n-Kirchhoff Choquard equations with exponential nonlinearity

R. Arora, J. Giacomoni,* T. Mukherjee† and K. Sreenadh‡

Abstract

This article deals with the study of the following Kirchhoff equation with exponential nonlinearity of Choquard type (see (KC) below). We use the variational method in the light of Moser-Trudinger inequality to show the existence of weak solutions to (KC). Moreover, analyzing the fibering maps and minimizing the energy functional over suitable subsets of the Nehari manifold, we prove existence and multiplicity of weak solutions to convex-concave problem $(\mathcal{P}_{\lambda,M})$ below.

Key words: Doubly non local equation, Kirchhoff equation, Choquard nonlinearity with critical growth, Moser-Trudinger inequality, Nehari Manifold.

2010 Mathematics Subject Classification: 35R11, 35R09, 35A15.

1 Introduction

This article is concerned with the study of the following Kirchhoff equation with exponential nonlinearity of Choquard type

$$(KC)\left\{\begin{array}{ll} -m(\int_\Omega |\nabla u|^n\,dx)\Delta_n u = \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u),\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{array}\right.$$

where $\Delta_n u = \nabla.(|\nabla u|^{n-2}\nabla u)$, $\mu \in (0,n)$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, $m : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \Omega\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying suitable assumptions specified in details below. The function F denotes the primitive of f with respect to the second variable (vanishing at 0). We also study the existence and multiplicity of solutions to

*LMAP (UMR E2S-UPPA CNRS 5142) Bat. IPRA, Avenue de l'Université F-64013 Pau, France. email: rakesh.arora@univ-pau.fr, jacques.giacomoni@univ-pau.fr

†Tata Institute of Fundamental Research(TIFR) Centre of Applicable Mathematics, Banglore, India. e-mail: tulimukh@gmail.com

‡Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-110016, India. e-mail: sreenadh@gmail.com

the following Kirchhoff equation with a convex-concave type nonlinearity:

$$(\mathcal{P}_{\lambda,M}) \begin{cases} -m\left(\int_\Omega |\nabla u|^n\, dx\right)\Delta_n u = (|x|^{-\mu} * F(u))f(u) + \lambda h(x)|u|^{q-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases}$$

where $\mu \in (0, n)$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $f(u) = u|u|^p exp(|u|^\beta)$, $0 < q < n-1 < 2n-1 < p+1 = \beta_0 + (n-1)$, $\beta \in \left(1, \frac{n}{n-1}\right)$ and $F(t) = \int_0^t f(s)\, ds$. We assume $m(t) = at + b$ where $a, b > 0$ and $h \in L^r(\Omega)$, with $r = \frac{p+2}{p-q+1}$, satisfying $h^+ \not\equiv 0$.
The main feature of these kind of problems is its doubly-nonlocal structure due to the presence of non-local Kirchhoff and Choquard terms which make the equation (KC) and $(\mathcal{P}_{\lambda,M})$ no longer a pointwise identity. The doubly non-local nature induces some further mathematical difficulties in the use of classical methods of nonlinear analysis.

The study of elliptic equations with nonlinearity having critical exponential growth is related to the following Trudinger-Moser inequality proved in [30]:

Theorem 1.1 *For $n \geq 2$, $u \in W_0^{1,n}(\Omega)$*

$$\sup_{\|u\|\leq 1} \int_\Omega \exp(\alpha |u|^{\frac{n}{n-1}})\, dx < \infty$$

if and only if $\alpha \leq \alpha_n$, where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and $\omega_{n-1} = (n-1)-$ dimensional surface area of $\mathbb{S}^{n-1}$.

The embedding $W_0^{1,n}(\Omega) \ni u \mapsto \exp(|u|^\beta) \in L^1(\Omega)$ is compact for all $\beta \in \left[1, \frac{n}{n-1}\right)$ and is continuous for $\beta = \frac{n}{n-1}$. Consequently the map $T : W_0^{1,n}(\Omega) \to L^q(\Omega)$, for $q \in [1, \infty)$, defined by $T(u) := \exp\left(|u|^{\frac{n}{n-1}}\right)$, is continuous with respect to the norm topology.
The study of Kirchhoff problems was initiated in 1883, when Kirchhoff [20] studied the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\, dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

where ρ, P_0, h, E, L represent physical quantities. This model extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. More general versions of these problems are termed as the Kirchhoff equations and has been extensively studied by researchers till date. Such equations also appear in biological systems where the function u describes a phenomenon which depends on the average of itself (such as a population density). We refer [3, 4] and references therein. We cite [10, 14, 15, 21, 23, 35] as references where the Kirchhoff equations have been treated by variational methods with no attempt to provide the complete list.

