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Nonconforming Least-Squares Spectral Element Method for European Options[☆]

Arbaz Khan[a,*], Pravir Dutt[a], Chandra Shekhar Upadhyay[b]

[a] Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, UP, 208016, India
[b] Department of Aerospace Engineering, Indian Institute of Technology Kanpur, UP, 208016, India

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ABSTRACT

Several methods have been proposed in the literature for solving the Black–Scholes equation for European Options. The method proposed in the current study achieves spectral accuracy in both space and time. The method is based on minimization of a functional given in terms of the sum of squares of the residuals in the partial differential equation and initial condition in different Sobolev norms, and a term which measures the jump in the function and its derivatives across inter-element boundaries in appropriate fractional Sobolev norms. To obtain values of the solution and its derivatives the initial condition is mollified and the computed solution is post processed. Error estimates are obtained for this method. Specific numerical examples are given to show the efficiency of this method.

1. Introduction

Consider the Black–Scholes (BS) equation [1,2] for European Option

$$V_\tau + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \quad \text{in } (0,\infty)\times[0,T] \tag{1.1}$$

where V, S, r and σ are respectively Option price, underlying asset price, risk-free interest rate and volatility.

Now, we define the European Call Option and the European Put Option.

Definition 1.1. In European Call Option the holder has the right, but not the obligation, to buy an asset at a prescribed price K (strike price) at maturity time T in future. The payoff function for European Call Option is

$$V_C(S,T) = \max(S-K, 0). \tag{1.2}$$

Definition 1.2. In European Put Option the holder has the right, but not the obligation, to sell an asset at a prescribed price K (strike price) at maturity time T in future. The payoff function for European Put Option is

$$V_P(S,T) = \max(K-S, 0). \tag{1.3}$$

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* Corresponding author.
E-mail addresses: arbaz@iitk.ac.in, arbazkha@gmail.com (A. Khan), pravir@iitk.ac.in (P. Dutt), shekhar@iitk.ac.in (C.S. Upadhyay).

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Recently, Spectral methods [3] have been used to solve Option Pricing problems. In 2000, Bunin et al. [4] proposed Chebyshev Collocation methods to solve the European Call Option problem on parallel computers. After this, Greenberg [5] solved American Options problem by Chebyshev Tau method. For smooth initial conditions, De Frutos [6] has presented a Laguerre–Galerkin Spectral Method to price bonds. More recently, Zhu et al. [7] have used a Spectral element method to price European Options. These methods give quadratic accuracy in time, while being spectrally accurate in space. Schötzau et al. [8] proposed *hp*-version of the Discontinuous Galerkin Finite Element Method to solve parabolic problems. In [9], Dutt et al. proposed Least-Squares Spectral Element Method for parabolic partial differential equations (PDE) on bounded domains and proved exponential accuracy for analytic data.

In this paper, we develop a Non-Conforming Least-Squares Spectral Element Method (LSSEM) for parabolic initial value problems with nonsmooth, unbounded initial data and variable coefficients on unbounded domains using parallel computers. One of the applications of this method is in finance, namely Black–Scholes equation for European Options. It will be shown that the proposed LSSEM is exponentially accurate in both space and time. Sobolev spaces of different orders in space and time are used for the results, as presented in [10]. If the data belong to certain Gevrey spaces then the solution also belongs to a Gevrey space [11].

The proposed method is a Least-Squares method as presented in [9]. The space domain is an interval which is divided into a number of sub-intervals. The functional is the sum of the squares of the residuals in the partial differential equation and initial condition in different Sobolev norms, and a term which measures the jump in the function and its derivatives across inter-element boundaries in appropriate fractional Sobolev norms. We minimize the functional on a given time interval. Hermite mollifiers, as described in [12,13], are used to resolve the difficulty of non-smooth initial data.

Now we describe the organization of this paper. In Section 2 the function spaces and a priori estimates for parabolic initial value problem, as presented in [14,10,11], are given. Discretization of the domain and stability estimates are discussed in Section 3. In Section 4 we describe the numerical scheme, parallelization and preconditioning for our method. Estimate for non smooth initial condition, in negative Sobolev norms, is presented in Section 5. In Section 6 error estimates are obtained for this method. Finally, in Section 7 specific numerical examples are provided to show the effectiveness of the method.

2. Function spaces

We consider $\Omega = \mathbb{R}$ as the domain of the logarithmic price $x = \log(S/K)$ and define $t = \frac{T-\tau}{T}$ on the time interval $I = [0, 1]$. We shall focus here on the Black–Scholes equation for the European call with the assumption that the rate of interest r and volatility σ are smooth (or even analytic) functions of x and t with bounded derivatives. The coefficients a, b and c belong to $\mathscr{D}_{2,1}(\Omega \times I)$ as defined in [11] and satisfy

$$\|D_x^i D_t^j a(x,t)\|_{L^\infty(\Omega\times I)} \le AB^{i+j}\ i!(j!)^2,$$

where A and B are positive numbers.

The price $u(x,t)$ has to satisfy the BS equation

$$\begin{aligned} \mathcal{L}u = u_t - au_{xx} - bu_x - cu = 0 \quad &\text{in } \Omega \times I, \\ u(x,0) = f(x) \quad &\text{in } \Omega \times \{0\}. \end{aligned} \tag{2.1}$$

Note that $f(x)$ may not be in $L^2(\Omega)$, for example

$$f(x) = (Ke^x - K)^+.$$

To resolve this difficulty, let us define

$$v(x,t) = u(x,t)\,\text{sech}(\eta x), \tag{2.2}$$

where $\eta > 0$ is sufficiently large so that the initial data

$$v(x,0) = u(x,0)\ \text{sech}(\eta x), \tag{2.3}$$

is such that $ve^{\mu x}, ve^{-\mu x} \in L^2(\Omega)$ for some $\mu > 0$.

Substituting $v(x,t)$ in Eq. (2.1), we get the partial differential equation that v satisfies, as:

$$\begin{aligned} \mathbb{L}v = v_t - \alpha\ v_{xx} - \beta\ v_x - \gamma\ v = 0 \quad &\text{in } \Omega \times I, \\ v(x,0) = f(x)\ \text{sech}(\eta x) = \ g(x) \quad &\text{in } \Omega \times \{0\}. \end{aligned} \tag{2.4}$$

We assume the coefficients $a,\ b$ and c in (2.1) are smooth or even analytic and all derivatives are bounded. Clearly the same assumption will continue to hold for the coefficients $\alpha = a,\ \beta = 2\ a\ \eta\ \tanh \eta x + b$ and $\gamma = \eta^2 a + b\eta \tanh \eta x + c$, since $\tanh \eta x$ has bounded derivatives of all orders. Moreover the coefficients belong to $\mathscr{D}_{2,1}(\bar{\Omega} \times [0,1])$.

However, the initial data $g(x) = f(x)\,\text{sech}\,\eta x$ is not smooth. To resolve this difficulty we use the Hermite mollifiers [12]:

$$\Phi(x) = e^{-\frac{x^2}{2}} \sum_{j=0}^{P} \frac{(-1)^j}{4^j j!} H_{2j}\left(\frac{x}{\sqrt{2}}\right). \tag{2.5}$$

Let us further define the following scaled functions:

$$\Phi_\delta(x) = \frac{1}{\delta}\Phi\left(\frac{x}{\delta}\right), \tag{2.6}$$

and

$$\theta_{N/\delta}(x) = \frac{N}{2\pi}\Phi_\delta(Nx) = \frac{N}{2\pi\delta}\Phi\left(\frac{Nx}{\delta}\right). \tag{2.7}$$

Using Eqs. (2.5) and (2.6), we get:

$$\Phi_\delta(Nx) = \frac{e^{-\frac{(Nx)^2}{2\delta^2}}}{\delta}\sum_{j=0}^{P}\frac{(-1)^j}{4^j j!}H_{2j}\left(\frac{Nx}{\sqrt{2}\delta}\right),$$

where $\delta = \sqrt{\beta_1 N}$, $\beta_1 = \theta_1 d_x$ and $P = \theta_1^2 d_x N$. Here

$$d_x = \frac{1}{\pi} dist\{x, \{c_1, \ldots, c_j\}\}[\mathrm{mod}\, \pi].$$

Further $0 < \theta_1 < 1$ and $c_1, \ldots, c_j$ are points around x where the initial function is not regular and $(x - \pi d_x, x + \pi d_x)$ is a neighborhood of analyticity around x.

We use the above mollifier to replace the initial function $g(x)$ by its mollified version $g_\delta(x)$ as:

$$g_\delta(x) = (g * \theta_{N/\delta})(x) = \int_{|y|\le \pi d_x} g(x-y)\theta_{N/\delta}(y)dy \quad \forall\, x \in \Omega. \tag{2.8}$$

Then we define $v_\delta(x,t)$ to be the solution of the following mollified IVP:

$$\begin{aligned} \mathscr{L} v_\delta &= 0 \quad \text{in } \Omega \times I, \\ v_\delta &= g_\delta \quad \text{on } \Omega \times \{0\}. \end{aligned} \tag{2.9}$$

We must now define some required Sobolev spaces. Letting $\omega(x,t)$ be a smooth function, the following norms can be defined:

$$\|\omega\|^2_{H^{r,s}(\Omega\times I)} = \int_I\int_\Omega \left(\sum_{\alpha\le r}|\partial_x^\alpha w|^2 + \sum_{0<\beta\le s}|\partial_t^\beta w|^2\right) dx\, dt. \tag{2.10}$$

Now, if $h(x)$ is a smooth function, with

$$\|h\|^2_{H^r(\Omega)} = \int_\Omega \sum_{\alpha\le r}|\partial_x^\alpha h|^2\, dx. \tag{2.11}$$

Then for smooth F and h, the following initial value problem can be defined:

$$\begin{aligned} \mathscr{L}\omega &= F \quad \text{in } \Omega \times I, \\ \omega &= h \quad \text{on } \Omega \times \{0\}. \end{aligned} \tag{2.12}$$

The solution $\omega(x,t)$ of Eq. (2.12), then satisfies the a-priori estimate:

$$\|\omega\|^2_{H^{2r+2,r+1}(\Omega\times I)} + \|\omega\|^2_{H^{2r+1}(\Omega\times\{1\})} \le C_r(\|\mathscr{L}\omega\|^2_{H^{2r,r}(\Omega\times I)} + \|\omega\|^2_{H^{2r+1}(\Omega\times\{0\})}), \tag{2.13}$$

where C_r is a constant which depends on r.

