

ON CONCENTRATION OF LEAST ENERGY SOLUTIONS FOR MAGNETIC CRITICAL CHOQUARD EQUATIONS

T. Mukherjee* and K. Sreenadh†

Abstract

In the present paper, we consider the following magnetic nonlinear Choquard equation

$$\begin{cases} (-i\nabla + A(x))^2 u + \mu g(x)u = \lambda u + (|x|^{-\alpha} * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u & \text{in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n, \mathbb{C}) \end{cases}$$

where $n \geq 4$, $2_\alpha^* = \frac{2n-\alpha}{n-2}$, $\alpha \in (0, n)$, $\mu > 0$, $\lambda > 0$ is a parameter, $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a magnetic vector potential and $g(x)$ is a real valued potential function on \mathbb{R}^n . Using variational methods, we establish the existence of least energy solution under some suitable conditions. Moreover, the concentration behavior of solutions is also studied as $\mu \rightarrow +\infty$.

Key words: Nonlinear Schrödinger equations, Magnetic potential, Choquard equation, Hardy-Littlewood-Sobolev critical exponent.

2010 Mathematics Subject Classification: 35R11, 35R09, 35A15.

1 Introduction

In this article, we study the existence and concentration behavior of nontrivial solutions of the following nonlinear Schrödinger equation with nonlocal nonlinearity of Choquard type

$$(P_{\lambda, \mu}) \begin{cases} (-i\nabla + A(x))^2 u + \mu g(x)u = \lambda u + (|x|^{-\alpha} * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u & \text{in } \mathbb{R}^n \\ u \in H^1(\mathbb{R}^n, \mathbb{C}) \end{cases}$$

where $n \geq 4$, $2_\alpha^* = \frac{2n-\alpha}{n-2}$, $\alpha \in (0, n)$, $\mu > 0$, $\lambda > 0$, $A = (A_1, A_2, \dots, A_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector(or magnetic) potential such that $A \in L_{loc}^n(\mathbb{R}^n, \mathbb{R}^n)$ and A is continuous at 0 and $g(x)$ satisfies the following assumptions:

- (g1) $g \in C(\mathbb{R}^n, \mathbb{R})$, $g \geq 0$ and $\Omega := \text{interior of } g^{-1}(0)$ is a nonempty bounded set with smooth boundary and $\bar{\Omega} = g^{-1}(0)$.

*Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-110016, India.
e-mail: tulimukh@gmail.com

†Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-110016, India.
e-mail: sreenadh@gmail.com

(g2) There exists $M > 0$ such that $\mathcal{L}\{x \in \mathbb{R}^n : g(x) \leq M\} < +\infty$, where \mathcal{L} denotes the Lebesgue measure in \mathbb{R}^n .

A more general form of the above problem is

$$(-i\nabla + A(x))^2 w + G(x)w = F(x, w), \quad w \in H^1(\mathbb{R}^n, \mathbb{C}) \quad (1.1)$$

which arises when we try to look for standing wave solution of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = (-i\hbar \nabla + A(x))^2 \phi + Q(x)\phi - n(x, |\phi|)\phi,$$

where \hbar is the Plank constant. A lot of attention has been paid to nonlinear Schrödinger equation in recent years. When $A \equiv 0$ (i.e. no magnetic potential) in (1.1), many authors studied the problem as in [7, 9, 16]. The problem of the type

$$-\Delta u + \mu a(x)u = \lambda u + |u|^{p-2}u, \quad (1.2)$$

where $a \geq 0$ is potential well, with subcritical growth i.e. $p < 2^* = 2n/(n-2)$ has been investigated extensively in [6, 10, 11, 34, 45]. In the critical case $p = 2^*$, Clapp and Ding in [19] established the existence and multiplicity of positive solutions for (1.2) using variational methods. For Schrödinger equations with critical nonlinearity, one may also refer [1, 13, 22, 39]. In [28], authors have studied the blow-up of radial solutions to a cubic nonlinear Schrödinger equation with a radial defect, located on the sphere of radius r_0 . We also suggest readers to refer [2, 37, 30] for further study.

When the magnetic vector potential $A \not\equiv 0$, the Schrödinger equation of the form

$$(-i\hbar \nabla + A(x))^2 u + V(x)u = |u|^{p-2}u \text{ in } \mathbb{R}^n,$$

where V is electric potential function, has been widely studied by many authors, we refer [12, 17, 29, 41, 42] for this and the references therein. Motivated by these results, very recently Lü [33] studied the problem

$$(-i\nabla + A(x))^2 u + (g_0(x) + \mu g(x))u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^n, \mathbb{C}), \quad (1.3)$$

where $n \geq 3$, $\alpha \in (0, n)$, $p \in \left(\frac{2n-\alpha}{n}, \frac{2n-\alpha}{n-2}\right)$, g_0 and g are real valued functions on \mathbb{R}^n satisfying some necessary conditions and $\mu > 0$. He proved the existence of ground state solution when $\mu \geq \mu^*$, for some $\mu^* > 0$ and concentration behavior of solutions as $\mu \rightarrow \infty$. The Hardy-Littlewood-Sobolev inequality (see Theorem 2.2) plays an important role for studying such problems and in that context, we call $2_\alpha^* = \frac{2n-\alpha}{n-2}$ as the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. When $A \equiv 0$, $g_0 \equiv 0$, $g \equiv 1$ and $\mu = 1$ in (1.3), that is

$$-\Delta u + u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^n)$$

the equations are generally called the Choquard equations which arise in various fields of physics, example quantum theory of large systems of nonrelativistic bosonic atoms and

molecules. Choquard equations are another topic of attraction for researchers now a days which in turn rendered a huge literature in this area, for instance refer [20, 27, 38]. In [31], Lieb proved the existence and uniqueness of solution, up to translations, for the problem

$$-\Delta u + u = (|x| * |u|^2)|u|^2 \text{ in } \mathbb{R}^n.$$

In [5, 25, 26], Gao and Yang showed existence and multiplicity results for Brezis-Nirenberg type problem for the nonlinear Choquard equation

$$-\Delta u = \left(\int_{\Omega} \frac{|u|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dy \right) |u|^{2^*_{\alpha}-2} u + \lambda g(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is smooth bounded domain in \mathbb{R}^n , $n > 2$, $\lambda > 0$, $0 < \alpha < n$ and $g(u)$ is a nonlinearity with certain necessary assumptions. Salazar in [40] showed existence of vortex type solutions for the stationary nonlinear magnetic Choquard equation

$$(-i\nabla + A(x))^2 u + W(x)u = (|x|^{-\alpha} * |u|^p)|u|^{p-2} u \text{ in } \mathbb{R}^n,$$

where $n \geq 3$, $\alpha \in (0, n)$, $p \in [2, 2^*_{\alpha})$, $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is magnetic potential and $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded electric potential (under some assumptions on decay of A and W at infinity). Cingloni, Sechi and Squassina showed existence of family of solutions for a Schrödinger equation in the presence of electric and magnetic potential and Hartree-type nonlinearity in [18]. Schrödinger equations with magnetic field and Choquard type nonlinearity has also been studied in [14, 15]. In this context, we also cite [3, 4, 44] with no attempt to provide the full list of references.

Now a very obvious question arises, what happens in the critical case i.e. when $p = 2^*_{\alpha}$ in (1.3)? Here in this paper, we consider the problem $(P_{\lambda, \mu})$ which is motivated by (1.2) and (1.3). The main difficulty for this problem is the presence of critical nonlinearity in the sense of Hardy-Littlewood-Sobolev inequality which is also nonlocal in nature. The critical exponent term being nonlocal adds on the difficulty to study the Palais-Smale level around a nontrivial critical point. We define $\nabla_A u := (-i\nabla + A(x))u$ and consider the minimization problem here by defining

$$S_A = \inf_{u \in H^1_A(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla_A u|^2 dx}{\int_{\mathbb{R}^n} (|x|^{\alpha} * |u|^{2^*_{\alpha}})|u|^{2^*_{\alpha}} dx}$$

and proved that it is attained under some necessary and sufficient conditions which is a new result. Also the other results proved here are completely new and there is no work concerning this problem till now to the best of our knowledge. Following the approach of [19], we show that $(P_{\lambda, \mu})$ has a solution. Also we show that the problem

$$(P_{\lambda}) \begin{cases} (-i\nabla + A(x))^2 u = \lambda u + (|x|^{\alpha} * |u|^{2^*_{\alpha}})|u|^{2^*_{\alpha}-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for small λ acts as a limit problem for $(P_{\lambda, \mu})$ as $\mu \rightarrow \infty$. We use the knowledge of (P_{λ}) to show the concentration behavior of solutions of $(P_{\lambda, \mu})$.

