ON THE EIGENVALUE PROBLEM INVOLVING THE WEIGHTED p-LAPLACIAN IN RADIALLY SYMMETRIC DOMAINS

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ABSTRACT. We investigate the following eigenvalue problem

$$\begin{cases} -\operatorname{div}\left(L(x)|\nabla u|^{p-2}\nabla u\right) = \lambda K(x)|u|^{p-2}u \quad \text{in } A_{R_1}^{R_2}, \\ u = 0 \quad \text{on } \partial A_{R_1}^{R_2}, \end{cases}$$

where $A_{R_1}^{R_2} := \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ $(0 < R_1 < R_2 \le \infty)$, $\lambda > 0$ is a parameter, the weights L and K are measurable with L positive a.e. in $A_{R_1}^{R_2}$ and K possibly sign-changing in $A_{R_1}^{R_2}$. We prove the existence of the first eigenpair and discuss the regularity and positiveness of eigenfunctions. The asymptotic estimates for $u(x)$ and $\nabla u(x)$ as $|x| \to R_1^+$ or R_2^- are also investigated.

1. Introduction and main results

In this paper we investigate the following eigenvalue problem

$$\begin{cases} -\operatorname{div}\left(L(x)|\nabla u|^{p-2}\nabla u\right) = \lambda K(x)|u|^{p-2}u \quad \text{in } A_{R_1}^{R_2}, \\ u = 0 \quad \text{on } \partial A_{R_1}^{R_2}, \end{cases} \tag{1.1}$$

where the weight L is measurable and positive a.e. in $A_{R_1}^{R_2} := \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ $(0 < R_1 < R_2 \le \infty)$ such that $L \in L^1_{\text{loc}}(A_{R_1}^{R_2})$; the weight K is measurable in $A_{R_1}^{R_2}$ such that $\text{meas}\{x \in A_{R_1}^{R_2} : K(x) > 0\} > 0$; λ is a spectral parameter. For the notational convenience we denote the operator $\operatorname{div}(L(x)|\nabla u|^{p-2}\nabla u)$ by $\Delta_{p,L}$ and by $|S|$ we denote the Lebesgue measure of $S \subset \mathbb{R}^N$. We note that K might change the sign in $A_{R_1}^{R_2}$.

(A) there exist functions v, w measurable and positive a.e. in (R_1, R_2), such that $v^{-\frac{1}{p-1}}, w \in L^1_{\text{loc}}(R_1, R_2)$ and
  (i) $P(r) := \min\left\{\left(\int_{R_1}^r \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}, \left(\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}\right\} < \infty$ for all $r \in (R_1, R_2)$ and $\int_{R_1}^{R_2} P(r)\sigma(r)\,\mathrm{d}r < \infty$, where $p' := \frac{p}{p-1}$, $\rho(r) := r^{N-1}v(r)$ and $\sigma(r) := r^{N-1}w(r)$;
  (ii) $L(x) \ge v(|x|)$ and $|K(x)| \le w(|x|)$ for a.e. $x \in A_{R_1}^{R_2}$.

Equation (1.1), which contains *weighted p-Laplacian* operator $\Delta_{p,L}$, describes several important phenomena which arise in Mathematical Physics, Riemannian geometry, Astrophysics, study of non-Newtonian fluids, subsonic motion of gases etc. (see e.g., [16, 22]). A weighted second order linear differential operator was basically introduced by Murthy and Stampacchia [18], being then extended to higher order linear weighted elliptic operators in the 80s and quasilinear elliptic equations including the weighted p-Laplacian in the 90s (see Drábek et al. [8]).

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The problem (1.1) in case of bounded domains or $\mathbb{R}^N$, was comprehensively investigated in [8], with suitable weights, and later studied by many authors, we mention Le-Schmitt [13], Lê-Schmitt [14], and references therein.

The weighted p-Laplacian eigenvalue problem in case of unbounded domains has got attention in the last two decades. In [17, 19], authors studied existence of an eigensolution with nonnegative weights on the right hand side for a nonlinear eigenvalue problem with mixed boundary condition. For an exterior domain B_1^c, the complement of the closed unit ball in $\mathbb{R}^N$ $(N \geq 2)$, Anoop et al. [2,3] studied the eigenvalue problem (1.1) with $L(x) \equiv 1$ and the weight K satisfying the following condition

(ADS) $K \in L^1_{\mathrm{loc}}(B_1^c)$, $\mathrm{meas}\{x \in B_1^c : K(x) > 0\} > 0$ and there exists a positive function w such that
 (i) $w \in \begin{cases} L^1((1,\infty); r^{p-1}), & p \neq N, \\ L^1((1,\infty); [r\log r]^{N-1}), & p = N; \end{cases}$
 (ii) $|K(x)| \leq w(|x|)$ for a.e. $x \in B_1^c$.

The authors proved the existence of a principal eigenvalue and discussed positivity and regularity of associated eigenfunctions when K satisfies some additional assumptions. It is worth mentioning that they allowed also the case $p \geq N$ and K possibly changing sign.

Another interesting aspect of qualitative properties is the behavior of solutions towards the boundary. The asymptotic estimates for solutions to problem (1.1) in exterior domains with $L(x) \equiv 1$ was obtained by several authors (see e.g., [2,4]). However, very few works deal with such kind of estimates for the weighted p-Laplacian. In the open ball B_R of radius R $(0 < R \leq \infty)$ centered at the origin with the convention that $B_R := \mathbb{R}^N$ when $R = \infty$, the authors in [1, 6] recently obtained the asymptotic estimates for solutions to (1.1) with radially symmetric weights $L(x) = v(|x|)$ and $K(x) = w(|x|)$ satisfying the following condition introduced in the book by Opic and Kufner [21]:

$$\text{(OK)} \quad \begin{cases} \text{either} & \left(\int_a^r \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_r^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} \to 0 \text{ as } r \to a^+, b^-, \\ \text{or} & \left(\int_r^b \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_a^r \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} \to 0 \text{ as } r \to a^+, b^-, -\infty \leq a < b \leq \infty, \end{cases}$$

with $a = 0$ and $b = R$.

The goal of this paper is twofold. First, we investigate the eigenvalue problem (1.1) with the weights L, K possibly not bounded and/or not separated away from zero in a general radially symmetric domain $A_{R_1}^{R_2}$. Second, we obtain the asymptotic estimates for solutions to problem (1.1) when the weights are radially symmetric. As in [2], there is no restriction on the dimension N in terms of p. We emphasize that for simplicity and clarity of statements of our results we are only concerned with two types of domains: annulus $(0 < R_1 < R_2 < \infty)$ and exterior of the ball of radius R_1 $(0 < R_1 < R_2 = \infty)$. In fact, some of our results also covers other two types of radially symmetric domains: bounded balls B_R $(0 < R < \infty)$ and the entire space $\mathbb{R}^N$ (see Remarks 2.9 and 3.3).

The novelty of this paper consists in considering (1.1) with new condition on the weights. Even when $L(x) = v(|x|) \equiv 1$, the condition (A) for the weight K is slightly weaker than the condition (ADS) introduced in [2] (see Remark 2.6 in Section 2). It is worth mentioning that there are weights v, w which satisfy (A) but do not satisfy (OK) (see Remark 2.7 in Section 2). We confess that we are not aware of weights v and w satisfying (OK) but not (A). Hence the class of weights satisfying (A) is a complement of the class of weights satisfying (OK) in order to study (1.1) with radially symmetric weights.

We look for solutions of (1.1) in the space $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$, which is the completion of $C_c^1(A_{R_1}^{R_2})$ (C^1 functions with compact support) with respect to the norm

$$\|u\| := \left(\int_{A_{R_1}^{R_2}} L(x) |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

We note that $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ is well defined uniformly convex Banach space under the assumption (A) (see Theorem 2.1 in Section 2). Moreover, we will prove in Section 2 that if (A) holds and $L^{-s} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ for some $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$, then $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ is compactly embedded in $L^p(A_{R_1}^{R_2}; w)$, the space of measurable functions u such that $\int_{A_{R_1}^{R_2}} w(|x|)|u|^p \, \mathrm{d}x < \infty$ (see Theorem 2.3).

Definition 1.1. By a *(weak) solution* of problem (1.1), we mean a function $u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ such that

$$\int_{A_{R_1}^{R_2}} L(x) |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_{A_{R_1}^{R_2}} K(x) |u|^{p-2} u v \, \mathrm{d}x, \quad \forall v \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L).$$

If problem (1.1) has a nontrivial solution u then λ is called an *eigenvalue* of $-\Delta_{p,L}$ in $A_{R_1}^{R_2}$ related to the weight K (an eigenvalue, for short) and such a solution u is called an *eigenfunction* corresponding to the eigenvalue λ.
Define

$$\lambda_1 := \inf \left\{ \int_{A_{R_1}^{R_2}} L(x) |\nabla u|^p \, \mathrm{d}x : \; u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L), \int_{A_{R_1}^{R_2}} K(x) |u|^p \, \mathrm{d}x = 1 \right\}. \tag{1.2}$$

We state our first main result of the existence of a principal eigenvalue and its simplicity.

Theorem 1.2 (Principal eigenpair)**.** *Assume that* (A) *holds and* $L^{-s} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ *for some* $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$. *Then* $\lambda_1 > 0$ *and* λ_1 *is a simple eigenvalue of* (1.1). *Moreover* λ_1 *is achieved at an eigenfunction* φ_1, *which is positive a.e. in* $A_{R_1}^{R_2}$.

Next, we state our results on the boundedness of solutions to problem (1.1) that will be utilized to obtain the C^1 regularity of solutions. The following theorems show that all eigenfunctions to eigenvalue problem (1.1) are locally bounded in $A_{R_1}^{R_2}$ if the weights satisfy some additional assumptions. In fact, in Section 4 we obtain the boundedness of solutions for a more general nonlinear term (see Theorem 4.2) via the De Giorgi type iteration technique. In the sequel, for $\alpha > 0$ we use the convention that $\frac{\alpha}{0} := \infty$ and define $p_\alpha := \frac{p\alpha}{\alpha+1}$ and $\alpha^* := \begin{cases} \frac{N\alpha}{N-\alpha} & \text{if } \alpha < N, \\ \infty & \text{if } \alpha \geq N. \end{cases}$

Theorem 1.3 (Boundedness I)**.** *Assume that* (A) *holds. Assume in addition that* $L^{-s}, L^{\frac{q}{q-p}}, |K|^{\frac{q}{q-p}} \in L^1(A_{R_1}^{R_1+2\epsilon})$ *for some* $\epsilon \in (0, \frac{R_2-R_1}{2})$, $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$ *and* $q \in [p, p_s^*)$. *Then for any solution* u *of problem* (1.1) *we have* $u \in L^q(A_{R_1}^{R_1+2\epsilon}) \cap L^\infty(A_{R_1}^{R_1+\epsilon})$ *and there exist* $C > 0$ *and* $\mu > 0$ *(independent of* u*) such that*

$$\|u\|_{L^\infty\left(A_{R_1}^{R_1+\epsilon}\right)} \leq C \left[1 + \left(\int_{A_{R_1}^{R_1+2\epsilon}} |u|^q \, \mathrm{d}x \right)^{\mu} \right].$$

Theorem 1.4 (Boundedness II)**.** *Assume that* (A) *holds. Assume in addition that* $L^{-s}, L^{\frac{q}{q-p}}, |K|^{\frac{q}{q-p}} \in L^1(B(x_0,r_0))$ *for some ball* $B(x_0,r_0) \subset A_{R_1}^{R_2}$, $s \in (\frac{N}{p},\infty) \cap [\frac{1}{p-1},\infty)$ *and* $q \in [p, p_s^*)$. *Then for any given* $\mu \in (0, 1-\frac{q}{p_s^*})$, *there exists* $C = C(\mu, r_0) > 0$ *such that for any solution* u *of problem* (1.1) *we have* $u \in L^q(B(x_0,r_0)) \cap L^\infty\big(B\big(x_0,\frac{r_0}{2}\big)\big)$ *and*

$$\|u\|_{L^\infty\left(B\left(x_0,\frac{r_0}{2}\right)\right)} \le C M_{L,K} \left(\int_{B(x_0,r_0)} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}}. \tag{1.3}$$

Here

$$M_{L,K} := \left(\int_{B(x_0,r_0)} L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{\mu s p}} \left[\|L\|_{L^{\frac{q}{q-p}}(B(x_0,r_0))} + \|K\|_{L^{\frac{q}{q-p}}(B(x_0,r_0))}\right]^{\frac{1}{\mu p}}.$$

In particular, if $L^{-s}, L^{\frac{q}{q-p}}$ *and* $|K|^{\frac{q}{q-p}} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$, *then* $u \in L^\infty_{\mathrm{loc}}(A_{R_1}^{R_2})$.

We now discuss certain smoothness properties of eigenfunctions. In the sequel, for an open set Ω in $\mathbb{R}^N$ we denote by $W^1(\Omega)$ the set of all $u \in L^1_{\mathrm{loc}}(\Omega)$ such that weak derivatives $\frac{\partial u}{\partial x_i}$ $(i = 1, \cdots, N)$ exist in Ω. We first have the C^1 regularity of eigenfunctions in $A_{R_1}^{R_2}$.

Theorem 1.5. *Assume that* (A) *holds. Assume in addition that* $L \in W^1(A_{R_1}^{R_2})$, $\operatorname*{ess\,inf}_{x \in A_{r_1}^{r_2}} L(x) > 0$ *for any* $R_1 < r_1 < r_2 < R_2$, $L, K \in L^{\frac{q}{q-p}}_{\mathrm{loc}}(A_{R_1}^{R_2})$ *for some* $q \in [p, p_s^*)$, *and* $|\frac{K}{L}| + |\frac{\nabla L}{L}|^p \in L^{\widetilde{q}}_{\mathrm{loc}}(A_{R_1}^{R_2})$ *for some* $\widetilde{q} > \frac{Np}{p-1}$. *Then for a (weak) solution* u *of* (1.1), *we have* $u \in C^1(A_{R_1}^{R_2})$.

The next result provides the regularity of eigenfunctions up to the inner boundary.

Theorem 1.6. *In addition to the assumptions of Theorem 1.5, we also assume that* $\operatorname*{ess\,inf}_{x \in A_{R_1}^{R_1+\epsilon}} L(x) > 0$, $L, K \in L^{\frac{q}{q-p}}(A_{R_1}^{R_1+\epsilon})$ *and* $|\frac{K}{L}| + |\frac{\nabla L}{L}| \in L^\infty(A_{R_1}^{R_1+\epsilon})$ *for some* $\epsilon \in (0, R_2 - R_1)$. *Then for a (weak) solution* u *of* (1.1) *and* $R \in (R_1, R_2)$, $u \in C^{1,\alpha(R)}(\overline{A_{R_1}^R})$ *for some* $\alpha(R) \in (0,1)$.

In view of the C^1 regularity of eigenfunctions above and the strong maximum principle we have the following result.

Theorem 1.7. *Assume that* (A) *holds. Assume in addition that* $K \in L^\infty_{\mathrm{loc}}(A_{R_1}^{R_2})$ *and* $L \in C^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ *such that* $\operatorname*{ess\,inf}_{x \in A_{r_1}^{r_2}} L(x) > 0$ *for all* $R_1 < r_1 < r_2 < R_2$. *Let* u *be a nonnegative eigenfunction of* (1.1). *Then,* $u \in C^1(A_{R_1}^{R_2})$ *and* $u > 0$ *everywhere in* $A_{R_1}^{R_2}$.

Finally, we discuss the decay of the solutions to problem (1.1) when $|x| \to R_1^+$ or R_2^-, that is important to obtain the asymptotic estimates near the boundary. Using the local behavior obtained in Theorem 1.4 we can obtain the decay of the solutions when $R_2 = \infty$ and L is non-degenerate at infinity.

Corollary 1.8. *Assume that* $1 < p < N$, $R_2 = \infty$ *and* (A) *holds. Assume in addition that there exists* $R \in (R_1, \infty)$ *such that* $\operatorname*{ess\,inf}_{x \in B_R^c} L(x) > 0$, $L, K \in L^{\frac{q}{q-p}}_{\mathrm{loc}}(B_R^c)$ *for some* $q \in [p, p^*)$ *and*

$$\operatorname*{ess\,sup}_{x \in B_R^c} \int_{B(x,r_0)} \left[L^{\frac{q}{q-p}}(y) + |K(y)|^{\frac{q}{q-p}}\right] \mathrm{d}y < \infty,$$

for some $r_0 \in (0, R-R_1)$*. Then, for any solution* u *to problem* (1.1)*, we have* $u(x) \to 0$ *uniformly as* $|x| \to \infty$*.*

The decay of solutions when $|x| \to R_1^+$ follows immediately if $u \in C^{1,\alpha}(\overline{A_{R_1}^R})$ for some $R > R_1$ and $\alpha \in (0,1)$.

Corollary 1.9. *Under the assumption of Theorem 1.6, for any solution* u *of* (1.1)*, we have* $u(x) \to 0$ *as* $|x| \to R_1^+$*.*

Next, we draw our attention to prove asymptotic behavior of a C^1 radially symmetric solution $u(x) = u(|x|)$ and its gradient to equation

$$-\operatorname{div}\left(v(|x|)|\nabla u|^{p-2}\nabla u\right) = \lambda w(|x|)|u|^{p-2}u \quad \text{in } A_{R_1}^{R_2}, \tag{1.4}$$

as $|x| \to R_1^+$ or $|x| \to R_2^-$ if $u(x) \to 0$ as $|x| \to R_1^+$ and $|x| \to R_2^-$. We assume

(W) v, w are positive a.e. in (R_1, R_2) such that v (resp. w) is continuous (resp. measurable) in (R_1, R_2) satisfying $v^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(R_1, R_2)$ (resp. $w \in L^1_{\text{loc}}(R_1, R_2)$).

