & & & \lambda x_2 \\ & & 1 & & z_1 \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b}{a}, x_i, z_1, w_0 \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & & & \lambda x_2 \\ & & 1 & & z_1 \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix}$

be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & & & \lambda a_2 \\ & & 1 & & c_1 \\ & & & 1 & d_0 \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a'_0 & a'_1 & a'_2 & a'_3 \\ & 1 & & & \lambda a'_2 \\ & & 1 & & c'_1 \\ & & & 1 & d'_0 \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_0 = a'_0$ $a_1 = a'_1$, $a_2 = a'_2$, $c_1 = c'_1$, $d_0 = d'_0$, and the following equation:

$$x_2 d_0 + c_1 x_1 + \lambda a_2 x_0 + a_3 = \lambda x_2 a'_0 + z_1 a'_1 + w_0 a'_2 + a'_3$$

We look at three cases, the first case is when $\lambda a_0 = d_0$ and $(a_1, a_2, c_1) = \mathbf{0}$. The second case is when $\lambda a_0 \neq d_0$ and the third case is when $\lambda a_0 = d_0$ but $(a_1, a_2, c_1) \neq \mathbf{0}$.

Case: $\lambda a_0 = d_0$ **and** $(a_1, a_2, c_1) = \mathbf{0}$. In this case, we get $a_3 = a'_3$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type B_5, and there are q^2 branches.

Case: $\lambda a_0 \neq d_0$ In this case, we can choose x_2 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is group of size q^5 isomorphic to centralizer of one of the type B_6. Thus (A, B) is of type B_6, and there are $q^4(q-1)$ branches of this type.

Case: $\lambda a_0 = d_0$ **and** $(a_1, a_2, c_1) \neq \mathbf{0}$. In this case, one of a_1, a_2 and c_1 is non-zero and depending on this, we can choose one of z_1, w_0 or x_1 suitably in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is group of size q^5 isomorphic to centralizer of one of the type B_6. Thus (A, B) is of type B_6, and there are $q(q-1)(q^2+q+1)$ branches of this type.

Therefore a matrix of type B_5 has q^2 branches of type B_5 and total $q(q^4-1)$ braches of type B_6. □

Proposition 7.12. *An upper unitriangular matrix of type B_6 has q^3 branches of type B_6, and $q^2(q^2-1)$ branches of regular type R_3.*

Proof. Let $A = \begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a, b \neq 0$ a matrix of type B_6. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & \lambda x_0 & & \\ & & 1 & & \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix} \mid \lambda = \frac{b}{a}, x_i, w_0 \in \mathbf{F}_q \right\}$. Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & \lambda x_0 & & \\ & & 1 & & \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix}$

be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & \lambda a_0 & & \\ & & 1 & & \\ & & & 1 & d_0 \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a'_0 & a'_1 & a'_2 & a'_3 \\ & 1 & \lambda a'_0 & & \\ & & 1 & & \\ & & & 1 & d'_0 \\ & & & & 1 \end{pmatrix}$ be a conjugate of B by X. Thus equating $XB = B'X$ gives us $a_0 = a'_0$, $a_1 = a'_1$, $a_2 = a'_2$, $d_0 = d'_0$, and the following equation:

$$x_2 d_0 + a_3 = w_0 a'_2 + a'_3$$

We look at two cases, when $(a_2, d_0) = (0,0)$ and $(a_2, d_0) \neq (0,0)$.

Case: $(a_2, d_0) = (0,0)$ In this case, we get $a_3 = a'_3$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A,B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A,B) is a branch of type B_6, and there are q branches.

Case: $(a_2, d_0) \neq (0,0)$ In this case, one of d_0 and a_2 is non-zero. We can choose x_2 or w_0 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A,B)$ is commutative group of size q^4. Thus (A,B) is of the regular type R_3, and there are $q^2(q^2-1)$ branches of this type. $\square$

7.3. **Branching of type D.** Now we look at the branching for type D.

Proposition 7.13. *An upper unitriangular matrix of type D_1 has the following branches:*

Branch	*No. of Branches*	*Branch*	*No. of Branches*
D_1	q^2	R_2	$q^2(q-1)$
B_4	$2q(q-1)$	R_3	$q^2(q^2-1)$.
UNT_3	$q(q-1)^2$		

Proof. An upper unitriangular matrix of type D_1 has the canonical form $A = \begin{pmatrix} 1 & & c_1 & & \\ & 1 & & c_2 & \\ & & 1 & & c_3 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, where $a, b, c \neq 0$. $Z_{UT_5(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & b_1 & d_1 \\ & 1 & a_3 & b_2 & d_2 \\ & & 1 & \frac{c_2}{c_1} a_1 & \frac{c_3}{c_1} a_2 \\ & & & 1 & \frac{c_3}{c_2} a_3 \\ & & & & 1 \end{pmatrix} \right\}$, which we rewrite as:

$$Z_{UT_5(\mathbf{F}_q)}(A) = \left\{ \begin{pmatrix} C & \overrightarrow{b} & \overrightarrow{d} \\ & 1 & \frac{c_3}{c_2} C_{23} \\ & & 1 \end{pmatrix} \mid C \in UT_3(\mathbf{F}_q),\ \overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \\ \frac{c_2}{c_1} C_{12} \end{pmatrix}, \overrightarrow{d} = \begin{pmatrix} d_1 \\ d_2 \\ \frac{c_3}{c_1} C_{13} \end{pmatrix} \right\}.$$

Let $B = \begin{pmatrix} C & \overrightarrow{b} & \overrightarrow{d} \\ & 1 & \frac{c_3}{c_2}C_{23} \\ & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} C' & \overrightarrow{b'} & \overrightarrow{d'} \\ & 1 & \frac{c_3}{c_2}C'_{23} \\ & & 1 \end{pmatrix}$ be a conjugate of B by a member $X = \begin{pmatrix} Z & \overrightarrow{y} & \overrightarrow{w} \\ & 1 & \frac{c_3}{c_2}Z_{23} \\ & & 1 \end{pmatrix} \in Z_{UT_5(\mathbf{F}_q)}(A)$, with $\overrightarrow{y} = \begin{pmatrix} y_1 \\ y_2 \\ \frac{c_2}{c_1}Z_{12} \end{pmatrix}$, and $\overrightarrow{w} = \begin{pmatrix} w_1 \\ w_2 \\ \frac{c_3}{c_1}Z_{13} \end{pmatrix}$. We thus have $XB = B'X$. First thing we see is that $ZC = C'Z$. So we can take C to be a conjugacy class representative in $UT_3(\mathbf{F}_q)$, and we thus have the following equations:

$$Z\overrightarrow{b} + \overrightarrow{y} = C\overrightarrow{y} + \overrightarrow{b'} \tag{7.15}$$

$$Z\overrightarrow{d} + \frac{c_3}{c_2}C_{23}\overrightarrow{y} + \overrightarrow{w} = C'\overrightarrow{w} + \frac{c_3}{c_2}Z_{23}\overrightarrow{b'} + \overrightarrow{d'} \tag{7.16}$$

When $C = I_3$: In this case $C_{12} = C_{13} = C_{23} = 0$. We have $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$.

Equation 7.15 becomes: $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ 0 \end{pmatrix}$. We look at two cases here: When $b_2 \neq 0$, and when $b_2 = 0$.