We divide our paper into 4 sections. Section 2 contains the variational setting and the main results of our work. We study the Palais-Smale sequences and proved some compactness results in section 3. Making use of these results, we establish the proof of main theorems in section 4.

2 Variational setting and main results

We assume that g satisfies the conditions (g1) and (g2) throughout this paper. Let us define

$$H_A^1(\mathbb{R}^n, \mathbb{C}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^n, \mathbb{C}^n)\}.$$

Then $H_A^1(\mathbb{R}^n, \mathbb{C})$ is a Hilbert space with the inner product

$$\langle u, v \rangle_A = \operatorname{Re} \left(\int_{\mathbb{R}^n} (\nabla_A u \overline{\nabla_A v} + u \bar{v}) \, dx \right),$$

where $\operatorname{Re}(w)$ denotes the real part of $w \in \mathbb{C}$ and \bar{w} denotes its complex conjugate. The associated norm $\|\cdot\|_A$ on the space $H_A^1(\mathbb{R}^n, \mathbb{C})$ is given by

$$\|u\|_A = \left(\int_{\mathbb{R}^n} (|\nabla_A u|^2 + |u|^2) \, dx \right)^{\frac{1}{2}}.$$

We call $H_A^1(\mathbb{R}^n, \mathbb{C})$ simply $H_A^1(\mathbb{R}^n)$. Let $H_A^{0,1}(\Omega, \mathbb{C})$ (denoted by $H_A^{0,1}(\Omega)$ for simplicity) be the Hilbert space defined by the closure of $C_c^\infty(\Omega, \mathbb{C})$ under the scalar product $\langle u, v \rangle_A = \operatorname{Re} \left(\int_{\Omega} (\nabla_A u \overline{\nabla_A v} + u \bar{v}) \, dx \right)$, where $\Omega = \text{interior of } g^{-1}(0)$. Thus norm on $H_A^{0,1}(\Omega)$ is given by

$$\|u\|_{H_A^{0,1}(\Omega)} = \left(\int_{\Omega} (|\nabla_A u|^2 + |u|^2) \, dx \right)^{\frac{1}{2}}.$$

Let $E = \{u \in H_A^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} g(x)|u|^2 \, dx < +\infty\}$ be the Hilbert space equipped with the inner product

$$\langle u, v \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^n} (\nabla_A u \overline{\nabla_A v} \, dx + g(x)u \bar{v}) \, dx \right)$$

and the associated norm $\|\cdot\|_E$, where

$$\|u\|_E^2 = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + g(x)|u|^2) \, dx.$$

Then $\|\cdot\|_E$ is clearly equivalent to each of the norm $\|\cdot\|_\mu$, where

$$\|u\|_\mu^2 = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2) \, dx$$

for $\mu > 0$. We have the following well known *diamagnetic inequality* (for detailed proof, see [32], Theorem 7.21).

Theorem 2.1 *If $u \in H_A^1(\mathbb{R}^n)$, then $|u| \in H^1(\mathbb{R}^n, \mathbb{R})$ and*

$$|\nabla|u|(x)| \leq |\nabla u(x) + iA(x)u(x)| \text{ for a.e. } x \in \mathbb{R}^n.$$

Proof. The outline of the proof is as follows: since $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we get

$$|\nabla|u|(x)| = \left| \operatorname{Re} \left(\nabla u \frac{\bar{u}}{|u|} \right) \right| = \left| \operatorname{Re} \left((\nabla u + iAu) \frac{\bar{u}}{|u|} \right) \right| \leq |\nabla u + iAu|.$$

■

So for each $q \in [2, 2^*]$, there exists constant $b_q > 0$ (independent of μ) such that

$$|u|_q \leq b_q \|u\|_\mu, \text{ for any } u \in E, \quad (2.1)$$

where $|\cdot|_q$ denotes the norm in $L^q(\mathbb{R}^n, \mathbb{C})$ and $2^* = \frac{2n}{n-2}$ is the Sobolev critical exponent. Also $H_A^1(\Omega) \hookrightarrow L^q(\Omega, \mathbb{C})$ is continuous for each $1 \leq q \leq 2^*$ and compact when $1 \leq q < 2^*$. Let us denote

$$B(u) = \int_{\mathbb{R}^n} (|x|^\alpha * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy. \quad (2.2)$$

To estimate the nonlocal term $B(u)$, we have the following Hardy-Littlewood-Sobolev inequality (refer [32], Theorem 4.3).

Proposition 2.2 *Let $t, r > 1$ and $0 < \alpha < n$ with $1/t + \alpha/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \alpha, r)$, independent of f, h such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\alpha} dx dy \leq C(t, n, \alpha, r) |f|_t |h|_r. \quad (2.3)$$

If $t = r = \frac{2n}{2n-\alpha}$ then

$$C(t, n, \alpha, r) = C(n, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{n}{2} - \frac{\alpha}{2})}{\Gamma(n - \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right\}^{-1 + \frac{\alpha}{n}}.$$

In this case there is equality in (2.3) if and only if $f \equiv (\text{constant})h$ and

$$h(x) = z(\gamma^2 + |x-a|^2)^{-\frac{(2n-\alpha)}{2}}$$

for some $z \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^n$.

Proposition 2.2 implies that

$$|B(u)| \leq C(n, \alpha) |u|_{2_\alpha^*}^{22_\alpha^*}, \quad (2.4)$$

where $C(n, \alpha)$ is as given in Proposition 2.2. By (2.1), we say that $B(u)$ is well defined for $u \in E$. Also $B(u) \in C^1(E, \mathbb{R})$, refer Lemma 2.5 of [44].

Definition 2.3 *We say that a function $u \in E$ is a weak solution of $(P_{\lambda, \mu})$ if*

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} (\mu g(x) - \lambda) u \bar{v} dx - \int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^* - 2} u \bar{v} dx \right) = 0$$

for all $v \in E$.

Definition 2.4 A solution u of $(P_{\lambda,\mu})$ is said to be a least energy solution if the energy functional

$$I_{\lambda,\mu}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} (|\nabla_A u|^2 + (\mu g(x) - \lambda)|u|^2) - \frac{1}{22_\alpha^*} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} \right) dx$$

achieves its minimum at u over all the nontrivial solutions of $(P_{\lambda,\mu})$.

Definition 2.5 A sequence of solutions $\{u_k\}$ of (P_{λ,μ_k}) is said to concentrate at a solution u of (P_λ) if a subsequence converges strongly to u in $H_A^1(\mathbb{R}^n)$ as $\mu_k \rightarrow \infty$.

The main idea to prove the existence of solution for the problem $(P_{\lambda,\mu})$ is using variational methods where the weak solutions for $(P_{\lambda,\mu})$ are obtained by finding the critical points of the energy functional $I_{\lambda,\mu} : H_A^1(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$I_{\lambda,\mu}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} (|\nabla_A u|^2 + (\mu g(x) - \lambda)|u|^2) - \frac{1}{22_\alpha^*} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} \right) dx.$$

Then $I_{\lambda,\mu} \in C^1(E, \mathbb{R})$ with

$$\langle I'_{\lambda,\mu}(u), v \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} (\mu g(x) - \lambda - (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^* - 2}) u \bar{v} dx \right)$$

for $u, v \in E$. Thus we characterize the weak solutions of $(P_{\lambda,\mu})$ as the critical points of $I_{\lambda,\mu}$. From now onwards, we denote $\lambda_1(\Omega) > 0$ as the best constant of the compact embedding $H_A^{0,1}(\Omega) \hookrightarrow L^2(\Omega, \mathbb{C})$ given by

$$\lambda_1(\Omega) = \inf_{u \in H_A^{0,1}(\Omega)} \left\{ \int_{\Omega} |\nabla_A u|^2 dx : \int_{\Omega} |u|^2 dx = 1 \right\}$$

which is also the first eigenvalue of $-\Delta_A := (-i\nabla + A)^2$ on Ω with boundary condition $u = 0$. Let S denote the best Sobolev constant of the embedding $H_0^1(\Omega, \mathbb{R}) \hookrightarrow L^{2^*}(\Omega, \mathbb{R})$ which is given by

$$S = \inf_{u \in H_0^1(\Omega, \mathbb{R})} \left\{ \int_{\Omega} |\nabla u|^2 dx : \int_{\Omega} |u|^{2^*} dx = 1 \right\}.$$

We know that S is independent of Ω and it is achieved if and only if $\Omega = \mathbb{R}^n$. We use $S_{H,L}$ to denote the best constant as

$$S_{H,L} = \inf_{u \in H^1(\mathbb{R}^n, \mathbb{R})} \left\{ \int_{\Omega} |\nabla u|^2 dx : B(u) = 1 \right\}.$$

By Lemma 1.2 of [25], we get that $S_{H,L}$ is achieved by functions of the form

$$U(x) = C \left(\frac{b}{b^2 + |x - a|^2} \right)^{\frac{n-2}{2}}$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^n$ and $b > 0$ are parameters. Now we state our main results :

Theorem 2.6 For every $\lambda \in (0, \lambda_1(\Omega))$ there exists a $\mu(\lambda) > 0$ such that $(P_{\lambda, \mu})$ has a least energy solution u_μ for each $\mu \geq \mu(\lambda)$.