Note that a similar problem in the case of a ball B_R $(0 < R \le \infty)$ was investigated in [8]. We write $u(R_1) = \lim_{r \to R_1^+} u(r)$ and $u(R_2) = \lim_{r \to R_2^-} u(r)$. Clearly, if $u(x) = u(|x|) \in C^1(A_{R_1}^{R_2})$ is a radially symmetric solution to problem (1.4) with $u(x) \to 0$ as $|x| \to R_1^+$ and $|x| \to R_2^-$, then $u \in C^1(R_1, R_2)$ satisfies

$$-\left(\rho(r)|u'(r)|^{p-2}u'(r)\right)' = \lambda\sigma(r)|u(r)|^{p-2}u(r) \quad \text{in } (R_1, R_2) \tag{1.5}$$

and $u(R_1) = u(R_2) = 0$. In two Theorems 1.10 and 1.11, we show that if the conditions on weights are made stronger than (A) near R_1 and R_2 (see Remark 5.1) then solutions obey certain decay properties. Namely, we assume

($\mathrm{A}_{\epsilon,\mathrm{L}}$) there exists $\xi \in (R_1, R_2)$ such that $\rho^{1-p'} \in L^1(R_1; \xi)$, and there exist $\epsilon \in (0, p-1)$ and $C > 0$ such that

$$\left(\int_r^{\xi} \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\epsilon} < C, \quad \forall r \in (R_1, \xi);$$

($\mathrm{A}_{\epsilon,\mathrm{R}}$) there exists $\xi \in (R_1, R_2)$ such that $\rho^{1-p'} \in L^1(\xi, R_2)$, and there exist $\epsilon \in (0, p-1)$ and $C > 0$ such that

$$\left(\int_{\xi}^{r} \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_{r}^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\epsilon} < C, \quad \forall r \in (\xi, R_2).$$

Theorem 1.10. *Assume that* (W) *and* $\left(\mathrm{A}_{\epsilon,\mathrm{L}}\right)$ *hold. Then for a radially symmetric solution* $u(x) = u(|x|) \in C^1(A_{R_1}^{R_2})$ *to problem* (1.4) *satisfying* $u(R_1) = u(R_2) = 0$*, there exist* $a \in (R_1, R_2)$ *and* $0 < C_1 < C_2, 0 < \widetilde{C}_1 < \widetilde{C}_2$ *such that*

$$C_1 \int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau \le |u(r)| \le C_2 \int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau, \quad \forall r \in (R_1, a), \tag{1.6}$$

and

$$\widetilde{C}_1 \rho^{1-p'}(r) \le |u'(r)| \le \widetilde{C}_2 \rho^{1-p'}(r), \quad \forall r \in (R_1, a). \tag{1.7}$$

Theorem 1.11. *Assume that* (W) *and* ($\mathrm{A}_{\epsilon,\mathrm{R}}$) *hold. Then for a radially symmetric solution* $u(x) = u(|x|) \in C^1(A_{R_1}^{R_2})$ *to problem* (1.4) *satisfying* $u(R_1) = u(R_2) = 0$*, there exist* $b \in (R_1, R_2)$ *and* $0 < C_1 < C_2, 0 < \widetilde{C}_1 < \widetilde{C}_2$ *such that*

$$C_1 \int_{r}^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau \le |u(r)| \le C_2 \int_{r}^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau, \quad \forall r \in (b, R_2),$$

and

$$\widetilde{C}_1\rho^{1-p'}(r) \leq |u'(r)| \leq \widetilde{C}_2\rho^{1-p'}(r), \quad \forall r \in (b, R_2).$$

The rest of the paper is organized as follows. In Section 2, we obtain some useful embeddings of the weighted Sobolev spaces into weighted Lebesgue spaces defined earlier. In Section 3, we prove the existence of the least positive eigenvalue and the corresponding positive eigenfunction associated to problem (1.1). The simplicity of such an eigenvalue is also discussed in this section. Section 4 deals with boundedness, smoothness and decay of solutions to problem (1.1). Section 5 is devoted to the investigation of the behavior of $u(x)$ and $\nabla u(x)$ as $|x| \to R_1^+$ or R_2^-, in the case of radially symmetric solutions. Finally, we provide a few concrete examples of weights L and K to illustrate our results in Section 6.

2. Weighted spaces

In this section we will obtain embeddings of certain weighted spaces and other properties. In what follows denote by S_1 the unit sphere $\{x \in \mathbb{R}^N : |x| = 1\}$ and for a function u defined on $A_{R_1}^{R_2}$, we write $u(x) = u(r,\omega)$, where $r = |x|$ and $\omega = x/r$. First, we prove the following continuous embedding.

Theorem 2.1. *Assume that* (A) *holds. Then, we have the following embedding*

$$\mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L) \hookrightarrow L^p(A_{R_1}^{R_2}; w).$$

Proof. Let $u \in C_c^1(A_{R_1}^{R_2})$ and $r \in (R_1, R_2)$. If $\int_{R_1}^r \rho^{1-p'}(\tau)\,\mathrm{d}\tau < \infty$, using Hölder's inequality we estimate

$$\begin{aligned}|u(r,\omega)| &= \left|\int_{R_1}^r \frac{\partial u}{\partial \tau}(\tau,\omega)d\tau\right| = \left|\int_{R_1}^r \rho^{-\frac{1}{p}}(\tau)\tau^{\frac{N-1}{p}}v^{\frac{1}{p}}(\tau)\frac{\partial u}{\partial \tau}(\tau,\omega)\,\mathrm{d}\tau\right| \\ &\leq \left(\int_{R_1}^r \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}}\left(\int_{R_1}^{R_2} \tau^{N-1}v(\tau)\left|\frac{\partial u}{\partial \tau}(\tau,\omega)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}}.\end{aligned}$$

Hence,

$$|u(r,\omega)|^p \leq \left(\int_{R_1}^r \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}\left(\int_{R_1}^{R_2} \tau^{N-1}v(\tau)\left|\frac{\partial u}{\partial \tau}(\tau,\omega)\right|^p \mathrm{d}\tau\right).$$

Analogously, if $\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau < \infty$, we have

$$|u(r,\omega)|^p \leq \left(\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}\left(\int_{R_1}^{R_2} \tau^{N-1}v(\tau)\left|\frac{\partial u}{\partial \tau}(\tau,\omega)\right|^p \mathrm{d}\tau\right).$$

In either case, we obtain

$$|u(r,\omega)|^p \leq P(r)\int_{R_1}^{R_2} \tau^{N-1}v(\tau)\left|\frac{\partial u}{\partial \tau}(\tau,\omega)\right|^p \mathrm{d}\tau.$$

Hence,

$$\begin{aligned}\int_{S_1} |u(r,\omega)|^p\,\mathrm{d}\omega &\leq P(r)\int_{S_1}\int_{R_1}^{R_2} \tau^{N-1}v(\tau)\left|\frac{\partial u}{\partial \tau}(\tau,\omega)\right|^p \mathrm{d}\tau\,\mathrm{d}\omega \\ &= P(r)\int_{A_{R_1}^{R_2}} v(|x|)|\nabla u(x)|^p dx.\end{aligned}$$

Combining this with the assumption (A) (ii), we get

$$\int_{S_1} |u(r,\omega)|^p \,\mathrm{d}\omega \le \|u\|^p P(r), \quad \forall r \in (R_1,R_2) \text{ and } \forall u \in C_c^1(A_{R_1}^{R_2}). \tag{2.1}$$

From this we deduce

$$\int_{R_1}^{R_2} r^{N-1} w(r) \int_{S_1} |u(r,\omega)|^p \,\mathrm{d}\omega \,\mathrm{d}r \le \|u\|^p \int_{R_1}^{R_2} r^{N-1} w(r) P(r)\,\mathrm{d}r.$$

That is,

$$\|u\|_{L^p\left(A_{R_1}^{R_2};w\right)} \le C\|u\|, \quad \forall u \in C_c^1(A_{R_1}^{R_2}), \tag{2.2}$$

where $C := \left(\int_{R_1}^{R_2} P(r)\sigma(r)\,\mathrm{d}r\right)^{\frac{1}{p}}$. By the density of $C_c^1(A_{R_1}^{R_2})$ in $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ we obtain (2.2) for all $u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ and it infers the continuity of the embedding. □

In what follows, for a normed space $(X, \|\cdot\|_X)$ of functions $u : \Omega \to \mathbb{R}$ with $\Omega \subseteq A_{R_1}^{R_2}$ such that $u|_\Omega \in X$ for all $u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$, we still denote $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow X$ if there is a constant $C > 0$ such that

$$\|u|_\Omega\|_X \le C\|u\|, \quad \forall u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L).$$

In fact such an embedding is not an injective map. In this sense the following embeddings are deduced from Theorem 2.1

Corollary 2.2. *Assume that the weight L satisfies*

(A1) *$L(x) \ge v(|x|) > 0$ for a.e. $x \in A_{R_1}^{R_2}$, where v is measurable in (R_1,R_2) such that $v, v^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(R_1,R_2)$ and $P(r) < \infty$ for all $r \in (R_1,R_2)$, where P is defined as in* (A).

For any given $R_1 < r_1 < r_2 < R_2$, the following embeddings hold:

(i) $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^p(A_{r_1}^{r_2})$;

(ii) $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow W^{1,p_s}(A_{r_1}^{r_2})$ *if* $L^{-s} \in L^1(A_{r_1}^{r_2})$ *for some* $s \in (\frac{N}{p},\infty) \cap [\frac{1}{p-1},\infty)$;

(iii) $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow W^{1,p}(A_{r_1}^{r_2})$ *if* $\operatorname*{ess\,inf}_{x \in A_{r_1}^{r_2}} L(x) > 0$.

Proof. (i) Let $R_1 < r_1 < r_2 < R_2$. Set $w(r) = P^{-1}(r)(r+1)^{-(N+1)}$ for $r \in (R_1,R_2)$. Then, $w \in L^1_{\mathrm{loc}}(R_1,R_2)$ and we also have

$$\int_{R_1}^{R_2} P(r)\sigma(r)\,\mathrm{d}r = \int_{R_1}^{R_2} \frac{r^{N-1}}{(r+1)^{N+1}}\,\mathrm{d}r < \infty.$$

From this and the hypothesis (A_1), we see that (A) holds. Thus, applying Theorem 2.1, we obtain

$$\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^p(A_{R_1}^{R_2};w). \tag{2.3}$$

It is easy to see that, for all $r \in (r_1,r_2)$, we have

$$0 < P(r) \le \min\left\{\left(\int_{R_1}^{r_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}, \left(\int_{r_1}^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}\right\} =: C_1 < \infty.$$

Thus,

$$w(r) \ge C_1^{-1}(r_2+1)^{-(N+1)} =: C_2 > 0, \quad \forall r \in (r_1,r_2),$$

and hence,

$$\|u\|_{L^p(A_{r_1}^{r_2})} \le C_2^{-1/p}\|u\|_{L^p(A_{R_1}^{R_2};w)}, \forall u \in L^p(A_{R_1}^{R_2};w).$$

From this and (2.3), it follows $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)\hookrightarrow L^p(A_{r_1}^{r_2})$.
(ii) Let $R_1<r_1<r_2<R_2$. For $u\in\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ we have

$$\int_{A_{r_1}^{r_2}}|\nabla u|^{p_s}\,\mathrm{d}x\le\left(\int_{A_{r_1}^{r_2}}L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{s+1}}\left(\int_{A_{r_1}^{r_2}}L(x)|\nabla u|^p\,\mathrm{d}x\right)^{\frac{s}{s+1}}.$$

From this and (i) we deduce the conclusion.
(iii) The conclusion can be deduced from (i) and the assumption on L. □

Next, we show the following compact embedding.

Theorem 2.3. *Assume that* (A) *holds and* $L^{-s}\in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ *for some* $s\in(\frac{N}{p},\infty)\cap\left[\frac{1}{p-1},\infty\right)$. *We have the following compact embedding*

$$\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)\hookrightarrow\hookrightarrow L^p(A_{R_1}^{R_2};w).$$

Proof. Let $u_n\rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ as $n\to\infty$. We will show that $u_n\to 0$ in $L^p(A_{R_1}^{R_2};w)$ as $n\to\infty$. To this end we will show that for any $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that

$$\int_{A_{R_1}^{R_2}}w(|x|)|u_n|^p\,\mathrm{d}x<\epsilon^p,\quad\forall n\ge n_\epsilon.\tag{2.4}$$

Without loss of generality we may assume that $\{u_n\}\subset C_c^1(A_{R_1}^{R_2})$ and $\|u_n\|\le 1$ for all $n\in\mathbb{N}$. Since $P(r)r^{N-1}w(r)\in L^1(R_1,R_2)$, there exists $g_\epsilon\in C_c^1(R_1,R_2)$ such that

$$\int_{R_1}^{R_2}|g_\epsilon(r)-P(r)r^{N-1}w(r)|\,\mathrm{d}r<\frac{\epsilon^p}{2}.$$

Set $w_\epsilon(r):=P^{-1}(r)r^{1-N}g_\epsilon(r)$ for all $r\in(R_1,R_2)$. Applying (2.1) and noticing $\|u_n\|\le 1$, we estimate

$$\begin{aligned}\int_{A_{R_1}^{R_2}}|(w-w_\epsilon)(|x|)|\,|u_n|^p\,\mathrm{d}x&=\int_{R_1}^{R_2}\left|r^{N-1}w(r)-r^{N-1}w_\epsilon(r)\right|\int_{S_1}|u_n(r,\omega)|^p\,\mathrm{d}\omega\,\mathrm{d}r\\&\le\int_{R_1}^{R_2}\left|P(r)r^{N-1}w(r)-g_\epsilon(r)\right|\mathrm{d}r\\&<\frac{\epsilon^p}{2},\quad\forall n\in\mathbb{N}.\end{aligned}\tag{2.5}$$

Let $R_1<r_1<r_2<R_2$ such that $\operatorname{supp}(g_\epsilon)\subset(r_1,r_2)$. Then for a.e. $x\in A_{r_1}^{r_2}$, we have

$$|w_\epsilon(|x|)|\le C_{r_1r_2}^{-1}r_1^{1-N}\|g_\epsilon\|_{L^\infty(R_1,R_2)}=:M_\epsilon,$$

where $C_{r_1r_2}:=\min\left\{\left(\int_{R_1}^{r_1}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1},\left(\int_{r_2}^{R_2}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}\right\}>0$. Thus, we infer

$$\int_{A_{R_1}^{R_2}}|w_\epsilon(|x|)|\,|u_n|^p\,\mathrm{d}x=\int_{A_{r_1}^{r_2}}|w_\epsilon(|x|)|\,|u_n|^p\,\mathrm{d}x\le M_\epsilon\int_{A_{r_1}^{r_2}}|u_n|^p\,\mathrm{d}x,\quad\forall n\in\mathbb{N}.\tag{2.6}$$

By (A), we have $L^{-\frac{1}{p-1}}\in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ and note that this condition guarantees that $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)\subset W^1(A_{R_1}^{R_2})$. By this and the embedding $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)\hookrightarrow L^p(A_{r_1}^{r_2})$ (see Corollary 2.2 (i)) we have

$$\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)\hookrightarrow W^{1,p}(A_{r_1}^{r_2};L),\tag{2.7}$$

where $W^{1,p}(A_{r_1}^{r_2};L) := \left\{u \in W^1(A_{r_1}^{r_2}) : \int_{A_{r_1}^{r_2}} \left[|u|^p + L(x)|\nabla u|^p\right] \mathrm{d}x < \infty\right\}$ endowed with the norm

$$\|u\|_{W^{1,p}(A_{r_1}^{r_2};L)} := \left(\int_{A_{r_1}^{r_2}} \left[|u|^p + L(x)|\nabla u|^p\right] \mathrm{d}x\right)^{\frac{1}{p}}.$$

Since $L^{-s} \in L^1(A_{r_1}^{r_2})$ for some $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$, we may apply a compact embedding result for weighted Sobolev spaces in [8, p. 26] to obtain

$$W^{1,p}(A_{r_1}^{r_2};L) \hookrightarrow\hookrightarrow L^p(A_{r_1}^{r_2}). \tag{2.8}$$

By (2.7), we have that $u_n|_{A_{r_1}^{r_2}} \rightharpoonup 0$ in $W^{1,p}(A_{r_1}^{r_2};L)$ as $n \to \infty$. Combining this with (2.8) we get $u_n|_{A_{r_1}^{r_2}} \to 0$ in $L^p(A_{r_1}^{r_2})$ as $n \to \infty$. Hence, there exists $n_\epsilon \in \mathbb{N}$ such that

$$M_\epsilon \int_{A_{r_1}^{r_2}} |u_n|^p \,\mathrm{d}x < \frac{\epsilon^p}{2}, \quad \forall n \geq n_\epsilon.$$

From this and (2.6) we obtain

$$\int_{A_{R_1}^{R_2}} |w_\epsilon(|x|)|\, |u_n|^p \,\mathrm{d}x < \frac{\epsilon^p}{2}, \quad \forall n \geq n_\epsilon.$$

Finally, combining the last estimate and (2.5) we obtain (2.4). Since $\epsilon > 0$ was chosen arbitrarily, we get $u_n \to 0$ in $L^p(A_{R_1}^{R_2};w)$ as $n \to \infty$ and the proof is complete. □

We now present several explicit consequences of Theorem 2.3. In the next two corollaries, we apply Theorem 2.3 for $L(x) = v(|x|)$ and write $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};v)$ instead of $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$. As in the assumption (A), we always denote $\rho(r) := r^{N-1}v(r)$ and $\sigma(r) := r^{N-1}w(r)$.

Corollary 2.4. *Let v, w be measurable and positive a.e. in (R_1, R_2) such that $v, v^{-s} \in L^1_{\mathrm{loc}}(R_1, R_2)$ for some $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$ and one of the following conditions holds true:*

(I) *there exists $\xi \in (R_1, R_2)$ such that $\int_\xi^{R_2} \rho^{1-p'}(r)\,\mathrm{d}r < \int_{R_1}^{\xi} \rho^{1-p'}(r)\,\mathrm{d}r = \infty$ and*

$$\int_{R_1}^{R_2} \left[\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right]^{p-1} \sigma(r)\,\mathrm{d}r < \infty;$$

(II) *there exists $\xi \in (R_1, R_2)$ such that $\int_{R_1}^{\xi} \rho^{1-p'}(r)\,\mathrm{d}r < \int_\xi^{R_2} \rho^{1-p'}(r)\,\mathrm{d}r = \infty$ and*

$$\int_{R_1}^{R_2} \left[\int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right]^{p-1} \sigma(r)\,\mathrm{d}r < \infty;$$

(III) *there exists $\xi \in (R_1, R_2)$ such that $\int_{R_1}^{R_2} \rho^{1-p'}(r)\,\mathrm{d}r < \infty$ and*

$$\int_{R_1}^{\xi} \left[\int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right]^{p-1} \sigma(r)\,\mathrm{d}r + \int_{\xi}^{R_2} \left[\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right]^{p-1} \sigma(r)\,\mathrm{d}r < \infty.$$

Then the following compact embedding holds

$$\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};v) \hookrightarrow\hookrightarrow L^p(A_{R_1}^{R_2};w).$$

Finally, we provide a simple special case of Theorem 2.3.