We now introduce the negative Sobolev norm on Ω as:

$$\|\omega\|_{H^{-m}(\Omega)} = \sup_{\Phi\in H^m(\Omega)}\frac{|(\omega,\Phi)_\Omega|}{\|\Phi\|_{H^m(\Omega)}}. \tag{2.14}$$

Similarly, over $(\Omega \times I)$ the negative Sobolev norm is given as:

$$\|\omega\|_{H^{-r,-s}(\Omega\times I)} = \sup_{\Phi\in H^{r,s}(\Omega\times I)}\frac{|(\omega,\Phi)_{\Omega\times I}|}{\|\Phi\|_{H^{r,s}(\Omega\times I)}}. \tag{2.15}$$

We further define some Gevrey Spaces, which are needed in the error analysis.

Definition 2.1. Let $\Phi(x) \in \mathscr{D}_1(\bar{\Omega})$, then Φ is an infinitely differentiable function in $\bar{\Omega}$ such that there exist two positive numbers A_1 and B_1 with:

$$\|D_x^\alpha \Phi(x)\|_{L^2(\bar{\Omega})} \le A_1(B_1)^i\, i!, \quad |\alpha| = i,\ i = 0, 1, 2, \ldots.$$

Definition 2.2. Let $\psi(x,t) \in \mathscr{D}_{2,1}(\bar{\Omega}\times[0,1])$, then $\psi(x,t)$ is an infinitely differentiable function in $\bar{\Omega}\times[0,1]$ such that there exist two positive numbers A_1 and B_1 with:

$$\|D_x^\alpha D_t^j \psi(x,t)\|_{L^2(\bar{\Omega}\times[0,1])} \le A_1(B_1)^{i+j}\, i!(j!)^2, \quad |\alpha| = i,\ \text{for all } i, j \ge 0.$$

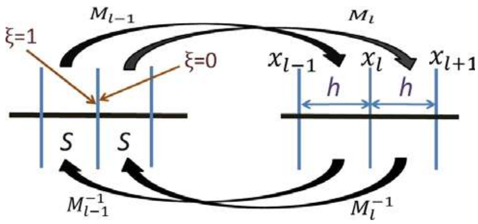

Fig. 1. Inter element boundary.

3. Discretization and stability estimates

3.1. Discretization

Let N and p be integers and p be proportional to N. We solve the initial value problem (2.9) for $I_N = [-N, N]$. First we divide the interval $I_N = [-N, N]$ into a number of sub-intervals $\{\Omega_l\}_{l=-N}^{N-1}$, where $\Omega_l = (l, l+1)$ and $I_N = \bigcup\{\bar{\Omega}_l\}_{l=-N}^{N-1}$. Each of these intervals $\{\Omega_l\}_{l=-N}^{N-1}$ is mapped to the standard element $S = (0, 1)$ by a set of smooth maps $\{M_l^{-1}\}_{l=-N}^{N-1}$, where M_l is a map from $S = (0, 1)$ to $\Omega_l = (l, l+1)$. The map M_l is affine and has the form:

$$M_l(\xi) = l + \xi,$$

where $\xi \in (0, 1)$.

The discretization uses results in a uniform mesh of interval size $h = 1$ and the corresponding time step k is proportional to h^2 (that is of order 1 here). Let $\{x_l\}_{l=-N,N}$ be the inter-element boundaries and boundary of $I_N = \bigcup\{\bar{\Omega}_l\}_{l=-N,N-1}$, which means that $x_l = l$. In all the results that follow, this nomenclature for the spatial and temporal discretization will be needed (see Fig. 1).

3.2. Stability estimates

Let $\check{v}_l^p(\xi, t)$ be the spectral element function which is defined to be a polynomial of degree p in the space variable ξ and of degree q in the time variable t, and is given by:

$$\check{v}_l^p(\xi, t) = \sum_{i=0}^{p}\sum_{j=0}^{q} \delta_{i,j}^l \xi^i t^j$$

for $\xi \in (0, 1), t \in [0, 1]$. Here $\delta_{i,j}^l$ are the coefficients and q is proportional to p^2. The corresponding function on the physical domain (x, t) is given by:

$$v_l^p(x, t) = \check{v}_l^p(M_l^{-1}(x), t).$$

$v^p(x, t)$ can thus be defined piecewise, as:

$$\begin{aligned} v^p(x, t) &= v_l^p(x, t), \quad \text{for } (x, t) \in \Omega_l \times I, \text{ for all } -N \le l \le N-1, \\ &= 0, \quad \text{for } (x, t) \in (I_N)^c \times I. \end{aligned} \tag{3.1}$$

Thus $v^p(x, t) = 0$ for $|x| \ge N$. Using the chain rule, we can write the derivative as:

$$\frac{\partial v_l^p}{\partial x} = (\check{v}_l^p)_\xi (\xi)_x. \tag{3.2}$$

Assume $(\hat{\xi})_x$ to be the polynomial of orthogonal projection of $(\xi)_x$ into the space of polynomials of degree p with respect to the inner product in $H^2(0, 1)$. Then the polynomial approximation of Eq. (3.2) at an interior point is defined by

$$\left(\frac{\partial v_l^p}{\partial x}\right)^a = (\check{v}_l^p)_\xi (\hat{\xi})_x. \tag{3.3}$$

Let x_l be the common interior point of the subintervals Ω_{l-1} and Ω_l, which are the image of $\xi = 1$ under the map M_{l-1} and the image of $\xi = 0$ under the map M_l respectively. Now we define the jump in the derivative at the inter element boundary x_l as follows:

$$\left\| \left[\left(\frac{\partial v^p}{\partial x} \right)^a \right] \right\|_{H^s(\{x_l\}\times I)}^2 = \left\| \left(\frac{\partial \check{v}_l^p}{\partial x} \right)^a (0, t) - \left(\frac{\partial \check{v}_{l-1}^p}{\partial x} \right)^a (1, t) \right\|_{H^s(\{x_l\}\times I)}^2 .$$

Also, the corresponding H^1-norm of v_l^p at initial time $(t = 0)$ of each sub-intervals (Ω_l) is given by:

$$\|v_l^p\|^2_{H^1(\Omega_l\times\{0\})} = \int_{(0,1)\times\{0\}} |\check{v}_l^p|^2 d\xi + \int_{(0,1)\times\{0\}} |D\check{v}_l^p|^2 d\xi = \|\check{v}_l^p\|^2_{H^1((0,1)\times\{0\})}, \tag{3.4}$$

and the L^2-norm of the residual in the PDE with zero data is as follow:

$$\int_{\Omega_l\times(0,1)} \|\mathscr{L} v_l^p\|^2 dxdt = \int_{(0,1)\times(0,1)} \|\mathscr{L}_l \check{v}_l^p\|^2 d\xi\, dt, \tag{3.5}$$

where $\mathscr{L}_l$ is the differential operator $\mathscr{L}$ in ξ and t coordinates. Now we take the orthogonal projection of the coefficients of the differential operator $\mathscr{L}_l$ into the space of polynomials with respect to the usual inner product in $H^{2,1}((0, 1) \times (0, 1))$ and define a new differential operator $\mathscr{L}_l^a$. The coefficients of the differential operator $\mathscr{L}_l^a$ are polynomials of degree p in ξ and of degree q in t. Hence

$$\int_{\Omega_l\times(0,1)} \|\mathscr{L} v_l^p\|^2 dxdt \approx \int_{(0,1)\times(0,1)} \|\mathscr{L}_l^a \check{v}_l^p\|^2 d\xi\, dt, \tag{3.6}$$

up to a negligible error term (see [15,9] for details).

We now state the stability theorem which is needed to formulate the numerical scheme. Define the quadratic form

$$\begin{aligned}\mathscr{V}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) = {} & \sum_{l=-N}^{N-1} \|(\mathscr{L}_l^a \check{v}_l^p)\|^2_{L^2((0,1)\times I)} + \sum_{x_l\in int(I_N)} (\|[\check{v}^p]\|^2_{H^{3/4}(\{x_l\}\times I)} + \|[(\check{v}_x^p)^a]\|^2_{H^{1/4}(\{x_l\}\times I)}) \\ & + \sum_{x_l\in\partial I_N} (\|\check{v}^p\|^2_{H^{3/4}(\{x_l\}\times I)} + \|(\check{v}_x^p)^a\|^2_{H^{1/4}(\{x_l\}\times I)}) + \sum_{l=-N}^{N-1} \|\check{v}_l^p\|^2_{H^1((0,1)\times\{0\})}.\end{aligned} \tag{3.7}$$

Here $int(I_N)$ denotes the interior of I_N and ∂I_N denotes the boundary of I_N, where $I_N = [-N, N]$.

Then, from Theorem 11 in [9], the following result holds.

Theorem 3.1. *There exists a constant C such that the estimate*

$$\sum_{l=-N}^{N-1} \|\check{v}_l^p\|^2_{H^{2,1}((0,1)\times I)} \le C\ (\ln\ p)^2\ \mathscr{V}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) \tag{3.8}$$

holds.