Theorem 2.7 Let $\{u_m\}$ be a sequence of non-trivial solutions of (P_{λ, μ_m}) with $\mu_m \rightarrow \infty$ and $I_{\lambda, \mu_m}(u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ as $m \rightarrow \infty$. Then u_m concentrates at a solution of (P_λ) .

3 Palais Smale analysis and compactness results

In this section, we find the Palais Smale critical threshold below which any Palais Smale $(PS)_c$ sequence has a convergent subsequence. We recall that a sequence $\{u_m\} \subset E$ is said to be a $(PS)_c$ sequence (for $I_{\lambda, \mu}$) if $I_{\lambda, \mu}(u_m) \rightarrow c$ and $I'_{\lambda, \mu}(u_m) \rightarrow 0$ as $m \rightarrow \infty$. We say that $I_{\lambda, \mu}$ satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence contains a convergent subsequence.

Lemma 3.1 Suppose $\mu_m \geq 1$ and $u_m \in E$ be such that $\mu_m \rightarrow \infty$ as $m \rightarrow \infty$ and there exists a $K > 0$ such that $\|u_m\|_{\mu_m} < K$, for all $m \in \mathbb{N}$. Then there exists a $u \in H_A^{0,1}(\Omega)$ such that (upto a subsequence), $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$.

Proof. Since the norms $\|\cdot\|_E$ and $\|\cdot\|_\mu$ are equivalent, we have $\|u_m\|_E^2 < K'$, for some constant $K' > 0$. So there exists $u \in E$ such that $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $m \rightarrow \infty$. Let $C_r = \{x : |x| \leq r, g(x) \geq 1/r\}$, $r \in \mathbb{N}$. Then we can easily see that

$$\int_{C_r} |u_m|^2 dx \leq r \int_{C_r} g(x) |u_m|^2 dx \leq \frac{r}{\mu_m} \|u_m\|_{\mu_m}^2 \leq \frac{rK}{\mu_m} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This holds for every r which implies that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Since $\partial\Omega$ is smooth, we have $u \in H_A^{0,1}(\Omega)$. The next step is to show that $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$. Let $D = \{x \in \mathbb{R}^n : g(x) \leq M\}$, where M is defined as in (g2). Then

$$\int_{\mathbb{R}^n \setminus D} u_m^2 dx \leq \frac{1}{\mu_m M} \int_{\mathbb{R}^n \setminus D} \mu_m g(x) u_m^2 dx \leq \frac{K'}{\mu_m M} \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.1)$$

Let $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$ and $q \in \left(1, \frac{n}{n-2}\right)$ such that $q' = \frac{q}{q-1}$. Then using (2.1) and equivalence of norms $\|\cdot\|_\mu$ and $\|\cdot\|_E$, we have

$$\int_{B_R^c \cap D} |u_m - u|^2 dx \leq |u_m - u|_{2q}^2 (\mathcal{L}(B_R^c \cap D))^{1/q'} \leq C_1 b_{2q}^2 \|u_m - u\|_E^2 (\mathcal{L}(B_R^c \cap D))^{1/q'},$$

where C_1 is a positive constant and $B_R^c = \mathbb{R}^n \setminus B_R$. Hence by (g2) we get

$$\int_{B_R^c \cap D} |u_m - u|^2 dx \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (3.2)$$

Lastly, as we know $u_m \rightarrow u$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n)$ we get

$$\int_{B_R} |u_m - u|^2 dx \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.3)$$

Therefore using (3.1), (3.2) and (3.3), we get $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$. \blacksquare

Now let $T_\mu := -\Delta_A + \mu g(x)$, where $-\Delta_A = (-i\nabla + A)^2$, be an operator defined on E . Also let $v \in E$ and denote (\cdot, \cdot) as L^2 inner product then we write

$$(T_\mu(u), v) = \operatorname{Re} \left(\int_{\mathbb{R}^n} (\nabla_A u \overline{\nabla_A v} + \mu g(x) u \bar{v}) \, dx \right).$$

Clearly T_μ is a self adjoint operator and if $a_\mu := \inf \sigma(T_\mu)$, i.e. the infimum of the spectrum of T_μ , then a_μ can be characterized as

$$0 \leq a_\mu = \inf \{ (T_\mu(u), u) : u \in E, |u|_2 = 1 \} = \inf \{ \|u\|_\mu^2 : u \in E, |u|_2 = 1 \}.$$

Thus a_μ is nondecreasing in μ . Therefore we get

$$((T_\mu - \lambda)u, u) = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2 - \lambda|u|^2) \, dx.$$

In the next lemma, we will show that the map $(T_\mu - \lambda)$ is coercive.

Lemma 3.2 *For each $\lambda \in (0, \lambda_1(\Omega))$, there exists a $\mu(\lambda) > 0$ such that $a_\mu \geq (\lambda + \lambda_1(\Omega))/2$ whenever $\mu \geq \mu(\lambda)$. As a consequence*

$$((T_\mu - \lambda)u, u) \geq \beta_\lambda \|u\|_\mu^2$$

for all $u \in E$, $\mu \geq \mu(\lambda)$, where $\beta_\lambda := (\lambda_1(\Omega) - \lambda)/(\lambda_1(\Omega) + \lambda)$.

Proof. Assume by contradiction that there exist a sequence $\mu_m \rightarrow \infty$ such that $a_{\mu_m} < (\lambda + \lambda_1(\Omega))/2$ for all m and $a_{\mu_m} \rightarrow \theta \leq (\lambda + \lambda_1(\Omega))/2$. Let us consider a minimizing sequence $\{u_m\} \in E$ such that $|u_m|_2 = 1$ and $((T_{\mu_m} - a_{\mu_m})u_m, u_m) \rightarrow 0$. This implies

$$\begin{aligned} \|u_m\|_{\mu_m}^2 &= \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu_m g(x)|u_m|^2) \, dx \\ &= ((T_{\mu_m} - a_{\mu_m})u_m, u_m) + a_{\mu_m}(u_m, u_m) \\ &\leq ((T_{\mu_m} - a_{\mu_m})u_m, u_m) + (1 + a_{\mu_m})|u_m|_2^2 \\ &\leq 2(1 + \lambda_1(\Omega)) \end{aligned}$$

for large m , using $\lambda < \lambda_1(\Omega)$ and $\theta \leq (\lambda + \lambda_1(\Omega))/2$. So using Lemma 3.1, we get $u \in H_A^{0,1}(\Omega)$ such that $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$. Therefore $|u|_2 = 1$ and $\liminf_{m \rightarrow \infty} |\nabla_A u_m|_2^2 \geq |\nabla_A u|_2^2$. Since $g \geq 0$ and $\mu_m \rightarrow \infty$, we have

$$\begin{aligned} \int_{\Omega} (|\nabla_A u|^2 - \theta|u|^2) &\leq \liminf_{m \rightarrow \infty} \int_{\mathbb{R}^n} (|\nabla_A u_m|^2 + \mu_m g(x)|u_m|^2 - a_{\mu_m}|u_m|^2) \, dx \\ &= \liminf_{m \rightarrow \infty} ((T_{\mu_m} - a_{\mu_m})u_m, u_m) = 0. \end{aligned}$$

Hence

$$\int_{\Omega} |\nabla_A u|^2 \, dx \leq \theta \leq \frac{\lambda + \lambda_1(\Omega)}{2} < \lambda_1(\Omega)$$

which is a contradiction to the definition of $\lambda_1(\Omega)$. Therefore there exists a $\mu(\lambda) > 0$ such that $a_\mu \geq (\lambda + \lambda_1(\Omega))/2$ whenever $\mu \geq \mu(\lambda)$. For the second part, let $u \in E$ and $\mu \geq \mu(\lambda)$ then $a_\mu \leq \frac{\|u\|_\mu^2}{|u|_2^2}$ which gives

$$\lambda|u|_2^2 \leq \frac{2\lambda\|u\|_\mu^2}{\lambda + \lambda_1(\Omega)}.$$

Therefore

$$\langle (T_\mu - \lambda)u, u \rangle \geq \|u\|_\mu^2 - \lambda|u|_2^2 \geq \frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega) + \lambda} \|u\|_\mu^2.$$

■

Our next lemma assures that all $(PS)_c$ sequences are bounded.