Corollary 2.5. *Let v, w are measurable and positive a.e. in (R, ∞) such that $v, v^{-s} \in L^1_{\mathrm{loc}}(R, \infty)$ for some $R \in (0, \infty)$, $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$ and one of the following conditions holds true:*

(W_1) *there exists* $\xi \in (R,\infty)$ *such that* $\operatorname*{ess\,inf}_{r\geq\xi} v(r) > 0$, $v^{-\frac{1}{p-1}} \in L^1(R,\xi)$ *and*

$$\begin{cases} \int_R^\xi \left[\int_R^r v^{-\frac{1}{p-1}}(\tau)\,\mathrm{d}\tau\right]^{p-1} w(r)\,\mathrm{d}r + \int_\xi^\infty r^{p-1}w(r)\,\mathrm{d}r < \infty, \ p \neq N, \\ \int_R^\xi \left[\int_R^r v^{-\frac{1}{N-1}}(\tau)\,\mathrm{d}\tau\right]^{N-1} w(r)\,\mathrm{d}r + \int_\xi^\infty [r\log r]^{N-1}w(r)\,\mathrm{d}r < \infty, \ p = N; \end{cases}$$

(W_2) *there exists* $\xi \in (R,\infty)$ *such that* $\operatorname*{ess\,inf}_{R\leq r\leq\xi} v(r) > 0$, $\left[r^{N-1}v\right]^{-\frac{1}{p-1}} \in L^1(\xi,\infty)$, *and*

$$\int_R^\xi (r-R)^{p-1}w(r)\,\mathrm{d}r + \int_\xi^\infty \left[\int_r^\infty \tau^{-\frac{N-1}{p-1}} v^{-\frac{1}{p-1}}(\tau)\,\mathrm{d}\tau\right]^{p-1} r^{N-1}w(r)\,\mathrm{d}r < \infty.$$

Then, we have the following embedding

$$\mathcal{D}_0^{1,p}(B_R^c;v) \hookrightarrow\hookrightarrow L^p(B_R^c;w).$$

Remark 2.6. In particular, (W_1) is a special case of (A). When v is a constant, say, $v \equiv 1$ and $R = 1$, then (W_1) becomes

$$(\mathrm{W}_{1,\mathrm{c}}) \quad w \in \begin{cases} L^1((1,\infty);(r-1)^{p-1}), \ p \neq N, \\ L^1((1,\infty);[r\log r]^{N-1}), \ p = N. \end{cases}$$

Clearly, a weight w satisfying (ADS) satisfies also ($W_{1,c}$). On the other hand, for $-p < \beta \leq -1$ and $p \neq N$ the weight

$$w(r) = \begin{cases} (r-1)^\beta, \quad 1 \leq r \leq 2, \\ \in L^1((2,\infty);r^{p-1}), \end{cases}$$

satisfies ($W_{1,c}$) but it does not satisfy (ADS). Therefore, the condition (A) is weaker than the condition (ADS).

Remark 2.7. It is worth noting that the condition (OK) does not include (W_1) and hence, does not include (A). For instance, let $1 < p < N$, $\alpha < p-1$, $\beta \geq 0$, $\alpha - p < \alpha_1 \leq -1$, and $-N \leq \beta_1 < -p$. Set

$$v(r) = \begin{cases} (r-1)^\alpha, \quad 1 \leq r \leq 2, \\ \in [1,3^\beta], \quad 2 \leq r \leq 3, \\ r^\beta, \quad 3 \leq r, \end{cases} \quad \text{and } w(r) = \begin{cases} (r-1)^{\alpha_1}, \quad 1 \leq r \leq 2, \\ \in [3^{\beta_1},1], \quad 2 \leq r \leq 3, \\ r^{\beta_1}, \quad 3 \leq r. \end{cases}$$

We can verify that v, w satisfy (W_1) with $R = 1$ but $\rho(r) = r^{N-1}v(r)$ and $\sigma(r) = r^{N-1}w(r)$ do not satisfy (OK) (with $a = 1$ and $b = \infty$) since $\int_1^r \sigma(\tau)\,\mathrm{d}\tau = \int_r^\infty \sigma(\tau)\,\mathrm{d}\tau = \infty$ for all $r \in (1,\infty)$. To find v and w which satisfy (OK) but do not satisfy (A) seems to be an open problem.

Finally, we state a property of $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$, that will be used in the next sections. In what follows, we denote $u^+ = \max\{u,0\}$ and $u^- = -\min\{u,0\}$.

Proposition 2.8. *If* $u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ *and* $k \geq 0$, *then* $(u-k)^+, (u+k)^- \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$.

Proof. Argument is standard and we only sketch the main idea. Since $(u+k)^- = (-u-k)^+$, it suffices to prove that $(u-k)^+ \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$. That is, we prove the existence of a sequence $\{u_n\} \subset C_c^1(A_{R_1}^{R_2})$ such that

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla u_n - \nabla(u-k)^+|^p\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{2.9}$$

To this end, let $\{\varphi_n\} \subset C_c^1(A_{R_1}^{R_2})$ such that $\|\varphi_n - u\| \to 0$ as $n \to \infty$. It is easy to see that

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla(\varphi_n - k)^+ - \nabla(u-k)^+|^p \,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{2.10}$$

For each $n \in \mathbb{N}$, set $\psi_n := (\varphi_n - k)^+$. Fix n and let $R_1 < r_1 < r_2 < R_2$ such that $\operatorname{supp}(\psi_n) \subset A_{r_1}^{r_2}$. For each $i \in \mathbb{N}$, define $\eta_i(x) := i^N \eta(ix)$, where η is a standard normalized mollifier in $\mathbb{R}^N$ and define

$$v_i^{(n)}(x) := (\eta_i * \psi_n)(x) = \int_{\mathbb{R}^N} \eta_i(x-y)\psi_n(y)\,\mathrm{d}y.$$

Thus, $v_i^{(n)} \in C^\infty(\mathbb{R}^N)$ for all i and $\operatorname{supp}(v_i^{(n)}) \subset A_{r_1}^{r_2}$ for i large. From this together with $L \in L^1(A_{r_1}^{r_2})$ and properties of mollifiers, we obtain

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla v_i^{(n)} - \nabla \psi_n|^p \,\mathrm{d}x \to 0 \quad \text{as} \quad i \to \infty.$$

Thus, we find i_n such that

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla v_{i_n}^{(n)} - \nabla \psi_n|^p \,\mathrm{d}x < \frac{1}{n} \quad \text{i.e.,} \quad \int_{A_{R_1}^{R_2}} L(x)|\nabla u_n - \nabla(\varphi_n - k)^+|^p \,\mathrm{d}x < \frac{1}{n},$$

where $u_n := v_{i_n}^{(n)}$ $(\in C_c^1(A_{R_1}^{R_2}))$. From here and (2.10), for such a sequence $\{u_n\}$ we obtain (2.9) and the proof is complete. $\square$

Remark 2.9. Obviously, in this section we can allow $R_1 = 0$, that is, $A_{R_1}^{R_2}$ is of the form $B_R \setminus \{0\}$ $(0 < R \le \infty)$. When $1 < p < N$ and $L \in L^1_{\mathrm{loc}}(B_R)$ such that $\lim_{r\to 0} \frac{1}{|B_r|}\int_{B_r} L(x)\,\mathrm{d}x < \infty$, then the space $\mathcal{D}_0^{1,p}(A_0^R; L)$ coincides with $\mathcal{D}_0^{1,p}(B_R; L)$, the completion of $C_c^1(B_R)$ with respect to the norm

$$\|u\| = \left(\int_{B_R} L(x)|\nabla u|^p \,\mathrm{d}x\right)^{1/p}.$$

That is, $\mathcal{D}_0^{1,p}(A_0^R; L)$ is the usual solution space for the Dirichlet problem in a ball B_R.

3. The eigenvalue problem involving the weighted p-Laplacian

In this section we discuss the existence and properties of the first eigenpair of the eigenvalue problem (1.1). If (A) holds and $L^{-s} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ for some $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$, then by the compact embedding $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L) \hookrightarrow\hookrightarrow L^p(A_{R_1}^{R_2}; w)$ and Proposition 2.8, arguing as in [2, Proof of Lemma 4.1], we obtain the existence of a principal eigenvalue as follows.

Lemma 3.1. *Assume that* (A) *holds and* $L^{-s} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ *for some* $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$. *Then* λ_1 *defined in* (1.2) *is positive, it is achieved at some* $\varphi_1 \ge 0$ *and* (λ_1, φ_1) *is an eigenpair of* (1.1).

The positivity of φ_1 and the simplicity of λ_1 can be obtained in the same fashion as in [11] with suitable modifications. However, the presence of the weight L in the main operator somehow makes the conclusions not to follow in a straightforward manner. For the reader's convenience, we sketch the proofs briefly. Note that under the assumption of Theorem 1.2 we have $u \in W^{1,p_s}_{\mathrm{loc}}(A^{R_2}_{R_1})$ for any (weak) solution u to problem (1.1) in view of Corollary 2.2. In fact, we work with the following representation of u, defined in $A^{R_2}_{R_1}$ by

$$u^*(x) := \begin{cases} \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,\mathrm{d}y & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

In the next lemma, we state a strong maximum principle type result, which is similar to [11, Proposition 3.2].

Lemma 3.2. *Assume that* (A) *holds and* $L^{-s} \in L^1_{\mathrm{loc}}(A^{R_2}_{R_1})$ *for some* $s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty)$. *Let* $V \in L^1_{\mathrm{loc}}(A^{R_2}_{R_1})$ *and* $V \geq 0$. *If a nontrivial nonnegative function* $u \in \mathcal{D}^{1,p}_0(A^{R_2}_{R_1}; L)$ *satisfies* $Vu^p \in L^1_{\mathrm{loc}}(A^{R_2}_{R_1})$ *and*

$$\int_{A^{R_2}_{R_1}} \left\{L(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\xi + Vu^{p-1}\xi\right\}\mathrm{d}x \geq 0, \quad \forall \xi \in C^\infty_c(A^{R_2}_{R_1}), \xi \geq 0, \tag{3.1}$$

then $\mathrm{Cap}_{p_s}(\mathcal{Z}) = 0$, *where* $\mathcal{Z} := \left\{x \in A^{R_2}_{R_1} : u(x) = 0\right\}$.

For the definition of the p-capacity $\mathrm{Cap}_p(\cdot)$ and related properties we refer to the book of Evans-Gariepy [9] (see also [11]).

Proof. We proceed as in [11, Proof of Proposition 3.2]. It is worth mentioning that in [11], the domain is required to be bounded when $N \leq p$. For each $n \in \mathbb{N}$, denote $\Omega_n := A^{R_1+n}_{R_1}$ when $R_2 = \infty$ and $\Omega_n := A^{R_2}_{R_1}$ when $R_2 < \infty$ and define $\mathcal{Z}_n := \big\{x \in \Omega_n : u(x) = 0\big\}$. Since $\mathcal{Z} = \bigcup_{n=1}^\infty \mathcal{Z}_n$, it suffices to show that $\mathrm{Cap}_{p_s}(\mathcal{Z}_n) = 0$ for all $n \in \mathbb{N}$. Let n be fixed. As in [11, Proof of Proposition 3.2], we will show for any $\xi \in C^\infty_c(\Omega_n)$ with $0 \leq \xi \leq 1$ there exits $C_0 = C_0(u,\xi) > 0$ such that

$$\int_{\Omega_n} \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^{p_s} \xi^{p_s}\,\mathrm{d}x \leq C_0, \quad \forall \delta > 0. \tag{3.2}$$

To obtain (3.2) we use the following identity

$$\int_{\Omega_n} L(x)\left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p\,\mathrm{d}x = \frac{1}{1-p}\int_{\Omega_n} L(x)|\nabla u|^{p-2}\nabla u\cdot\left[\nabla\left(\frac{\xi^p}{(u+\delta)^{p-1}}\right) - p\xi^{p-1}(\nabla\xi)(u+\delta)^{1-p}\right]\mathrm{d}x.$$

Then, we use the same argument as in [11, Proof of Proposition 3.2], and employing (3.1), to obtain

$$\int_{\Omega_n} L(x)\left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p\,\mathrm{d}x \leq \int_{\Omega_n} V(x)(1+|u|^p)\xi^p\,\mathrm{d}x + p^{p-1}\int_{\Omega_n} L(x)|\nabla\xi|^p\,\mathrm{d}x.$$

Combining this and the estimate

$$\int_{\Omega_n} \left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^{p_s} \xi^{p_s}\,\mathrm{d}x \leq \left(\int_{\mathrm{supp}(\xi)} L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{s+1}} \left(\int_{\Omega_n} L(x)\left|\nabla \log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p\,\mathrm{d}x\right)^{\frac{s}{s+1}},$$

we obtain (3.2). The rest of the proof is similar to that of [11, Proof of Proposition 3.2]. □

Finally, we sketch the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.1, we have λ_1 is a positive eigenvalue of (1.1) and there is a nonnegative eigenfunction φ_1 associated with λ_1. Since

$$\int_{A_{R_1}^{R_2}} \left\{L(x)|\nabla\varphi_1|^{p-2}\varphi_1\cdot\nabla\xi+\lambda_1 K^-\varphi_1^{p-1}\xi\right\}\mathrm{d}x=\lambda_1\int_{A_{R_1}^{R_2}} K^+\varphi_1^{p-1}\xi\,\mathrm{d}x\geq 0$$

for all $\xi\in C_c^\infty(A_{R_1}^{R_2}),\xi\geq 0$, we get $\varphi_1>0$ a.e. in $A_{R_1}^{R_2}$ in view of Lemma 3.2. The simplicity of λ_1 can be proved by the same argument as [11, Proof of Theorem 1.3] for which we invoke Lemma 3.2 and use p_s-capacity instead of p-capacity. □

Remark 3.3. Similarly to Section 2, in this section we can also allow $R_1=0$. As shown in Remark 2.9, when $1<p<N$ and $L\in L^1_{\mathrm{loc}}(B_R)$ such that $\lim_{r\to 0}\frac{1}{|B_r|}\int_{B_r}L(x)\,\mathrm{d}x<\infty$ also in this section we recover results for a ball B_R $(0<R\leq\infty)$.

4. Qualitative properties of solutions

In this section we prove qualitative properties of solutions mentioned in Section 1 (Theorems 1.3–1.7 and Corollaries 1.8–1.9).

4.1. Boundedness of solutions.

In this subsection, we obtain the (local) boundedness of solutions to problem (1.1). As we mentioned in Section 1, the boundedness of solutions can be obtained for more general nonlinear term via the De Giorgi type iterations technique. More precisely, consider the following problem

$$-\operatorname{div}\left(L(x)|\nabla u|^{p-2}\nabla u\right)=f(x,u)\quad\text{a.e. in } A_{R_1}^{R_2},\tag{4.1}$$

where the weight L satisfies the condition (A1) in the Corollary 2.2 and the nonlinear term f satisfies

(F) $f: A_{R_1}^{R_2}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $|f(x,\tau)|\leq a(x)|\tau|^{p-1}+b(x)$ for a.e. $x\in A_{R_1}^{R_2}$ and all $\tau\in\mathbb{R}$, where a,b are nonnegative measurable functions in $A_{R_1}^{R_2}$.

Definition 4.1. By a weak solution of problem (4.1), we mean a function $u\in\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ such that $f(\cdot,u)\in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ and

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\xi\,\mathrm{d}x=\int_{A_{R_1}^{R_2}} f(x,u)\xi\,\mathrm{d}x,\quad\forall\xi\in C_c^1(A_{R_1}^{R_2}).$$

Theorem 4.2. *Assume that* (A1) *and* (F) *hold.*

(i) *Assume in addition that* $L,a\in L^{\frac{q}{q-p}}(A_{R_1}^{R_1+2\epsilon})$, $b\in L^{\frac{t}{t-1}}(A_{R_1}^{R_1+2\epsilon})$ *and* $L^{-s}\in L^1(A_{R_1}^{R_1+2\epsilon})$ *for some* $\epsilon\in(0,\frac{R_2-R_1}{2})$, $s\in(\frac{N}{p},\infty)\cap[\frac{1}{p-1},\infty)$, $q\in[p,p_s^*)$ *and* $t\in[1,q]\cap[1,\frac{p_s^*}{p})$. *Then for any weak solution* u *of problem* (4.1), *we have* $u\in L^q(A_{R_1}^{R_1+2\epsilon})\cap L^\infty(A_{R_1}^{R_1+\epsilon})$ *and*

$$\|u\|_{L^\infty\left(A_{R_1}^{R_1+\epsilon}\right)}\leq C\left[1+\left(\int_{A_{R_1}^{R_1+2\epsilon}}|u|^q\,\mathrm{d}x\right)^{\mu}\right],\tag{4.2}$$

where $C,\mu>0$ *are independent of* u.

(ii) *Assume in addition that* $L, a \in L^{\frac{q}{q-p}}(B(x_0,r_0))$, $b \in L^{\frac{t}{t-1}}(B(x_0,r_0))$ *and* $L^{-s} \in L^1(B(x_0,r_0))$ *for some ball* $B(x_0,r_0) \subset A_{R_1}^{R_2}$, $s \in (\frac{N}{p},\infty) \cap [\frac{1}{p-1},\infty)$, $q \in [p,p_s^*)$ *and* $t \in [1,q] \cap [1,\frac{p_s^*}{p})$. *Then for any weak solution* u *of problem* (4.1), *we have* $u \in L^q(B(x_0,r_0)) \cap L^\infty(B(x_0,\frac{r_0}{2}))$ *and*

$$\|u\|_{L^\infty\left(B\left(x_0,\frac{r_0}{2}\right)\right)} \le C\left[1+\left(\int_{B(x_0,r_0)}|u|^q\,\mathrm{d}x\right)^\mu\right],$$

where $C,\mu > 0$ *are independent of* u. *In particular, if* $L, a \in L_{\mathrm{loc}}^{\frac{q}{q-p}}(A_{R_1}^{R_2})$, $b \in L_{\mathrm{loc}}^{\frac{t}{t-1}}(A_{R_1}^{R_2})$ *and* $L^{-s} \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$ *then* $u \in L^\infty_{\mathrm{loc}}(A_{R_1}^{R_2})$.