We now define a modified version of the quadratic form, $\mathscr{W}^p(\check{v}_l^p)$, which is given by:

$$\begin{aligned}\mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) = {} & \sum_{l=-N}^{N-1} \|(\mathscr{L}_l^a \check{v}_l^p)\|^2_{L^2((0,1)\times I)} + \sum_{x_l\in int(I_N)\cup\partial I_N} (\|[\check{v}^p]\|^2_{H^{3/4}(\{x_l\}\times I)} + \|[(\check{v}_x^p)^a]\|^2_{H^{1/4}(\{x_l\}\times I)}) \\ & + \sum_{l=-N}^{N-1} \|\check{v}_l^p\|^2_{H^1((0,1)\times\{0\})}.\end{aligned} \tag{3.9}$$

Then, from Theorem 3.1, the following result follows immediately.

Theorem 3.2. *There exists a constant C such that the estimate*

$$\sum_{l=-N}^{N-1} \|\check{v}_l^p\|^2_{H^{2,1}((0,1)\times I)} \le C\ (\ln\ p)^2\ \mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) \tag{3.10}$$

holds.

4. Numerical scheme and parallelization

Let $(g_\delta)_l(\xi) = g_\delta(M_l(\xi))$, where g_δ is as defined in (2.9) and let $(\tilde{g}_\delta)_l(\xi)$ be the orthogonal projection of $g_\delta(\xi)$ into the space of polynomials of degree p in ξ with respect to the usual inner product in $H^1(S)$.

Following definition (3.1), we define our approximate solution to be the unique ω^p (where $\omega^p = \{\omega_l^p\}_{l=-N}^{N-1}$ in $\Omega_l \times I$ and zero for $(I_N)^c \times I$) which minimizes the functional

$$\begin{aligned}\mathscr{R}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) = {} & \sum_{l=-N}^{N-1} \|(\mathscr{L}_l^a \check{v}_l^p)\|^2_{L^2((0,1)\times I)} + \sum_{l=-N}^{N} (\|[\check{v}^p]\|^2_{H^{3/4}(\{x_l\}\times I)} + \|[(\check{v}_x^p)^a]\|^2_{H^{1/4}(\{x_l\}\times I)}) \\ & + \sum_{l=-N}^{N-1} \|(\check{v}_l^p - (\tilde{g}_\delta)_l)\|^2_{H^1((0,1)\times\{0\})},\end{aligned} \tag{4.1}$$

over all $\{\check{v}_l^p\}_{-N\le l\le N-1}$.

The mollified IVP, which is defined in Eq. (2.9), is as follows:

$$\mathscr{L} v_\delta = 0 \quad \text{in } \Omega \times I, \tag{4.2}$$
$$v_\delta = g_\delta \quad \text{on } \Omega \times \{0\}. \tag{4.3}$$

Clearly, $g_\delta \in \mathscr{D}_1(\Omega \times \{0\})$. Hence $v_\delta \in \mathscr{D}_{2,1}(\Omega \times I)$. Now, for $t = 0$, we get

$$v_\delta(x, 0) = g_\delta(x) = g(x) * \theta_{N/\delta}(x), \tag{4.4}$$

with the following boundedness property:

$$\|v_\delta\|_{H^s(\Omega\times\{0\})} = \|g(x) * \theta_{N/\delta}(x)\|_{H^s(\Omega)} \le \|g\|_{L^1(\Omega)} \|\theta_{N/\delta}(x)\|_{H^s(\Omega)}. \tag{4.5}$$

Lemma 4.1. *The estimate*

$$|\theta_{N/\delta}|_s \le B_N\, s! \tag{4.6}$$

holds. Here $\theta_{N/\delta}(x)$ is the Hermite mollifier which is defined in (2.7) and $B_N = N\sqrt{e}\,(N e^3/\beta_1)^{\frac{(\sqrt{N e/\beta_1})}{4}} \sim K N d^{a\sqrt{N}\log N}$, where K, d and a are constants.

Proof. From Eq. (2.7), we know that

$$\theta_{N/\delta}(x) = \frac{N}{2\pi}\Phi_\delta(Nx) = \frac{N}{2\pi\delta}\Phi\left(\frac{Nx}{\delta}\right),$$

where $\delta = \sqrt{\beta_1 N}$.

Let $\phi(\zeta)$ be the Fourier transform of $\Phi(x)$, which is given by:

$$\hat{\Phi}(\zeta) = \phi(\zeta) = e^{-\frac{\zeta^2}{2}}\left(\sum_{j=0}^{N}\frac{\zeta^{2j}}{2^j j!}\right). \tag{4.7}$$

$\phi(\zeta)$ satisfies:

$$|\phi(\zeta)|_s \le \sum_{j=0}^{N}\left\| e^{-\frac{\zeta^2}{2}}\frac{\zeta^{2j+s}}{2^j j!}\right\|_{L^2(\mathbb{R})}. \tag{4.8}$$

The right hand side of Eq. (4.8) is given by:

$$\left\| e^{-\frac{\zeta^2}{2}}\frac{\zeta^{2j+s}}{2^j j!}\right\|_{L^2(\mathbb{R})} = \frac{1}{2^j j!}\left(\int_{-\infty}^{\infty} e^{-\zeta^2}\zeta^{4j+2s}\,d\zeta\right)^{1/2}. \tag{4.9}$$

Substituting $\eta = \zeta^2$, we obtain,

$$\left\| e^{-\frac{\zeta^2}{2}}\frac{\zeta^{2j+s}}{2^j j!}\right\|_{L^2(\mathbb{R})} = \frac{1}{2^j j!}\left(\int_{0}^{\infty} e^{-\eta}\eta^{2j+s-1/2}\,d\eta\right)^{1/2}, \tag{4.10}$$
$$= \frac{1}{2^j j!}\sqrt{\Gamma(2j+s+1/2)}. \tag{4.11}$$

The maximum of $\frac{\sqrt{\Gamma(2j+s+1/2)}}{2^j j!}$ is achieved when $\frac{\sqrt{(2j+s-1/2)}}{2j} \sim 1$. This happens when

$$j \sim \frac{\sqrt{s}}{2}.$$

Hence

$$\frac{|\theta_{N/\delta}|_s}{s!} \le \frac{N}{2^{\frac{\sqrt{s}}{2}}\left(\frac{\sqrt{s}}{2}\right)!}\frac{((\sqrt{s}+s)!)^{1/2}}{s!}(\sqrt{N/\beta_1})^s. \tag{4.12}$$

Let $A_s = \frac{|\theta_{N/\delta}|_s}{s!}$. Since, by Stirling's formula,

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

This leads to the estimate

$$A_s = \frac{|\theta_{N/\delta}|_s}{s!} \sim \frac{N}{2^{\frac{\sqrt{s}}{2}} \left(\frac{\sqrt{s}}{2}\right)^{\frac{\sqrt{s}}{2}} e^{-\frac{\sqrt{s}}{2}}} \frac{((\sqrt{s}+s)^{(\sqrt{s}+s)} e^{-(\sqrt{s}+s)})^{1/2}}{s^s e^{-s}} (\sqrt{N/\beta_1})^s.$$

The above estimate can be rewritten as:

$$A_s \sim \frac{N}{s^{\frac{\sqrt{s}}{4}} e^{-\frac{\sqrt{s}}{2}}} \frac{\left(1+\frac{1}{\sqrt{s}}\right)^{\frac{\sqrt{s}(1+\sqrt{s})}{2}} e^{-\frac{(\sqrt{s}+s)}{2}}}{s^{\frac{s}{2}-\frac{\sqrt{s}}{2}} e^{-s}} (\sqrt{N/\beta_1})^s.$$

Therefore

$$A_s \sim N\sqrt{e}\,(\sqrt{N\,e/(\beta_1\, s)}\,)^s\, e^{\frac{\sqrt{s}}{2}}\, s^{\frac{\sqrt{s}}{4}}. \tag{4.13}$$

Now the maximum of A_s is achieved when $\frac{N\,e}{\beta_1\, s} \sim 1$ or $s \sim \frac{N\,e}{\beta_1}$. Hence, we get

$$A_s \sim N\sqrt{e}\,(e)^{\frac{\sqrt{\frac{N\,e}{\beta_1}}}{2}}\,(N\,e/\beta_1)^{\frac{(\sqrt{N\,e/\beta_1})}{4}}. \tag{4.14}$$

This gives a bound on A_s, as:

$$A_s \le N\sqrt{e}\;(N\,e^3/\beta_1)^{\frac{(\sqrt{N\,e/\beta_1})}{4}} = B_N. \tag{4.15}$$

Hence

$$|\theta_{N/\delta}|_s \le B_N\, s!. \quad \Box \tag{4.16}$$

Using Lemma 4.1, the estimate for Eq. (4.5) is given by:

$$\|g_\delta\|_{H^s(\Omega\times\{0\})} \lesssim B_N\; s! = K\; N\; d^{a\sqrt{N}\log N}\; s!, \tag{4.17}$$

where K, d and a are constants. Here $B_N \lesssim KN\; d^{a\sqrt{N}\log N}$.

Following Lions et al. [10,11], the fundamental results for Gevrey spaces can be written as:

$$\|D_x^i\, D_t^j\; v_\delta\|_{L^2(\Omega\times I)} \lesssim C\; B_N\, i!(j!)^2(\alpha_1)^{i+j}, \tag{4.18}$$

where C and α_1 are constants.

Moreover

$$\|g_\delta\; e^{\mu x}\|_{H^1(\Omega\times\{0\})} \le C, \tag{4.19}$$

for some $\mu > 0$. This means that

$$\|v_\delta\; e^{\mu x}\|_{H^1(\Omega\times\{0\})} \le C. \tag{4.20}$$

Further

$$\|g_\delta\; e^{-\mu x}\|_{H^1(\Omega\times\{0\})} \le C. \tag{4.21}$$

Hence, we can conclude that

$$\|v_\delta\; e^{-\mu x}\|_{H^1(\Omega\times\{0\})} \le C. \tag{4.22}$$

Eqs. (4.18), (4.20) and (4.22) lead to the following estimates for the local region and the exterior region:

$$\|v_\delta e^{\mu|x|}\|_{H^{2,1}(\Omega\times I)} \le C \quad \text{and} \quad \|v_\delta\|_{H^{2,1}(I_N^c\times I)} \le Ke^{-\rho N}. \tag{4.23}$$

Here K and ρ are generic constants and $I_N^c = \mathbb{R}\setminus I_N$, $I_N = [-N, N]$.