Lemma 3.3 *Let $\{u_m\}$ be a $(PS)_c$ sequence for $I_{\lambda,\mu}$. If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then $\{u_m\}$ is bounded in E and*

$$\lim_{m \rightarrow \infty} \langle (T_\mu - \lambda)u_m, u_m \rangle = \lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n - \alpha)}{(n + 2 - \alpha)},$$

where $B(\cdot)$ is defined in (2.2).

Proof. Using definitions of $I_{\lambda,\mu}$ and T_μ , we get

$$\begin{aligned} I_{\lambda,\mu}(u_m) - \frac{1}{22_\alpha^*} \langle I'_{\lambda,\mu}(u_m), u_m \rangle &= \left(\frac{2_\alpha^* - 1}{22_\alpha^*} \right) \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u_m|^2 - \lambda|u_m|^2) \, dx \\ &= \frac{n + 2 - \alpha}{2(2n - \alpha)} \langle (T_\mu - \lambda)u_m, u_m \rangle \end{aligned} \quad (3.4)$$

and

$$I_{\lambda,\mu}(u_m) - \frac{1}{2} \langle I'_{\lambda,\mu}(u_m), u_m \rangle = \left(\frac{2_\alpha^* - 1}{22_\alpha^*} \right) B(u_m) = \frac{n + 2 - \alpha}{2(2n - \alpha)} B(u_m). \quad (3.5)$$

Using Lemma 3.2 and (3.4), we get

$$c - \frac{1}{22_\alpha^*} o(\|u_m\|_\mu) \geq \frac{n + 2 - \alpha}{2(2n - \alpha)} \langle (T_\mu - \lambda)u_m, u_m \rangle \geq \frac{n + 2 - \alpha}{2(2n - \alpha)} \beta_\lambda \|u_m\|_\mu^2.$$

This implies $\{u_m\}$ is a bounded sequence in E . Taking limit $m \rightarrow \infty$ in (3.4), we get

$$\lim_{m \rightarrow \infty} \langle (T_\mu - \lambda)u_m, u_m \rangle = \frac{2c(2n - \alpha)}{n + 2 - \alpha},$$

and taking limit $m \rightarrow \infty$ in (3.5), we get

$$\lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n - \alpha)}{n + 2 - \alpha}.$$

This completes the proof. ■

Let

$$S_A := \inf_{u \in H_A^1(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx}{B(u)^{\frac{n-2}{2n-\alpha}}}.$$

For the preceding sections, enlarging $\mu(\lambda)$ if necessary, we assume $\mu(\lambda) \geq \lambda/M$, where M is defined in (g2). Thus,

$$\mu M - \lambda \geq 0, \text{ for all } \mu \geq \mu(\lambda). \quad (3.6)$$

Proposition 3.4 *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then the functional $I_{\lambda, \mu}$ satisfies the $(PS)_c$ condition, for all*

$$c \in \left(-\infty, \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}} \right).$$

Proof. Let $\{u_m\} \subset E$ be a sequence such that $I_{\lambda, \mu}(u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ and $I'_{\lambda, \mu}(u_m) \rightarrow 0$ as $m \rightarrow \infty$. By Lemma 3.3, $\{u_m\}$ is bounded in E that is $\|u_m\|_{\mu} \leq K_1$ for some constant $K_1 > 0$ and for all m . Therefore, there exists a subsequence (still denoted by $\{u_m\}$) such that $u_m \rightharpoonup u$ weakly in E , $u_m \rightharpoonup u$ weakly in $L^{2^*}(\mathbb{R}^n)$, $u_m \rightarrow u$ strongly in $L^2_{loc}(\mathbb{R}^n)$ and $u_m(x) \rightarrow u(x)$ a.e. for $x \in \mathbb{R}^n$. This implies that as $m \rightarrow \infty$

$$|u_m|^{2^*_{\alpha}} \rightharpoonup |u|^{2^*_{\alpha}} \text{ in } L^{\frac{2n}{2n-\alpha}}(\mathbb{R}^n) \text{ and } |u_m|^{2^*_{\alpha}-2} u_m \rightharpoonup |u|^{2^*_{\alpha}-2} u \text{ in } L^{\frac{2n}{n+2-\alpha}}(\mathbb{R}^n).$$

Therefore by Hardy-Littlewood-Sobolev inequality we get

$$|x|^{\alpha} * |u_m|^{2^*_{\alpha}} \rightharpoonup |x|^{\alpha} * |u|^{2^*_{\alpha}} \text{ in } L^{\frac{2n}{\alpha}}(\mathbb{R}^n) \text{ as } m \rightarrow \infty.$$

Hence we obtain that

$$(|x|^{\alpha} * |u_m|^{2^*_{\alpha}}) |u_m|^{2^*_{\alpha}-2} u_m \rightharpoonup (|x|^{\alpha} * |u|^{2^*_{\alpha}}) |u|^{2^*_{\alpha}-2} u \text{ in } L^{\frac{2n}{n+2}}(\mathbb{R}^n) \text{ as } m \rightarrow \infty.$$

This implies that for any $\phi \in E \subset L^{2^*}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (|x|^{\alpha} * |u_m|^{2^*_{\alpha}}) |u_m|^{2^*_{\alpha}-2} u_m \phi \, dx \rightarrow \int_{\mathbb{R}^n} (|x|^{\alpha} * |u|^{2^*_{\alpha}}) |u|^{2^*_{\alpha}-2} u \phi \, dx \text{ as } m \rightarrow \infty.$$

Thus we get $\langle I'_{\lambda, \mu}(u), \phi \rangle = \lim_{m \rightarrow \infty} \langle I'_{\lambda, \mu}(u_m), \phi \rangle = 0$. Therefore u is a weak solution of $(P_{\lambda, \mu})$.

Let $\widetilde{u}_m = u_m - u$, so by Lemma 2.3 of [25] we have

$$B(u_m) - B(\widetilde{u}_m) \rightarrow B(u) \text{ as } m \rightarrow \infty. \quad (3.7)$$

Also since $I'_{\lambda, \mu}(u_m) \rightarrow 0$, we get

$$((T_{\mu} - \lambda)u_m, u_m) - B(u_m) \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.8)$$

Then using (3.7) and (3.8), we get

$$\lim_{m \rightarrow \infty} (((T_{\mu} - \lambda)\widetilde{u}_m, \widetilde{u}_m) - B(\widetilde{u}_m)) = 0.$$

Let $\lim_{m \rightarrow \infty} ((T_{\mu} - \lambda)\widetilde{u}_m, \widetilde{u}_m) = \lim_{m \rightarrow \infty} B(\widetilde{u}_m) = d$ (say). Then it is easy to show that $I_{\lambda, \mu}(u) \geq 0$ and using this we get

$$\frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}} > c = \lim_{m \rightarrow \infty} I_{\lambda, \mu}(u_m) \geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_A \widetilde{u}_m|^2 \, dx - \frac{1}{22^*_{\alpha}} B(\widetilde{u}_m) + o_m(1).$$

This implies

$$d \leq \frac{2c(2n-\alpha)}{n+2-\alpha} < S_A^{\frac{2n-\alpha}{n+2-\alpha}}. \quad (3.9)$$

Let $D = \{x \in \mathbb{R}^n : g(x) \leq M\}$, where M is defined in (g2). Then using similar arguments as in Lemma 3.1, we can show that

$$\int_D |\widetilde{u}_m|^2 dx \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.10)$$

Then using (3.6), definition of S_A and (3.10) we get

$$\begin{aligned} S_A B(\widetilde{u}_m)^{\frac{1}{2\alpha}} &\leq \int_{\mathbb{R}^n} |\nabla_A \widetilde{u}_m|^2 dx \leq \int_{\mathbb{R}^n} |\nabla_A \widetilde{u}_m|^2 dx + \int_{\mathbb{R}^n \setminus D} (\mu g(x) - \lambda) |\widetilde{u}_m|^2 dx \\ &\leq ((T_\mu - \lambda) \widetilde{u}_m, \widetilde{u}_m) + \lambda \int_D |\widetilde{u}_m|^2 dx = ((T_\mu - \lambda) \widetilde{u}_m, \widetilde{u}_m) + o_m(1). \end{aligned}$$

Passing on the limits we get $d \geq S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ which is a contradiction to (3.9). Therefore, $d = 0$ that is $u_m \rightarrow u$ strongly in E as $m \rightarrow \infty$. \blacksquare

4 Proof of main Theorems

Before proving the main theorems, we prove some results that will help us to achieve our goal. The theorem below is similar to Theorem 1.1 of [8].