To prove Theorem 4.2 we first prove the following lemma.

Lemma 4.3. *Assume that* (A1) *holds.*

(i) *If* $L^{-s} \in L^1(A_{R_1}^{R_1+2\epsilon})$ *for some* $\epsilon \in (0,\frac{R_2-R_1}{2})$, *then* $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow W^{1,p_s}(A_{R_1}^{R_1+2\epsilon})$ *and hence* $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^q(A_{R_1}^{R_1+2\epsilon})$ *for* $q \in [1,p_s^*)$.

(ii) *If* $L^{-s} \in L^1(B(x_0,r_0))$ *for some ball* $B(x_0,r_0) \subset A_{R_1}^{R_2}$, *then* $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow W^{1,p_s}(B(x_0,r_0))$ *and hence* $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^q(B(x_0,r_0))$ *for* $q \in [1,p_s^*)$.

Proof. (i) Let $u \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ and let $\{u_n\} \subset C_c^1(A_{R_1}^{R_2})$ such that $u_n \to u$ in $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L)$ as $n \to \infty$. By Corollary 2.2 (i), up to a subsequence we have $u_n \to u$ a.e. in $A_{R_1}^{R_2}$. Let $\phi \in C^\infty(\mathbb{R}^N)$ such that $\chi_{B_{R_1+\epsilon}} \le \phi \le \chi_{B_{R_1+\frac{3\epsilon}{2}}}$, where χ_Ω denotes the characteristic function on the set Ω. Then $\phi u_n \in C_c^1(A_{R_1}^{R_1+2\epsilon})$. Thus, by Poincaré's inequality there exists a positive constant C such that

$$\int_{A_{R_1}^{R_1+2\epsilon}}|\phi u_n|^{p_s}\,\mathrm{d}x \le C\int_{A_{R_1}^{R_1+2\epsilon}}|\nabla(\phi u_n)|^{p_s}\,\mathrm{d}x, \quad \forall n \in \mathbb{N}.$$

Hence, applying Hölder's inequality and the embedding $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^p(A_{R_1+\epsilon}^{R_1+\frac{3\epsilon}{2}})$ we obtain from the last inequality that

$$\begin{aligned}
\int_{A_{R_1}^{R_1+\epsilon}}|u_n|^{p_s}\,\mathrm{d}x &\le C_1\int_{A_{R_1}^{R_1+2\epsilon}}|\nabla u_n|^{p_s}\,\mathrm{d}x + C_1\int_{A_{R_1+\epsilon}^{R_1+\frac{3\epsilon}{2}}}|u_n|^{p_s}\,\mathrm{d}x\\
&\le C_1\left(\int_{A_{R_1}^{R_1+2\epsilon}}L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{s+1}}\left(\int_{A_{R_1}^{R_1+2\epsilon}}L(x)|\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{s}{s+1}} + C_2\left(\int_{A_{R_1+\epsilon}^{R_1+\frac{3\epsilon}{2}}}|u_n|^p\,\mathrm{d}x\right)^{\frac{s}{s+1}}\\
&\le C_3\left(\int_{A_{R_1}^{R_2}}L(x)|\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{s}{s+1}}, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Letting $n \to \infty$ and invoking Fatou's lemma we obtain the above estimate for $u_n = u$. Combining this with the embedding $\mathcal{D}_0^{1,p}(A_{R_1}^{R_2};L) \hookrightarrow L^p(A_{R_1+\epsilon}^{R_1+2\epsilon})$ and the estimate

$$\int_{A_{R_1}^{R_1+2\epsilon}}|\nabla u|^{p_s}\,\mathrm{d}x \le \left(\int_{A_{R_1}^{R_1+2\epsilon}}L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{s+1}}\left(\int_{A_{R_1}^{R_1+2\epsilon}}L(x)|\nabla u|^p\,\mathrm{d}x\right)^{\frac{s}{s+1}},$$

we deduce $\|u\|_{W^{1,p_s}(A_{R_1}^{R_1+2\epsilon})} \le C_4\|u\|$ for some constant C_4 independent of u.

(ii) The conclusion is clear in view of [8, p. 25, the embedding (1.22)]. □

To employ the De Giorgi iteration, we need the following key lemma. The special case $\delta_1 = \delta_2$ was obtained in [12, Ch.2, lemma 4.7].

Lemma 4.4. *([10, Lemma 4.3]) Let $\{J_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the recursion inequality*

$$J_{n+1} \leq K\eta^n \left(J_n^{1+\delta_1} + J_n^{1+\delta_2}\right), \quad n = 0, 1, 2, \cdots, \tag{4.3}$$

for some $\eta > 1$, $K > 0$ and $\delta_2 \geq \delta_1 > 0$. If $J_0 \leq \min\left(1, (2K)^{\frac{-1}{\delta_1}} \eta^{\frac{-1}{\delta_1^2}}\right)$ or

$$J_0 \leq \min\left((2K)^{\frac{-1}{\delta_1}} \eta^{\frac{-1}{\delta_1^2}}, (2K)^{\frac{-1}{\delta_2}} \eta^{-\frac{1}{\delta_1\delta_2} - \frac{\delta_2-\delta_1}{\delta_2^2}}\right),$$

then there exists $n \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$ such that $J_n \leq 1$. Moreover,

$$J_n \leq \min\left(1, (2K)^{\frac{-1}{\delta_1}} \eta^{\frac{-1}{\delta_1^2}} \eta^{\frac{-n}{\delta_1}}\right), \ \forall n \geq n_0,$$

where n_0 is the smallest $n \in \mathbb{N}_0$ for which $J_n \leq 1$. In particular, $J_n \to 0$ as $n \to \infty$.

Proof of Theorem 4.2. (i) Let u be a weak solution of problem (4.1). In the rest of the proof of the theorem, the constant C might vary from line to line, but will be always independent of L, a, b, ϵ and u. Without loss of generality we may assume that $t > \frac{q}{p}$.

Step 1: Caccioppoli-type inequality. Denote

$$\alpha := \|L\|_{L^{\frac{q}{q-p}}\left(A_{R_1}^{R_1+2\epsilon}\right)}, \ \beta := \|a\|_{L^{\frac{q}{q-p}}\left(A_{R_1}^{R_1+2\epsilon}\right)} \text{ and } \gamma := \|b\|_{L^{\frac{t}{t-1}}\left(A_{R_1}^{R_1+2\epsilon}\right)}, \tag{4.4}$$

and for $k > 0$, $r \in (R_1, R_2)$, denote

$$A_{k,r} := \{x \in A_{R_1}^r : u(x) > k\}.$$

We claim that there exists a positive constant C such that, for any r_1, r_2 satisfying $R_1 + \epsilon \leq r_1 < r_2 \leq R_1 + 2\epsilon$ and for any $k > 0$ we have

$$\int_{A_{k,r_1}} L(x)|\nabla u|^p \, \mathrm{d}x \leq C(\alpha + \beta\epsilon^p)\left(\int_{A_{k,r_2}} \left(\frac{u-k}{r_2 - r_1}\right)^q \mathrm{d}x\right)^{\frac{p}{q}} +$$

$$+ p\gamma\left(\int_{A_{k,r_2}} (u-k)^q \, \mathrm{d}x\right)^{\frac{1}{q}} |A_{k,r_2}|^{\frac{q-t}{qt}} + C\beta k^p |A_{k,r_2}|^{\frac{p}{q}}. \tag{4.5}$$

To this end, let $\xi \in C^1(\mathbb{R}^N)$ such that

$$\chi_{B_{r_1}} \leq \xi \leq \chi_{B_{r_2}} \text{ and } |\nabla\xi| \leq \frac{2}{r_2 - r_1}.$$

By an approximation argument, we can show that for $\widetilde{u} \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ and $\widetilde{\xi} \in C^1(\mathbb{R}^N)$ with $\chi_{B_{r_1}} \leq \widetilde{\xi} \leq \chi_{B_{r_2}}$, we have $\widetilde{u}\widetilde{\xi} \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ and $\widetilde{u}\widetilde{\xi}$ is a test function for (4.1). By this and Proposition 2.8, we can use $(u-k)^+\xi^p$ as a test function in (4.1) and get

$$\int_{A_{R_1}^{R_2}} L(x)|\nabla u|^{p-2}\nabla u \cdot \nabla((u-k)^+\xi^p) \, \mathrm{d}x = \int_{A_{R_1}^{R_2}} f(x,u)(u-k)^+\xi^p \, \mathrm{d}x.$$

By the assumption on f, the last equality leads to

$$\int_{A_{k,r_2}} L(x)|\nabla u|^p\xi^p \, \mathrm{d}x \leq -\, p\int_{A_{k,r_2}} L(x)|\nabla u|^{p-2}(\nabla u \cdot \nabla\xi)(u-k)\xi^{p-1} \, \mathrm{d}x$$

$$+\int_{A_{k,r_2}} a(x)|u|^{p-1}(u-k)\xi^p\,\mathrm{d}x + \int_{A_{k,r_2}} b(x)(u-k)\xi^p\,\mathrm{d}x.$$

That is

$$\int_{A_{k,r_2}} L(x)|\nabla u|^p\xi^p\,\mathrm{d}x \le p\int_{A_{k,r_2}} L(x)|\nabla u|^{p-1}\xi^{p-1}|\nabla\xi|(u-k)\,\mathrm{d}x + \int_{A_{k,r_2}} a(x)u^p\,\mathrm{d}x + \int_{A_{k,r_2}} b(x)(u-k)\,\mathrm{d}x. \tag{4.6}$$

Now we estimate three integrals on the right hand side (RHS for short) of (4.6) separately. For simplicity, denote

$$J := \int_{A_{k,r_2}} L(x)|\nabla u|^p\xi^p\,\mathrm{d}x \text{ and } Q := \int_{A_{k,r_2}} \left(\frac{u-k}{r_2-r_1}\right)^q \mathrm{d}x.$$

We estimate the first integral on RHS of (4.6), using Young's inequality and Hölder's inequality, as follows

$$\begin{aligned}
&\int_{A_{k,r_2}} L(x)|\nabla u|^{p-1}\xi^{p-1}|\nabla\xi|(u-k)\,\mathrm{d}x\\
&\le \frac{p-1}{p}\int_{A_{k,r_2}} L(x)\frac{1}{p}|\nabla u|^p\xi^p\,\mathrm{d}x + \frac{1}{p}\int_{A_{k,r_2}} L(x)p^{p-1}(|\nabla\xi|(u-k))^p\,\mathrm{d}x\\
&\le \frac{p-1}{p^2}J + 2^p p^{p-2}\int_{A_{k,r_2}} L(x)\left(\frac{u-k}{r_2-r_1}\right)^p \mathrm{d}x\\
&\le \frac{p-1}{p^2}J + 2^p p^{p-2}\|L\|_{L^{\frac{q}{q-p}}\left(A_{R_1}^{R_1+2\epsilon}\right)}\left(\int_{A_{k,r_2}}\left(\frac{u-k}{r_2-r_1}\right)^q \mathrm{d}x\right)^{\frac{p}{q}}\\
&= \frac{p-1}{p^2}J + 2^p p^{p-2}\alpha Q^{\frac{p}{q}}.
\end{aligned} \tag{4.7}$$

Using Hölder's inequality, we estimate the second integral on RHS of (4.6)

$$\begin{aligned}
\int_{A_{k,r_2}} a(x)u^p\,\mathrm{d}x &\le \|a\|_{L^{\frac{q}{q-p}}\left(A_{R_1}^{R_1+2\epsilon}\right)}\left(\int_{A_{k,r_2}} u^q\,\mathrm{d}x\right)^{\frac{p}{q}}\\
&\le \beta\left[\int_{A_{k,r_2}} 2^q\big((u-k)^q + k^q\big)\,\mathrm{d}x\right]^{\frac{p}{q}}\\
&\le C\beta\epsilon^p Q^{\frac{p}{q}} + C\beta k^p|A_{k,r_2}|^{\frac{p}{q}}.
\end{aligned} \tag{4.8}$$

Using Hölder's inequality again, we estimate the third integral on RHS of (4.6)

$$\begin{aligned}
\int_{A_{k,r_2}} b(x)(u-k)\,\mathrm{d}x &\le \|b\|_{L^{\frac{t}{t-1}}\left(A_{R_1}^{R_1+2\epsilon}\right)}\left(\int_{A_{k,r_2}} (u-k)^t\,\mathrm{d}x\right)^{\frac{1}{t}}\\
&\le \gamma\left(\int_{A_{k,r_2}} (u-k)^q\,\mathrm{d}x\right)^{\frac{1}{q}}|A_{k,r_2}|^{\frac{q-t}{qt}}.
\end{aligned} \tag{4.9}$$

From (4.6)–(4.9), we obtain

$$J \le \frac{p-1}{p}J + 2^p p^{p-1}\alpha Q^{\frac{p}{q}} + C\beta\epsilon^p Q^{\frac{p}{q}} + C\beta k^p|A_{k,r_2}|^{\frac{p}{q}} + \gamma\left(\int_{A_{k,r_2}} (u-k)^q\,\mathrm{d}x\right)^{\frac{1}{q}}|A_{k,r_2}|^{\frac{q-t}{qt}}.$$

Hence

$$J \le C(\alpha+\beta\epsilon^p)Q^{\frac{p}{q}} + p\gamma\Big(\int_{A_{k,r_2}} (u-k)^q \,\mathrm{d}x\Big)^{\frac{1}{q}} |A_{k,r_2}|^{\frac{q-t}{qt}} + C\beta k^p |A_{k,r_2}|^{\frac{p}{q}}.$$

From this and the definitions of J, Q and ξ we obtain (4.5).

Step 2: Definition of recursive sequence and recursion inequality. Define the recursive sequence $\{J_n\}$ as

$$J_n := \int_{A_{k_n,\rho_n}} (u-k_n)^q \,\mathrm{d}x, \quad \forall n \in \mathbb{N}_0,$$

where $\rho_n := R_1+\epsilon+\frac{\epsilon}{2^n}$ and $k_n := k_*\big(1-\frac{1}{2^{n+1}}\big)$ for some $k_*>1$, to be specified later. We also denote $\bar\rho_n := \frac{\rho_n+\rho_{n+1}}{2}$ $(n\in\mathbb{N}_0)$. Clearly, $\rho_n \downarrow R_1+\epsilon$, $k_n \uparrow k_*$, $R_1+\epsilon<\rho_n\le R_1+2\epsilon$ and $\frac{k_*}{2}\le k_n<k_*$ for all $n\in\mathbb{N}_0$. Moreover, notice that

$$\rho_n-\bar\rho_n = \frac{\epsilon}{2^{n+2}}, \quad k_{n+1}-k_n = \frac{k_*}{2^{n+2}}, \quad \forall n\in\mathbb{N}_0.$$

Next, we obtain a recursion inequality of the form (4.3). Fix $\zeta\in C^1(\mathbb{R})$, such that $\chi_{(-\infty,1)}\le\zeta\le\chi_{\left(-\infty,\frac{3}{2}\right)}$ and $|\zeta'|\le 4$. Define

$$\zeta_n(x) = \zeta\left(\frac{2^{n+1}}{\epsilon}(|x|-R_1-\epsilon)\right), \quad \forall n\in\mathbb{N}_0.$$

Thus, $\zeta_n\in C^1(\mathbb{R}^N)$ and satisfies

$$\chi_{B_{\rho_{n+1}}}\le\zeta_n\le\chi_{B_{\bar\rho_n}} \text{ and } |\nabla\zeta_n|\le\frac{2^{n+3}}{\epsilon}, \quad \forall n\in\mathbb{N}_0.$$

Before estimating J_{n+1} in terms of J_n we note that

$$\int_{A_{k_{n+1},\bar\rho_n}} (u-k_{n+1})^q \,\mathrm{d}x \le \int_{A_{k_{n+1},\rho_n}} (u-k_{n+1})^q \,\mathrm{d}x \le J_n, \tag{4.10}$$

also

$$\left|A_{k_{n+1},\rho_{n+1}}\right| \le \left|A_{k_{n+1},\bar\rho_n}\right| \le \left|A_{k_{n+1},\rho_n}\right| \le \int_{A_{k_{n+1},\rho_n}} \left(\frac{u-k_n}{k_{n+1}-k_n}\right)^q \mathrm{d}x \le 2^{(n+2)q}k_*^{-q}J_n. \tag{4.11}$$

Furthermore, we will need the following simple inequality

$$(x+y)^m \le C_m(x^m+y^m), \quad \forall x,y\ge 0 \quad (m\ge 0). \tag{4.12}$$

Now, fix $\bar q\in(tp,p_s^*)$. Using Hölder's inequality we estimate

$$J_{n+1} = \int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^q \,\mathrm{d}x \le \left(\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q} \,\mathrm{d}x\right)^{\frac{q}{\bar q}} \left|A_{k_{n+1},\rho_{n+1}}\right|^{\frac{\bar q-q}{\bar q}}. \tag{4.13}$$

On the other hand, in view of Lemma 4.3 and Sobolev's embedding, we get

$$\begin{aligned}\left(\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q} \,\mathrm{d}x\right)^{\frac{1}{\bar q}} &= \left(\int_{A_{k_{n+1},\rho_{n+1}}} \left((u-k_{n+1})\zeta_n\right)^{\bar q} \mathrm{d}x\right)^{\frac{1}{\bar q}} \\ &\le \left(\int_{A_{R_1}^{R_1+2\epsilon}} \left((u-k_{n+1})^+\zeta_n\right)^{\bar q} \mathrm{d}x\right)^{\frac{1}{\bar q}}\end{aligned}$$

$$\leq C_\epsilon\Bigg[\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}\big((u-k_{n+1})^+\zeta_n\big)^{p_s}\,\mathrm{d}x\bigg)^{\dfrac{1}{p_s}}+$$

$$+\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}|\nabla\big((u-k_{n+1})^+\zeta_n\big)|^{p_s}\,\mathrm{d}x\bigg)^{\dfrac{1}{p_s}}\Bigg], \tag{4.14}$$

here C_ϵ is the embedding constant for $W^{1,p_s}(A_{R_1}^{R_1+2\epsilon})\hookrightarrow L^{\bar{q}}(A_{R_1}^{R_1+2\epsilon})$. Using Hölder's inequality, we have

$$\int_{A_{R_1}^{R_1+2\epsilon}}\big((u-k_{n+1})^+\zeta_n\big)^{p_s}\,\mathrm{d}x\leq\int_{A_{k_{n+1},\bar{\rho}_n}}(u-k_{n+1})^{p_s}\,\mathrm{d}x$$

$$\leq\bigg(\int_{A_{k_{n+1},\bar{\rho}_n}}(u-k_{n+1})^{q}\,\mathrm{d}x\bigg)^{\dfrac{p_s}{q}}\big|A_{k_{n+1},\bar{\rho}_n}\big|^{\dfrac{q-p_s}{q}}.$$