On a similar note, researchers have recently payed a lot of attention on nonlocal problems involving the nonlinearity of convolution type. They are termed as Hartree type or the Choquard type nonlinearities. Consider the problem

$$(C): \quad -\Delta u + V(x)u = (|x|^{-\mu} * F(x,u))f(x,u) \text{ in } \mathbb{R}^n$$

where $\mu \in (0,n)$, F is the primitive of f with respect to second variable, V, f are continuous functions satisfying certain assumptions. The starting point of studying such problems was the work of S. Pekar (see [31]) in 1954 where he used such equation to describe the quantum theory of a polaron at rest. Later, P. Choquard (see [24]) in 1976 used it to model an electron trapped in its own hole while a certain approximation to Hartree-Fock theory of component plasma is performed. The problem (C) also appears when we look for standing waves of the nonlinear nonlocal Schrödinger equation which is known to influence the propagation of electromagnetic waves in plasma [8]. Moreover, such problems play a key role in the Bose-Einstein condensation ([11]). For interested readers, we refer the survey paper on Choquard equations by Moroz and Schaftingen [29]. In 2015, Lü [27] studied the following Choquard equation involving Kirchhoff operator

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + (1+\mu g(x))u = \left(|x|^{-\alpha} * |u|^p\right)|u|^{p-2}u \text{ in } \mathbb{R}^3$$

where $a > 0,\ b \geq 0$ are constants, $\alpha \in (0,3), p \in (2, 6-\alpha), \mu > 0$ is a parameter and g is a nonnegative continuous potential satisfying some conditions. By using the Nehari manifold and the concentration compactness principle, he establishes the existence of ground state solutions when μ is large enough and studies the concentration behavior of these solutions as $\mu \to +\infty$. Recently, Li, Gao and Zu [22] studied the existence and the concentration of sign-changing solutions to a class of Kirchhoff-type systems with Hartree-type nonlinearity in $\mathbb{R}^3$ using minimization argument on the sign-changing Nehari manifold and a quantitative deformation lemma. Pucci et al. [32] also studied existence of nonnegative solutions of a Schrödinger-Choquard-Kirchhoff type fractional p-equation via variational methods.

An important question now arises is the case of critical dimension $n = 2$. But there is not much literature concerning problem (C) when $n = 2$ except the articles by Alves et al. [5, 7]. In [5], authors studied a singularly perturbed nonlocal Schrödinger equation using variational methods. We point out that there is no work on Kirchhoff equations involving Choquard equations when $n = 2$ till date. So our work is new in this regard where we have considered the problem with a more general quasilinear elliptic operator, the n-laplace operator, in the dimension $n \geq 2$. As pointed out in the beginning, the critical growth of the nonlinearity in this case is of exponential type, motivated by the Trudinger-Moser inequality. The problem of the type (KC) for $n=2$ without the convolution term, that is

$$-m(\int_\Omega |\nabla u|^2)\Delta u \; dx = f(x,u) \text{ in } \Omega, \;\; u=0 \text{ on } \partial\Omega,$$

was studied by Figueiredo and Severo [16]. This result was later extended for the n-Laplace operator by Goyal et al. in [17]. It is then a natural question to investigate the existence results for a Kirchhoff equation involving a Choquard nonlinearity with exponential growth.

Precisely, in the first part of the present work, we prove Adimurthi [1] type existence results for the n-Kirchhoff Choquard problem (KC) with nonlinearity $f(x,u)$ that has an exponential critical growth and a superlinear behavior at 0. The nonlinear nature of the second order operator $-\Delta_n$ requires to show the pointwise convergence of gradients for the Palais-Smale sequences. For that, we analyze the occurrence of concentration phenomena for any Palais Smale sequence associated to (KC). This concentration compactness analysis is further used to establish the Palais Smale condition for Palais Smale sequences whose energy levels are strictly below some determined critical level. Due to the doubly nonlocal feature of the problem, the task appeals new non trivial estimates with the help of the Riesz potential semi-group property and the Lions' compactness lemma (see Lemma 3.8). Since the energy functional posseses the Moutain pass geometry, we then prove the existence of a Palais Smale sequence with subcritical energy level and consequently the existence of at least one solution to (KC). For that we need crucially that the nonlinearity satisfies a growth condition given by (2.4) (see Lemma 3.3).