When $b_2 = 0$, We have $b'_1 = b_1$, and Equation 7.16 becomes:

$$\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d'_1 + \frac{c_3}{c_2}z_3 b_1 \\ d'_2 + \frac{c_3 z_3}{c_2} b'_2 \\ 0 \end{pmatrix}$$

We have $d'_2 = d_2$.

When $b_2 = b_1 = d_2 = 0$: We have $d'_1 = d_1$. Thus B is reduced to $\begin{pmatrix} 1 & & & & d_1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$.

So $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. Hence (A, B) is a branch of type D_1, and there are q such branches.

When $b_2 = b_1 = 0$, and $d_2 \neq 0$, we can choose z_1 such that $d_1 = 0$. Thus, B is reduced to $\begin{pmatrix} 1 & & & & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \frac{c_3}{c_1}z_2 \\ & & & 1 & \frac{c_3}{c_2}z_3 \\ & & & & 1 \end{pmatrix} \right\}$, which is of type . So (A, B) is a branch of type B_4, as $Z_{UT_5(\mathbf{F}_q)}(A, B)$ can be conjugated by the elementary matrix that swaps rows and columns 1 and 2 to get the centralizer subgroup of one of the canonical matrices of type B_4, and there are $(q-1)$ branches of this type.

When $b_1 \neq 0$, in Equation 7.16, we choose z_3 so that $d_1 = 0$. Thus B is reduced to $\begin{pmatrix} 1 & & & b_1 & \\ & 1 & & & d_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{c_2 d_2}{c_3 b_1} z_1 & y_2 & w_2 \\ & & 1 & \frac{c_2}{c_1} z_1 & \frac{c_3}{c_1} z_2 \\ & & & 1 & \frac{d_2}{b_1} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. Again, we have 2 cases here:

When $d_2 = 0$, $B = \begin{pmatrix} 1 & & & b_1 & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. Here $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & & y_2 & w_2 \\ & & 1 & \frac{c_2}{c_1} z_1 & \frac{c_3}{c_1} z_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$.

On conjugating by an elementary matrix, which swaps rows and columns 2 and 3 of each element of $Z_{UT_5(\mathbf{F}_q)}(A,B)$, we get the centralizer of one of the canonical matrices of the type B_4. Thus there are $q-1$ branches of type B_4.

When $d_2 \neq 0$, we have $Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{c_2 d_2}{c_3 b_1} z_1 & y_2 & w_2 \\ & & 1 & \frac{c_2}{c_1} z_1 & \frac{c_3}{c_1} z_2 \\ & & & 1 & \frac{d_2}{b_1} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. Thus this branch is of the new type UNT_3, and there are $(q-1)^2$ such branches.

When $b_2 \neq 0$, choose z_1 such that $b_1' = 0$. Thus equating Equation 7.15 with b_1 replaced by 0, we get that $z_1 = 0$. Thus with $b_1 = 0$ and $z_1 = 0$, we get from Equation 7.16, $d_1' = d_1$, and with a nice choice of z_3, we can reduce d_2' to 0. Hence, B is reduced to $\begin{pmatrix} 1 & & & & d_1 \\ & 1 & & b_2 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & & y_2 & w_2 \\ & & 1 & & \frac{c_3}{c_1} z_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is a centralizer of type R_2. Thus (A,B) is a branch of type R_2, and there are $q(q-1)$ such branches.

When $C = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}, c \neq 0$: Here Equation 7.15 becomes: $\begin{pmatrix} b_1 + z_1 b_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1' + \frac{c_3 c}{c_2} z_1 \\ b_2' \\ 0 \end{pmatrix}$.

So we have $b_2' = b_2$. We see 2 cases here: $b_2 = \frac{c_2}{c_1} c$, and $b_2 \neq \frac{c_2}{c_1} c$.

When $b_2 \neq \frac{c_2}{c_1} c$. In the above equation, we choose z_1 such that $b_1' = 0$. Thus, with substituting b_1 with $b_1' = 0$ in the above equation, we get $z_1 = 0$. Thus, with this, Equation 7.16 becomes $\begin{pmatrix} d_1 \\ d_2 + \frac{c_3}{c_1} c z_3 \\ \frac{c_3}{c_1} c \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 + \frac{c_3}{c_2} b_2 z_3 \\ \frac{c_3}{c_1} c \end{pmatrix}$. As $b_2 \neq \frac{c_2}{c_1} c$, we can choose a z_3 so that $d'_2 = 0$, and we have $d'_1 = d_1$. So B boils down to $\begin{pmatrix} 1 & & c & & d_1 \\ & 1 & & b_2 & \\ & & 1 & & \frac{c_3}{c_1} c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & & y_2 & w_2 \\ & & 1 & & \frac{c_3}{c_1} z_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. Thus (A, B) too is a branch of type R_2, and there are $q(q-1)^2$ such branches.

When $b_2 = \frac{c_2}{c_1} c$, we get from Equation 7.15, $b'_1 = b_1$. Equation 7.16 boils down to: $\begin{pmatrix} d_1 + z_1 d_2 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d'_1 + \frac{c_3}{c_2} z_3 b_1 \\ d'_2 \\ 0 \end{pmatrix}$. So we have $d'_2 = d_2$. We look first at $b_1 = d_2 = 0$. B is reduced to $\begin{pmatrix} 1 & & c & & d_1 \\ & 1 & & \frac{c_2}{c_1} c & \\ & & 1 & & \frac{c_3}{c_1} c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. Thus (A, B) is a branch of type D_1, and there are $q(q-1)$ such branches.

When $b_1 \neq 0$ choose z_3 such that $d'_1 = 0$. So, B becomes: $\begin{pmatrix} 1 & & c & b_1 & \\ & 1 & & \frac{c_2}{c_1} c & d_2 \\ & & 1 & & \frac{c_3}{c_1} c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. We have two cases here:

When $d_2 = 0$, we have $B = \begin{pmatrix} 1 & & c & b_1 & \\ & 1 & & \frac{c_2}{c_1}c & \\ & & 1 & & \frac{c_3}{c_1}c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ and

$$Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & & y_2 & w_2 \\ & & 1 & \frac{c_2}{c_1}z_1 & \frac{c_3}{c_1}z_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\},$$

thus (A,B) is of a type B_4, and there are $(q-1)^2$ such branches.

When $d_2 \neq 0$, $B = \begin{pmatrix} 1 & & c & b_1 & \\ & 1 & & \frac{c_2}{c_1}c & d_2 \\ & & 1 & & \frac{c_3}{c_1}c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$$Z_{UT_5(\mathbf{F}_q)}(A,B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{c_2 d_2}{c_3 b_1}z_1 & y_2 & w_2 \\ & & 1 & \frac{c_2}{c_1}z_1 & \frac{c_3}{c_1}z_2 \\ & & & 1 & \frac{d_2}{b_1}z_1 \\ & & & & 1 \end{pmatrix} \right\},$$

so, this branch too is of the type UNT_3. Thus there are $(q-1)^3$ branches of this new type.