Combining this with (4.10) and (4.11) we obtain

$$\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}\big((u-k_{n+1})^+\zeta_n\big)^{p_s}\,\mathrm{d}x\bigg)^{\dfrac{1}{p_s}}\leq 2^{\dfrac{2(q-p_s)}{p_s}}\,2^{\dfrac{n(q-p_s)}{p_s}}\,k_*^{-\dfrac{q-p_s}{p_s}}\,J_n^{\dfrac{1}{p_s}}. \tag{4.15}$$

We also have

$$\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}|\nabla\big((u-k_{n+1})^+\zeta_n\big)|^{p_s}\,\mathrm{d}x\bigg)^{\frac{1}{p_s}}\leq\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}L^{-s}(x)\,\mathrm{d}x\bigg)^{\frac{1}{sp}}\times$$

$$\times\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}L(x)|\nabla\big((u-k_{n+1})^+\zeta_n\big)|^{p}\,\mathrm{d}x\bigg)^{\frac{1}{p}}$$

$$\leq 2\delta\bigg[\int_{A_{k_{n+1},\bar{\rho}_n}}L(x)|\nabla u|^p\,\mathrm{d}x+2^{(n+3)p}\epsilon^{-p}\int_{A_{k_{n+1},\bar{\rho}_n}}L(x)(u-k_{n+1})^p\,\mathrm{d}x\bigg]^{\frac{1}{p}}$$

$$\leq 2\delta\bigg[\int_{A_{k_{n+1},\bar{\rho}_n}}L(x)|\nabla u|^p\,\mathrm{d}x+2^{(n+3)p}\epsilon^{-p}\alpha\bigg(\int_{A_{k_{n+1},\bar{\rho}_n}}(u-k_{n+1})^q\,\mathrm{d}x\bigg)^{\frac{p}{q}}\bigg]^{\frac{1}{p}}, \tag{4.16}$$

where $\delta:=\bigg(\int_{A_{R_1}^{R_1+2\epsilon}}L^{-s}(x)\,\mathrm{d}x\bigg)^{\frac{1}{sp}}$ and α is as in (4.4). From (4.10) and (4.14)-(4.16), invoking (4.12), we get

$$\bigg(\int_{A_{k_{n+1},\rho_{n+1}}}(u-k_{n+1})^{\bar{q}}\,\mathrm{d}x\bigg)^{\frac{q}{\bar{q}}}\leq CC_\epsilon^q\bigg\{2^{\frac{n(q-p_s)}{p_s}}k_*^{-\frac{q-p_s}{p_s}}J_n^{\frac{1}{p_s}}+2^n\epsilon^{-1}\alpha^{\frac{1}{p}}\delta J_n^{\frac{1}{q}}$$

$$+\delta\bigg(\int_{A_{k_{n+1},\bar{\rho}_n}}L(x)|\nabla u|^p\,\mathrm{d}x\bigg)^{\frac{1}{p}}\bigg\}^q. \tag{4.17}$$

Applying (4.5) with $r_1=\bar{\rho}_n,\ r_2=\rho_n$ and $k=k_{n+1}$, we get

$$\int_{A_{k_{n+1},\bar{\rho}_n}}L(x)|\nabla u|^p\,\mathrm{d}x\leq C(\alpha+\beta\epsilon^p)\epsilon^{-p}2^{np}\bigg(\int_{A_{k_{n+1},\rho_n}}(u-k_{n+1})^q\,\mathrm{d}x\bigg)^{\frac{p}{q}}+$$

$$+ p\gamma\Big(\int_{A_{k_{n+1},\rho_n}} (u-k_{n+1})^q \,\mathrm{d}x\Big)^{\frac{1}{q}} |A_{k_{n+1},\rho_n}|^{\frac{q-t}{qt}} + C\beta k_*^p |A_{k_{n+1},\rho_n}|^{\frac{p}{q}}.$$

Using (4.10) and (4.11) again, we deduce from the last inequality that

$$\int_{A_{k_{n+1},\bar\rho_n}} L(x)|\nabla u|^p \,\mathrm{d}x \le C(\epsilon^{-p}\alpha+\beta)2^{np} J_n^{\frac{p}{q}} + C\gamma 2^{\frac{n(q-t)}{t}} k_*^{-\frac{q-t}{t}} J_n^{\frac{1}{t}}.$$

Invoking (4.12) the last inequality yields

$$\Big(\int_{A_{k_{n+1},\bar\rho_n}} L(x)|\nabla u|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \le C(\epsilon^{-1}\alpha^{\frac{1}{p}}+\beta^{\frac{1}{p}})2^n J_n^{\frac{1}{q}} + C\gamma^{\frac{1}{p}} 2^{\frac{n(q-t)}{tp}} k_*^{-\frac{q-t}{tp}} J_n^{\frac{1}{tp}}.$$

From this and (4.17), we obtain

$$\Big(\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q} \,\mathrm{d}x\Big)^{\frac{q}{\bar q}} \le CC_\epsilon^q \Big\{ 2^{\frac{n(q-p_s)}{p_s}} k_*^{-\frac{q-p_s}{p_s}} J_n^{\frac{1}{p_s}} + \delta(\epsilon^{-1}\alpha^{\frac{1}{p}}+\beta^{\frac{1}{p}})2^n J_n^{\frac{1}{q}}$$
$$+ \gamma^{\frac{1}{p}} 2^{\frac{n(q-t)}{tp}} k_*^{-\frac{q-t}{tp}} J_n^{\frac{1}{tp}} \Big\}^q. \tag{4.18}$$

It follows from (4.18) and (4.12), noticing $k^* > 1$ and $J_n^{\frac{1}{p_s}} + J_n^{\frac{1}{q}} + J_n^{\frac{1}{tp}} \le 2(J_n^{\frac{1}{p_s}} + J_n^{\frac{1}{tp}})$ due to $p_s < q < tp$, that

$$\Big(\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q} \,\mathrm{d}x\Big)^{\frac{q}{\bar q}} \le \widetilde C(\epsilon,\alpha,\beta,\gamma,\delta) 2^{\frac{nq^2}{p_s}} (J_n^{\frac{q}{p_s}} + J_n^{\frac{q}{tp}}). \tag{4.19}$$

From (4.13), (4.11) and (4.19), we obtain

$$J_{n+1} \le C(\epsilon,\alpha,\beta,\gamma,\delta) 2^{\frac{nq^2}{p_s}} \Big(J_n^{\frac{q}{p_s}} + J_n^{\frac{q}{tp}}\Big) 2^{\frac{nq(\bar q-q)}{\bar q}} k_*^{-\frac{q(\bar q-q)}{\bar q}} J_n^{\frac{\bar q-q}{\bar q}}.$$

That is,

$$J_{n+1} \le C(\epsilon,\alpha,\beta,\gamma,\delta) k_*^{-\frac{q(\bar q-q)}{\bar q}} \eta^n \Big(J_n^{1+\delta_1} + J_n^{1+\delta_2}\Big), \tag{4.20}$$

where

$$0 < \delta_1 := \frac{q}{tp} - \frac{q}{\bar q} < \delta_2 := \frac{q}{p_s} - \frac{q}{\bar q} \text{ and } \eta := 2^{\frac{q^2}{p_s} + \frac{q(\bar q-q)}{\bar q}} > 1.$$

Step 3: A-priori bounds. Invoking Lemma 4.4, we deduce from (4.20) that $J_n \to 0$ as $n \to \infty$, provided

$$J_0 \le \min\Big((2\widetilde k)^{-\frac{1}{\delta_1}} \eta^{-\frac{1}{\delta_1^2}}, (2\widetilde k)^{-\frac{1}{\delta_2}} \eta^{-\frac{1}{\delta_1\delta_2} - \frac{\delta_2-\delta_1}{\delta_2^2}} \Big), \tag{4.21}$$

where $\widetilde k := C(\epsilon,\alpha,\beta,\gamma,\delta) k_*^{-\frac{q(\bar q-q)}{\bar q}}$. We have

$$J_0 = \int_{A_{k_0,\rho_0}} (u-k_0)^q \,\mathrm{d}x = \int_{A_{R_1}^{\rho_0}} \big((u-k_0)^+\big)^q \,\mathrm{d}x \le \int_{A_{R_1}^{R_1+2\epsilon}} (u^+)^q \,\mathrm{d}x.$$

On the other hand, the inequality

$$\int_{A_{R_1}^{R_1+2\epsilon}} (u^+)^q \,\mathrm{d}x \le \Big(2C(\epsilon,\alpha,\beta,\gamma,\delta) k_*^{-\frac{q(\bar q-q)}{\bar q}}\Big)^{-\frac{1}{\delta_1}} \eta^{-\frac{1}{\delta_1^2}}$$

is equivalent to

$$k_* \geq \big(2C(\epsilon,\alpha,\beta,\gamma,\delta)\big)^{\frac{\bar q}{q(\bar q-q)}}\eta^{\frac{\bar q}{\delta_1 q(\bar q-q)}}\left(\int_{A_{R_1}^{R_1+2\epsilon}}(u^+)^q\,\mathrm{d}x\right)^{\frac{\bar q\delta_1}{q(\bar q-q)}}.$$

We also have that the following inequality

$$\int_{A_{R_1}^{R_1+2\epsilon}}(u^+)^q\,\mathrm{d}x \leq \left(2C(\epsilon,\alpha,\beta,\gamma,\delta)k_*^{-\frac{q(\bar q-q)}{\bar q}}\right)^{-\frac{1}{\delta_2}}\eta^{-\frac{1}{\delta_1\delta_2}-\frac{\delta_2-\delta_1}{\delta_2^2}}$$

is equivalent to

$$k_* \geq \big(2C(\epsilon,\alpha,\beta,\gamma,\delta)\big)^{\frac{\bar q}{q(\bar q-q)}}\eta^{\left(\frac{1}{\delta_1}+\frac{\delta_2-\delta_1}{\delta_2}\right)\frac{\bar q}{q(\bar q-q)}}\left(\int_{A_{R_1}^{R_1+2\epsilon}}(u^+)^q\,\mathrm{d}x\right)^{\frac{\bar q\delta_2}{q(\bar q-q)}}.$$

So if we choose

$$k_* = \left[1+(2C(\epsilon,\alpha,\beta,\gamma,\delta))^{\frac{\bar q}{q(\bar q-q)}}\eta^{\left(\frac{1}{\delta_1}+\frac{\delta_2-\delta_1}{\delta_2}\right)\frac{\bar q}{q(\bar q-q)}}\right]\left[1+\left(\int_{A_{R_1}^{R_1+2\epsilon}}|u|^q\,\mathrm{d}x\right)^{\frac{\bar q\delta_2}{q(\bar q-q)}}\right], \tag{4.22}$$

then, we obtain (4.21), and hence, thanks to Lemma 4.4

$$J_n = \int_{A_{R_1}^{R_1+2\epsilon}}\big((u-k_n)^+\big)^q\chi_{A_{R_1}^{\rho_n}}\,\mathrm{d}x \to 0 \text{ as } n\to\infty.$$

Note that, due to Lebesgue's dominated convergence theorem we have

$$J_n \to \int_{A_{R_1}^{R_1+2\epsilon}}\big((u-k_*)^+\big)^q\chi_{A_{R_1}^{R_1+\epsilon}}\,\mathrm{d}x = \int_{A_{R_1}^{R_1+\epsilon}}\big((u-k_*)^+\big)^q\,\mathrm{d}x \text{ as } n\to\infty.$$

Thus, $\int_{A_{R_1}^{R_1+\epsilon}}\big((u-k_*)^+\big)^q\,\mathrm{d}x = 0$ and hence, $(u-k_*)^+ = 0$ a.e. in $A_{R_1}^{R_1+\epsilon}$, i.e.,

$$\operatorname*{ess\,sup}_{A_{R_1}^{R_1+\epsilon}} u \leq k_*. \tag{4.23}$$

Replacing u by $-u$ in Steps 1 and 2 and arguing as above, we get

$$\operatorname*{ess\,sup}_{A_{R_1}^{R_1+\epsilon}}(-u) \leq k_*. \tag{4.24}$$

It follows from (4.23) and (4.24) that

$$\|u\|_{L^\infty(A_{R_1}^{R_1+\epsilon})} \leq k_*. \tag{4.25}$$

Note that by Lemma 4.3, we have

$$u \in L^q\big(A_{R_1}^{R_1+2\epsilon}\big). \tag{4.26}$$

Combining (4.22), (4.25) and (4.26) there exist $C>0$ and $\mu>0$, both independent of u, such that (4.2) holds. This completes the proof of part (i).

(ii) We proceed in the same fashion as in part (i) of this proof. Let u be a weak solution of problem (4.1). Without loss of generality we may assume that $t > \frac{q}{p}$. Denote

$$A := \|L\|_{L^{\frac{q}{q-p}}\left(B(x_0,r_0)\right)},\ B := \|a\|_{L^{\frac{q}{q-p}}\left(B(x_0,r_0)\right)} \text{ and } M := \|b\|_{L^{\frac{t}{t-1}}\left(B(x_0,r_0)\right)},$$

and for $k>0$, $\delta\in(0,r_0)$, denote

$$A_{k,\delta} := \{x\in B(x_0,\delta) : u(x) > k\}.$$

We will prove that there exists a positive constant C independent of u such that for any $0 < r_1 < r_2 < r_0$ and $k > 0$, the following Caccioppoli-type inequality holds true:

$$\int_{A_{k,r_1}} L(x)|\nabla u|^p \,\mathrm{d}x \le C(A + Br_0^p)\bigg(\int_{A_{k,r_2}} \left(\frac{u-k}{r_2-r_1}\right)^q \mathrm{d}x\bigg)^{\frac{p}{q}} + \\ + pM\bigg(\int_{A_{k,r_2}} (u-k)^q \,\mathrm{d}x\bigg)^{\frac{1}{q}} |A_{k,r_2}|^{\frac{q-t}{qt}} + CBk^p|A_{k,r_2}|^{\frac{p}{q}}. \quad (4.27)$$

Indeed, let $\xi \in C^\infty(\mathbb{R}^N)$, such that $\chi_{B(x_0,r_1)} \le \xi \le \chi_{B(x_0,r_2)}$ and $|\nabla\xi| \le \frac{2}{r_2-r_1}$. We note that for $\widetilde{u} \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ and $\widetilde{\xi} \in C^1(\mathbb{R}^N)$ with $\chi_{B(x_0,r_1)} \le \widetilde{\xi} \le \chi_{B(x_0,r_2)}$, we have $\widetilde{u}\widetilde{\xi} \in \mathcal{D}_0^{1,p}(A_{R_1}^{R_2}; L)$ and $\widetilde{u}\widetilde{\xi}$ is a test function for (4.1). By this and Proposition 2.8, we can use $(u-k)^+\xi^p$ as a test function in (4.1) and then repeating the arguments used in the proof of part (i), we easily obtain (4.27).

Next, we define the recursive sequence $\{J_n\}$ as follows. For each $n \in \mathbb{N}_0$, define

$$J_n := \int_{A_{k_n,\rho_n}} (u-k_n)^q \,\mathrm{d}x,$$

where

$$\rho_n := \frac{r_0}{2} + \frac{r_0}{2^{n+1}} \quad \text{and} \quad k_n := k_*\left(1 - \frac{1}{2^{n+1}}\right)$$

with $k_* > 0$ to be specified later. Note that

$$\rho_n \downarrow \frac{r_0}{2}, \ k_n \uparrow k_* \text{ and } \frac{r_0}{2} < \rho_n \le r_0, \ \frac{k_*}{2} \le k_n < k_*, \ \forall n \in \mathbb{N}_0.$$

Denote $\bar{\rho}_n := \frac{\rho_n+\rho_{n+1}}{2}$ $(n \in \mathbb{N}_0)$ and fix $\zeta \in C^1(\mathbb{R})$, such that $\chi_{(-\infty,\frac{1}{2})} \le \zeta \le \chi_{(-\infty,\frac{3}{4})}$ and $|\zeta'| \le 8$. Define

$$\zeta_n(x) := \zeta\left(\frac{2^{n+1}}{r_0}\left(|x-x_0| - \frac{r_0}{2}\right)\right), \quad x \in \mathbb{R}^N.$$

Then $\zeta_n \in C^1(\mathbb{R}^N)$, $\chi_{B(x_0,\rho_{n+1})} \le \zeta_n \le \chi_{B(x_0,\bar{\rho}_n)}$ and $|\nabla\zeta_n| \le \frac{2^{n+4}}{r_0}$ for all $n \in \mathbb{N}_0$. Fix $\bar{q} \in (tp, p_s^*)$. Using Höder's inequality, we have

$$J_{n+1} = \int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^q \,\mathrm{d}x \le \left(\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar{q}} \,\mathrm{d}x\right)^{\frac{q}{\bar{q}}} \left|A_{k_{n+1},\rho_{n+1}}\right|^{\frac{\bar{q}-q}{\bar{q}}}. \quad (4.28)$$

It is easy to see that

$$\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar{q}} \,\mathrm{d}x \le \int_{B(x_0,r_0)} \left((u-k_{n+1})^+\zeta_n\right)^{\bar{q}} \mathrm{d}x. \quad (4.29)$$

By the assumption on L, $W^{1,p}(B(x_0,r_0); L) := \{u \in W^1(B(x_0,r_0)) : \int_{B(x_0,r_0)} \left[|u|^p + L(x)|\nabla u|^p\right] \mathrm{d}x < \infty\}$ is a Sobolev space with respect to the norm

$$\|u\|_{W^{1,p}(B(x_0,r_0);L)} := \left(\int_{B(x_0,r_0)} \left[|u|^p + L(x)|\nabla u|^p\right] \mathrm{d}x\right)^{\frac{1}{p}}.$$