Next question that arises is the multiplicity of such Kirchhoff-Choquard equations with exponential nonlinearities. So in the second part of the present work, we study the existence and multiplicity results for problems with an extra n-sublinear sign changing term by using the Nehari manifold techniques. Precisely, we study $(\mathcal{P}_{\lambda,M})$ to obtain in the subcritical case $(\beta < \frac{n}{n-1})$ the multiplicity of the solutions with respect to the parameter λ by extracting the Palais Smale sequences in the natural decomposition of the Nehari Manifold. This requires very accurate estimates on the energy functional restricted to the non-degenerate components of the Nehari manifold. In the critical case $(\beta = \frac{n}{n-1})$, we use again the concentration compactness together with an accurate analysis of the energy levels on the Nehari manifold to determine potential concentration phenomena for associated Palais Smale sequences. Based on this analysis, we show for λ small enough the existence of a relatively compact Palais Smale sequence that yields at least one solution to $(\mathcal{P}_{\lambda,M})$.

During last few decades, several authors such as in [6, 9, 12, 13, 36, 37, 38] used the Nehari manifold and associated fiber maps approach to study the multiplicity results with polynomial type nonlinearity and sign changing weight functions whereas the n-Laplace problems with exponential type nonlinearity has been addressed in [17, 18, 19]. In case of Kirchhoff equations with a Choquard nonlinearity, we highlight that no result is avalaible in the current literature. In this regard, the results proved in the present paper are completely new.

2 Main results

First, we consider the problem (KC). The function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

(m1) There exists $m_0 > 0$ such that $m(t) \geq m_0$ for all $t \geq 0$ and $M(t) = \displaystyle\int_0^t m(s)ds$ satisfies

$$M(t+s) \geq M(t) + M(s), \text{ for all } t, s \geq 0.$$

(m2) There exist constants $b_1, b_2 > 0$ and $\hat{t} > 0$ such that for some $r \in \mathbb{R}$

$$m(t) \leq b_1 + b_2 t^r, \text{ for all } t \geq \hat{t}.$$

(m3) The function $\frac{m(t)}{t}$ is non-increasing for $t > 0$.

Example 1 *An example of a function satisfying (m1), (m2) and (m3) is* $m(t) = m_0 + bt^\beta$ *where* $m_0 > 0$, $\beta < 1$ *and* $b \geq 0$. *Also* $m(t) = m_0 + \log(1+t)$ *with* $m_0 > 0$ *verifies (m1)-(m3).*

Using (m3), one can easily deduce that the function

$$(m3)' \qquad \frac{1}{n}M(t) - \frac{1}{\theta}m(t)t \text{ is non-negative and non-decreasing for } t \geq 0 \text{ and } \theta \geq 2n.$$

The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by $f(x,t) = h(x,t)\exp(|t|^{\frac{n}{n-1}})$. In the frame of problem (KC), $h \in C(\bar{\Omega} \times \mathbb{R})$ satisfies the following conditions:

(h1) $h(x,t) = 0$ for $t \leq 0$ and $h(x,t) > 0$ for $t > 0$.

(h2) For any $\epsilon > 0$, $\lim\limits_{t\to\infty} \sup_{x\in\bar{\Omega}} h(x,t)\exp(-\epsilon|t|^{\frac{n}{n-1}}) = 0$ and $\lim\limits_{t\to\infty} \inf_{x\in\bar{\Omega}} h(x,t)\exp(\epsilon|t|^{\frac{n}{n-1}}) = \infty$.

(h3) There exists $\ell > n-1$ such that $t \to \frac{f(x,t)}{t^\ell}$ is increasing on $\mathbb{R}^+ \setminus \{0\}$, uniformly in $x \in \Omega$.

(h4) There exist $T, T_0 > 0$ and $\gamma_0 > 0$ such that $0 < t^{\gamma_0}F(x,t) \leq T_0 f(x,t)$ for all $|t| \geq T$ and uniformly in $x \in \Omega$.

The condition (h3) implies that $\frac{f(x,t)}{t^{n-1}}$ is increasing for each $t > 0$ and $\lim\limits_{t\to 0^+} \dfrac{f(x,t)}{t^{n-1}} = 0$ uniformly in $x \in \Omega$.