When $b_1 = 0$, and $d_2 \neq 0$. We choose z_1 so that $d_1' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & & c & & \\ & 1 & & \frac{c_2}{c_1}c & d_2 \\ & & 1 & & \frac{c_3}{c_1}c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)} = \left\{ \begin{pmatrix} 1 & & z_2 & y_1 & w_1 \\ & 1 & z_3 & y_2 & w_2 \\ & & 1 & & \frac{c_3}{c_1}z_2 \\ & & & 1 & \frac{c_3}{c_2}z_3 \\ & & & & 1 \end{pmatrix} \right\}$. This is of type B_4. (A,B) is a branch of type B_4, and there are $(q-1)^2$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), c \neq 0$: Here, $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \\ & & 1 \end{pmatrix}$. Equation 7.15 boils down to $\begin{pmatrix} b_1 + b_2 z_1 + \frac{c_2}{c_1}cz_2 \\ b_2 \\ \frac{c_2}{c_1}c \end{pmatrix} = \begin{pmatrix} cy_2 + b_1' \\ b_2' \\ \frac{c_2}{c_1}c \end{pmatrix}$. So $b_2' = b_2$. As $c \neq 0$, we choose y_2 such that

$b_1' = 0$. Equation 7.16 becomes: $\begin{pmatrix} d_1 + d_2 z_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} cw_2 + d_1' \\ d_2' \\ 0 \end{pmatrix}$. We have $d_2' = d_2$.

Take w_2 such that $d_1' = 0$. So B is reduced to $\begin{pmatrix} 1 & c & & & \\ & 1 & & b_2 & d_2 \\ & & 1 & \frac{c_2}{c_1}c & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and therefore

$Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & y_2 \\ & 1 & & \frac{b_2}{c}z_1 + \frac{c_2}{c_1}z_2 & \frac{d_2}{c}z_1 \\ & & 1 & \frac{c_2}{c_1}z_1 & \frac{c_3}{c_1}z_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$, which is of size q^4. It is routine to check that this centralizer is commutative. Thus this is a centralzier of type R_3. Thus (A, B) is a branch of type R_3, and there are $q^2(q-1)$ such branches.

When $C = \left(\begin{smallmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{smallmatrix}\right), c \neq 0$**:** In this case $Z = \begin{pmatrix} 1 & & z_2 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}$. With this, Equation 7.15 becomes $\begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1' \\ b_2' \\ 0 \end{pmatrix}$. So, our focus thus is solely on Equation 7.16. The equation is reduced to $\begin{pmatrix} d_1 + \frac{c_3}{c_2}cy_1 \\ d_2 + \frac{c_3}{c_2}cy_2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1' + \frac{c_3}{c_2}b_1 z_3 \\ d_2' + \frac{c_3}{c_1}cz_2 + \frac{c_3}{c_2}b_2 z_3 \\ 0 \end{pmatrix}$

As $\frac{c_3}{c_2}c \neq 0$, choose y_1, y_2 so that $d_1' = d_2' = 0$. Thus B is reduced to $\begin{pmatrix} 1 & & & b_1 & \\ & 1 & c & b_2 & \\ & & 1 & & \\ & & & 1 & \frac{c_3}{c_2}c \\ & & & & 1 \end{pmatrix}$,

and $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & & z_2 & \frac{b_1}{c}z_3 & w_1 \\ & 1 & z_3 & \frac{c_2}{c_1}z_2 + \frac{b_2}{c}z_3 & w_2 \\ & & 1 & & \frac{c_3}{c_1}z_2 \\ & & & 1 & \frac{c_3}{c_2}z_3 \\ & & & & 1 \end{pmatrix} \right\}$. This is of size q^4, and with a routine check we see that it is commutative. This is a centralizer of type R_3, hence (A, B) is a branch of type R_3, and there are $q^2(q-1)$ such branches.

When $C = \left(\begin{smallmatrix} 1 & c_0 & \\ & 1 & d_0 \\ & & 1 \end{smallmatrix}\right), c_0, d_0 \neq 0$: Here $Z = \begin{pmatrix} 1 & z_1 & z_2 \\ & 1 & \lambda_0 z_1 \\ & & 1 \end{pmatrix}$, where $\lambda_0 = \frac{d_0}{c_0}$. Equation 7.15 becomes: $\begin{pmatrix} b_1 + z_1 b_2 + z_2 \frac{c_2}{c_1} c_0 \\ b_2 + \frac{c_2}{c_1} d_0 z_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_0 y_2 + b_1' \\ \frac{c_2}{c_1} d_0 z_1 + b_2' \\ 0 \end{pmatrix}$. As c_0 and d_0 are nonzero, we have $b_2' = b_2$. We choose y_2 such that $b_1' = 0$. Hence, on replacing b_1 with 0 in the above equation we get $y_2 = \frac{b_2}{c_0} z_1 + \frac{c_2}{c_1} z_2$. With these, Equation 7.16 boils down to $\begin{pmatrix} d_1 + z_1 d_2 + \frac{c_3}{c_2} d_0 y_1 \\ d_2 \\ 0 \end{pmatrix} = \begin{pmatrix} c_0 w_2 + d_1' \\ d_2' \\ 0 \end{pmatrix}$. So $d_2' = d_2$, and choose w_2 such that $d_1' = 0$. Hence, B is reduced to $\begin{pmatrix} 1 & c_0 & & & \\ & 1 & d_0 & b_2 & d_2 \\ & & 1 & \frac{c_2}{c_1} c_0 & \\ & & & 1 & \frac{c_3}{c_2} d_0 \\ & & & & 1 \end{pmatrix}$, with $Z_{UT_5(\mathbf{F}_q)}(A, B) =$ $\left\{ \begin{pmatrix} 1 & z_1 & z_2 & y_1 & w_1 \\ & 1 & \frac{d_0}{c_0} z_1 & \frac{b_2}{c_0} z_1 + \frac{c_2}{c_1} z_2 & \frac{c_3}{c_2 c_0} y_1 + \frac{d_2 c_0}{d_0} z_1 \\ & & 1 & \frac{c_2}{c_1} z_1 & \frac{c_3}{c_1} z_2 \\ & & & 1 & \frac{c_3 d_0}{c_2 c_0} z_1 \\ & & & & 1 \end{pmatrix} \right\}$. This too is of type R_3. So (A, B) is a brach of type R_3, and there are $q^2(q-1)^2$ such branches.

So, on adding up the branches of each of the types, we have

- q^2 branches of type D_1,
- $2q(q-1)$ branches of type B_4,
- $q^2(q-1)$ branches of type R_2,
- $q^2(q^2-1)$ branches of type R_3, and
- $q(q-1)^2$ branches of type UNT_3.

These match with the estimations done for $q = 3$ in GAP. □

Proposition 7.14. *An upper unitriangular matrix of type D_2 has q^3 branches of type D_2, and $q^2(q^2-1)$ branches of regular type R_3.*

Proof. Let $A = \begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a, b, c \neq 0$ a matrix of type D_2. The centralizer $Z_{UT_5}(A)$ of A is $\left\{ \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & \lambda_1 x_0 & & \lambda_2 x_1 \\ & & 1 & & \lambda_2 x_0 \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix} \mid \lambda_1 = \frac{b}{a}, \lambda_2 = \frac{c}{a}, x_i, w_0 \in \mathbf{F}_q \right\}$.