Moreover, $W^{1,p}(B(x_0,r_0); L) \hookrightarrow W^{1,p_s}(B(x_0,r_0)) \hookrightarrow L^{\bar{q}}(B(x_0,r_0))$ in view of [8, Theorem 1.3 and the embedding (1.22)]. Denote by $W_0^{1,p}(B(x_0,r_0); L)$ the closure of $C_c^\infty(B(x_0,r_0))$ in $W^{1,p}(B(x_0,r_0); L)$ with respect to the norm $\|\cdot\|_{W^{1,p}(B(x_0,r_0);L)}$. For any

$\widetilde{u} \in C_c^\infty(B(x_0,r_0))$ using the change of variable of the form $x = x_0+y$, $\widetilde{v}(y) = \widetilde{u}(x_0+y)$, and employing Sobolev's embedding and Poincaré's inequality we obtain

$$\begin{aligned}
\left(\int_{B(x_0,r_0)} |\widetilde{u}(x)|^{\bar{q}}\,\mathrm{d}x\right)^{\frac{1}{\bar{q}}} &= \left(\int_{B(0,r_0)} |\widetilde{v}(y)|^{\bar{q}}\,\mathrm{d}y\right)^{\frac{1}{\bar{q}}}\\
&\le C_1(r_0)\left(\int_{B(0,r_0)} \left(|\widetilde{v}(y)|^{p_s} + |\nabla\widetilde{v}(y)|^{p_s}\right)\mathrm{d}y\right)^{\frac{1}{p_s}}\\
&\le C_2(r_0)\left(\int_{B(0,r_0)} |\nabla\widetilde{v}(y)|^{p_s}\,\mathrm{d}y\right)^{\frac{1}{p_s}} = C_2(r_0)\left(\int_{B(x_0,r_0)} |\nabla\widetilde{u}(x)|^{p_s}\,\mathrm{d}x\right)^{\frac{1}{p_s}}\\
&\le C_2(r_0)\left(\int_{B(x_0,r_0)} L^{-s}(x)\,\mathrm{d}x\right)^{\frac{1}{sp}}\left(\int_{B(x_0,r_0)} L(x)|\nabla\widetilde{u}(x)|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

Here, and in what follows, $C_i(r_0)$ $(i \in \mathbb{N})$ depend only on r_0. Thus we obtain

$$\int_{B(x_0,r_0)} |\widetilde{u}(x)|^{\bar{q}}\,\mathrm{d}x \le C_3(r_0)D\left(\int_{B(x_0,r_0)} L(x)|\nabla\widetilde{u}(x)|^{p}\,\mathrm{d}x,\right)^{\frac{\bar{q}}{p}}$$

where $D := \left(\int_{B(x_0,r_0)} L^{-s}(x)\,\mathrm{d}x\right)^{\frac{\bar{q}}{sp}}$, for all $\widetilde{u} \in C_c^\infty(B(x_0,r_0))$. By the density argument, it holds for all $\widetilde{u} \in W_0^{1,p}(B(x_0,r_0);L)$. It is easy to see that $(u-k_{n+1})^+\zeta_n \in W_0^{1,p}(B(x_0,r_0),L)$. Thus, applying the last inequality for $\widetilde{u} = (u-k_{n+1})^+\zeta_n$ and combining this with (4.29) we obtain

$$\begin{aligned}
\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar{q}}\,\mathrm{d}x &\le C_3(r_0)D\left(\int_{B(x_0,r_0)} L(x)|\nabla((u-k_{n+1})^+\zeta_n)|^p\,\mathrm{d}x\right)^{\frac{\bar{q}}{p}}\\
&\le C_4(r_0)D\Bigg[\int_{B(x_0,r_0)} L(x)|\nabla(u-k_{n+1})^+|^p\zeta_n^p\,\mathrm{d}x\\
&\qquad + \int_{B(x_0,r_0)} L(x)((u-k_{n+1})^+)^p|\nabla\zeta_n|^p\,\mathrm{d}x\Bigg]^{\frac{\bar{q}}{p}}\\
&\le C_4(r_0)D\Bigg[\int_{A_{k_{n+1},\bar{\rho}_n}} L(x)|\nabla u|^p\,\mathrm{d}x\\
&\qquad + 2^{(n+4)p}r_0^{-p}\int_{A_{k_{n+1},\bar{\rho}_n}} L(x)(u-k_{n+1})^p\,\mathrm{d}x\Bigg]^{\frac{\bar{q}}{p}}\\
&\le C_5(r_0)D\Bigg[\int_{A_{k_{n+1},\bar{\rho}_n}} L(x)|\nabla u|^p\,\mathrm{d}x\\
&\qquad + 2^{np}A\left(\int_{A_{k_{n+1},\bar{\rho}_n}} (u-k_{n+1})^q\,\mathrm{d}x\right)^{\frac{p}{q}}\Bigg]^{\frac{\bar{q}}{p}}.
\end{aligned}$$

This yields

$$\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q}\,\mathrm{d}x \le C_5(r_0)D\Bigg[\int_{A_{k_{n+1},\bar\rho_n}} L(x)|\nabla u|^p\,\mathrm{d}x + 2^{np}AJ_n^{\frac{p}{q}}\Bigg]^{\frac{\bar q}{p}}. \tag{4.30}$$

Applying (4.27) for $k=k_{n+1},\ r_1=\bar\rho_n$ and $r_2=\rho_n$, we get

$$\int_{A_{k_{n+1},\bar\rho_n}} L(x)|\nabla u|^p\,\mathrm{d}x \le C2^{(n+3)p}r_0^{-p}(A+Br_0^p)\Big(\int_{A_{k_{n+1},\rho_n}}(u-k_{n+1})^q\Big)^{\frac{p}{q}} +$$

$$+pM\Big(\int_{A_{k_{n+1},\rho_n}}(u-k_{n+1})^q\,\mathrm{d}x\Big)^{\frac1q}|A_{k_{n+1},\rho_n}|^{\frac{q-t}{qt}} + CBk_*^p|A_{k_{n+1},\rho_n}|^{\frac{p}{q}}.$$

Combining this and (4.30), we obtain

$$\int_{A_{k_{n+1},\rho_{n+1}}} (u-k_{n+1})^{\bar q}\,\mathrm{d}x \le C_6(r_0)D\Big[(A+B)2^{np}J_n^{\frac{p}{q}} + MJ_n^{\frac1q}|A_{k_{n+1},\rho_n}|^{\frac{q-t}{qt}}$$

$$+Bk_*^p\big|A_{k_{n+1},\rho_n}\big|^{\frac{p}{q}} + 2^{np}AJ_n^{\frac{p}{q}}\Big]^{\frac{\bar q}{p}}.$$

From this and the estimate

$$\big|A_{k_{n+1},\rho_{n+1}}\big| \le \big|A_{k_{n+1},\rho_n}\big| \le \int_{A_{k_{n+1},\rho_n}}\Big(\frac{u-k_n}{k_{n+1}-k_n}\Big)^q\,\mathrm{d}x \le \frac{2^{(n+2)q}}{k_*^q}J_n,$$

we obtain from (4.28) that

$$J_{n+1} \le C_7(r_0)D^{\frac{q}{\bar q}}\Big[(A+B)2^{np}J_n^{\frac{p}{q}} + Mk_*^{1-\frac{q}{t}}2^{\frac{n(q-t)}{t}}J_n^{\frac1q+\frac{q-t}{qt}}\Big]^{\frac{q}{p}}\frac{2^{\frac{nq(\bar q-q)}{\bar q}}J_n^{\frac{\bar q-q}{\bar q}}}{k_*^{\frac{q(\bar q-q)}{\bar q}}}. \tag{4.31}$$

So if we choose $k_*>1$ then (4.31) implies

$$J_{n+1} \le C(A,B,M,D,r_0)k_*^{-\frac{q(\bar q-q)}{\bar q}}\eta^n\Big(J_n^{1+\delta_1}+J_n^{1+\delta_2}\Big),$$

where

$$0<\delta_1 := \frac{q}{tp}-\frac{q}{\bar q} < \delta_2 := \frac{\bar q-q}{\bar q} \text{ and } \eta := 2^{q+\frac{q(\bar q-q)}{\bar q}} > 1.$$

Finally, arguing as in Step 3 of the proof of part (i) we get the desired conclusion. □

Obviously, Theorem 1.3 is a special case of Theorem 4.2. Now we give the proof of Theorem 1.4. Since this proof is similar to that of Theorem 4.2 (ii), we only sketch it.

Proof of Theorem 1.4. Let u be a solution of problem (1.1) and let $\mu\in(0,1-\frac{q}{p_s^*})$. We follow the argument in the proof of Theorem 4.2 (ii) with the choice $a=K$, $b=0$ and $\bar q := \frac{q}{1-\mu}$ to obtain (4.31) of the form

$$J_{n+1} \le C(r_0)D^{\frac{q}{\bar q}}\Big[(A+B)2^{np}J_n^{\frac{p}{q}}\Big]^{\frac{q}{p}}\frac{2^{\frac{nq(\bar q-q)}{\bar q}}J_n^{\frac{\bar q-q}{\bar q}}}{k_*^{\frac{q(\bar q-q)}{\bar q}}},$$

where

$A := \|L\|_{L^{\frac{q}{q-p}}\left(B(x_0,r_0)\right)}$, $B := \|K\|_{L^{\frac{q}{q-p}}\left(B(x_0,r_0)\right)}$, $D := \left(\int_{B(x_0,r_0)} L^{-s}(x)\,\mathrm{d}x\right)^{\frac{\bar{q}}{sp}}$, and $C(r_0) > 0$ depends only on r_0. This implies

$$J_{n+1} \le C(r_0) D^{\frac{q}{\bar{q}}} (A+B)^{\frac{q}{p}} k_*^{-q\mu} \eta^n J_n^{1+\mu}, \tag{4.32}$$

where $\eta := 2^{q(1+\mu)} > 1$. Invoking Lemma 4.4 with $\delta_1 = \delta_2 = \mu$, we deduce from (4.32) that $J_n \to 0$ as $n \to \infty$, provided

$$J_0 \le \left[C(r_0) D^{\frac{q}{\bar{q}}} (A+B)^{\frac{q}{p}} k_*^{-q\mu}\right]^{-\frac{1}{\mu}} \eta^{-\frac{1}{\mu^2}}. \tag{4.33}$$

We have

$$J_0 = \int_{A_{k_0,\rho_0}} (u-k_0)^q\,\mathrm{d}x = \int_{B(x_0,\rho_0)} \left((u-k_0)^+\right)^q \mathrm{d}x \le \int_{B(x_0,r_0)} (u^+)^q\,\mathrm{d}x.$$

So if we choose

$$k_* = \left[C(r_0)\eta^{\frac{1}{\mu}}\right]^{\frac{1}{q\mu}} D^{\frac{1}{\bar{q}\mu}} (A+B)^{\frac{1}{p\mu}} \left(\int_{B(x_0,r_0)} (u^+)^q\,\mathrm{d}x\right)^{\frac{1}{q}}, \tag{4.34}$$

then, we obtain (4.33), and hence, thanks to Lemma 4.4

$$J_n = \int_{B(x_0,r_0)} \left((u-k_n)^+\right)^q \chi_{B(x_0,\rho_n)}\,\mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Note that, due to Lebesgue's dominated convergence theorem we have

$$J_n \to \int_{B(x_0,r_0)} \left((u-k_*)^+\right)^q \chi_{B(x_0,\frac{r_0}{2})}\,\mathrm{d}x = \int_{B(x_0,\frac{r_0}{2})} \left((u-k_*)^+\right)^q \mathrm{d}x \text{ as } n \to \infty.$$

Thus, $\int_{B(x_0,\frac{r_0}{2})} \left((u-k_*)^+\right)^q \mathrm{d}x = 0$ and hence, $(u-k_*)^+ = 0$ a.e. in $B(x_0,\frac{r_0}{2})$, i.e.,

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r_0}{2})} u \le k_*. \tag{4.35}$$

Replacing u by $-u$ in the arguments above, we get

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r_0}{2})} (-u) \le k_*. \tag{4.36}$$

It follows from (4.35) and (4.36) that

$$\|u\|_{L^\infty(B(x_0,\frac{r_0}{2}))} \le k_*. \tag{4.37}$$

Note that by Lemma 4.3, we have

$$u \in L^q\big(B(x_0,r_0)\big). \tag{4.38}$$

Combining (4.34), (4.37) and (4.38) there exists $C = C(\mu, r_0) > 0$ independent of u, such that (1.3) holds. The proof is complete. □

4.2. **The smoothness of solutions.** In this subsection we prove the results on smoothness of solutions.

Proof of Theorem 1.5. We rewrite (1.1) as

$$-L\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)-|\nabla u|^{p-2}(\nabla u\cdot\nabla L)=\lambda K|u|^{p-2}u,$$

i.e.,

$$-\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)=\lambda\frac{K}{L}|u|^{p-2}u+|\nabla u|^{p-2}(\nabla u\cdot\frac{\nabla L}{L}).$$

Thus, $\phi=u$ is a weak solution to

$$-\,\mathrm{div}\,\vec{a}(x,\phi,\nabla\phi)+b(x,\phi,\nabla\phi)=0,$$

where $\vec{a}(x,\phi,\nabla\phi)=-|\nabla\phi|^{p-2}\nabla\phi$ and $b(x,\phi,\nabla\phi)=\lambda\frac{K}{L}|u|^{p-2}u+|\nabla\phi|^{p-2}(\nabla\phi\cdot\frac{\nabla L}{L})$. In view of Corollary 2.2 and Theorem 1.4 we have $u\in W^{1,p}_{\mathrm{loc}}(A_{R_1}^{R_2})\cap L^\infty_{\mathrm{loc}}(A_{R_1}^{R_2})$. Using Young's inequality, for any $R_1<r_1<r_2<R_2$ we have

$$|b(x,\phi,\nabla\phi)|\le\lambda\|u\|^{p-1}_{L^\infty(A_{r_1}^{r_2})}\left|\frac{K}{L}\right|+\frac{p-1}{p}|\nabla\phi|^p+\frac{1}{p}\left|\frac{\nabla L}{L}\right|^p.$$

Hence

$$|b(x,\phi,\nabla\phi)|\le\frac{p-1}{p}|\nabla\phi|^p+\left(\lambda\|u\|^{p-1}_{L^\infty(A_{r_1}^{r_2})}+\frac{1}{p}\right)\left(\left|\frac{K}{L}\right|+\left|\frac{\nabla L}{L}\right|^p\right).$$

Thus by [5, Theorem 2 and its Remark] we obtain $C^{1,\alpha}_{\mathrm{loc}}(A_{r_1}^{r_2})$ for any $R_2<r_1<r_2<R_2$ and hence the proof is completed. ☐

Finally we conclude this subsection by proving Hölder regularity of eigenfunctions up to inner boundary.

Proof of Theorem 1.6. By Theorem 1.3, we have $u\in L^\infty\big(A_{R_1}^{R_1+\epsilon}\big)$. From this and the estimate

$$\int_{A_{R_1}^{R_1+\epsilon}}|\nabla u|^p\,\mathrm{d}x\le\frac{1}{\operatorname*{ess\,inf}_{x\in A_{R_1}^{R_1+\epsilon}}L(x)}\int_{A_{R_1}^{R_1+\epsilon}}L(x)|\nabla u|^p\,\mathrm{d}x<\infty,$$

we obtain $u\in W^{1,p}(A_{R_1}^{R_1+\epsilon})\cap L^\infty\big(A_{R_1}^{R_1+\epsilon}\big)$. As in the proof of Theorem 1.5, we have

$$-\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)=\lambda\frac{K}{L}|u|^{p-2}u+|\nabla u|^{p-2}(\nabla u\cdot\frac{\nabla L}{L}).$$

Thus $\phi=u\in W^{1,p}(A_{R_1}^{R_1+\epsilon})\cap L^\infty\big(A_{R_1}^{R_1+\epsilon}\big)$ is a weak solution to the following problem

$$\begin{cases}-\,\mathrm{div}(|\nabla\phi|^{p-2}\nabla\phi)=\lambda\frac{K}{L}|\phi|^{p-2}\phi+|\nabla\phi|^{p-2}(\nabla\phi\cdot\frac{\nabla L}{L})\ \text{ in }\ A_{R_1}^{R_1+\epsilon},\\ \phi=0\ \text{ on }\ \partial B_{R_1}\ \text{ and }\ \phi=u\text{ on }\partial B_{R_1+\epsilon}.\end{cases}$$

By Theorem 1.5, we have $u\in C^{1,\alpha}(\partial B_{R_1+\epsilon})$. From this and $|\frac{\nabla L}{L}|+|\frac{K}{L}|\in L^\infty\big(A_{R_1}^{R_1+\epsilon}\big)$, we have $\phi=u\in C^{1,\beta_\epsilon}(\overline{A_{R_1}^{R_1+\epsilon}})$ for some $\beta_\epsilon\in(0,1)$ in view of [15, Theorem 1]. ☐

4.3. **Positivity and decay of solutions.** In this subsection we prove the positivity and decay of solutions. First, we prove Theorem 1.7, which states that a nonnegative C^1 solution is positive everywhere.

Proof of Theorem 1.7. By Theorem 1.5, we have $u \in C^1(A_{R_1}^{R_2})$. The conclusion of the theorem then follows from [20, Theorem 8.1]. □

Finally, we show the decay of solutions at infinity when the domain is unbounded.