Theorem 4.1 *If $g \geq 0$ and $A \in L_{loc}^n(\mathbb{R}^n, \mathbb{R}^n)$, then the infimum S_A is attained if and only if $\text{curl } A \equiv 0$.*

Proof. At first, we prove that $S_A = S_{H,L}$. By Proposition 2.2 and Theorem 2.1, for $u \in H_A^1(\mathbb{R}^n)$ we have

$$S_{H,L} \leq \frac{\int_{\mathbb{R}^n} |\nabla |u||^2 dx}{B(u)^{\frac{n-2}{2n-\alpha}}} \leq \frac{\int_{\mathbb{R}^n} |\nabla_A u|^2 dx}{B(u)^{\frac{n-2}{2n-\alpha}}}.$$

This implies $S_{H,L} \leq S_A$. Without loss of generality, we assume $0 \in \Omega$ and for some $\delta > 0$, $B(0, \delta) \subset \Omega \subset B(0, 2\delta)$ ($B(0, r)$ is an open ball of radius r and center 0). Let

$$U_\epsilon(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{4}}$$

and $u_\epsilon(x) = \psi(x)U_\epsilon(x)$, where $\psi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ such that $\psi \equiv 1$ in $B(0, \delta)$ and $\psi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. We recall the following asymptotic estimates from section 3 of [25]-

$$(1) \quad |\nabla u_\epsilon|_2^2 = C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + O(\epsilon^{n-2}).$$

(2)

$$|u_\epsilon|_2^2 \geq \begin{cases} d\epsilon^2 |\ln \epsilon| + O(\epsilon^2) & \text{if } n = 4 \\ d\epsilon^2 + O(\epsilon^{n-2}) & \text{if } n \geq 5, \end{cases}$$

where d is a positive constant.

$$(3) \quad B(u_\epsilon)^{(n-2)/(2n-\alpha)} \geq \left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - O(\epsilon^{(2n-\alpha)/2}) \right)^{(n-2)/(2n-\alpha)}.$$

$$(4) \quad |u_\epsilon|_{2^*}^{2^*} = S^{n/2} + O(\epsilon^n).$$

We have u_ϵ is bounded in $L^{2^*}(\mathbb{R}^n)$ and $u_\epsilon(x) \rightarrow 0$ a.e. in \mathbb{R}^n as $\epsilon \rightarrow 0$. Since $A \in L_{\text{loc}}^n(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |Au_\epsilon|^2 dx = \langle |A|^2, |u_\epsilon|^2 \rangle \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

where the duality product is taken with respect to $L^{n/2}(\mathbb{R}^n)$ and $L^{2^*/2}(\mathbb{R}^n)$. Now, let $\delta' > 0$ be given then choosing $\epsilon > 0$ small enough and using (1) and (3) we get

$$\begin{aligned} \frac{\int_{\mathbb{R}^n} |\nabla_A u_\epsilon|^2 dx}{B(u_\epsilon)^{\frac{n-2}{2n-\alpha}}} &\leq \frac{\int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 + |Au_\epsilon|^2 dx}{B(u_\epsilon)^{\frac{n-2}{2n-\alpha}}} \\ &\leq \frac{C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{n/2} + O(\epsilon)}{\left(C(n, \alpha)^{\frac{n}{2}} S_{H,L}^{\frac{2n-\alpha}{2}} - O(\epsilon^{\frac{2n-\alpha}{2}}) \right)^{\frac{n-2}{2n-\alpha}}} \\ &\leq S_{H,L} + \delta'. \end{aligned}$$

This implies $S_A \leq S_{H,L}$, therefore $S_A = S_{H,L}$. Let u be minimizer of S_A normalized by $B(u) = 1$. Then

$$S_A = \int_{\mathbb{R}^n} |\nabla_A u|^2 \geq \int_{\mathbb{R}^n} |\nabla |u||^2 dx \geq S_{H,L}. \quad (4.1)$$

Consequently, $|u(x)| = U_\epsilon(x-a)/B(U_\epsilon)$, for some $a \in \mathbb{R}^n$ because the minimizers of $S_{H,L}$ are of the form U_ϵ which are invariant under translation and dilation (Lemma 1.2, 1.3 of [25]). We can take $|u| > 0$ and the equality in (4.1) occurs when the diamagnetic inequality in Theorem 2.1 has an equality a.e. Therefore $\text{Im}((\nabla_A u)\bar{u}/|u|) = 0$ that is $A = -\text{Im}(\nabla u/u)$. Since $\text{curl}(\nabla u/u) = 0$, we are done. The condition is also sufficient, the proof follows similarly as in Theorem 1.1 of [8]. \blacksquare

The next step to prove our main theorem is introducing the Nehari manifold. Let

$$\mathcal{N}_{\lambda,\mu} = \{u \in E \setminus \{0\} : \langle I'_{\lambda,\mu}(u), u \rangle = 0\} = \{u \in E \setminus \{0\} : ((T_\mu - \lambda)u, u) = B(u)\}.$$

Then the critical points of $I_{\lambda,\mu}$ lie in $\mathcal{N}_{\lambda,\mu}$. Let $X = \{v \in E : B(v) = 1\}$ then using fibering map analysis, we say that for each $u \in E$, there exist

$$t_u = \left(\frac{((T_\mu - \lambda)(u), u)}{B(u)} \right)^{\frac{n-2}{2(n+2-\alpha)}}$$

such that $t_u u \in \mathcal{N}_{\lambda,\mu}$. Using Proposition 1.1 of [43], we get $\mathcal{N}_{\lambda,\mu}$ is radially diffeomorphic to X via the map

$$u \mapsto \left(\frac{((T_\mu - \lambda)(u), u)}{B(u)} \right)^{\frac{n-2}{2(n+2-\alpha)}} u.$$

On $\mathcal{N}_{\lambda,\mu}$,

$$I_{\lambda,\mu}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} ((T_\mu - \lambda)(u), u),$$

so we get

$$k_{\lambda,\mu} := \inf_{u \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} \inf_{v \in X} ((T_\mu - \lambda)(v), v)^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Now consider any domain $\mathcal{Q} \subset \mathbb{R}^n$. As we defined $I_{\lambda,\mu}$, in a similar manner we define

$$\begin{aligned} I_{\mu,\mathcal{Q}}(u) &= \frac{1}{2} \int_{\mathcal{Q}} (|\nabla_A u|^2 + \lambda|u|^2) \, dx - \frac{1}{22_\alpha^*} \int_{\mathcal{Q}} \int_{\mathcal{Q}} \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \, dx dy \\ &= \frac{1}{2} ((T_0 - \lambda)(u), u) - \frac{1}{22_\alpha^*} B(u) \end{aligned}$$

for $u \in H_A^{0,1}(\mathcal{Q})$. This is an energy functional associated to the problem

$$(P_\lambda) \begin{cases} (-i\nabla + A(x))^2 u = \lambda u + (|x|^\alpha * |u|^{2_\alpha^*}) |u|^{2_\alpha^* - 2} u, & u > 0 & \text{in } \mathcal{Q} \\ u = 0 & & \text{on } \partial\mathcal{Q}. \end{cases}$$

The Nehari manifold associated to (P_λ) with $\mathcal{Q} = \Omega$ is given by

$$\mathcal{N}_{\lambda,\mathcal{Q}} = \left\{ u \in H_A^{0,1}(\mathcal{Q}) \setminus \{0\} : ((T_0 - \lambda)(u), u) = B(u) \right\}$$

which is radially diffeomorphic to $X_{\mathcal{Q}} = \{v \in H_A^{0,1}(\mathcal{Q}) : B(v) = 1\}$. We set

$$k_{\mu,\mathcal{Q}} := \inf_{u \in \mathcal{N}_{\lambda,\mathcal{Q}}} I_{\lambda,\mathcal{Q}}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} \inf_{v \in X_{\mathcal{Q}}} ((T_0 - \lambda)(u), u)^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Lemma 4.2 *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then the following holds*

$$\frac{n+2-\alpha}{2(2n-\alpha)} (\beta_\lambda S_A)^{\frac{2n-\alpha}{n+2-\alpha}} \leq k_{\lambda,\mu} \leq k_{\lambda,\Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$$

where β_λ is defined as in Lemma 3.2.