Let $X = \begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ & 1 & \lambda_1 x_0 & & \lambda_2 x_1 \\ & & 1 & & \lambda_2 x_0 \\ & & & 1 & w_0 \\ & & & & 1 \end{pmatrix}$ be an element of $Z_{UT_5(\mathbf{F}_q)}(A)$. Let $B = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ & 1 & \lambda_1 a_0 & & \lambda_2 a_1 \\ & & 1 & & \lambda_2 a_0 \\ & & & 1 & d_0 \\ & & & & 1 \end{pmatrix}$, and $B' = \begin{pmatrix} 1 & a_0' & a_1' & a_2' & a_3' \\ & 1 & \lambda_1 a_0' & & \lambda_2 a_1' \\ & & 1 & & \lambda_2 a_0' \\ & & & 1 & d_0' \\ & & & & 1 \end{pmatrix} = XBX^{-1}$. Thus equating $XB = B'X$ gives us $a_0 = a_0'$, $a_1 = a_1'$, $a_2 = a_2'$, $d_0 = d_0'$, and the following equation:

$$x_2 d_0 + a_3 = w_0 a_2' + a_3'$$

We look at two cases, when $(a_2, d_0) = (0, 0)$ and $(a_2, d_0) \neq (0, 0)$.

Case: $(a_2, d_0) = (0, 0)$ In this case, we get $a_3 = a_3'$. Therefore $Z_{UT_5(\mathbf{F}_q)}(A, B) = Z_{UT_5(\mathbf{F}_q)}(A)$. So (A, B) is a branch of type D_2, and there are q branches.

Case: $(a_2, d_0) \neq (0, 0)$ In this case, one of d_0 and a_2 is non-zero. We can choose x_2 or w_0 in such a way that we get $a_3 = 0$. By routine check, we get $Z_{UT_5(\mathbf{F}_q)}(A, B)$ is commutative group of size q^4. Thus (A, B) is of the regular type R_3, and there are $q^2(q^2 - 1)$ branches of this type. □

Proposition 7.15. *A matrix of the R_1 type has q^6 branches of type R_1, a matrix of the R_2 type has q^5 branches of type R_2, and a matrix of the R_3 type has q^4 branches of type R_3.*

Proof. The type R_1, R_2 and R_3 are Regular types, hence the centralizer of matrices of such a type is a commutative. □

7.4. **Branching Rules for the New Types.** While determining the branching rules for the types in $UT_5(\mathbf{F}_q)$, we observed that there are some commuting pairs of elements of

$UT_5(\mathbf{F}_q)$, which are not isomorphic to the centralizers of any of the elements in $UT_5(\mathbf{F}_q)$. Thus, giving rise to what we call "new types". The new types, we have seen so far are UNT_1 (first observed in Proposition 7.4), UNT_2 (observed in Proposition 7.7) and UNT_3 (observed in Propositions 7.7). Now, we compute the branching for these cases and we see that no further new types occur.

Proposition 7.16. *The new type UNT_1 has q^3 branches of type UNT_1, $q^2(q^2-1)$ branches of type R_2, and $q^4 - q^3$ branches of type R_3.*

Proof. For some pair (A, B) of commuting elements in $UT_5(\mathbf{F}_q)$, of type UNT_1, the centralizer subgroup is $Z_{UT_5(\mathbf{F}_q)}(A, B) = \left\{ \begin{pmatrix} 1 & x_0 & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \right\}$, where $\lambda \neq 0$ is fixed. Let $C = \begin{pmatrix} 1 & a_0 & a_1 & \lambda c_0 & a_3 \\ & 1 & b_0 & & b_2 \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix}$, and let $C' = \begin{pmatrix} 1 & a'_0 & a'_1 & \lambda c'_0 & a'_3 \\ & 1 & b'_0 & & b'_2 \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix} = XCX^{-1}$, where $X = \begin{pmatrix} 1 & x_0 & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix}$. On equation $XC = CX$, we get $a'_0 = a_0$, $b'_0 = b_0$, $b'_2 = b_2$, $c'_0 = c_0$, and the following equations:

$$a_1 + x_0 b_0 = a'_1 + a_0 y_0 \tag{7.17}$$

$$a_3 + x_0 b_2 = a'_3 + y_2 a_0 \tag{7.18}$$

We look at two main cases: $(a_0, b_2) = (0, 0)$, and $(a_0, b_2) \neq (0, 0)$.

When $a_0 = b_2 = 0$: Equation 7.18 becomes $a'_3 = a_3$, Equation 7.17 becomes $a'_1 = a_1 + x_0 b_0$. We have two subcases here:

When $b_0 = 0$, then we get $a'_1 = a_1$. Thus C boils down to $\begin{pmatrix} 1 & & a_1 & \lambda c_0 & a_3 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B, C) = Z_{UT_5(\mathbf{F}_q)}(A, B)$. (A, B, C) is therefore of type UNT_1, and there are q^3 such branches.

When $b_0 \neq 0$, in Equation 7.17, we can choose x_0 such that $a_1' = 0$. Hence C is reduced to $\begin{pmatrix} 1 & & & \lambda c_0 & a_3 \\ & 1 & b_0 & & \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} 1 & & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \right\}$. Easy to see that this is a commutative group of size q^5. (A,B,C) is a branch of type R_2, and there are $q^2(q-1)$ such branches.

When $(a_0, b_2) \neq (0,0)$**:** When $a_0 \neq 0$, in Equation 7.17, we choose y_0 such that $a_1' = 0$. Thus, on replacing a_1 with $a_1' = 0$ in that equation, we get $y_0 = \frac{b_0}{a_0} x_0$. In Equation 7.18 choose y_2 so that $a_3' = 0$. Thus C is reduced to $\begin{pmatrix} 1 & a_0 & & \lambda c_0 & \\ & 1 & b_0 & & b_2 \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix}$, and thus $Z_{UT_5(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} 1 & x_0 & x_1 & \lambda z_0 & x_3 \\ & 1 & \frac{b_0}{a_0} x_0 & & \frac{b_2}{a_0} x_0 \\ & & 1 & & y_1 \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \right\}$. Easy to see that this subgroup is a commutative one of size q^4. Thus (A,B,C) is a branch of type R_3, and there are $(q-1)q^3 = q^4 - q^3$ such branches.

When $a_0 = 0$, and $b_2 \neq 0$. Equation 7.18 becomes $a_3 + x_0 b_2 = a_3'$, and Choose x_0 such that $a_3' = 0$. Then, on replacing a_3 with $a_3' = 0$ in Equation 7.18, we get $x_0 = 0$. With these, Equation 7.17 becomes $a_1' = a_1$. C thus boils down to $\begin{pmatrix} 1 & & a_1 & \lambda c_0 & \\ & 1 & b_0 & & b_2 \\ & & 1 & & \\ & & & 1 & c_0 \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B,C) = \left\{ \begin{pmatrix} 1 & & x_1 & \lambda z_0 & x_3 \\ & 1 & y_0 & & y_2 \\ & & 1 & & \\ & & & 1 & z_0 \\ & & & & 1 \end{pmatrix} \right\}$. This branch too is of type R_2, and there are $q^3(q-1)$ such branches. So, in total there are $q^3(q-1) + q^2(q-1) = q^4 - q^2$ branches of type R_2. □

Proposition 7.17. *The new type UNT_2 has q^3 branches of type UNT_2, $q^5 - q^2$ branches of type R_1, and $q^4 - q^3$ branches of type R_3.*

Proof. A commuting pair (A,B) of type UNT_2 has the centralizer $\left\{\begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \lambda x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}\right\}$.