Proof of Corollary 1.8. Denote

$$\alpha_1 := \operatorname*{ess\,inf}_{x \in B_R^c} L(x) \text{ and } \beta_1 := \operatorname*{ess\,sup}_{x \in B_{R+r_0}^c} \left[\|L\|_{L^{\frac{q}{q-p}}(B(x,r_0))} + \|K\|_{L^{\frac{q}{q-p}}(B(x,r_0))} \right],$$

then $0 < \alpha_1, \beta_1 < \infty$ by the assumptions of the corollary. Let u be a solution to problem (1.1). We first show that $u \in L^{p^*}(B_{R+\epsilon}^c)$ for all $\epsilon > 0$. Indeed, fix an $\epsilon > 0$ and let $\{u_n\} \subset C_c^1(B_{R_1}^c)$ such that

$$\int_{B_{R_1}^c} L(x)|\nabla u_n - \nabla u|^p \,\mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{4.39}$$

Since $\mathcal{D}_0^{1,p}(B_{R_1}^c; L) \hookrightarrow L^p_{\text{loc}}(B_{R_1}^c)$, up to a subsequence, we have

$$\begin{cases} u_n \to u \text{ a.e. in } B_{R_1}^c, \\ u_n \to u \text{ in } L^p(A_R^{R+\epsilon}). \end{cases} \tag{4.40}$$

Let $\phi \in C^\infty(\mathbb{R}^N)$ such that $\chi_{B_{R+\epsilon}^c} \le \phi \le \chi_{B_R^c}$ and $|\nabla \phi| \le \frac{2}{\epsilon}$. Since $\phi u_n \in C_c^1(\mathbb{R}^N)$, by Sobolev's embedding we have

$$\left(\int_{\mathbb{R}^N} |\phi u_n|^{p^*} \,\mathrm{d}x \right)^{\frac{p}{p^*}} \le C \int_{\mathbb{R}^N} \left|\nabla(\phi u_n)\right|^p \mathrm{d}x, \quad \forall n \in \mathbb{N},$$

where $C > 0$ is independent of n. Hence

$$\begin{aligned} \left(\int_{B_{R+\epsilon}^c} |u_n|^{p^*} \,\mathrm{d}x \right)^{\frac{p}{p^*}} &\le 2^{p-1} C \left(\int_{\mathbb{R}^N} \phi^p |\nabla u_n|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^p |\nabla \phi|^p \,\mathrm{d}x \right) \\ &\le 2^{p-1} C \left[\frac{1}{\alpha_1} \int_{B_R^c} L(x) |\nabla u_n|^p \,\mathrm{d}x + \left(\frac{2}{\epsilon}\right)^p \int_{A_R^{R+\epsilon}} |u_n|^p \,\mathrm{d}x \right]. \end{aligned}$$

Letting $n \to \infty$ in the last estimate, using (4.39), (4.40) and Fatou's lemma, we get

$$\left(\int_{B_{R+\epsilon}^c} |u|^{p^*} \,\mathrm{d}x \right)^{\frac{p}{p^*}} \le \frac{2^{p-1} C}{\alpha_1} \int_{B_R^c} L(x) |\nabla u|^p \,\mathrm{d}x + \frac{2^{2p-1} C}{\epsilon^p} \int_{A_R^{R+\epsilon}} |u|^p \,\mathrm{d}x < \infty.$$

Thus $u \in L^{p^*}(B_{R+\epsilon}^c)$. Hence for a fixed $x \in B_{R+r_0+\epsilon}$, we get

$$\int_{B(x,r_0)} |u|^q \,\mathrm{d}y \le |B(x,r_0)|^{\frac{p^*-q}{p^*}} \left(\int_{B(x,r_0)} |u|^{p^*} \,\mathrm{d}y \right)^{\frac{q}{p^*}}. \tag{4.41}$$

Let $s > \frac{N}{p} + \frac{1}{p-1}$ be sufficiently large such that $q < p_s^* < p^*$. Fix such s and $\mu \in (0, 1 - \frac{q}{p_s^*})$. Clearly, all the assumptions of Theorem 1.4 are satisfied so we obtain (1.3) for any ball $B(x, r_0)$. From this estimate and (4.41), for all $|x| > R + r_0 + \epsilon$, we have

$$\|u\|_{L^\infty\left(B\left(x,\frac{r_0}{2}\right)\right)} \le C(r_0,\mu)(\alpha_1^{-s}|B(x,r_0)|)^{\frac{1}{sp\mu}} \beta_1^{\frac{1}{\mu p}} |B(0,r_0)|^{\frac{p^*-q}{qp^*}} \left(\int_{B(x,r_0)} |u|^{p^*} \,\mathrm{d}y \right)^{\frac{1}{p^*}}.$$

That is,

$$\|u\|_{L^\infty\left(B\left(x,\frac{r_0}{2}\right)\right)} \leq C(r_0,\mu,\alpha_1,\beta_1)\left(\int_{B(x,r_0)} |u|^{p^*}\,\mathrm{d}y\right)^{\frac{1}{p^*}}.$$

where $C(r_0,\mu,\alpha_1,\beta_1)$ is independent of x. Since $u \in L^{p^*}(B^c_{R+\epsilon})$, we deduce from the last inequality that $u(x) \to 0$ uniformly as $|x| \to \infty$. □

5. The asymptotic estimates of solutions towards the boundary

In this section we prove the asymptotic estimates of solutions towards the boundary stated in Theorems 1.10 and 1.11. Such asymptotic estimates are obtained due to strengthened versions of (A) near R_1 and R_2.

Remark 5.1. Note that in the condition (A), when $v, w \in L^1_{\mathrm{loc}}(R_1,R_2)$ and $P(r) < \infty$ for all $r \in (R_1,R_2)$, then $\int_{R_1}^{R_2} P(r)\sigma(r)\,\mathrm{d}r < \infty$ is equivalent to $\int_{R_1}^{r_1} P(r)\sigma(r)\,\mathrm{d}r < \infty$ and $\int_{r_2}^{R_2} P(r)\sigma(r)\,\mathrm{d}r < \infty$ for some $R_1 < r_1 < r_2 < R_2$. Note that $(\mathrm{A}_{\epsilon,\mathrm{L}})$ implies that $\int_{R_1}^{r_1} P(r)\sigma(r)\,\mathrm{d}r < \infty$ for some $r_1 \in (R_1,\xi)$. Indeed, since $\rho^{1-p'} \in L^1(R_1;\xi)$, we have $\int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau \to 0$ as $r \to R_1^+$. Thus, there exists $r_1 \in (R_1,\xi)$ such that $P(r) = \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}$ for all $r \in (R_1,r_1)$. Hence, by $(\mathrm{A}_{\epsilon,\mathrm{L}})$ we have

$$P(r) < C^{\frac{p-1}{\epsilon}}\left(\int_r^{\xi} \sigma(\tau)\,\mathrm{d}\tau\right)^{-\frac{p-1}{\epsilon}}, \quad \forall r \in (R_1,r_1).$$

Therefore

$$\begin{aligned}\int_{R_1}^{r_1} P(r)\sigma(r)\,\mathrm{d}r &< C^{\frac{p-1}{\epsilon}} \int_{R_1}^{r_1}\left(\int_r^{\xi} \sigma(\tau)\,\mathrm{d}\tau\right)^{-\frac{p-1}{\epsilon}} \sigma(r)\,\mathrm{d}r \\ &< \frac{C^{\frac{p-1}{\epsilon}}\epsilon}{p-1-\epsilon}\left(\int_{r_1}^{\xi} \sigma(\tau)\,\mathrm{d}\tau\right)^{-\frac{p-1-\epsilon}{\epsilon}} < \infty.\end{aligned}$$

Similarly, it is easy to see that $(\mathrm{A}_{\epsilon,\mathrm{R}})$ implies that $\int_{r_2}^{R_2} P(r)\sigma(r)\,\mathrm{d}r < \infty$ for some $r_2 \in (\xi,R_2)$.

We start the proof of Theorem 1.10 by stating nonoscillatory property of the radial solution in the right neighborhood of R_1. This fact can be obtained by applying [21, Theorem 1.14] and using a similar argument to that of [7, Proof of Proposition 4.3]. Therefore, we omit it.

Lemma 5.2 (Nonoscillatory I)**.** *Assume that* $(\mathrm{A}_{\epsilon,\mathrm{L}})$ *holds. Then for a solution* $u \in C^1(R_1,R_2)$ *of* (1.5) *with* $u(R_1) = 0$*, there exists* $a \in (R_1,\xi)$ *such that* $u(r) \neq 0$ *and* $u'(r) \neq 0$ *for all* $r \in (R_1,a)$*.*

Thanks to Lemma 5.2 and the technique used in [6, Proof of Theorem 1.1], we now prove the behavior of $u(x)$ and $\nabla u(x)$ as $|x| \to R_1^+$, provided hypothesis of Theorem 1.10 is satisfied.

Proof of Theorem 1.10. Since $u \in C^1(R_1,R_2)$ and $u(R_1) = u(R_2) = 0$, there exists $r_0 \in (R_1,R_2)$ such that $u'(r_0) = 0$. Take $\tilde{a} := \min\{r \in (R_1,R_2) : u'(r) = 0\}$. Then, $\tilde{a} \in (R_1,R_2)$ in view of Lemma 5.2. Clearly, $u(r)$ satisfies

$$\begin{cases} -(\rho(r)|u'(r)|^{p-2}u'(r))' = \lambda\sigma(r)|u(r)|^{p-2}u(r), & r \in (R_1,R_2), \\ u(R_1) = 0 = u'(\tilde{a}). \end{cases}$$

Then,

$$\rho(r)|u'(r)|^{p-2}u'(r) = \lambda \int_r^{\tilde{a}} \sigma(r)|u(\tau)|^{p-2}u(\tau)\,\mathrm{d}\tau, \quad \forall r \in (R_1, \tilde{a})..$$

We may assume that $u'(r) > 0$ in $(R_1, \tilde{a})$ and hence $u(r) > 0$ in $(R_1, \tilde{a})$. Thus, we have

$$u'(r) = \lambda^{\frac{1}{p-1}} \rho^{1-p'}(r) \left(\int_r^{\tilde{a}} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}}, \quad \forall r \in (R_1, \tilde{a}). \tag{5.1}$$

Hence

$$u(r) = \lambda^{\frac{1}{p-1}} \int_{R_1}^{r} \rho^{1-p'}(t) \left(\int_t^{\tilde{a}} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}t, \quad \forall r \in (R_1, \tilde{a}). \tag{5.2}$$

Estimates from below: Fix $a \in (R_1, \tilde{a})$, then

$$u(r) \geq \lambda^{\frac{1}{p-1}} \left(\int_a^{\tilde{a}} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \int_{R_1}^{r} \rho^{1-p'}(t)\,\mathrm{d}t, \quad \forall r \in (R_1, a),$$

i.e.,

$$u(r) \geq C_1 \int_{R_1}^{r} \rho^{1-p'}(t)\,\mathrm{d}t, \quad \forall r \in (R_1, a),$$

where $C_1 := \lambda^{\frac{1}{p-1}} \left(\int_a^{\tilde{a}} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}}$.

To obtain an estimate from below of the derivative of solution, we use (5.1) to get

$$u'(r) \geq \lambda^{\frac{1}{p-1}} \rho^{1-p'}(r) \left(\int_a^{\tilde{a}} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}}, \ \forall r \in (R_1, a),$$

i.e.,

$$u'(r) \geq C_1 \rho^{1-p'}(r), \quad \forall r \in (R_1, a).$$

Estimates from above: We proceed with an iteration argument.

1st Step: From (5.2) and Hölder's inequality, for all $r \in (R_1, \tilde{a})$, we have

$$\begin{aligned} u(r) &\leq \lambda^{\frac{1}{p-1}} \int_{R_1}^{r} \rho^{1-p'}(t) \left(\int_t^{\tilde{a}} \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p(p-1)}} \left(\int_t^{\tilde{a}} \sigma(\tau) u^p(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p}} \mathrm{d}t \\ &\leq \lambda^{\frac{1}{p-1}} \left(\int_{R_1}^{\tilde{a}} \sigma(\tau) u^p(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p}} \int_{R_1}^{r} \rho^{1-p'}(t) \left(\int_t^{\tilde{a}} \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p(p-1)}} \mathrm{d}t, \end{aligned}$$

i.e.,

$$u(r) \leq c_1 \int_{R_1}^{r} \rho^{1-p'}(t) I_1^{\frac{1}{p-1}}(t)\,\mathrm{d}t, \quad \forall r \in (R_1, \tilde{a}), \tag{5.3}$$

where $c_1 := \lambda^{\frac{1}{p-1}} \left(\int_{R_1}^{\tilde{a}} \sigma(\tau) u^p(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p}}$ and

$$I_1(t) := \left(\int_t^{\tilde{a}} \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p}}, \quad \forall t \in (R_1, \tilde{a}).$$

Here we note that $c_1 \in (0, \infty)$ since

$$\int_{A_{R_1}^{R_2}} w(|x|)|u|^p\,\mathrm{d}x = \frac{1}{\lambda} \int_{A_{R_1}^{R_2}} v(|x|)|\nabla u|^p\,\mathrm{d}x < \infty.$$

2nd Step: Using (5.3) in (5.2), we get

$$u(r) \le \lambda^{\frac{1}{p-1}} \int_{R_1}^{r} \rho^{1-p'}(t) \left[\int_t^{\tilde{a}} \sigma(\tau) \left(c_1 \int_{R_1}^{\tau} \rho^{1-p'}(t_1) I_1^{\frac{1}{p-1}}(t_1) dt_1 \right)^{p-1} \mathrm{d}\tau \right]^{\frac{1}{p-1}} \mathrm{d}t$$

i.e.,

$$u(r) \le c_2 \int_{R_1}^{r} \rho^{1-p'}(t) I_2^{\frac{1}{p-1}}(t)\, \mathrm{d}t, \quad \forall r \in (R_1, \tilde{a}),$$

where $c_2 := \lambda^{\frac{1}{p-1}} c_1$ and

$$I_2(t) := \int_t^{\tilde{a}} \sigma(\tau) \left(\int_{R_1}^{\tau} \rho^{1-p'}(t_1) I_1^{\frac{1}{p-1}}(t_1)\, \mathrm{d}t_1 \right)^{p-1} \mathrm{d}\tau.$$

nth Step: By induction, we obtain the following estimate for arbitrary n,

$$u(r) \le c_n \int_{R_1}^{r} \rho^{1-p'}(t) I_n^{\frac{1}{p-1}}(t)\, \mathrm{d}t, \quad \forall r \in (R_1, \tilde{a}), \tag{5.4}$$

where $c_n := \lambda^{\frac{1}{p-1}} c_{n-1}$ and

$$I_n(t) := \int_t^{\tilde{a}} \sigma(\tau) \left(\int_{R_1}^{\tau} \rho^{1-p'}(t_{n-1}) I_{n-1}^{\frac{1}{p-1}}(t_{n-1})\, \mathrm{d}t_{n-1} \right)^{p-1} \mathrm{d}\tau, \quad \forall t \in (R_1, \tilde{a}). \tag{5.5}$$

By (5.4), to prove upper estimate for the solution u near ∂B_{R_1} it is sufficient to show that there exists $n \in \mathbb{N}$ and a constant $C > 0$ such that

$$I_n(t) < C, \quad \forall t \in (R_1, \tilde{a}).$$

To this end, fix $\tilde{\xi} \in (\max\{\xi, \tilde{a}\}, R_2)$, where ξ appears in $(\mathrm{A}_{\epsilon,\mathrm{L}})$. By (W) and $(\mathrm{A}_{\epsilon,\mathrm{L}})$, there exists a constant $\bar{C} > 0$ such that

$$\left(\int_r^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} < \bar{C}, \quad \forall r \in (R_1, \tilde{\xi}). \tag{5.6}$$

Indeed, for $r \in (R_1, \xi)$, we have

$$\begin{aligned} &\left(\int_r^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} \\ &= \left(\int_r^{\xi} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} + \left(\int_{\xi}^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} \\ &\le C + \left(\int_{\xi}^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{\xi} \rho^{1-p'}(\tau)\, \mathrm{d}s \right)^{\epsilon} := \bar{C}_1. \end{aligned}$$

For $r \in [\xi, \tilde{\xi}]$, we have

$$\left(\int_r^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{r} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} \le \left(\int_{\xi}^{\tilde{\xi}} \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_{R_1}^{\tilde{\xi}} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{\epsilon} =: \bar{C}_2.$$

Take $\bar{C} = \max\{\bar{C}_1, \bar{C}_2\}$, we obtain (5.6).
Similarly, we may also assume that (5.6) holds for ϵ satisfying

$$\epsilon \ne \frac{kp(p-1)}{kp+1}, \forall k \in \mathbb{N}_0 \quad \text{i.e.,} \quad \frac{1}{p} - k\frac{p-1-\epsilon}{\epsilon} \ne 0,\ \forall k \in \mathbb{N}_0. \tag{5.7}$$

We now use (5.6) and (5.7) to estimate $I_n(t)$. Let $n_0 \in \mathbb{N}$ such that

$$\frac{\epsilon}{p(p-1-\epsilon)}+1<n_0<\frac{\epsilon}{p(p-1-\epsilon)}+2,$$

i.e., n_0 is the smallest integer n such that

$$\frac{1}{p}-(n-1)\frac{p-1-\epsilon}{\epsilon}<0.$$

Clearly, $n_0\geq 2$. We first prove the following estimate for I_n.