Let $s_l^p(x,t)$ be the approximate representation of $v_\delta(x,t)$ on Ω_l defined in Theorem 13 in [9] and Theorem 4.2.1 in [15]. Then, we obtain the error estimate

$$\sum_{l=-N}^{N-1} \|v_\delta - s_l^p\|_{H^{2,1}(\Omega_l\times I)} \le K\, e^{-\rho\, N}, \tag{4.24}$$

provided q is proportional to p^2 and N is proportional to p.

Now $s^p(x,t) = 0$ for $(x,t) \in I_N^c \times I$. Then from (4.18) and (4.23),

$$\begin{aligned} \|(v_\delta)_x - (s_l^p)_x\|_{H^{1/4}(\{x_l\}\times I)} &\le K\, e^{-\rho N}, \quad \text{for } x_l = \pm N, \\ \|v_\delta - s_l^p\|_{H^{3/4}(\{x_l\}\times I)} &\le K\, e^{-\rho N}, \quad \text{for } x_l = \pm N. \end{aligned} \tag{4.25}$$

Hence

$$\mathcal{R}^p(\{s_l^p(x,t)\}_{-N\le l\le N-1}) \le K\, e^{-\rho N}, \tag{4.26}$$

provided p is proportional to N and N is large enough.

Moreover, the residual of the approximate solution decays as:

$$\mathcal{R}^p(\{\omega_l^p(x,t)\}_{-N\le l\le N-1}) \le \mathcal{R}^p(\{s_l^p(x,t)\}_{-N\le l\le N-1}) \le K\, e^{-\rho N}. \tag{4.27}$$

Using Eqs. (4.26) and (4.27), we obtain:

$$\mathcal{R}^p(\{(\omega_l^p - s_l^p)(x,t)\}_{-N\le l\le N-1}) \le K\, e^{-\rho N}. \tag{4.28}$$

Therefore by Theorem 3.2, we can conclude that

$$\left(\sum_{l=-N}^{N-1} \|\omega_l^p - s_l^p\|^2_{H^{2,1}(\Omega_l\times I)}\right)^{1/2} \le K\, e^{-\rho N}. \tag{4.29}$$

Moreover, $\omega^p(x,t) = 0$ and $s^p(x,t) = 0$ for $(x,t) \in I_N^c \times I$.

Combining the above with (4.23), (4.24) and (4.29) we obtain

$$\sum_{l=-N}^{N-1} \|v_\delta - \omega_l^p\|_{H^{2,1}(\Omega_l\times I)} + \|v_\delta - \omega^p\|_{H^{2,1}(I_N^c\times I)} \le K\, e^{-\rho N}. \tag{4.30}$$

Here $I_N^c = \mathbb{R} \setminus I_N$ and $I_N = [-N, N]$.

4.1. Symmetric formulation

As defined in (4.1) we choose our approximate solution to be the unique $\{\omega_l^p\}_{l=-N}^{N-1}$ which minimizes the functional $\mathcal{R}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1})$ over all $\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}$. Let the above overdetermined system [9,16] of Eq. (4.1), be of the form

$$AW = G. \tag{4.31}$$

Then the Normal Equations are

$$A^T AW = A^T G, \tag{4.32}$$

where W is a vector assembled from the values of $\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}$, and G is assembled from the data. Here A is a matrix.

Since, our method is a Least-Squares method [17], we use the preconditioned conjugate gradient method (PCGM) for solving the Normal Equations. Now from [16]

$$R^p(U + \epsilon W) = R^p(U) + 2\epsilon (W)^T (SU - TG) + O(\epsilon^2),$$

for all W. Here U is the vector assembled from the values of $\{\omega_l^p\}_{l=-N,N-1}$ and S, T are matrices which contain valuation of integrals in Eq. (4.1) using quadrature rule.

Define

$$U^{p,q}_{l,(p+1)k+i} = \omega_l^p(\xi_i^p, t_k^q) \quad \text{for } 0 \le i \le p,\ 0 \le k \le q.$$

Similarly

$$U^{2p,2q}_{l,(2p+1)k+i} = \omega_l^p(\xi_i^{2p}, t_k^{2q}) \quad \text{for } 0 \le i \le 2p,\ 0 \le k \le 2q.$$

The integrals, which arise in the above minimization formulation, are computed by the Gauss–Lobatto–Legendre (GLL) quadrature formula. Then the minimization formulation is represented as:

$$(V_l^{2p,2q})^T\, O_l^{2p,2q}, \tag{4.33}$$

where $O_l^{2p,2q}$ is a $(2p+1)(2q+1)$ vector which can be easily computed. Now we can always find a matrix $F_l^{p,q}$ such that

$$V_l^{2p,2q} = F_l^{p,q} V_l^{p,q}.$$

Fig. 2. Parallelization.

Thus, expression (4.33) can be rewritten as:

$$(V_l^{2p,2q})^T\ O_l^{2p,2q} = (V_l^{p,q})^T (F_l^{p,q})^T O_l^{2p,2q}. \tag{4.34}$$

Hence the residuals satisfy the relation

$$R^p = (F_l^{p,q})^T O_l^{2p,2q}.$$

Note that, neither is a need to compute any mass and stiffness matrices, (as we can calculate the residuals in the normal equations inexpensively and efficiently) nor do we need to filter the coefficients and data. A detailed description can be found in [15,16].

4.2. Parallelization and preconditioning

From (3.9) and (4.1), we can conclude that the quadratic form $\mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{l=-N}^{N-1})$ (which is defined in (3.9)) is obtained from the functional $\mathscr{R}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1})$ with zero data. For the quadratic form $\mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1})$, we define the preconditioner which is denoted by $\mathscr{U}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1})$, as:

$$\mathscr{U}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) = \sum_{l=-N}^{N-1} \|\check{v}_l^p\|^2_{H^{2,1}((0,1)\times I)}. \tag{4.35}$$

Then from [9] the following result holds:

$$\mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) \le K\ \mathscr{U}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}), \tag{4.36}$$

where K is a constant. By Theorem 3.2, we get the following result:

$$\frac{1}{C(\log p)^2}\mathscr{U}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}) \le\ \mathscr{W}^p(\{\check{v}_l^p(\xi,t)\}_{-N\le l\le N-1}). \tag{4.37}$$

By (4.36) and (4.37), we conclude that the condition number of the preconditioned system is $O((\log p)^2)$. Assume that $\check{v}_l^p$ is defined in terms of Legendre polynomials in ξ, of degree p, and in t, of degree q, for each element Ω_l, $-N \le l \le N-1$. Then $\check{v}_l^p$ can be written as:

$$\check{v}_l^p(\xi,t) = \sum_{i=0}^{p}\sum_{j=0}^{q} a_{i,j} L_i(2\xi-1)L_j(2t-1), \tag{4.38}$$

where the coefficients $a_{i,j}$ are arranged lexicographically in i and j.

Therefore, we obtain a $((p+1)(q+1)\times(p+1)(q+1))$ matrix corresponding to the quadratic form $\|\check{v}_l^p\|^2_{H^{2,1}((0,1)\times I)}$. Using separation of variables technique this preconditioner can be diagonalized in a new set of basis functions which is given in [18]. In Section 3.1, the discretization of the domain has already been discussed. Each element is mapped to a single processor for ease of parallelism. During the PCGM process, communication between neighboring processors is confined to the interchange of information of the value of function and its derivatives at inter-element boundaries on which $\check{v}_l^p$ is defined. Moreover we need to compute two global scalars to update the approximate solution and the search direction. Hence inter-processor communication is quite small [15,16,19] (see Fig. 2).

5. Estimates in negative norms

Lemma 5.1. *Assume g is piecewise analytic in Ω and g_δ is the mollified representation of g such that*

$$g_\delta(x) = (g * \theta_{N/\delta})(x)$$

with

$$\theta_{N/\delta}(x) = \frac{N}{2\pi}\Phi_\delta(Nx) = \frac{N}{2\pi\delta}\Phi(Nx/\delta),$$

where $\delta = \sqrt{\beta_1 N}$. Then the estimate

$$\|g - g_\delta\|_{H^{-2N-2}(\Omega)} \le \frac{C\,\beta_1^{N+1}}{2^{N+1}(N+1)!N^{N+1}} \tag{5.1}$$

holds, where C is a positive constant.

Proof. Let $\phi(\zeta)$ be the Fourier Transform of $\Phi(x)$, which is given by:

$$\hat{\Phi}(\zeta) = \phi(\zeta) = e^{-\frac{\zeta^2}{2}} \left(\sum_{j=0}^{N} \frac{\zeta^{2j}}{2^j j!} \right). \tag{5.2}$$

Now

$$\langle g - g_\delta, \psi \rangle = \langle \hat{g} - \hat{g}_\delta, \hat{\psi} \rangle. \tag{5.3}$$

Using the Convolution Theorem for Fourier transforms, we deduce that

$$\hat{g}_\delta = \widehat{g * \theta_{N/\delta}} = \hat{g}\,\hat{\theta}_{N/\delta}. \tag{5.4}$$

Substituting Eq. (5.4) in Eq. (5.3) and using the property of inner product, we obtain the relation

$$\langle g - g_\delta, \psi \rangle = \langle \hat{g} - \hat{g}\,\hat{\theta}_{N/\delta}, \hat{\psi} \rangle = \langle \hat{g}, (1 - \hat{\theta}_{N/\delta})\hat{\psi} \rangle. \tag{5.5}$$

Here $\hat{\theta}_{N/\delta}(\zeta) = \phi(\delta\zeta/N)$. To estimate Eq. (5.5), we use the Taylor expansion of exponential function, which is given by:

$$e^{\mu} = 1 + \mu + \cdots + \frac{\mu^N}{N!} + e^{\eta} \frac{\mu^{N+1}}{(N+1)!},$$

where $0 \le \eta \le \mu$. Hence

$$\left| 1 - e^{-\mu} \left(1 + \mu + \cdots + \frac{\mu^N}{N!} \right) \right| \le \frac{\mu^{N+1}}{(N+1)!}. \tag{5.6}$$