Proof. Using Lemma 3.2, we have $\beta_\lambda \|v\|_A^2 \leq \beta_\lambda \|v\|_\mu^2 \leq ((T_\mu - \lambda)(u), u)$. This implies, taking infimum over X , we get

$$\frac{n+2-\alpha}{2(2n-\alpha)} (\beta_\lambda S_A)^{\frac{2n-\alpha}{n+2-\alpha}} \leq k_{\lambda,\mu}.$$

This gives the first inequality. Now, for the second inequality, since

$$X_\Omega = \left\{ u \in H_A^{0,1}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \, dx dy = 1 \right\} \subset X,$$

where $\Omega = \text{interior of } g^{-1}(0)$, we get $k_{\lambda,\mu} \leq k_{\lambda,\Omega}$. We aim to show that

$$k_{\lambda,\Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Let U_ϵ and u_ϵ be as defined in Lemma 4.1. Define

$$J_\lambda(u) = \frac{\int_\Omega (|\nabla_A u|^2 - \lambda|u|^2) \, dx}{\left(\int_\Omega \int_\Omega \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \, dx dy \right)^{\frac{n-2}{2n-\alpha}}}.$$

Let A be continuous at 0 and $\gamma(x) := -\sum A_j(0)x_j$. Then it is easy to check that $(A+\nabla\gamma)(0) = 0$ and therefore by continuity of A at 0 we get a $\delta_1 > 0$ such that

$$|(A + \nabla\gamma)(x)|^2 \leq \tilde{k} < \lambda, \text{ for all } |x| < \delta_1.$$

Also let $\delta_2 = \min\{\delta, \delta_1\}$ and define $v_\epsilon(x) = \psi(x)U_\epsilon(x) \exp(i\gamma(x))$, where

$$\psi(x) = \begin{cases} 1 & \text{in } B(0, \delta_2/2) \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then using (1) of Lemma 4.1, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (|\nabla_A v_\epsilon|^2 - \lambda|v_\epsilon|^2) dx &= \int_{\mathbb{R}^n} (|(-i\nabla + A)(\psi U_\epsilon \exp(i\gamma))|^2 - \lambda\psi^2 U_\epsilon^2) dx \\ &\leq \int_{\mathbb{R}^n} (|\nabla(\psi U_\epsilon)|^2 + \psi^2 U_\epsilon^2 |\nabla\gamma + A|^2 - \lambda\psi^2 U_\epsilon^2) dx \\ &\leq C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + O(\epsilon^{n-2}) + (\tilde{k} - \lambda) \int_{B(0, \frac{\delta_2}{2})} U_\epsilon^2 dx. \end{aligned}$$

Moreover, using (3) of Lemma 4.1, we get

$$B(v_\epsilon) = B(u_\epsilon) \geq C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - o(\epsilon^{(2n-\alpha)/2}).$$

It is a standard result that for $\epsilon > 0$ small enough

$$\int_{B(0, \frac{\delta_2}{2})} U_\epsilon^2 dx \geq \begin{cases} C\epsilon^2 |\log \epsilon| & \text{if } n = 4 \\ C\epsilon^2 & \text{if } n \geq 5, \end{cases}$$

where $C > 0$ is a constant depending only on n . Therefore since $\tilde{k} - \lambda < 0$, when $n \geq 5$, for $\epsilon > 0$ small enough we get

$$J_\lambda(v_\epsilon) \leq \frac{C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + O(\epsilon^{n-2}) + (\tilde{k} - \lambda)C\epsilon^2}{\left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - O(\epsilon^{(2n-\alpha)/2})\right)^{(n-2)/(2n-\alpha)}} < S_{H,L} = S_A. \quad (4.2)$$

This implies

$$k_{\lambda, \Omega} \leq \frac{n+2-\alpha}{2(2n-\alpha)} J_\lambda(v_\epsilon)^{\frac{2n-\alpha}{n+2-\alpha}} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$$

that is the last inequality holds for $n \geq 5$. Similarly the result follows for $n = 4$. \blacksquare

Remark 4.3 For the case $n = 3$, (4.2) becomes

$$J_\lambda(v_\epsilon) \leq S_{H,L} - \frac{((\lambda - \tilde{k}) - O(1))C\epsilon}{\left(C(\alpha)^{3/2} S_{H,L}^{(6-\alpha)/2} - O(\epsilon^{(6-\alpha)/2})\right)^{1/(6-\alpha)}} \quad (4.3)$$

where the right hand side of (4.3) becomes less than $S_{H,L}$ if $\lambda > 0$ is chosen large enough. But since $\lambda \in (0, \lambda_1)$ this can not be possible. So we remark that the question of existence and concentration of solutions of $(P_{\lambda, \mu})$ remains open in the case $n = 3$.

Proof of Theorem 2.6: Let $\{u_m\}$ be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Then by Ekeland Variational Principle [23], $\{u_m\}$ becomes a Palais-Smale sequence. Using Proposition 3.4 and Lemma 4.2, we conclude that there exist a subsequence of $\{u_m\}$ that converges to least energy solution, say u_μ of $(P_{\lambda,\mu})$. \blacksquare

Proof of Theorem 2.7: Let $\{u_m\}$ be a sequence of solution for the problem (P_{λ,μ_m}) such that $\lambda \in (0, \lambda_1(\Omega))$, $\mu_m \rightarrow \infty$ and

$$I_{\lambda,\mu_m}(u_m) = ((T_{\mu_m} - \lambda)(u_m), u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_{H,L}^{\frac{2n-\alpha}{n+2-\alpha}}.$$

By Lemma 3.2, $((T_{\mu_m} - \lambda)u_m, u_m) \geq \beta_\lambda \|u_m\|_{\mu_m}^2$ for sufficiently large m which implies that $\beta_\lambda \|u_m\|_{\mu_m}^2 \leq c + o(1)$. So, $\{u_m\}$ is bounded in E and using Lemma 3.1, we say that there exists $u \in H_A^{0,1}(\Omega)$ such that, upto a subsequence, $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$. Since u_{μ_m} solves (P_{λ,μ_m}) , we have

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u_m \overline{\nabla_A v} \, dx + \int_{\mathbb{R}^n} \left(\mu_m g(x) - \lambda \right) u_m \bar{v} - (|x|^{-\alpha} * |u_m|^{2_\alpha^*}) |u_m|^{2_\alpha^* - 2} u_m \bar{v} \, dx \right) = 0 \quad (4.4)$$

for every $v \in E$. Since $\Omega = \{x \in \mathbb{R}^n : g(x) = 0\}$, for any $v \in H_A^{0,1}(\Omega)$, $\mu_m \int_{\mathbb{R}^n} g(x) u_m \bar{v} \, dx = 0$. Letting $m \rightarrow \infty$ in (4.4), we obtain

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} \, dx - \lambda \int_{\mathbb{R}^n} u \bar{v} \, dx - \int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^* - 2} u \bar{v} \, dx \right) = 0,$$

for all $v \in H_A^{0,1}(\Omega)$ which implies that u is a weak solution of (P_λ) . Let $\tilde{u}_m = u_m - u$, then $\tilde{u}_m \rightharpoonup 0$ weakly in E and $\tilde{u}_m \rightarrow 0$ strongly in $L^2(\mathbb{R}^n)$. Therefore,

$$((T_{\mu_m} - \lambda)u_m, u_m) = ((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) + ((T_0 - \lambda)u, u) + o(1). \quad (4.5)$$