Let $C = \begin{pmatrix} 1 & a_1 & b_1 & b_2 & a_2 \\ & 1 & c_1 & c_2 & d_1 \\ & & 1 & & \lambda a_1 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, $C' = \begin{pmatrix} 1 & a'_1 & b'_1 & b'_2 & a'_2 \\ & 1 & c'_1 & c'_2 & d_1 \\ & & 1 & & \lambda a'_1 \\ & & & 1 & a'_1 \\ & & & & 1 \end{pmatrix}$ be a conjugate of C, and let $X = \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \lambda x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}$ such that $XC = C'X$. Equating $XC = C'X$ gives us $a'_1 = a_1$, $c'_1 = c_1$ and $c'_2 = c_2$, and the following bunch of equations:

$$\begin{pmatrix} a_1 & b_1 + x_1c_1 & b_2 + x_1c_2 \end{pmatrix} = \begin{pmatrix} a_1 & b'_1 + a_1z_1 & b'_2 + a_1z_2 \end{pmatrix} \tag{7.19}$$

$$d_1 + (\lambda z_1 + z_2)a_1 = d'_1 + (\lambda c_1 + c_2)x_1 \tag{7.20}$$

$$a_2 + x_1d_1 + (\lambda y_1 + y_2)a_1 = a'_2 + (\lambda b'_1 + b'_2)x_1 + w_1a_1 \tag{7.21}$$

There are two main cases here:

Case: $a_1 = 0$

When $c_1 = c_2 = 0$, Equation 7.19 leads us to $b'_1 = b_1$, $b'_2 = b_2$, and from Equation 7.20 $d'_1 = d_1$. With these, Equation 7.21 becomes $a_2 + x_1d_1 = a'_2 + (\lambda b_1 + b_2)x_1$.

When $d_1 = \lambda b_1 + b_2$, we get $a'_2 = a_2$. Thus C is reduced to $\begin{pmatrix} 1 & & b_1 & b_2 & a_2 \\ & 1 & & & \lambda b_1 + b_2 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_{A,B,C}(\mathbf{F}_q)} = Z_{UT_5(\mathbf{F}_q)}(A,B)$. Thus, (A,B,C) is of type UNT_2, and there are q^3 such branches.

When $d_1 \neq \lambda b_1 + b_2$, we can choose x_1 such that $a'_2 = 0$. Thus C is reduced to $\begin{pmatrix} 1 & & b_1 & b_2 & \\ & 1 & & & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A,B,C) = \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. Thus (A,B,C) is of type R_1, and there are $q^2(q-1)$ such branches.

When $c_1 \neq 0$, in Equation 7.19, we can choose x_1 so that $b_1' = 0$. Thus on replacing b_1 with $b_1' = 0$, we get $x_1 = 0$, and thus $b_2' = b_2$. And Equation 7.20 reduces to $d_1' = d_1$, and Equation 7.21 boils down to $a_2' = a_2$. C is reduced to $\begin{pmatrix} 1 & & & b_2 & a_2 \\ & 1 & c_1 & c_2 & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and

$Z_{UT_5(\mathbf{F}_q)}(A, B, C) = \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$. (A, B, C) is thus of type R_2, and there are $(q-1)q^4$ such branches.

When $c_1 = 0$, and $c_2 \neq 0$. In Equation 7.19, we get $b_1' = b_1$, and choose x_1 such that $b_2' = 0$. Hence on substituting b_2 with $b_2' = 0$ and equating Equation 7.19, we get $x_1 = 0$. With this Equation 7.20 boils down to $d_1' = d_1$, and Equation 7.21 boils down to $a_2' = a_2$. C is reduced to $\begin{pmatrix} 1 & & b_1 & & a_2 \\ & 1 & & c_2 & d_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z(A, B, C) = \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & z_1 & z_2 & w_1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$.

(A, B, C) is a branch of type R_1, and there are $q^3(q-1)$ such branches.

Case $a_1 \neq 0$: In this case, in Equation 7.19, we choose z_1 and z_2 such that $b_1' = 0$ and $b_2' = 0$ respectively. Thus, on replacing b_1 by $b_1' = 0$, and b_2 by $b_2' = 0$ in Equation 7.19, and equating, we get $z_1 = \frac{c_1}{a_1}x_1$ and $z_2 = \frac{c_2}{a_1}x_1$. Putting these in Equation 7.20 leads us to $d_1 + \left(\lambda\frac{c_1}{a_1}x_1 + \frac{c_2}{a_1}x_1\right)a_1 = d_2' + (\lambda c_1 + c_2)x_1$. Thus $d_1' = d_1$.

With all this, Equation 7.21 boils down to $a_2 + x_1 d_1 + (\lambda y_1 + y_2)a_1 = a_2' + w_1 a_1$. Choose w_1 so that $a_2' = 0$. Hence C is reduced to $\begin{pmatrix} 1 & a_1 & & & \\ & 1 & c_1 & c_2 & d_1 \\ & & 1 & & \lambda a_1 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and

$$Z_{UT_5(\mathbf{F}_q)}(A, B) = \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \frac{c_1}{a_1}x_1 & \frac{c_2}{a_1}x_1 & \lambda y_1 + y_2 + \frac{d_1}{a_1}x_1 \\ & & 1 & & \lambda x_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}.$$

Easy to check that the above centralizer subgroup is a commutative one, of size q^4. Thus (A, B, C) is of type R_3, and there are $(q-1)q^3$ such branches.

Adding up all the branches of type R_1 gives us $q^2(q-1)+q^3(q-1)+q^4(q-1) = q^5 - q^2$ branches of type R_1. $\square$

Proposition 7.18. *The new type UNT_3 has q^3 branches of type UNT_3, $q^4 - q^2$ branches of type R_2, and $q^4 - q^3$ branches of type R_3.*

Proof. A commuting pair (A, B) of matrices in $UT_5(\mathbf{F}_q)$ of type UNT_3 has as its centralizer: $\left\{\begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \lambda_1 x_1 & z_2 & w_1 \\ & & 1 & \lambda_2 x_1 & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}\right\}$.

Let $C = \begin{pmatrix} 1 & a_1 & b_1 & b_2 & a_2 \\ & 1 & \lambda_1 a_1 & c_2 & d_1 \\ & & 1 & \lambda_2 a_1 & \frac{\lambda_2}{\lambda_1} b_1 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and $C' = \begin{pmatrix} 1 & a'_1 & b'_1 & b'_2 & a'_2 \\ & 1 & \lambda_1 a'_1 & c'_2 & d'_1 \\ & & 1 & \lambda_2 a'_1 & \frac{\lambda_2}{\lambda_1} b'_1 \\ & & & 1 & a'_1 \\ & & & & 1 \end{pmatrix} =$ XCX^{-1}, where $X = \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \lambda_1 x_1 & z_2 & w_1 \\ & & 1 & \lambda_2 x_1 & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix}$. From $XC = C'X$, we get $a'_1 = a_1$, $b'_1 = b_1$, $c'_2 = c_2$, and the following equations:

$$b_2 + x_1 c_2 + \lambda_2 y_1 a_1 = b'_2 + z_2 a_1 + \lambda_2 x_1 b_1 \tag{7.22}$$
$$d_1 + \lambda_2 x_1 b_1 + z_2 a_1 = d'_1 + \lambda_2 y_1 a_1 + x_1 c_2 \tag{7.23}$$
$$a_2 + x_1 d_1 + y_2 a_1 = a'_2 + w_1 a_1 + x_1 b'_2. \tag{7.24}$$

Case $a_1 = 0$: Equation 7.22 becomes $b_1 + x_1 c_2 = b'_1 + x_1 \lambda_2 b_1$. When $c_2 = \lambda_2 b_1$, then $b'_2 = b_2$, and similarly in Equation 7.23, $d'_1 = d_1$. Here, if $b_2 = d_1$, we get from Equation 7.24, $a'_2 = a_2$. Hence C is reduced to $\begin{pmatrix} 1 & & b_1 & b_2 & a_2 \\ & 1 & & \lambda_2 b_1 & b_2 \\ & & 1 & & \frac{\lambda_2}{\lambda_1} b_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B, C) = Z_{UT_5(\mathbf{F}_q)}(A, B)$. (A, B, C) is a branch of type UNT_3, and there are q^3 such branches.