From Eqs. (5.2) and (5.6), we obtain

$$|1 - \hat{\theta}_{N/\delta}(\zeta)| \le \frac{\beta_1^{N+1} \zeta^{2(N+1)}}{2^{N+1}(N+1)!N^{N+1}}. \tag{5.7}$$

Now, Eqs. (5.5) and (5.7) lead to the following estimate

$$|\langle g - g_\delta, \psi \rangle| = |\langle \hat{g}, (1 - \hat{\theta}_{N/\delta})\hat{\psi} \rangle| \le \|\hat{g}\|_{L^2} \frac{\beta_1^{N+1} \|\zeta^{2(N+1)}\hat{\psi}\|_{L^2}}{2^{N+1}\,(N+1)!\,N^{N+1}} \le \frac{C\,\beta_1^{N+1} |\psi|_{H^{2N+2}}}{2^{N+1}(N+1)!N^{N+1}}. \tag{5.8}$$

Using the above estimate, we obtain the desired result

$$\|g - g_\delta\|_{H^{-2N-2}(\Omega)} \le \sup_{\psi \in H^{2N+2}(\Omega)} \frac{|(g - g_\delta, \psi)|_\Omega}{\|\psi\|_{H^{2N+2}(\Omega)}} \le \frac{C\,\beta_1^{N+1}}{2^{N+1}(N+1)!N^{N+1}}. \quad \square$$

6. Error estimates

In this section we recover point-wise values with spectral accuracy. We use the exponentially accurate mollifier which was proposed by Tanner in his seminal paper [13] and obtain the error estimate for the solution at a point $(x_0, 1)$. Tadmor has also examined the exponentially accurate mollifier in his erudite exposition [12].

Lemma 6.1. *Let $\epsilon = \sqrt{\gamma_1 N}$, where $\gamma_1 = \epsilon_1 d_x$. Here $0 < \epsilon_1 < 1$ and $d_x = 1$. The estimate*

$$|(v - v * \theta_{N/\epsilon})(x_0, 1)| \le Ce^{-\rho N} \tag{6.1}$$

holds. Here C and ρ are constants and $\theta_{N/\epsilon}(x)$ is the Hermite mollifier which is defined in (2.7).

Proof. Note that $v(x,t)$ is analytic for $t>0$ and satisfies [8]

$$D_t^j D_x^\alpha v(x,t) \le \frac{K}{(\sqrt{t})^{\alpha+2j}}(j!)^2\alpha!,$$

where K, j and α are positive numbers.

From Eqs. (2.2) and (2.3), we have

$$\|v(x,t)e^{\mu|x|}\|_{L^2(\Omega\times\{t_0\})} \le C, \tag{6.2}$$

for some $\mu>0$ and all t_0.

We rewrite the left hand side of Eq. (6.1) and apply the triangle inequality, to obtain

$$|v - v*\theta_{N/\epsilon}| = \left| v(x_0,1) - \int_{-\infty}^{\infty} v(x_0-y,1)\Psi_{N/\epsilon}(y)dy \right|, \tag{6.3}$$

$$\le \underbrace{\left| v(x_0,1) - \int_{-\pi}^{\pi} v(x_0-y,1)\theta_{N/\epsilon}(y)dy \right|}_{J_1} + \underbrace{\left| \int_{|y|\ge\pi} v(x_0-y,1)\theta_{N/\epsilon}(y)dy \right|}_{J_2}. \tag{6.4}$$

Adding and subtracting $\int_{-\pi}^{\pi} v(x_0-y,1)\theta_{N/\epsilon}(y)dy$ in J_1, we get

$$J_1 = \underbrace{v(x_0,1) - \int_{-\pi}^{\pi} v(x_0-y,1)\Psi_{N/\epsilon}(y)dy}_{L_1} + \underbrace{\int_{-\pi}^{\pi} v(x_0-y,1)\Psi_{N/\epsilon}(y)dy - \int_{-\pi}^{\pi} v(x_0-y,1)\theta_{N/\epsilon}(y)dy}_{L_2}. \tag{6.5}$$

Here as in [12], we define

$$\Psi_{N/\epsilon}(x) = \frac{N}{2\pi}\sum_{j=-\infty}^{+\infty}\Phi_\epsilon(N(x+2\pi j)). \tag{6.6}$$

Applying the triangle inequality in (6.5), the following estimate holds:

$$|J_1| \le |L_1| + |L_2|.$$

Using the bound on the regularization error (I_1+I_2), in Theorem 11.6 of [12], with

$$I_1 = \left| \int_{\epsilon_1 d_x \le |y| \le \pi} \Psi_{N/\epsilon}(y)\,(v(x_0,1) - v(x_0-y,1))\,dy \right| \le Ce^{-\eta N}$$

and

$$I_2 = \left| \int_{|y|\le\epsilon_1 d_x} \Psi_{N/\epsilon}(y)\,(v(x_0,1) - v(x_0-y,1))\,dy \right| \le Ce^{-\eta N},$$

we get:

$$|L_1| \lesssim Ce^{-\eta N}, \tag{6.7}$$

where η is a positive constant.

Similarly, L_2 in Eq. (6.5), can be estimated by the process given below.

$$L_2 = \int_{-\pi}^{\pi} v(x_0-y,1)\Psi_{N/\epsilon}(y)dy - \int_{-\pi}^{\pi} v(x_0-y,1)\theta_{N/\epsilon}(y)dy,$$

$$= \int_{-\pi}^{\pi} v(x_0-y,1)\left(\sum_{j=-\infty, j\ne 0}^{j=+\infty} \frac{N}{2\pi\epsilon}\Phi(N(y+2\pi j)/\epsilon)\right) dy. \tag{6.8}$$

Following Eq. (2.14a) of Lemma 2.2 in [12], we can deduce that:

$$\sum_{j=-\infty, j\ne 0}^{j=+\infty} \left| \frac{N}{2\pi\epsilon}\Phi(N(y+2\pi j)/\epsilon) \right| \lesssim \frac{2^P}{\epsilon}\sum_{j=1}^{\infty} e^{-\frac{((2j-1)\pi N)^2}{4\epsilon^2}}.$$

Moreover, we obtain

$$\sum_{j=-\infty, j\ne 0}^{j=+\infty} \left| \frac{N}{2\pi\epsilon}\Phi(N(y+2\pi j)/\epsilon) \right| \lesssim \frac{2^P}{\sqrt{\gamma_1 N}} e^{-\eta_2 N/\gamma_1}, \quad |x|\le\pi.$$

Here $P = \epsilon_1^2 d_x N = \epsilon_1^2 N$. Hence we have $2^P \le \exp(\kappa\epsilon_1^2 N)$ with $\kappa := \log(2)$. Then

$$\sum_{j=-\infty, j\neq 0}^{j=+\infty} \left| \frac{N}{2\pi\epsilon} \Phi(N(y+2\pi j)/\epsilon) \right| \lesssim \frac{1}{\sqrt{\gamma_1 N}} e^{\left(\kappa\epsilon_1^2 N - \frac{\eta_2 N}{\gamma_1}\right)}, \quad |x| \le \pi.$$

Substituting for $\gamma_1 = \epsilon_1 d_x = \epsilon_1$, we get

$$\sum_{j=-\infty, j\neq 0}^{j=+\infty} \left| \frac{N}{2\pi\epsilon} \Phi(N(y+2\pi j)/\epsilon) \right| \lesssim \frac{1}{\sqrt{\gamma_1 N}} e^{\left(\kappa\epsilon_1^2 - \frac{\eta_2}{\epsilon_1}\right)N}, \quad |x| \le \pi. \tag{6.9}$$

For sufficiently small $\epsilon_1 < 1$, the above estimate is exponentially accurate.

Eqs. (6.8) and (6.9) lead to the result:

$$|L_2| \lesssim Ce^{-\eta' N}, \tag{6.10}$$

where η' is a positive constant.

Choose $\rho_1 = \min\{\eta, \eta'\}$. Then, the bound for $|J_1| \lesssim |L_1| + |L_2|$ satisfies

$$|J_1| \lesssim Ce^{-\rho_1 N}. \tag{6.11}$$

By (2.14b) in [12], we have

$$|\theta_{N/\epsilon}(y)| \lesssim \frac{2^P}{\sqrt{\gamma_1 N}} e^{-\eta_1 N} \quad \text{for } |y| \ge \pi, \tag{6.12}$$

where η_1 is a positive constant. Further, it can be shown that:

$$\|v(x,t)\|_{L^1(\Omega\times\{t_0\})} \le \|v(x,t)e^{\mu|x|}\|_{L^2(\Omega\times\{t_0\})} \|e^{-\mu|x|}\|_{L^2(\Omega\times\{t_0\})} \le C. \tag{6.13}$$

Now from (6.12) and (6.13), an estimate for J_2 is obtained as

$$|J_2| \lesssim Ce^{-\rho_2 N}, \tag{6.14}$$

with ρ_2, a positive constant.

Choosing $\rho = \min\{\rho_1, \rho_2\}$ and combining Eqs. (6.3), (6.11) and (6.14), the final estimate is as follows:

$$|v - v * \theta_{N/\epsilon}| \lesssim Ce^{-\rho N}. \quad \square \tag{6.15}$$

Lemma 6.2. *The estimate*

$$|(v * \theta_{N/\epsilon} - v_\delta * \theta_{N/\epsilon})(x_0, 1)| \le Ce^{-\rho N}, \tag{6.16}$$

holds. Here C and ρ are constants and $\theta_{N/\epsilon}(x)$ is the Hermite mollifier which is defined in (2.7) and $\epsilon = \sqrt{\gamma_1 N}$.