Claim 1. For each $n\in\{1,\cdots,n_0-1\}$, there exists $\tilde{c}_n>0$ such that

$$I_n(t)\leq \tilde{c}_n\left[\int_t^{\tilde{\xi}}\sigma(\tau)\,\mathrm{d}\tau\right]^{\frac{1}{p}-(n-1)\frac{p-1-\epsilon}{\epsilon}},\quad \forall t\in(R_1,\tilde{a}). \tag{5.8}$$

We prove the Claim 1 by induction. The conclusion is obvious if $n_0=2$. Suppose that $n_0\geq 3$ and (5.8) holds for some n with $1\leq n<n_0-1$. We prove that (5.8) holds for $n+1$ too. Indeed, from (5.5), (5.6) and (5.8) we have

$$\begin{aligned}
I_{n+1}(t)&\leq \int_t^{\tilde{a}}\sigma(\tau)\left[\int_{R_1}^{\tau}\rho^{1-p'}(t_n)\tilde{c}_n^{\frac{1}{p-1}}\left(\int_{t_n}^{\tilde{\xi}}\sigma(t_{n-1})\,\mathrm{d}t_{n-1}\right)^{\frac{1}{p(p-1)}-\frac{(n-1)(p-1-\epsilon)}{\epsilon(p-1)}}\mathrm{d}t_n\right]^{p-1}\mathrm{d}\tau\\
&\leq \tilde{c}^1_{n+1}\int_t^{\tilde{a}}\sigma(\tau)\left[\int_{R_1}^{\tau}\rho^{1-p'}(t_n)\left(\int_{R_1}^{t_n}\rho^{1-p'}(t_{n-1})\,\mathrm{d}t_{n-1}\right)^{-\frac{\epsilon}{p(p-1)}+\frac{(n-1)(p-1-\epsilon)}{p-1}}\mathrm{d}t_n\right]^{p-1}\mathrm{d}\tau\\
&=\tilde{c}^2_{n+1}\int_t^{\tilde{a}}\sigma(\tau)\left(\int_{R_1}^{\tau}\rho^{1-p'}(t_n)\,\mathrm{d}t_n\right)^{-\frac{\epsilon}{p}+(n-1)(p-1-\epsilon)+p-1}\mathrm{d}\tau,\quad\forall t\in(R_1,\tilde{a}).
\end{aligned}\tag{5.9}$$

Here we note that $-\frac{\epsilon}{p(p-1)}+\frac{(n-1)(p-1-\epsilon)}{p-1}+1\geq 1-\frac{\epsilon}{p-1}>0$. From (5.6), (5.9) and noticing $\frac{1}{p}-\frac{n(p-1-\epsilon)}{\epsilon}>0$, we have

$$\begin{aligned}
I_{n+1}(t)&\leq \tilde{c}^3_{n+1}\int_t^{\tilde{a}}\sigma(\tau)\left[\int_{\tau}^{\tilde{\xi}}\sigma(t_n)\,\mathrm{d}t_n\right]^{\frac{1}{p}-\frac{(n-1)(p-1-\epsilon)}{\epsilon}-\frac{p-1}{\epsilon}}\mathrm{d}\tau\\
&=-\tilde{c}^3_{n+1}\int_t^{\tilde{a}}\left[\int_{\tau}^{\tilde{\xi}}\sigma(t_n)\,\mathrm{d}t_n\right]^{\frac{1}{p}-\frac{(n-1)(p-1-\epsilon)}{\epsilon}-\frac{p-1}{\epsilon}}\mathrm{d}\left(\int_{\tau}^{\tilde{\xi}}\sigma(t_n)\,\mathrm{d}t_n\right)\\
&=-\frac{\tilde{c}^3_{n+1}}{\frac{1}{p}-\frac{n(p-1-\epsilon)}{\epsilon}}\left[\int_{\tau}^{\tilde{\xi}}\sigma(t_n)\,\mathrm{d}t_n\right]^{\frac{1}{p}-\frac{n(p-1-\epsilon)}{\epsilon}}\Bigg|_{\tau=t}^{\tau=\tilde{a}}\\
&\leq \tilde{c}_{n+1}\left[\int_t^{\tilde{\xi}}\sigma(t_n)\,\mathrm{d}t_n\right]^{\frac{1}{p}-\frac{n(p-1-\epsilon)}{\epsilon}},\quad\forall t\in(R_1,\tilde{a}),
\end{aligned}$$

where

$$\tilde{c}_{n+1}:=\frac{\tilde{c}^3_{n+1}}{\frac{1}{p}-\frac{n(p-1-\epsilon)}{\epsilon}}.$$

Therefore, (5.8) also holds for $n+1$ and hence, Claim 1 is proved.

Claim 2. There exists $\tilde{c}_{n_0}>0$ such that $I_{n_0}(t)<\tilde{c}_{n_0}$ for all $t\in(R_1,\tilde{a})$.

Indeed, from (5.5), (5.6) and applying (5.8) for $n = n_0 - 1$, we obtain

$$\begin{aligned} I_{n_0}(t) \le& \int_t^{\tilde a} \sigma(\tau)\Big[\int_{R_1}^{\tau} \rho^{1-p'}(t_{n_0-1})\tilde c_{n_0-1}^{\frac{1}{p-1}}\times \\ &\times\left(\int_{t_{n_0-1}}^{\tilde\xi} \sigma(t_{n_0-2})\,\mathrm{d}t_{n_0-2}\right)^{\frac{1}{p(p-1)}-\frac{(n_0-2)(p-1-\epsilon)}{\epsilon(p-1)}} \mathrm{d}t_{n_0-1}\Big]^{p-1}\mathrm{d}\tau \\ \le& \tilde c_{n_0}^1 \int_t^{\tilde a}\sigma(\tau)\Big[\int_{R_1}^{\tau}\rho^{1-p'}(t_{n_0-1})\times \\ &\times\left(\int_{R_1}^{t_{n_0-1}}\rho^{1-p'}(t_{n_0-2})\,\mathrm{d}t_{n_0-2}\right)^{-\frac{\epsilon}{p(p-1)}+\frac{(n_0-2)(p-1-\epsilon)}{p-1}}\mathrm{d}t_{n_0-1}\Big]^{p-1}\mathrm{d}\tau \end{aligned}$$

for all $t \in (R_1, \tilde a)$. Taking into account $-\frac{\epsilon}{p(p-1)} + \frac{(n_0-2)(p-1-\epsilon)}{p-1} + 1 \ge 1 - \frac{\epsilon}{p-1} > 0$, we obtain from the last estimate that there exists $\tilde c_{n_0}^2 > 0$ such that

$$I_{n_0}(t) \le \tilde c_{n_0}^2 \int_t^{\tilde a}\sigma(\tau)\left(\int_{R_1}^{\tau}\rho^{1-p'}(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right)^{-\frac{\epsilon}{p}+(n_0-2)(p-1-\epsilon)+p-1}\mathrm{d}\tau, \quad \forall t\in(R_1,\tilde a). \tag{5.10}$$

From (5.6), (5.10) and noticing $\frac{1}{p} - (n_0-1)\frac{p-1-\epsilon}{\epsilon} < 0$, there is $\tilde c_{n_0}^3$ such that

$$\begin{aligned} I_{n_0}(t) &\le \tilde c_{n_0}^3\int_t^{\tilde a}\sigma(\tau)\left[\int_\tau^{\tilde\xi}\sigma(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right]^{\frac1p-\frac{(n_0-1)(p-1-\epsilon)}{\epsilon}}\mathrm{d}\tau \\ &= -\tilde c_{n_0}^3\int_t^{\tilde a}\left[\int_\tau^{\tilde\xi}\sigma(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right]^{\frac1p-\frac{(n_0-2)(p-1-\epsilon)}{\epsilon}-\frac{p-1}{\epsilon}}\mathrm{d}\left(\int_\tau^{\tilde\xi}\sigma(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right) \\ &= -\frac{\tilde c_{n_0}^3}{\frac1p-\frac{(n_0-1)(p-1-\epsilon)}{\epsilon}}\left[\int_\tau^{\tilde\xi}\sigma(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right]^{\frac1p-\frac{(n_0-1)(p-1-\epsilon)}{\epsilon}}\Bigg|_{\tau=t}^{\tau=\tilde a} \\ &\le -\frac{\tilde c_{n_0}^3}{\frac1p-\frac{(n_0-1)(p-1-\epsilon)}{\epsilon}}\left[\int_{\tilde a}^{\tilde\xi}\sigma(t_{n_0-1})\,\mathrm{d}t_{n_0-1}\right]^{\frac1p-\frac{(n_0-1)(p-1-\epsilon)}{\epsilon}} =: \tilde c_{n_0}, \quad \forall t\in(R_1,\tilde a). \end{aligned}$$

Thus, we have proved Claim 2.
By Claim 2, we get from (5.4) that

$$u(r) \le C_2\int_{R_1}^{r}\rho^{1-p'}(t)\,\mathrm{d}t, \quad \forall r\in(R_1,\tilde a), \tag{5.11}$$

where $C_2 := c_{n_0}\tilde c_{n_0}^{\frac{1}{p-1}}$.

Finally, we look for the estimate of u' from above. By (5.6) and (5.11), we have

$$\begin{aligned} \int_r^{\tilde a}\sigma(\tau)u^{p-1}(\tau)\,\mathrm{d}\tau &\le C_2^{p-1}\int_t^{\tilde a}\sigma(\tau)\Big(\int_{R_1}^{\tau}\rho^{1-p'}(t)\,\mathrm{d}t\Big)^{p-1}\mathrm{d}\tau \\ &\le \bar C_2\int_r^{\tilde a}\sigma(\tau)\Big(\int_\tau^{\tilde\xi}\sigma(t)\,\mathrm{d}t\Big)^{-\frac{p-1}{\epsilon}}\mathrm{d}\tau \\ &= \bar C_2\Big(\frac{\epsilon}{p-1-\epsilon}\Big)\Big(\int_\tau^{\tilde\xi}\sigma(t)\,\mathrm{d}t\Big)^{-\frac{p-1-\epsilon}{\epsilon}}\Bigg|_{\tau=t}^{\tilde a} \end{aligned}$$

$$\le \frac{\epsilon \bar{C}_2}{p-1-\epsilon}\left(\int_{\tilde{a}}^{\tilde{\xi}} \sigma(\tau)\,\mathrm{d}\tau\right)^{-\frac{p-1-\epsilon}{\epsilon}}, \quad \forall r \in (R_1, \tilde{a}).$$

Combining this and (5.1) we deduce

$$u'(r) \le \tilde{C}_2 \rho^{1-p'}(r), \quad \forall r \in (R_1, a).$$

□

The asymptotic estimates of solutions towards the boundary ∂B_{R_2} are obtained in the same manner. As before, we need the following nonoscillatory property and its proof can be obtained by invoking [21, Theorem 6.2] and using a similar argument to that of [7, Proof of Proposition 4.3]. Therefore, we omit it.

Lemma 5.3 (Nonoscillatory II)**.** *Assume that* $(\mathrm{A}_{\epsilon,\mathrm{R}})$ *holds. Then for a solution* $u \in C^1(R_1, R_2)$ *of* (1.5) *with* $u(R_2) = 0$, *there exists* $b \in (\xi, R_2)$ *such that* $u(r) \neq 0$ *and* $u'(r) \neq 0$ *for all* $r \in (b, R_2)$.

Using Lemma 5.3 and similar argument as in the proof of Theorem 1.10 we prove Theorem 1.11 as follows.

Proof of Theorem 1.11. Let $\tilde{b} := \max\{r \in (R_1, R_2) : u'(r) = 0\}$. Then $\tilde{b} \in (R_1, R_2)$ in view of Lemma 5.3. We have $u \in C^1(R_1, R_2)$ satisfies

$$\begin{cases} -(\rho(r)|u'(r)|^{p-2}u'(r))' = \lambda\sigma(r)|u(r)|^{p-2}u(r), & r \in (R_1, R_2), \\ u'(\tilde{b}) = 0 = u(R_2). \end{cases}$$

We may assume that $u'(r) < 0$ in $(\tilde{b}, R_2)$ and hence $u(r) > 0$ in $(\tilde{b}, R_2)$. Thus, we have

$$-u'(r) = \lambda^{\frac{1}{p-1}} \rho^{1-p'}(r) \left(\int_{\tilde{b}}^{r} \sigma(t) u^{p-1}(t)\,\mathrm{d}t\right)^{\frac{1}{p-1}}, \quad r \in (\tilde{b}, R_2).$$

Using this and the fact that $u(R_2) = 0$, we get

$$u(r) = \lambda^{\frac{1}{p-1}} \int_r^{R_2} \rho^{1-p'}(t) \left(\int_{\tilde{b}}^{t} \sigma(\tau) u^{p-1}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}t, \quad \forall r \in (\tilde{b}, R_2).$$

The rest of the proof is similar to that of the proof of Theorem 1.10 for which we mofdify $(\mathrm{A}_{\epsilon,\mathrm{R}})$ as

$$\left(\int_{\tilde{\xi}}^{r} \sigma(\tau)\,\mathrm{d}\tau\right) \left(\int_r^{R_2} \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{\epsilon} \le \bar{C}, \quad \forall r \in (\tilde{\xi}, R_2)$$

for some fixed $\tilde{\xi} \in (R_1, \min\{\tilde{b}, \xi\})$. □

6. Applications

In this section we give concrete examples to illustrate our main results. Consider the following equation

$$-\operatorname{div}\left(v(|x|)|\nabla u|^{p-2}\nabla u\right) = \lambda w(|x|)|u|^{p-2}u \quad \text{in } B_1^c \tag{6.1}$$

with $v(|x|) = (|x| - 1)^{\alpha}$ and $w \in L^1_{\mathrm{loc}}(1, \infty)$ such that $w > 0$ a.e. in $(1, \infty)$. Note that for such weights v, w, the condition (W) is clearly satisfied.

Example 6.1 (Degenerate weight)**.** Let $0 \le \alpha < p - 1$.

- If $p \neq N$ and $w \in L^1\big((1,\infty);(r-1)^{p-1-\alpha}\big) \cap L^1\big((1,\infty);(r-1)^{p-1}\big)$, then v, w satisfy (W$_1$) of Corollary 2.5 and hence,
$$\mathcal{D}_0^{1,p}(B_1^c;v) \hookrightarrow\hookrightarrow L^p(B_1^c;w).$$
In this case, the eigenvalue problem (6.1) has a principal eigenpair due to Theorem 1.2.
- If $w \in L^1\big((1,\xi);(r-1)^{\delta}\big)$ for some $\xi \in (1,\infty)$ and $\delta < p-1-\alpha$, then (A$_{\epsilon,\mathrm{L}}$) holds for $\epsilon \in (\frac{(p-1)\delta}{p-1-\alpha}, p-1) \cap (0,\infty)$. By Theorem 1.10, if $u(x) = u(|x|) \in C^1(B_1^c)$ is a solution to equation (6.1) with $u(1) = u(\infty) = 0$, there exist $a \in (1,\infty)$, $0 < C_1 < C_2$ and $0 < \tilde{C}_1 < \tilde{C}_2$, such that
$$C_1(r-1)^{\frac{p-1-\alpha}{p-1}} \leq |u(r)| \leq C_2(r-1)^{\frac{p-1-\alpha}{p-1}}, \quad \forall r \in (1,a) \text{ and}$$
$$\tilde{C}_1(r-1)^{-\frac{\alpha}{p-1}} \leq |u'(r)| \leq \tilde{C}_2(r-1)^{-\frac{\alpha}{p-1}}, \quad \forall r \in (1,a).$$
Since
$$u'_+(1) = \lim_{r\to 1^+} \frac{u(r)-u(1)}{r-1},$$
we have $0 < |u'_+(1)| < \infty$ when $\alpha = 0$ and $|u'_+(1)| = \infty$, when $\alpha > 0$.
- If $p < N + \alpha$ and $w \in L^1\big((\bar{\xi},\infty);(r-1)^{\bar{\delta}}\big)$ for some $\bar{\xi} \in (1,\infty)$ and $\bar{\delta} = p-1$ when $\alpha \in (0,p-1)$ and $\bar{\delta} \in (p-1,N-1)$ when $\alpha = 0$, then (A$_{\epsilon,\mathrm{L}}$) holds for some $\epsilon \in (0,p-1)$. By Theorem 1.11, if $u(x) = u(|x|) \in C^1(B_1^c)$ is a solution to equation (6.1) with $u(1) = u(\infty) = 0$, there exist $b \in (1,\infty)$, $0 < C_1 < C_2$ and $0 < \tilde{C}_1 < \tilde{C}_2$, such that
$$C_1 r^{-\frac{N-p+\alpha}{p-1}} \leq |u(r)| \leq C_2 r^{-\frac{N-p+\alpha}{p-1}}, \quad \forall r \in (b,\infty), \text{ and}$$
$$\tilde{C}_1 r^{-\frac{N-1-\alpha}{p-1}} \leq |u'(r)| \leq \tilde{C}_2 r^{-\frac{N-1-\alpha}{p-1}}, \quad \forall r \in (b,\infty).$$

Remark 6.2. For instance, let $v(r) = 1$ and $0 < w(r) < Cr^{-\gamma}$ $(\gamma > p)$, we obtain better estimates for u, u' at infinity than that of [4] and [2], by putting $\alpha = 0$ in Example 6.1.

Example 6.3 (Singular weight)**.** Consider $1 < p < N$ and let $p - N < \alpha < 0$.

- If $w \in L^1\big((1,\infty);(r-1)^{p-1}\big) \cap L^1\big((1,\infty);(r-1)^{p-1-\alpha}\big)$, then the weights v, w satisfy (W$_2$) of Corollary 2.5 and we get
$$\mathcal{D}_0^{1,p}(B_1^c;v) \hookrightarrow\hookrightarrow L^p(B_1^c;w).$$
Hence, the eigenvalue problem (6.1) has a principal eigenpair due to Theorem 1.2.
- If $w \in L^1\big((1,\xi);(r-1)^{p-1}\big)$ for some $\xi \in (1,\infty)$, then (A$_{\epsilon,\mathrm{L}}$) holds for $\epsilon \in (\frac{(p-1)^2}{p-1-\alpha}, p-1)$. By Theorem 1.10, if $u(x) = u(|x|) \in C^1(B_1^c)$ is a solution to equation (6.1) with $u(1) = u(\infty) = 0$, there exist $a \in (1,\infty)$, $0 < C_1 < C_2$ and $0 < \tilde{C}_1 < \tilde{C}_2$, such that
$$C_1(r-1)^{\frac{p-1-\alpha}{p-1}} \leq |u(r)| \leq C_2(r-1)^{\frac{p-1-\alpha}{p-1}}, \quad \forall r \in (1,a) \text{ and}$$
$$\tilde{C}_1(r-1)^{-\frac{\alpha}{p-1}} \leq |u'(r)| \leq \tilde{C}_2(r-1)^{-\frac{\alpha}{p-1}}, \quad \forall r \in (1,a).$$
In this case, we have $u'_+(1) = 0$.

- If $w \in L^1\big((\bar{\xi},\infty);(r-1)^{\bar{\delta}}\big)$ for some $\bar{\xi} \in (1,\infty)$ and $\bar{\delta} \in (p-1-\alpha, N-1)$, then $(\mathrm{A}_{\epsilon,\mathrm{L}})$ holds for some $\epsilon \in (0, p-1)$. By Theorem 1.11, if $u(x) = u(|x|) \in C^1(B_1^c)$ is a solution to equation (6.1) with $u(1) = u(\infty) = 0$, there exist $b \in (1,\infty)$, $0 < C_1 < C_2$ and $0 < \tilde{C}_1 < \tilde{C}_2$, such that

$$C_1 r^{-\frac{N-p+\alpha}{p-1}} \le |u(r)| \le C_2 r^{-\frac{N-p+\alpha}{p-1}}, \quad \forall r \in (b,\infty), \text{ and}$$
$$\tilde{C}_1 r^{-\frac{N-1-\alpha}{p-1}} \le |u'(r)| \le \tilde{C}_2 r^{-\frac{N-1-\alpha}{p-1}}, \quad \forall r \in (b,\infty).$$

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