Example 2 *An example of functions satisfying* $(h1)-(h4)$ *is* $f(x,t) = t^{\beta_0+(n-1)}\exp(t^p)\exp(|t|^{\frac{n}{n-1}})$ *for* $t \geq 0$ *and* $f(x,t) = 0$ *for* $t < 0$ *where* $0 \leq p < \frac{n}{n-1}$ *and* $\beta_0 > 0$.

We equip $W_0^{1,n}(\Omega)$ with the natural Banach norm $\|u\| := \left(\int_\Omega |\nabla u|^n \, dx\right)^{1/n}$. Then,

Definition 2.1 *We call a function $u \in W_0^{1,n}(\Omega)$ to be a solution of (KC) if*

$$m(\|u\|^n)\int_\Omega |\nabla u|^{n-2}\nabla u.\nabla\varphi \, dx = \int_\Omega \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u)\varphi \, dx, \text{ for all } \varphi \in W_0^{1,n}(\Omega).$$

The energy functional $E : W_0^{1,n}(\Omega) \to \mathbb{R}$ associated to (KC) is given by

$$E(u) = \frac{1}{n}M(\|u\|^n) - \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) F(x,u) \, dx.$$

Under the assumptions on f, we get that for any $\epsilon > 0$, $p \geq 1$ and $0 \leq \beta_0 < \ell$, there exists $C(\epsilon, n, \mu) > 0$ such that for each $x \in \Omega$

$$|F(x,t)| \leq \epsilon |t|^{\beta_0+1} + C(\epsilon, n, \mu)|t|^p \exp((1+\epsilon)|t|^{\frac{n}{n-1}}), \text{ for all } t \in \mathbb{R}. \tag{2.1}$$

For any $u \in W_0^{1,n}(\Omega)$, by virtue of Sobolev embedding we get that $u \in L^q(\Omega)$ for all $q \in [1,\infty)$. This also implies that

$$F(x,u) \in L^q(\Omega) \text{ for any } q \geq 1. \tag{2.2}$$

Now we recall the well known Hardy-Littlewood-Sobolev inequality.

Proposition 2.2 ***(Hardy-Littlewood-Sobolev inequality)*** *[25, Theorem 4.3, p.106] Let $t, r > 1$ and $0 < \mu < n$ with $1/t + \mu/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t,n,\mu,r)$, independent of f, h such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(t,n,\mu,r)\|f\|_{L^t(\mathbb{R}^n)}\|h\|_{L^r(\mathbb{R}^n)}. \tag{2.3}$$

If $t = r = \frac{2n}{2n-\mu}$ then

$$C(t,n,\mu,r) = C(n,\mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{n}{2}-\frac{\mu}{2}\right)}{\Gamma\left(n-\frac{\mu}{2}\right)} \left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right\}^{-1+\frac{\mu}{n}}.$$

In this case there is equality in (2.3) if and only if $f \equiv (constant)h$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{\frac{-(2n-\mu)}{2}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^n$.

Taking $t = r = \frac{2n}{2n-\mu}$ in Proposition 2.2 and using (2.2), we get that E is well defined. Also $E \in C^1(W_0^{1,n}(\Omega), \mathbb{R})$. Naturally, the critical points of E corresponds to weak solutions of (KC) and for any $u \in W_0^{1,n}(\Omega)$ we have

$$\langle E'(u), \varphi\rangle = m(\|u\|^n)\int_\Omega |\nabla u|^{n-2}\nabla u.\nabla\varphi \, dx - \int_\Omega \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u)\varphi \, dx$$

for all $\varphi \in W_0^{1,n}(\Omega)$. The following theorem is the main result concerning (KC) proved in this article.

Theorem 2.3 *Assume (m1)-(m3) and (h1)-(h4) holds. Assume in addition*

$$\lim_{s\to+\infty} \frac{sf(x,s)F(x,s)}{\exp\left(2|s|^{\frac{n}{n-1}}\right)} = \infty, \text{ uniformly in } x \in \overline{\Omega}. \tag{2.4}$$

Then the problem (KC) admits a weak solution.

Example 3 *f defined by $f(x,t) = g(x)t^p \exp(t^{\frac{n}{n-1}})$ for $t \geq 0$, $x \in \Omega$ with $0 \not\equiv g \in L^\infty(\Omega)$ non-negative and $p > n-1$ satisfy (h1)-(h4) and* (2.4).