By Lemma 2.3 of [25], we get

$$B(u_m) - B(\tilde{u}_m) \rightarrow B(u) \text{ as } m \rightarrow \infty.$$

Since u is a weak solution of (P_λ) it is easy to see that

$$((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) - B(\tilde{u}_m) = o(1). \quad (4.6)$$

We claim that $B(\tilde{u}_m) \rightarrow 0$ as $m \rightarrow \infty$. Suppose not, that is $B(\tilde{u}_m) \rightarrow d > 0$ as $m \rightarrow \infty$. Using (4.6) and arguments as in Lemma 3.4, we get

$$S_A B(\tilde{u}_m)^{\frac{n-2}{2n-\alpha}} \leq \int_{\mathbb{R}^n} |\nabla_A \tilde{u}_m|^2 \, dx \leq ((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) + o(1) = B(\tilde{u}_m) + o(1).$$

Consequently, we get that $S_A \leq B(\tilde{u}_m)^{\frac{n+2-\alpha}{2n-\alpha}} + o(1)$ which implies

$$S_A^{\frac{2n-\alpha}{n+2-\alpha}} \leq \lim_{m \rightarrow \infty} B(u_m). \quad (4.7)$$

It is not hard to find that $\lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n-\alpha)}{(n+2-\alpha)}$. Thus from (4.7) we get

$$S_A^{\frac{2n-\alpha}{n+2-\alpha}} \leq \lim_{m \rightarrow \infty} B(u_m) < S_A^{\frac{2n-\alpha}{n+2-\alpha}}$$

which is a contradiction. Hence, $\lim_{m \rightarrow \infty} B(\tilde{u}_m) = 0$ and $((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) \rightarrow 0$ as $m \rightarrow \infty$. Using (4.5), we get

$$((T_0 - \lambda)u, u) = \lim_{m \rightarrow \infty} ((T_{\mu_m} - \lambda)u_m, u_m). \quad (4.8)$$

Since $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, so $u_m = \tilde{u}_m$ in $\mathbb{R}^n \setminus \Omega$. Also since $g \equiv 0$ in Ω , for sufficiently large m we get

$$\int_{\mathbb{R}^n} g(x)u_m^2 \, dx \leq \mu_m \int_{\mathbb{R}^n \setminus \Omega} g(x)\tilde{u}_m^2 \, dx \leq ((T_{\mu_m} - \lambda)(\tilde{u}_m), \tilde{u}_m) + o(1).$$

Therefore, $\int_{\mathbb{R}^n} g(x)u_m^2 \, dx = 0$ and since $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$, (4.8) implies

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla_A u_m|^2 \, dx = \int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx$$

that is $u_m \rightarrow u$ strongly in E as $m \rightarrow \infty$. ■

Corollary 4.4 *If $\lambda \in (0, \lambda_1(\Omega))$, then $\lim_{\mu \rightarrow \infty} k_{\lambda, \mu} = k_{\lambda, \Omega}$.*

Proof. By Lemma 4.2, $k_{\lambda, \mu} \rightarrow a \leq k_{\lambda, \Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ as $\mu \rightarrow \infty$. Theorem 2.6 implies that $k_{\lambda, \mu}$ is achieved for $\mu \geq \mu(\lambda)$. Therefore, Theorem 2.7 says a must be achieved by $I_{\mu, \Omega}$ on $\mathcal{N}_{\lambda, \Omega}$. Hence $a \geq k_{\mu, \Omega}$. ■

5 A Remark on nonlocal counterpart of $(P_{\lambda, \mu})$

In this section, we brief the nonlocal extension of the problem $(P_{\lambda, \mu})$ given by

$$(P_{\lambda, \mu}^s) \begin{cases} (-\Delta)_A^s u + \mu g(x)u = \lambda u + (|x|^{-\alpha} * |u|^{2_{\alpha, s}^*})|u|^{2_{\alpha, s}^* - 2}u \text{ in } \mathbb{R}^n, \\ u \in H_A^s(\mathbb{R}^n, \mathbb{C}) \end{cases}$$

where $n \geq 4s$, $s \in (0, 1)$ and $\alpha \in (0, n)$. Here $2_{\alpha, s}^* = \frac{2n-\alpha}{n-2s}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. We assume the same conditions on A and g as before. For $u \in C_c^\infty(\Omega)$, the fractional magnetic operator $(-\Delta)_A^s$, up to a normalization constant, is defined by

$$(-\Delta)_A^s u(x) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x-y|^{n+2s}} dy$$

for all $x \in \mathbb{R}^n$. Up to correcting the operator by the factor $(1-s)$, it is true that $(-\Delta)_A^s$ converges to $-\Delta_A$ as $s \uparrow 1$. For further details we refer [21] and references therein.

Functional Setting- Let $L_g^2(\mathbb{R}^n, \mathbb{C})$ denote the Lebesgue space of complex valued functions with $\int_{\mathbb{R}^n} g|u|^2 < +\infty$ endowed with the real scalar product

$$\langle u, v \rangle_{L_g^2} := \operatorname{Re} \left(\int_{\mathbb{R}^n} g(x) u \bar{v} \, dx \right), \text{ for } u, v \in L_g^2(\mathbb{R}^n, \mathbb{C}).$$

We consider the magnetic Gagliardo semi-norm defined by

$$[u]_{s,A}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| u(x) - e^{i(x-y) \cdot A \left(\frac{x+y}{2} \right)} u(y) \right|^2}{|x-y|^{n+2s}} \, dx dy.$$

The scalar product is defined as

$$\begin{aligned} & \langle u, v \rangle_{s,A} \\ & := \langle u, v \rangle_{L_g^2} + \operatorname{Re} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(u(x) - e^{i(x-y) \cdot A \left(\frac{x+y}{2} \right)} u(y) \right) \overline{\left(v(x) - e^{i(x-y) \cdot A \left(\frac{x+y}{2} \right)} v(y) \right)}}{|x-y|^{n+2s}} \, dx dy \right) \end{aligned}$$

and the corresponding norm is given by

$$\|u\|_{s,A} = \left(\|u\|_{L_g^2}^2 + [u]_{s,A}^2 \right)^{\frac{1}{2}}.$$

We define $H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$ as the closure of $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm $\|\cdot\|_{s,A}$. Then we have the following properties regarding the function space $H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$ -

- (i) $(H_{A,g}^s(\mathbb{R}^n, \mathbb{C}), \langle \cdot, \cdot \rangle_{s,A})$ is a real Hilbert space.
- (ii) The embedding $H_{A,g}^s(\mathbb{R}^n, \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^n, \mathbb{C})$ is continuous for all $p \in [2, 2_s^*]$ where $2_s^* = \frac{2n}{n-2s}$. Furthermore, for any $K \Subset \mathbb{R}^n$ and $p \in [1, 2_s^*)$, the embedding $H_{A,g}^s(\mathbb{R}^n, \mathbb{C}) \hookrightarrow L^p(K, \mathbb{C})$ is compact.
- (iii) (*Diamagnetic inequality*) For each $u \in H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$

$$|u| \in H_g^s(\mathbb{R}^n, \mathbb{C}) \text{ and } \| |u| \|_s \leq \|u\|_{s,A}$$

where $H_g^s(\mathbb{R}^n, \mathbb{C}) = H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$ with $A \equiv 0$.

For further details related to this topic, we refer [21, 35] and the references therein.

Definition 5.1 We say that $u \in H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$ is a weak solution of $(P_{\lambda,\mu}^s)$ if

$$\begin{aligned} & \operatorname{Re} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(u(x) - e^{i(x-y) \cdot A \left(\frac{x+y}{2} \right)} u(y) \right) \overline{\left(v(x) - e^{i(x-y) \cdot A \left(\frac{x+y}{2} \right)} v(y) \right)}}{|x-y|^{n+2s}} \, dx dy + \mu \int_{\mathbb{R}^n} g(x) u \bar{v} \, dx \right. \\ & \quad \left. - \lambda \int_{\mathbb{R}^n} u \bar{v} \, dx - \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^{2_{\alpha,s}^*}) |u|^{2_{\alpha,s}^* - 2} u \bar{v} \, dx \right) = 0 \end{aligned}$$

for all $v \in H_{A,g}^s(\mathbb{R}^n, \mathbb{C})$.