When $b_2 \neq d_1$, choose x_1 such that $a_2' = 0$. C is reduced to $\begin{pmatrix} 1 & & b_1 & b_2 & \\ & 1 & & \lambda_2 b_1 & d_1 \\ & & 1 & & \frac{\lambda_2}{\lambda_1} b_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$,

and $Z_{UT_5(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. (A, B, C) is thus of type R_2, and there are $q^2(q-1)$ such branches.

When $c_2 \neq \lambda_2 b_1$. In this case, in equation 7.22 itself, we choose x_1 such that $b_2' = 0$. And on substituting b_2 with 0 in this equation and equating, we get $x_1 = 0$. Thus, Equation 7.23 becomes $d_1' = d_1$, and from Equation 7.24, we get $a_2' = a_2$. Thus C is reduced to $\begin{pmatrix} 1 & & b_1 & & a_2 \\ & 1 & & c_2 & d_1 \\ & & 1 & & \frac{\lambda_2}{\lambda_1} b_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, and $Z_{UT_5(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} 1 & & y_1 & y_2 & x_2 \\ & 1 & & z_2 & w_1 \\ & & 1 & & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}$. This too is a branch of type R_2, and there are $q^3(q-1)$.

Case $a_1 \neq 0$**:** In Equation 7.22 choose z_2 such that $b_2' = 0$. Thus, substituting b_2 with $b_2' = 0$ in this equation, leads us to $z_2 = \lambda_2 y_1 + \frac{(c_2 - \lambda_2 b_1)}{a_1} x_1$. With these Equation 7.23 becomes $d_1' = d_1$. Thus Equation 7.24 becomes $a_2 + x_1 d_1 + y_2 a_1 = a_2' + w_1 a_1$. Choose w_1 such that $a_2' = 0$. Thus C is reduced to $\begin{pmatrix} 1 & a_1 & b_1 & & \\ & 1 & \lambda_1 a_1 & c_2 & d_1 \\ & & 1 & \lambda_2 a_1 & \frac{\lambda_2}{\lambda_1} b_1 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}$, and

$$Z_{UT_5(\mathbf{F}_q)}(A, B, C) = \left\{ \begin{pmatrix} 1 & x_1 & y_1 & y_2 & x_2 \\ & 1 & \lambda_1 x_1 & \lambda_2 y_1 + \frac{(c_2 - \lambda_2 b_1)}{a_1} x_1 & y_2 + \frac{d_1}{a_1} x_1 \\ & & 1 & \lambda_2 x_1 & \frac{\lambda_2}{\lambda_1} y_1 \\ & & & 1 & x_1 \\ & & & & 1 \end{pmatrix} \right\}.$$

By a routine check, one can see that this centralizer group is commutative. Thus we have a branch of type R_3, and there are $(q-1)q^3$ such branches.

Adding up the branches of type R_2, there is a total of $q^2(q-1) + q^3(q-1) = q^4 - q^2$ branches of type R_2. □

8. Commuting Probabilities

The number of simultaneous conjugacy classes of commuting k-tuples in $UT_n(\mathbf{F}_q)$ is denoted by $c_{UT}(n,k,q)$ and the same for $GT_n(\mathbf{F}_q)$ is denoted by $c_{GT}(n,k,q)$. From Lemma 7.1 [SS], it follows that $c_{GT}(n,k,q) = \mathbf{1}.B^k_{GT_n(\mathbf{F}_q)}.e_1$ and $c_{UT}(n,k,q) = \mathbf{1}.B^k_{UT_n(\mathbf{F}_q)}.e_1$ where $\mathbf{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$, and $\mathbf{e}_1 = {}^t\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$. We note that all of the branching matrices computed in this paper for triangular and unitriangular groups have entries polynomial in q with integer coefficients. Thus, $c_{UT}(n,k,q)$ for $n = 3,4,5$ and $c_{GT}(n,k,q)$ for $n = 2,3,4$ are polynomials in q with integer coefficients.

From Theorem 1.1 in [SS], for $k \geq 2$, and any finite group G, the probability that a k-tuple commutes is $cp_k(G) = \dfrac{c_G(k-1)}{|G|^{k-1}} = \dfrac{\mathbf{1}B_G^{k-1}.e_1}{|G|^{k-1}}$. Now, that we have determined the branching matrix for the groups $GT_i(\mathbf{F}_q)$ for $i = 2,3,4$, and $UT_j(\mathbf{F}_q)$ for $j = 3,4,5$, for each of the groups, we will mention the commuting probabilities for $k \leq 5$. This computation is done using Sage [SA].

For the triangular groups we have:

k	$cp_k(GT_2(\mathbf{F}_q))$	k	$cp_k(GT_2(\mathbf{F}_q))$
2	$\frac{1}{q-1}$	4	$\frac{q^2-2q+4}{q^5-3q^4+3q^3-q^2}$
3	$\frac{q^2-q+2}{q^4-2q^3+q^2}$	5	$\frac{q^4-3q^3+7q^2-3q+2}{q^8-4q^7+6q^6-4q^5+q^4}$

k	$cp_k(GT_3(\mathbf{F}_q))$	k	$cp_k(GT_3(\mathbf{F}_q))$
2	$\frac{q^2+q-1}{q^3(q-1)^2}$	4	$\frac{q^5-3q^4+7q^3-5q^2+11q+4}{q^8(q-1)^6}$
3	$\frac{q^3-q^2+q+5}{q^5(q-1)^4}$	5	$\frac{q^7-5q^6+17q^5-32q^4+54q^3-34q^2+25q+2}{q^{11}(q-1)^8}$

k	$cp_k(GT_4(\mathbf{F}_q))$
2	$\frac{q^3+3q^2-2q-1}{q^{10}(q-1)^3}$
3	$\frac{12q^5-52q^4+116q^3-97q^2+63q-37}{q^{20}(q-1)^6}$
4	$\frac{6q^8-16q^7+3q^6+195q^5-593q^4+1105q^3-1129q^2+912q-477}{q^{30}(q-1)^9}$
5	$\frac{7q^{11}-32q^{10}+122q^9-192q^8+342q^7-714q^6+2038q^5-3954q^4+6136q^3-6304q^2+4596q-2213}{q^{40}(q-1)^{12}}$

In the case of unitriangular group we have:

k	$cp_k(UT_3(\mathbf{F}_q))$	k	$cp_k(UT_3(\mathbf{F}_q))$
2	$\frac{q^2+q-1}{q^3}$	3	$\frac{q^3+q^2-1}{q^5}$
4	$\frac{q^4+q^3-1}{q^7}$	5	$\frac{q^5+q^4-1}{q^9}$.