We also study the existence of positive solutions to the perturbed quasilinear Kirchhoff equation $(\mathcal{P}_{\lambda,M})$ defined above. Using the Nehari manifold technique, we show existence and multiplicity of solutions with respect to the parameter λ. Precisely, we show the following main results in the subcritical and critical case:

Theorem 2.4 *Let $\beta \in \left(1, \frac{n}{n-1}\right)$. Then there exists λ_0 such that $(\mathcal{P}_{\lambda,M})$ admits at least two solutions for $\lambda \in (0, \lambda_0)$.*

In the critical case, we show the following existence result.

Theorem 2.5 *Let $\beta = \frac{n}{n-1}$, then there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $(\mathcal{P}_{\lambda,M})$ admits a solution.*

Motivated by the famous Ambrosetti-Brezis-Cerami result (see [2]), a natural question is the (local) multiplicity of weak solutions to $(\mathcal{P}_{\lambda,M})$ in the critical case. Accurate estimates of the energy levels on the non degenerate and closed component $N^-_{\lambda,M}$ are required to get the compactness of Palais Smale sequences. Due to the double non local nature of problem $(\mathcal{P}_{\lambda,M})$, we believe that this task appeals new kind of testing functions.

3 Existence of a positive weak solution to (KC)

In this section, we study problem (KC) and for that we use the mountain pass Theorem and analyze accurately the compactness of Palais Smale sequences for E. First we show that the energy functional E possesses the mountain pass geometry.

Lemma 3.1 *Assume the assumptions (m1), (m2) and (h1)-(h4). Then, E has the mountain pass geometry around* 0.

Proof. Let $u \in W_0^{1,n}(\Omega)$ such that $\|u\|$ small enough. Let $0 < \beta_0 < \ell$. Then from Proposition 2.2, (h3) and (2.1), for any $\epsilon > 0$ and $p > 1$ we know that there exists a $C(\epsilon) > 0$ such that

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu} dy\right) F(x,u)\, dx \leq C(n,\mu)\|F(x,u)\|^2_{L^{\frac{2n}{2n-\mu}}(\Omega)} \\
&\leq C(n,\mu) 2^{\frac{2n}{2n-\mu}} \left(\epsilon \int_\Omega |u|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C(\epsilon)\int_\Omega |u|^{\frac{2pn}{2n-\mu}} \exp\left(\frac{2n(1+\epsilon)}{2n-\mu}|u|^{\frac{n}{n-1}}\right)\right)^{\frac{2n-\mu}{n}} \\
&\leq C_1 \left(\epsilon \int_\Omega |u|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C_2(\epsilon)\|u\|^{\frac{2pn}{2n-\mu}} \left(\int_\Omega \exp\left(\frac{4n(1+\epsilon)\|u\|^{\frac{n}{n-1}}}{2n-\mu}\left(\frac{|u|}{\|u\|}\right)^{\frac{n}{n-1}}\right)\right)^{\frac{1}{2}}\right)^{\frac{2n-\mu}{n}}
\end{aligned} \tag{3.1}$$

where we used Sobolev and Hölder inequality. So if we choose $\epsilon > 0$ small enough and u such that $\dfrac{4n(1+\epsilon)\|u\|^{\frac{n}{n-1}}}{2n-\mu} \leq \alpha_n$ then using Theorem 1.1 in (3.1) we get

$$\begin{aligned}
\int_\Omega \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu} dy\right) F(x,u)\, dx &\leq C_3 \left(\epsilon \|u\|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C(\epsilon)\|u\|^{\frac{2pn}{2n-\mu}}\right)^{\frac{2n-\mu}{n}} \\
&\leq C_4 \left(\epsilon \|u\|^{2(\beta_0+1)} + C(\epsilon)\|u\|^{2p}\right).
\end{aligned}$$

Hence from (m1) and above estimate, we deduce that for $\|u\| = \rho$ where $\rho < \left(\frac{\alpha_n(2n-\mu)}{4n(1+\epsilon)}\right)^{\frac{n-1}{n}}$

$$E(u) \geq m_0 \frac{\|u\|^n}{n} - C_4 \left(\epsilon \|u\|^{2(\beta_0+1)} + C(\epsilon)\|u\|^{2p}\right).$$