Proof. To verify the above bound, we define

$$I_2 = v * \theta_{N/\epsilon} - v_\delta * \theta_{N/\epsilon} = (v - v_\delta) * \theta_{N/\epsilon}.$$

Hence, we get

$$|I_2| \le \|v - v_\delta\|_{H^{-2N-2}} \|\theta_{N/\epsilon}\|_{H^{2N+2}}. \tag{6.17}$$

Now, consider the adjoint problem

$$L^*\psi = 0 \quad \text{in } \Omega \times I, \tag{6.18}$$

with initial condition

$$\psi = \theta_{N/\epsilon}(x) = \frac{N}{2\pi\epsilon} \Phi\left(\frac{Nx}{\epsilon}\right) \quad \text{on } \Omega \times \{1\}.$$

Then, the following result follows immediately:

$$(v * \psi)(x_0, 1) = (v * \psi)(x_0, 0).$$

Moreover, we have the relation

$$((v - v_\delta) * \psi)(x_0, 1) = ((g - g_\delta) * \psi)(x_0, 0). \tag{6.19}$$

From Lemma 4.1, the following estimate holds:

$$\|\psi\|_{H^s(\Omega\times\{1\})} \le B_N\ s!\ \alpha_1^s.$$

Here $B_N = C\,N\,\sqrt{e}\,(N\,e^3/\gamma_1)^{\frac{(\sqrt{N\,e/\gamma_1})}{4}} \sim K'Nd'^{a'\sqrt{N}\log N}$.

From this result, as a consequence, we get:

$$\|\psi\|_{H^s(\Omega\times\{0\})} \le CB_N\ s!\ \alpha_1^s, \tag{6.20}$$

for some constants C and α_1. Substituting Eqs. (6.19) and (6.20) in Eq. (6.17), and applying Lemma 5.1, we can deduce that

$$|I_2| \le \|g - g_\delta\|_{H^{-2N-2}}\|\psi\|_{H^{2N+2}} \le \frac{C\,\beta_1^{N+1}}{2^{N+1}(N+1)!N^{N+1}} B_N(2N+2)!\ \alpha_1^{2N+2}. \tag{6.21}$$

Using Stirling's Formula, we obtain the desired result

$$|I_2| \sim C(2\beta_1\alpha_1^2)^{N+1}\ B_N\ \frac{2}{e^{(N+1)}} \sim C_1 e^{-\rho N} \tag{6.22}$$

provided β_1 is small enough and satisfies $\left(\frac{4\beta_1(\alpha_1)^2}{e}\right) < 1$. Here $\delta = \sqrt{\beta_1 N}$ and $\epsilon = \sqrt{\gamma_1 N}$. □

Theorem 6.3. *Define*

$$\omega^p = \omega_l^p \quad \text{in } \Omega_l \times I \text{ for } -N \le l \le N-1, \tag{6.23}$$
$$= 0, \quad \text{otherwise}. \tag{6.24}$$

Let $\epsilon = \sqrt{\gamma_1 N}$, *where* $\gamma_1 = \epsilon_1 d_x$. *Here* $0 < \epsilon_1 < 1$ *and* $d_x = 1$. *If* $v_\delta \in \mathscr{D}_{2,1}(\bar{\Omega} \times [0, 1])$ *then the following error estimate holds*

$$|v(x_0, 1) - (\omega^p * \psi)(x_0, 1)| \le C_1\,e^{-\rho N}, \tag{6.25}$$

for any $x_0 \in I_N = [-N, N]$, *provided q is proportional to* p^2, *as p tends to infinity and N is proportional to p. Here* C_1 *and* ρ *are constants and* $\psi = \theta_{N/\epsilon}(x)$ *is the Hermite mollifier which is defined in* (2.7).

Proof. Firstly, the left hand side of (6.25) is rewritten as follow:

$$|(v - \omega^p * \psi)(x_0, 1)| = |(\underbrace{(v - v * \psi)}_{I_1} + \underbrace{(v * \psi - v_\delta * \psi)}_{I_2} + \underbrace{(v_\delta * \psi - \omega^p * \psi)}_{I_3})(x_0, 1)|. \tag{6.26}$$

Applying triangle inequality, the above estimate satisfies

$$|(v - \omega^p * \psi)(x_0, 1)| \le |I_1| + |I_2| + |I_3|. \tag{6.27}$$

Then, from Lemma 6.1, the following result can be established:

$$|I_1| = |(v - v * \psi)(x_0, 1)| \le C_1\,e^{-\rho N}. \tag{6.28}$$

Using Lemma 6.2, we have

$$|I_2| = |(v * \psi - v_\delta * \psi)(x_0, 1)| = |((v - v_\delta) * \psi)(x_0, 1)| \le C_1 e^{-\rho N}. \tag{6.29}$$

From Eq. (4.30) and Lemma 4.1, the following result holds

$$|I_3| = |((v_\delta - \omega^p) * \psi)(x_0, 1)| \le \|(v_\delta - \omega^p)\|_{L^2}\|\psi\|_{L^2} \le C_1\,e^{-\rho_1 N}. \tag{6.30}$$

Combining Eqs. (6.28)–(6.30), we obtain

$$|(v - \omega^p * \psi)(x_0, 1)| \le C_1\,e^{-\rho N}. \quad \square$$

Now we want to recover point-wise values at an interior point (x_0, t_0) with spectral accuracy. Assume that $\omega^p(x, t) \in \mathscr{D}_{2,1}(O)$, where the set O is

$$O = \{(x, t) : |x - x_0| \le \delta_1, |t - t_0| \le \epsilon_1\} \subseteq \mathbb{R} \times (0, 1).$$

Here we use the Hermite mollifier, which is defined in (2.7), to recover the value in space direction. We use the root exponential accurate mollifier [12] to recover the value in time direction. Define the root exponential accurate mollifier

$$\Theta_{Q,\delta_2}(t) = \frac{1}{\delta_2}\eta_1\left(\frac{t}{\delta_2}\right)D_Q\left(\frac{t}{\delta_2}\right); \qquad \eta_2 := e^{\left(\frac{ct^2}{t^2-\pi^2}\right)}1_{(-\pi,\pi)}(t), \quad c > 0, \tag{6.31}$$

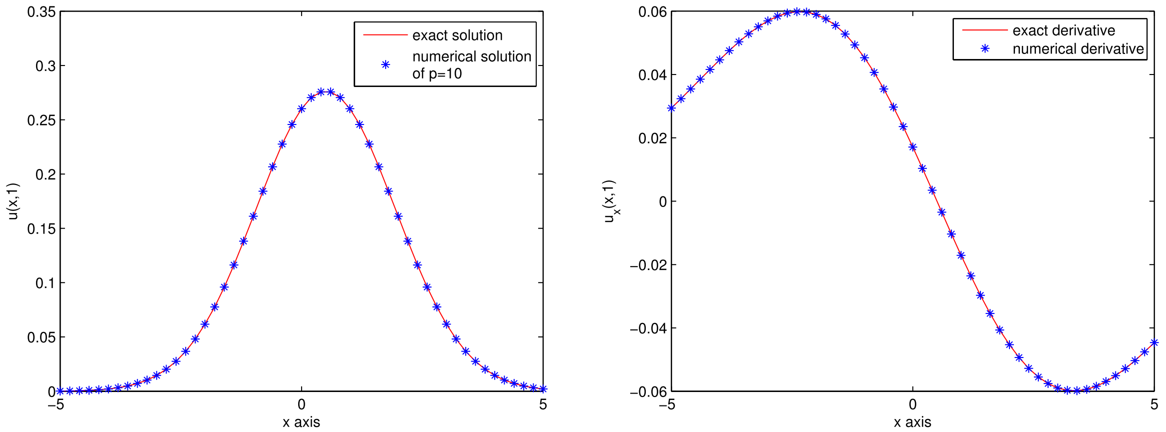

Fig. 3. (Left) Numerical solution and exact solution at $t = 1$, (Right) Derivative (u_x) of numerical solution and exact solution at $t = 1$.

with adaptive parameterization, $\delta_2 = d_t := \frac{1}{\pi} dist\{t, \{0, 1\}\}[\text{mod}\pi]$ and $Q \sim d_t N/\sqrt{e}$. Here $D_Q(t)$ denotes the Dirichlet kernel

$$D_Q(t) := \begin{cases} \dfrac{\sin(Q + 1/2)t}{2\pi \sin(t/2)} & t \neq 2m\pi, \\ 2Q + 1 & t = 2m\pi. \end{cases} \tag{6.32}$$

Now we define the regularized version of ω^p at (x_0, t_0) as:

$$R\omega^p(x_0, t_0) = \int_{-\pi}^{\pi} \int_{-\pi d_t}^{\pi d_t} \theta_{N/\epsilon}(x) \Theta_{Q,\delta_2}(t) \omega^p(x_0 - x, t_0 - t) dxdt,$$

and

$$RD_x^\alpha D_t^j \omega^p(x_0, t_0) = (-1)^{\alpha+j} \int_{-\pi}^{\pi} \int_{-\pi d_t}^{\pi d_t} D_x^\alpha \theta_{N/\epsilon}(x)\ D_t^j \Theta_{Q,\delta_2}(t)\ \omega^p(x_0 - x, t_0 - t) dxdt.$$

Once again it can be shown that this regularized version of $w^p(x_0, t_0)$ approximates $v(x_0, t_0)$ with exponential accuracy.

7. Computational results

The efficacy of the proposed computational strategy is established through numerical examples. All computations have been done on 372-node HPC cluster which is based on *n* Intel Xeon Quadcore processors with a total of 2944 cores and high-speed Infiniband network and it has a peak performance of 34.5 TF. The details of the configuration of Intel Xeon CPU X5570 @ 2.93 GHz are as follows: Number of CPU (Physically)-2, Cores per CPU (Physically and after Hyper-Threading)-4, Total CPU cores (Physically)-8, Number of CPU (after Hyper-Threading)-4, Total CPU cores (after Hyper-Threading)-16, RAM-24 GB, HDD Capacity-2 *X* 500 GB.