k	$cp_k(UT_4(\mathbf{F}_q))$	k	$cp_k(UT_4(\mathbf{F}_q))$
2	$\frac{2q^3-1}{q^6}$	4	$\frac{q^7+3q^6-3q^5+5q^4-4q^3-3q+2}{q^{15}}$
3	$\frac{2q^4+3q^3-2q^2-3q+1}{q^{10}}$	5	$\frac{q^{10}+2q^9-2q^8+3q^7-q^6+q^4-3q^3-2q+2}{q^{20}}$

k	$cp_k(UT_5(\mathbf{F}_q))$
2	$\frac{5q^4-4q^3+9q^2-14q+5}{q^{10}}$
3	$\frac{11q^8-7q^7+23q^6-41q^5+5q^4+11q^3+3q^2-7q+3}{q^{20}}$
4	$\frac{2q^{13}+3q^{12}+5q^{11}+10q^{10}-6q^9-20q^8+8q^7-27q^6+42q^5-24q^4+9q^3+q^2-5q+3}{q^{29}}$
5	$\frac{\left(\begin{array}{c}2q^{18}+5q^{16}-5q^{15}+23q^{14}-25q^{13}+28q^{12}-41q^{11}+23q^{10}-17q^9+\\ 10q^8-25q^7+18q^6+23q^5-26q^4+7q^3+3q^2-5q+3\end{array}\right)}{q^{38}}$

Appendix A. Conjugacy classes of $GT_4(\mathbf{F}_q)$

The conjugacy classes for upper triangular group can be algorathmically computed following Belitskii's algorithm as described in [Ko] and in the appendix of [Bh]. We list them here for the convenience of reader and also to set the notation for types.

Class Representatives	Number of Classes	Order of Centralizer	Name of Type
$a_0 I_4, a_0 \neq 0$	$(q-1)$	$(q-1)^4 q^6$	C
$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} a & & & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$ $a \neq 0$	$2(q-1)$	$(q-1)^3 q^4$	A_1
$\begin{pmatrix} a & & & \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^3 q^4$	A_1'
$\begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} a & & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$ $a \neq 0$	$2(q-1)$	$(q-1)^3 q^5$	A_2
$\begin{pmatrix} a & & & 1 \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^3 q^6$	A_3

$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^2q^4$	A_4
$\begin{pmatrix} a & & & 1 \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^2q^4$	A_5
$\begin{pmatrix} a & & 1 & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^2q^5$	A_6
$\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} a & & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{pmatrix}$ $a \neq 0$	$2(q-1)$	$(q-1)^2q^3$	A_7
$\begin{pmatrix} a & 1 & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$(q-1)^2q^4$	A_8
$\begin{pmatrix} a & 1 & 1 & \\ & a & & \\ & & a & 1 \\ & & & a \end{pmatrix}$ $a \neq 0$	$q-1$	$q(q-1)q^4$	A_9
$\begin{pmatrix} a & & & \\ & a & & \\ & & b & \\ & & & b \end{pmatrix}, \begin{pmatrix} a & & & \\ & b & & \\ & & a & \\ & & & b \end{pmatrix}$ $\begin{pmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{pmatrix}; a \neq b$	$3(q-1)(q-2)$	$(q-1)^4q^2$	B_1
$\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}, \begin{pmatrix} a & & & \\ & a & & \\ & & b & \\ & & & a \end{pmatrix}$ $\begin{pmatrix} a & & & \\ & b & & \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} b & & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix}; a \neq b$	$4(q-1)(q-2)$	$(q-1)^4q^3$	B_2
$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}$, and 3 more $\begin{pmatrix} a & & & \\ & a & 1 & \\ & & a & \\ & & & b \end{pmatrix}$, and 3 more; $a \neq b$	$8(q-1)(q-2)$	$(q-1)^3q^2$	B_3
$\begin{pmatrix} a & & 1 & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}, \begin{pmatrix} a & & & 1 \\ & a & & \\ & & b & \\ & & & a \end{pmatrix}$ $\begin{pmatrix} a & & & 1 \\ & b & & \\ & & a & \\ & & & a \end{pmatrix}, \begin{pmatrix} b & & & \\ & a & & 1 \\ & & a & \\ & & & a \end{pmatrix}; a \neq b$	$4(q-1)(q-2)$	$(q-1)^3q^3$	B_4
$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & b \end{pmatrix}$, and 5 more; $a \neq b$	$6(q-1)(q-2)$	$(q-1)^3q^2$	B_5
$\begin{pmatrix} a & & & \\ & a & & \\ & & b & \\ & & & c \end{pmatrix}$, and 5 more; $a \neq b \neq c \neq a$	$6(q-1)(q-2)(q-3)$	$(q-1)^4q$	B_6
The Regular types			

$\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{pmatrix}$, $a \neq 0$	$q-1$	$(q-1)q^3$	R_1
$\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{pmatrix}$, and 3 more; $a \neq b$	$4(q-1)(q-2)$	$(q-1)^2q^2$	R_2
$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{pmatrix}$, $\begin{pmatrix} a & & 1 & \\ & b & & 1 \\ & & a & \\ & & & b \end{pmatrix}$ $\begin{pmatrix} a & & & 1 \\ & b & 1 & \\ & & b & \\ & & & a \end{pmatrix}$; $a \neq b$	$3(q-1)(q-2)$	$(q-1)^2q^2$	R_3
$\begin{pmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{pmatrix}$, and 5 others; $a \neq b \neq c \neq a$	$6(q-1)(q-2)(q-3)$	$(q-1)^3q$	R_4
$\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}$, $a \neq b \neq c \neq a$ $a, b, c \neq d$	$(q-1).(q-2).$ $(q-3).q-4)$	$(q-1)^4$	R_5

Appendix B. Conjugacy classes of $UT_4(\mathbf{F}_q)$ and $UT_5(\mathbf{F}_q)$

Understanding conjugacy classes in unitriangular group is a challenging problem. We refer a reader to [VA1, VA2] for the reference. We list down the same for $UT_4(\mathbf{F}_q)$ and $UT_5(\mathbf{F}_q)$, what we need for our purpose.