The functional $I_s : H_{A,g}^s(\mathbb{R}^n, \mathbb{C}) \rightarrow \mathbb{R}$, associated to $(P_{\lambda,\mu}^s)$, is defined by

$$I_s(u) = \frac{\|u\|_{s,A}^2}{2} - \frac{\lambda}{2} \int_{\mathbb{R}^n} |u|^2 \, dx - \frac{1}{22_{\alpha,s}^*} \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^{2_{\alpha,s}^*}) |u|^{2_{\alpha,s}^*} \, dx.$$

Then $I_s \in C^1(H_{A,g}^s(\mathbb{R}^n, \mathbb{C}), \mathbb{R})$ and the critical points of I_s are exactly the weak solutions of $(P_{\lambda,\mu}^s)$. Based on this setting, we expect that Theorem 2.6 and 2.7 type of results can as well as obtained for the problem $(P_{\lambda,\mu}^s)$ employing the same arguments as in this article. Lastly, we cite [36, 24] for readers as very recent articles concerning this topic.

Acknowledgments

We thank the anonymous referee for advising us to mention the possibility of investigating our results in the framework of nonlocal magnetic operator which we have addressed in section 5 and bringing into our notice references [21, 35].

References

- [1] C. O. Alves and G. M. Figueiredo, *Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field*, Milan J. Math., 82 (2) (2014), 389-405.
- [2] C. O. Alves, G. Figueiredo and M. Yang, *Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity*, Adv. Nonlinear Anal., 5 (4) (2016), 331-345.
- [3] C. O. Alves, G. M. Figueiredo and M. Yang, *Multiple semiclassical solutions for a nonlinear Choquard equation with magnetic field*, Asymptotic Analysis, 96 (2) (2016), 135–159.
- [4] C. O. Alves, F. Gao, M. Squassina and M. Yang, *Singularly perturbed critical Choquard equations*, J. Differential Equations, 263 (7) (2017), 3943–3988.
- [5] C. O. Alves and M. Yang, *Existence of semiclassical ground state solutions for a generalized Choquard equation*, J. Differential Equations, 257 (11) (2014), 4133-4164.
- [6] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal., 140 (3) (1997), 285–300.
- [7] A. Ambrosetti, A. Malchiodi and S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rational Mech. Anal., 159 (3) (2001), 253–271.
- [8] G. Arioli and A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Arch. Rational Mech. Anal., 170 (4) (2003), 277–295.
- [9] N. Ba, Y. Deng and S. Peng, *Multi-peak bound states for Schrödinger equations with compactly supported or unbounded potentials*, Annales de l’Institut Henri Poincaré Analyse Non Linéaire, 27 (5) (2010), 1205–1226.

- [10] T. Bartsch and Z.-Q. Wang, *Multiple positive solutions for a nonlinear Schrödinger equation*, Z. Angew. Math. Phys., 51 (2000), 366–384.
- [11] T. Bartsch, A. Pankov and Z.Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Commun. Contemp. Math., 3 (2001), 549–569.
- [12] D. Cao and Z. Tang, *Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields*, J. Differential Equations, 222 (2) (2006), 381–424.
- [13] J. Chabrowski and A. Szulkin, *On the Schrödinger equation involving a critical Sobolev exponent and magnetic field*, Topol. Methods Nonlinear Anal., 25 (2005), 3-21.
- [14] S. Cingolani, M. Clapp and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys., 63(2) (2012), 233–248.
- [15] S. Cingolani, M. Clapp and S. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Contin. Dyn. Syst. Ser. S, 6(4) (2013), 891–908.
- [16] S. Cingolani and M. Lazzo, *Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions*, J. Differential Equations, 160 (1) (2000), 118–138.
- [17] S. Cingolani and S. Secchi, *Semiclassical limit for nonlinear Schrödinger equation with electromagnetic fields*, J. Math. Anal. Appl., 275 (1) (2002), 108–130.
- [18] S. Cingolani, S. Secchi and M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A, 140 (5) (2010), 973–1009.
- [19] M. Clapp and Y. Ding, *Positive solutions of a Schrödinger equation with critical nonlinearity*, Z. Angew. Math. Phys., 55 (4) (2004), 592–605.
- [20] M. Clapp and D. Salazar, *Positive and sign changing solutions to a nonlinear Choquard equation*, J. Math. Anal. Appl., 407 (1) (2013), 1-15.
- [21] P. d’Avenia and M. Squassina, *Ground states for fractional magnetic operators*, ESAIM Control Optim. Calc. Var., 24 (2018), 1-24.
- [22] Y. Ding and X. Liu, *Semiclassical solutions of Schrödinger equations with magnetic fields and critical nonlinearities*, Manuscripta Math., 140 (2013), 51-82.
- [23] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 47 (1974), 324–353.
- [24] A. Fiscella, A. Pinamonti and E. Vecchi, *Multiplicity results for magnetic fractional problems*, J. Differential Equations, 263 (8) (2017), 4617-4633.

- [25] F. Gao and M. Yang, *On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation*, Science China Mathematics, doi: 10.1007/s11425-016-9067-5.
- [26] F. Gao and M. Yang, *On nonlocal Choquard equations with Hardy-Littlewood-Sobolev critical exponents*, J. Math. Anal. Appl., 448 (2017), 1006-1041.
- [27] M. Ghimenti and J. V. Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal., 271 (1) (2016), 107-135.
- [28] O. Goubet and E. Hamraoui, *Blow-up of solutions to a cubic nonlinear Schrödinger equations with defect: the radial case*, Adv. Nonlinear Anal., 6 (2) (2017), 183-197.
- [29] G. Li, S. Peng and C. Wang, *Infinitely many solutions for nonlinear Schrödinger equations with electromagnetic fields*, J. Differential Equations, 251 (12) (2011), 3500–3521.
- [30] M. Holzleitner, A. Kostenko and G. Teschl, *Dispersion estimates for spherical Schrödinger equations: the effect of boundary conditions*, Opuscula Math., 36 (6) (2016), 769-786.
- [31] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math., 57 (2) (1976/77), 93–105.
- [32] E.H. Lieb and M. Loss, *Analysis*, 2nd Edition, AMS, 2001.
- [33] D. Lü, *Existence and concentration behavior of ground state solutions for magnetic nonlinear Choquard equations*, Communications on Pure and Applied Analysis, 15 (5) (2016), 1781–1795.
- [34] Y.G. Oh, *Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$* , Comm. Partial Differential Equations, 13 (1988), 1499–1519.
- [35] X. Mingqi, P. Pucci, M. Squassina and B. Zhang, *Nonlocal Schrödinger-Kirchhoff equations with external magnetic field*, Discrete Contin. Dyn. Syst., 37 (3) (2017), 1631–1649.
- [36] X. Mingqi, V. Rădulescu and Binlin Zhang, *A critical fractional Choquard-Kirchhoff problem with magnetic field*, Commun. Contemp. Math., <https://doi.org/10.1142/S0219199718500049>.
- [37] G. Molica Bisci and V. Rădulescu, *Ground state solutions of scalar field fractional Schrödinger equations*, Calc. Var. Partial Differential Equations, 54 (3) (2015), 2985-3008.
- [38] T. Mukherjee and K. Sreenadh, *Fractional Choquard equation with critical nonlinearities*, Nonlinear Differ. Equ. Appl., (2017) 24: 63.
- [39] T. Mukherjee and K. Sreenadh, *Positive solutions for nonlinear Choquard equation with singular nonlinearity*, Complex Variables and Elliptic Equations, 62 (2017), 1044-1071.

- [40] D. Salazar, *Vortex-type solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys., 66 (3) (2015), 663–675.
- [41] S. Secchi and M. Squassina, *On the location of spikes for the Schrödinger equation with electromagnetic field*, Commun. Contemp. Math., 7 (2) (2005), 251-268.
- [42] M. Squassina, *Soliton dynamics for the nonlinear Schrödinger equation with magnetic field*, Manuscripta Math., 130 (4) (2009), 461-494.
- [43] A. Szulkin, *The method of Nehari manifold revisited*, Kyoto: Research Institute for Mathematical Sciences, Kyoto University, 1740 (2011), 89–102.
- [44] M. Yang and Y. Wei, *Existence and multiplicity of solutions for nonlinear Schrödinger equations with magnetic field and Hartree type nonlinearities*, J. Math. Anal. Appl., 403 (2) (2013), 680-694.
- [45] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Comm. Math. Phys., 153 (2) (1993), 229–244.