Example 7.1 (*Nonsmooth Initial Data*)**.** Consider the problem

$$u_t - u_{xx} = 0 \quad \text{in } \Omega \times (0, 1), \tag{7.1}$$

$$u(x, 0) = f(x) \quad \text{on } \Omega \times \{0\}, \tag{7.2}$$

where

$$f(x) = \begin{cases} 1 & x \in (0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

From the numerical results given in Table 1, Figs. 3–5, it can be seen that the point-wise error of the solution and its derivative decay rapidly with polynomial order *p*. Further, from Table 1 it is observed that the number of iterations, using the PCGM method, increases marginally with *p*, though the computational time increases due to increased matrix size as *p* is increased. This example validates the efficacy of the proposed method (i.e. LSSEM). In the next two examples, the European options problem is dealt with.

Table 1
Point-wise error as function of p.

p	q	Error(1, 1)	Error(0, 1)	Error(−1, 1)	Iterations	No. of cores	CPU (s)
5	25	7.25×10^{-5}	7.34×10^{-5}	6.98×10^{-5}	144	10	1.9
6	36	1.88×10^{-5}	1.87×10^{-5}	9.27×10^{-6}	163	12	3.1
7	49	1.22×10^{-6}	1.14×10^{-6}	9.35×10^{-7}	175	14	26.7
8	64	1.99×10^{-7}	1.99×10^{-7}	9.97×10^{-8}	186	16	36.9
9	81	9.87×10^{-9}	9.75×10^{-9}	8.75×10^{-9}	195	18	49.8
10	100	9.92×10^{-10}	9.92×10^{-10}	8.87×10^{-10}	204	20	61.2

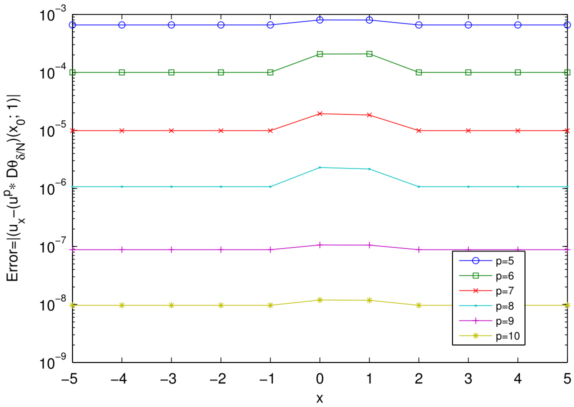

Fig. 4. Point-wise error between derivative of numerical solution and exact solution.

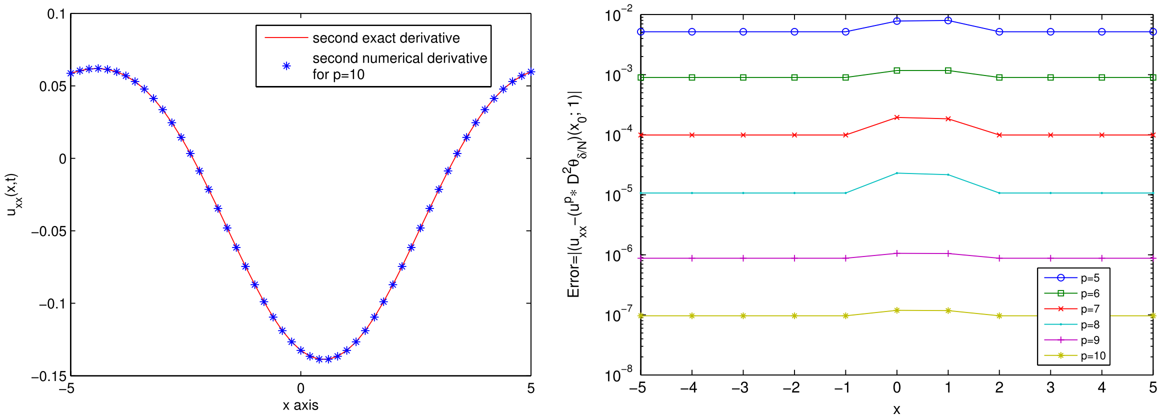

Fig. 5. (Left) Second derivative (u_{xx}) of numerical solution and exact solution at $t = 1$, (Right) Point-wise error between second derivative (u_{xx}) of numerical solution and exact solution.

Example 7.2 (*European Black–Scholes Put Options Problem*)**.** Here a problem of the ''European Black–Scholes Put option'' is considered. The method is used to solve this problem and the results are compared with those due to Zhu et al. [7]. Consider the problem:

$$V_\tau - \frac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S + rV = 0 \quad \text{in } (0, \infty) \times [0, T],$$
$$V(S, 0) = \max(K - S, 0) \quad \text{on } \Omega \times \{0\}.$$

Here V, S, K, r and σ are respectively option price, underlying asset price, strike price, risk-free interest rate and volatility (see Table 2).

The results obtained using the proposed method are given in Table 3. From Table 3, it can be observed that:

1. In order to achieve an accuracy of 10^{-6}, LSSEM requires $p = 6, q = 36$ and the computational time required is only 3.2 s.
2. LSSEM can easily obtain high accuracies. For examples, an accuracy of 10^{-10} is obtained with only $q = 100$.
3. From Figs. 7 and 8, we observe that the errors of derivatives also decay exponentially with polynomial order p.

Table 2
Put option problem: variable value from Zhu et al. [7].

K	r	σ	T
100	0.05	0.15	0.25

Table 3
Put option problem: point-wise error as function of p for LSSEM.

p	q	Error(0, 1)	Error(−1, 1)	Error(−2, 1)	Iteration	No. of cores	CPU (s)
5	25	$6.81 \cdot 10^{-5}$	$5.98 \cdot 10^{-5}$	$5.92 \cdot 10^{-5}$	151	10	2.1
6	36	$6.12 \cdot 10^{-6}$	$5.32 \cdot 10^{-6}$	$5.29 \cdot 10^{-6}$	178	12	3.2
7	49	$5.87 \cdot 10^{-7}$	$5.23 \cdot 10^{-7}$	$5.14 \cdot 10^{-7}$	190	14	27.5
8	64	$5.96 \cdot 10^{-8}$	$4.99 \cdot 10^{-8}$	$4.88 \cdot 10^{-8}$	202	16	38.1
9	81	$6.67 \cdot 10^{-9}$	$5.87 \cdot 10^{-9}$	$5.57 \cdot 10^{-9}$	213	18	51.3
10	100	$6.24 \cdot 10^{-10}$	$5.22 \cdot 10^{-10}$	$5.22 \cdot 10^{-10}$	226	20	61.6

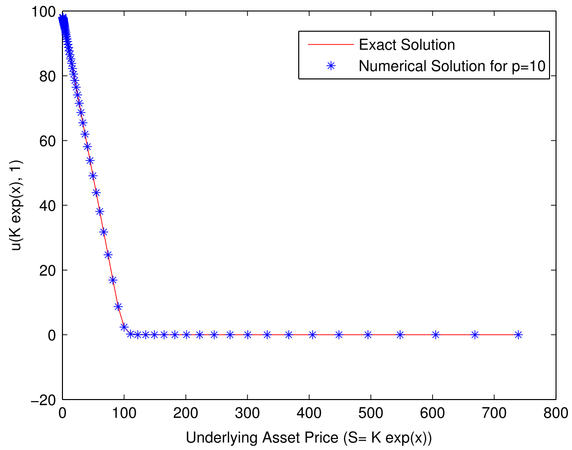

Fig. 6. Numerical solution and exact solution at $t = 1$.

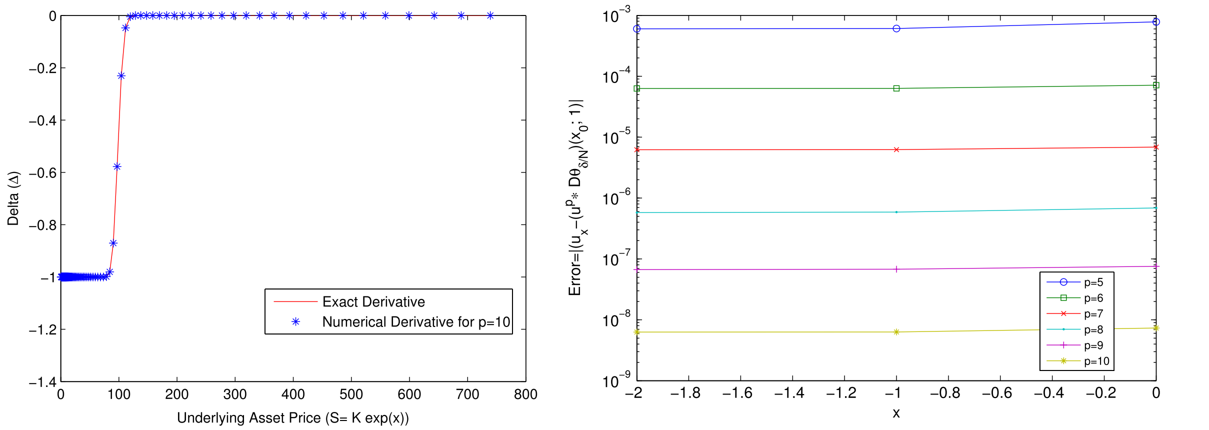

Fig. 7. (Left) Derivative (Δ) of numerical solution and exact solution at $t = 1$, (Right) Point-wise error between derivative (Δ) of numerical solution and exact solution.

4. Number of iterations for PCGM increases marginally with p.
5. LSSEM is exponentially accurate theoretically as well as numerically (see Fig. 6).