Class Representatives	Number of Classes	Centralizer size in $UT_4(\mathbf{F}_q)$	Name of Type
$\begin{pmatrix} 1 & & & a \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $a \in \mathbf{F}_q$	q	q^6	C
$\begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $a \in \mathbf{F}_q^*$	$(q-1), (q-1)$	q^5	A_1
$\begin{pmatrix} 1 & & a & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $a, b \in \mathbf{F}_q^*$	$(q-1)^2$	q^5	A_2
$\begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & a \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & b \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & b \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & b & \\ & 1 & & \\ & & 1 & c \\ & & & 1 \end{pmatrix}$, $a, b, c \in \mathbf{F}_q^*$	$(q-1), (q-1),$ $(q-1)^2, (q-1)^2,$ $(q-1)^2, (q-1)^3$	q^4	A_3

$\begin{pmatrix} 1&&&\\ &1&a&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 1&&&b\\ &1&a&\\ &&1&\\ &&&1 \end{pmatrix},$ $a,b \in \mathbf{F}_q^*$	$(q-1),(q-1)^2$	q^4	R_1
$\begin{pmatrix} 1&a&&\\ &1&b&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 1&&&\\ &1&b&\\ &&1&a\\ &&&1 \end{pmatrix},$ $\begin{pmatrix} 1&a&&\\ &1&b&\\ &&1&c\\ &&&1 \end{pmatrix}, a,b,c \in \mathbf{F}_q^*$	$(q-1)^2,(q-1)^2,$ $(q-1)^3$	q^3	R_2

Class Representatives	Number of Classes	Order of Centralizer in $UT_5(\mathbf{F}_q)$	Name of Type
$\begin{pmatrix} 1&&&&a\\ &1&&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, a \in \mathbf{F}_q$	q	q^{10}	C
$\begin{pmatrix} 1&&&a&\\ &1&&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&\\ &1&&&a\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}$ $a \in \mathbf{F}_q^*$	$(q-1),(q-1)$	q^9	A_1
$\begin{pmatrix} 1&&&&\\ &1&&a&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&b\\ &1&&a&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}$ $a,b \in \mathbf{F}_q^*$	$(q-1),(q-1)^2$	q^8	A_2
$\begin{pmatrix} 1&&a&&\\ &1&&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&\\ &1&&&\\ &&1&&a\\ &&&1&\\ &&&&1 \end{pmatrix},$ $a,b \in \mathbf{F}_q^*$	$(q-1),(q-1)^2,$ $(q-1),(q-1)^2$	q^8	A_3
$\begin{pmatrix} 1&&&&\\ &1&a&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&b\\ &1&a&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix},$ $\begin{pmatrix} 1&&&&\\ &1&&&\\ &&1&a&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&b\\ &1&&&\\ &&1&a&\\ &&&1&\\ &&&&1 \end{pmatrix}$ $a,b \in \mathbf{F}_q^*$	$(q-1),(q-1)^2,$ $(q-1),(q-1)^2$	q^7	A_4
$\begin{pmatrix} 1&a&&&\\ &1&&&\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&a&&&\\ &1&&&b\\ &&1&&\\ &&&1&\\ &&&&1 \end{pmatrix},$ $\begin{pmatrix} 1&&a&&\\ &1&&&\\ &&1&&b\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&a&&&\\ &1&&&\\ &&1&&b\\ &&&1&\\ &&&&1 \end{pmatrix},$ $\begin{pmatrix} 1&a&c&&\\ &1&&&\\ &&1&&b\\ &&&1&\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&&\\ &1&&&\\ &&1&&\\ &&&1&a\\ &&&&1 \end{pmatrix},$ $\begin{pmatrix} 1&&&b&\\ &1&&&\\ &&1&&\\ &&&1&a\\ &&&&1 \end{pmatrix}, \begin{pmatrix} 1&&b&&\\ &1&&&\\ &&1&&\\ &&&1&a\\ &&&&1 \end{pmatrix},$ $\begin{pmatrix} 1&&b&c&\\ &1&&&\\ &&1&&\\ &&&1&a\\ &&&&1 \end{pmatrix}, a,b,c \in \mathbf{F}_q^*$	$(q-1),(q-1)^2,$ $(q-1)^2,(q-1)^2,$ $(q-1)^3,(q-1),$ $(q-1)^2,(q-1)^2,$ $(q-1)^3$	q^7	A_5

$\begin{pmatrix} 1 & & & a & \\ & 1 & & & b \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, a,b \in \mathbf{F}_q^*$	$(q-1)^2$	q^9	B_1
$\begin{pmatrix} 1 & & a & & \\ & 1 & & & b \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & b & \\ & 1 & & & \\ & & 1 & & a \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ $a,b \in \mathbf{F}_q^*$	$(q-1),(q-1)^2,$ $(q-1),(q-1)^2$	q^8	B_2
$\begin{pmatrix} 1 & & a & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & a & \\ & & 1 & & b \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $a,b \in \mathbf{F}_q^*$	$(q-1)^2,(q-1)^2$	q^7	B_3
$\begin{pmatrix} 1 & a & & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & & & \\ & 1 & & b & \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & & & \\ & 1 & a & & \\ & & 1 & & b \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & c & \\ & 1 & a & & \\ & & 1 & & b \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & b & & \\ & 1 & & & \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & b & & & \\ & 1 & & & \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & b & c & & \\ & 1 & & & \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & b & & \\ & 1 & & & c \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & b & & & \\ & 1 & & & c \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & b & c & & \\ & 1 & & & d \\ & & 1 & a & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & c & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & & & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & c & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & & & \\ & 1 & b & c & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & d & \\ & 1 & b & c & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$ $a,b,c,d \in \mathbf{F}_q^*$	$(q-1)^2,(q-1)^3,$ $(q-1)^2,(q-1)^3,$ $(q-1)^2,(q-1)^2,$ $(q-1)^3,(q-1)^3,$ $(q-1)^3,(q-1)^4$ $(q-1)^2,(q-1)^3,$ $(q-1)^2,(q-1)^3,$ $(q-1)^3,(q-1)^4$	q^6	B_4
$\begin{pmatrix} 1 & b & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & b & & c & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}$ $a,b,c \in \mathbf{F}_q^*$	$(q-1)^2,(q-1)^3$	q^6	B_5
$\begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & c \\ & & & & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & a & \\ & & & 1 & b \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & c & & & \\ & 1 & & & \\ & & 1 & a & \\ & & & 1 & b \\ & & & & 1 \end{pmatrix}$ $a,b,c \in \mathbf{F}_q^*$	$(q-1)^2,(q-1)^3,$ $(q-1)^2,(q-1)^3$	q^5	B_6
$\begin{pmatrix} 1 & & a & & \\ & 1 & & b & \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix},\ a,b,c \in \mathbf{F}_q^*$	$(q-1)^3$	q^7	D_1

$\begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & & c \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & & & \\ & 1 & & b & \\ & & 1 & & \\ & & & 1 & c \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & & & \\ & 1 & b & c & \\ & & 1 & & \\ & & & 1 & d \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & a & & \\ & 1 & & & \\ & & 1 & b & \\ & & & 1 & c \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & b & & \\ & 1 & & & \\ & & 1 & c & \\ & & & 1 & d \\ & & & & 1 \end{pmatrix}$, $a, b, c, d \in \mathbf{F}_q^*$	$(q-1)^3, (q-1)^3,$ $(q-1)^4, (q-1)^3$ $(q-1)^4$	q^5	D_2
$\begin{pmatrix} 1 & & & a & \\ & 1 & b & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & & \\ & 1 & & & a \\ & & 1 & b & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $a, b \in \mathbf{F}_q^*$	$(q-1)^2, (q-1)^2$	q^6	R_1
$\begin{pmatrix} 1 & & & & \\ & 1 & a & & \\ & & 1 & b & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & & c \\ & 1 & a & & \\ & & 1 & b & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ $a, b, c \in \mathbf{F}_q^*$	$(q-1)^2, (q-1)^3$	q^5	R_2
$\begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & & \\ & 1 & a & & \\ & & 1 & b & \\ & & & 1 & c \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & a & & & \\ & 1 & b & & \\ & & 1 & c & \\ & & & 1 & d \\ & & & & 1 \end{pmatrix}$, $a, b, c, d \in \mathbf{F}_q^*$	$(q-1)^3, (q-1)^3$ $(q-1)^4$	q^4	R_3

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