Example 7.3 (*European Black–Scholes Call Options Problem*). Usually, in the literature, the "European Black–Scholes Put option" problem is solved. Few researchers, e.g. Bunnin et al. [4] have addressed the "European Black–Scholes Call option" problem. The difficulty is due to an unbounded initial state. In the following the Call option problem is solved and

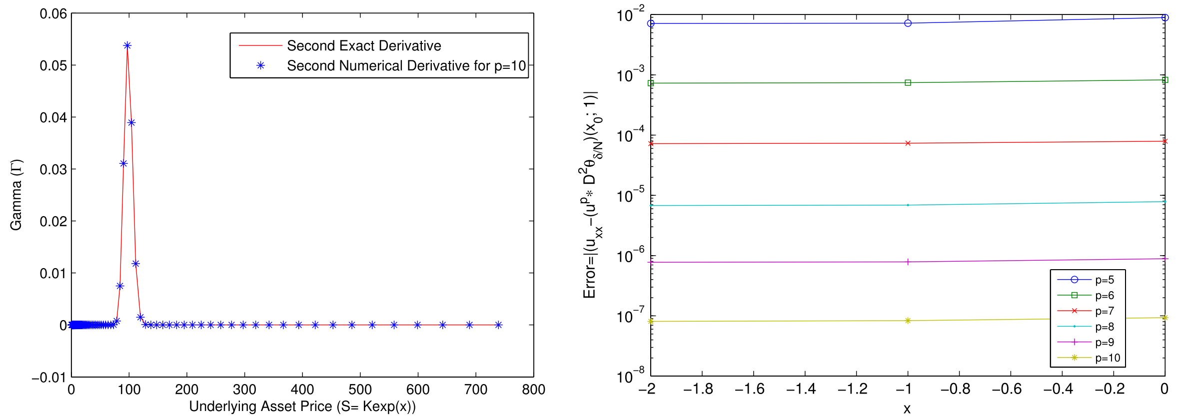

Fig. 8. (Left) Second derivative (Γ) of numerical solution and exact solution at $t = 1$, (Right) Point-wise error between second derivative (Γ) of numerical solution and exact solution.

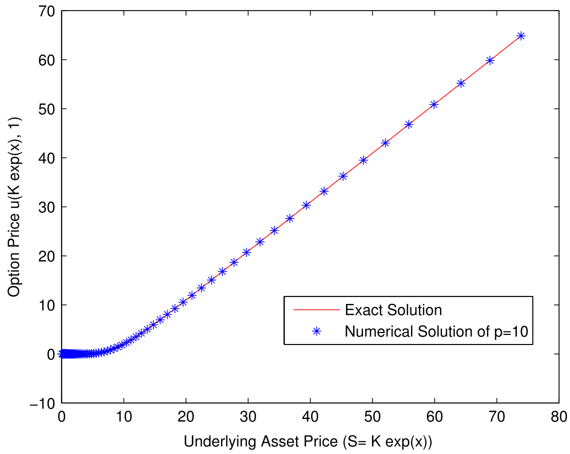

Fig. 9. Numerical solution and exact solution at $t = 1$.

Table 4
Call option problem: variable value from Bunnin et al. [4].

K	r	σ	T
10	0.1	0.4	1

the results are compared with those due to Bunnin et al. [4]. Consider the problem

$$V_\tau - \frac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S + rV = 0 \quad \text{in } (0, \infty) \times [0, T],$$
$$V(S, 0) = \max(S - K, 0) \quad \text{on } \Omega \times \{0\}.$$

Here V, S, K, r and σ are respectively option price, underlying asset price, strike price, risk-free interest rate and volatility (see Table 4).

In Tables 5 and 6 the results are presented. From these results it can be seen that

1. In [4] an accuracy of 10^{-3} is achieved for $N = 100$, while LSSEM achieves an accuracy of 10^{-5} with $p = 5, q = 25$.
2. LSSEM achieves an accuracy of 10^{-10} for $p = 10, q = 100$.
3. In Figs. 10 and 11, the errors of derivatives also decay rapidly.
4. LSSEM achieves exponential accuracy (see Fig. 9).

8. Conclusion

In this paper we have presented a non-conforming least squares spectral element method for Black–Scholes equation. Hermite mollifier has been used to resolve the difficulty of non-smooth initial conditions. We have provided error estimates

Table 5
Call option: point-wise error as function of p for LSSEM.

p	q	Error(1, 1)	Error(2, 1)	Error(3, 1)	Iteration	No. of cores	CPU (s)
5	25	$7.12 \cdot 10^{-5}$	$7.10 \cdot 10^{-5}$	$7.03 \cdot 10^{-5}$	246	10	2.5
6	36	$7.09 \cdot 10^{-6}$	$7.09 \cdot 10^{-6}$	$7.14 \cdot 10^{-6}$	283	12	3.9
7	49	$6.96 \cdot 10^{-7}$	$6.96 \cdot 10^{-7}$	$6.98 \cdot 10^{-7}$	319	14	29.6
8	64	$6.03 \cdot 10^{-8}$	$6.03 \cdot 10^{-8}$	$6.06 \cdot 10^{-8}$	356	16	40.3
9	81	$7.18 \cdot 10^{-9}$	$7.18 \cdot 10^{-9}$	$7.23 \cdot 10^{-9}$	389	18	54
10	100	$7.96 \cdot 10^{-10}$	$7.96 \cdot 10^{-10}$	$7.92 \cdot 10^{-10}$	412	20	65.7

Table 6
CALL option problem: point-wise error, as reported in Bunnin et al. [4].

N	Stock price	Error
100	3	−0.1059
100	6	−0.0021
100	9	0.0020
100	12	0.0012
100	15	0.0014
100	20	0.0043

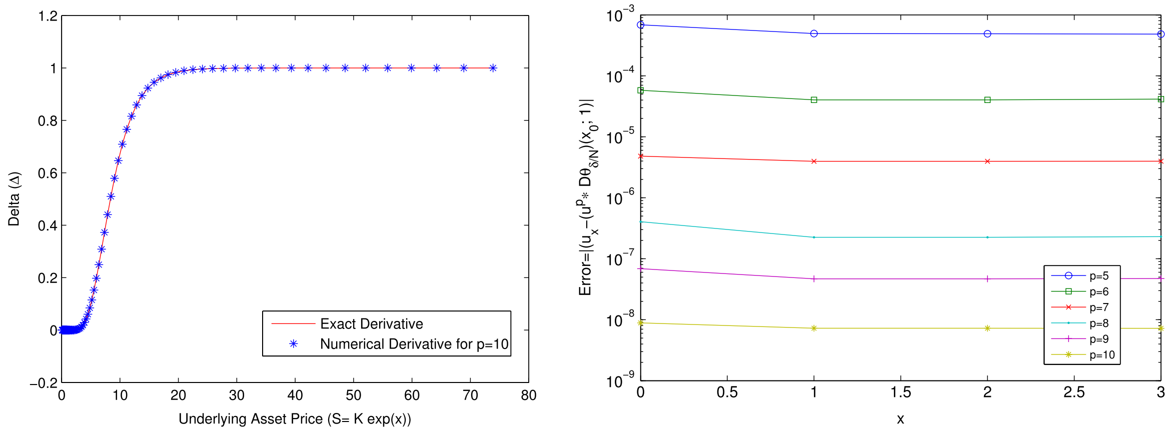

Fig. 10. (Left) Derivative (Δ) of numerical solution and exact solution at $t = 1$, (Right) Point-wise error between derivative (Δ) of numerical solution and exact solution.

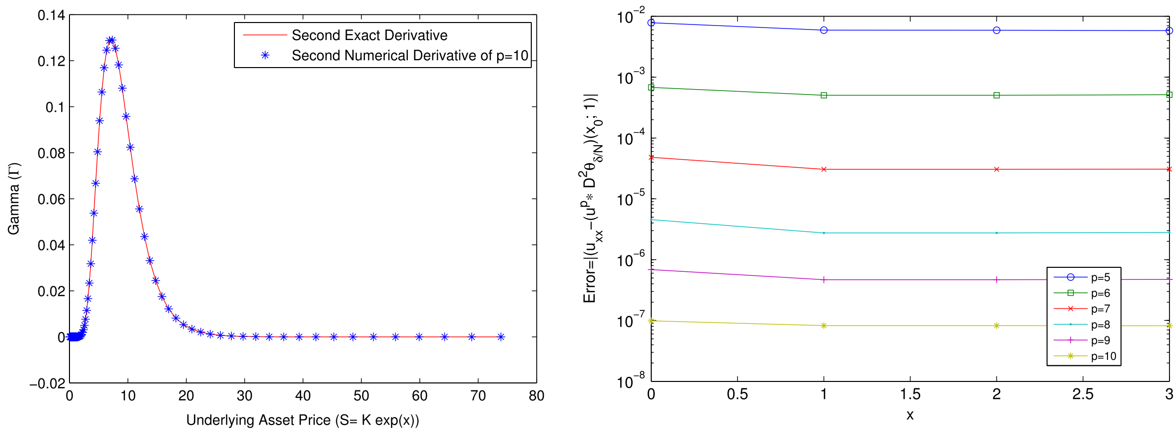

Fig. 11. (Left) Second derivative (Γ) of numerical solution and exact solution at $t = 1$, (Right) Point-wise error between second derivative (Γ) of numerical solution and exact solution.

to establish the exponential accuracy of the method theoretically. Specific numerical examples have been given to validate the error estimate. In the first example we have shown the point-wise exponential accuracy of the proposed method. The second example is the European Black–Scholes Put Option problem. LSSEM can easily obtain very high accuracies. European Black–Scholes Call Option problem has been chosen as the third example. The numerical solution of this problem has been compared to that obtained by Bunnin et al. [4]. Bunnin et al. [4] have achieved a maximum accuracy of 10^{-3} for $N = 100$, while LSSEM achieves an accuracy of 10^{-10} with $p = 10$, $q = 100$. From the three examples, and the theoretical results, it has been demonstrated that LSSEM is an exponentially accurate method in space and time. Further, the method is non-conforming and hence is parallelizable. The LSSEM seems to be superior to any of the existing methods.

The method can also be used to solve jump diffusion problems and higher dimension problems of Options Pricing. We intend to study the application of this method to these problems in future work.

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