Taking $\beta_0 > 0$ such that $2(\beta_0 + 1) > n$ and $2p > n$, we can choose ρ small enough so that $E(u) \geq \sigma$ for some $\sigma > 0$ (depending on ρ) when $\|u\| = \rho$. Furthermore, under the assumption (m2), for some $a_1,\ a_2 > 0$ and $t_0 > 0$ we have $m(t) \leq a_1 + a_2 t^r$ and

$$M(t) \leq \begin{cases} a_0 + a_1 t + \dfrac{a_2 t^{r+1}}{r+1},\ r \neq -1 \\ a_0 + a_1 t + a_2 \ln t,\ r = -1 \end{cases}$$

when $t \geq \hat{t}$ and where

$$a_0 = \begin{cases} M(t_0) - a_1 t_0 - a_2 \dfrac{t_0^{r+1}}{r+1},\ r \neq -1 \\ M(t_0) - a_1 t_0 - a_2 \ln t_0,\ r = -1. \end{cases}$$

Let $u_0 \in W_0^{1,n}(\Omega)$ such that $u_0 \geq 0$ and $\|u_0\| = 1$. Then (h3) implies that there exists $K_1 \geq \max\{\frac{n}{2}, \frac{n(r+1)}{2}\}$ such that $F(x,s) \geq C_1 s^{K_1} - C_2$ for all $(x,s) \in \Omega \times [0,\infty)$ and for some positive constants C_1 and C_2. Using this, we obtain

$$\begin{aligned}\int_\Omega \left(\int_\Omega \frac{F(y,tu_0)}{|x-y|^\mu}dy\right) F(x,tu_0)\ dx &\geq \int_\Omega\int_\Omega \frac{(C_1(tu_0)^{K_1}(y)-C_2)(C_1(tu_0)^{K_1}(x)-C_2)}{|x-y|^\mu}\ dxdy\\ &= C_1^2t^{2K_1}\int_\Omega\int_\Omega \frac{u_0^{K_1}(y)u_0^{K_1}(x)}{|x-y|^\mu}\ dxdy\\ &\quad - 2C_1C_2t^{K_1}\int_\Omega\int_\Omega \frac{u_0^{K_1}(y)}{|x-y|^\mu}\ dxdy + C_2^2\int_\Omega\int_\Omega |x-y|^{-\mu}\ dxdy.\end{aligned}$$

Therefore from above we obtain

$$\begin{aligned}E(tu_0) &\leq \frac{M(\|tu_0\|^n)}{n} - \int_\Omega \left(\int_\Omega \frac{F(y,tu_0)}{|x-y|^\mu}dy\right) F(x,tu_0)\ dx\\ &\leq C_3 + C_4t^n + C_5t^{n(r+1)} - C_4t^{2K_1} + C_6t^{K_1}\end{aligned}$$

where $C_i's$ are positive constants for $i=4,5,6$. This implies that $E(tu_0)\to-\infty$ as $t\to\infty$. Thus there exists a $v_0\in W_0^{1,n}(\Omega)$ with $\|v_0\|>\sigma$ such that $E(v_0)<0$. ∎

Lemma 3.2 *Every Palais Smale sequence is bounded in* $W_0^{1,n}(\Omega)$.

Proof. Let $\{u_k\}\subset W_0^{1,n}(\Omega)$ denotes a $(PS)_c$ sequence of E that is

$$E(u_k)\to c \text{ and } E'(u_k)\to 0 \text{ as } k\to\infty$$

for some $c\in\mathbb{R}$. This implies

$$\begin{aligned}&\frac{M(\|u_k\|^n)}{n} - \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) F(x,u_k)\ dx \to c \text{ as } k\to\infty,\\ &\left| m(\|u_k\|^n)\int_\Omega |\nabla u_k|^{n-2}\nabla u_k\nabla\phi - \int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\phi\ dx\right| \leq \epsilon_k\|\phi\|\end{aligned} \tag{3.2}$$

where $\epsilon_k\to 0$ as $k\to\infty$. In particular, taking $\phi=u_k$ we get

$$\left| m(\|u_k\|^n)\int_\Omega |\nabla u_k|^n - \int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(u_k)u_k\ dx\right| \leq \epsilon_k\|u_k\|. \tag{3.3}$$

From the assumption (h3), there exists $\alpha>n$ such that $\alpha F(x,t)\leq tf(x,t)$ for any $t>0$ and $x\in\Omega$ which yields

$$\alpha\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) F(u_k)\ dx \leq \int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(u_k)u_k\ dx. \tag{3.4}$$

Using (3.2), (3.3) along with above inequality and $(m3)'$, we get

$$\begin{aligned}&E(u_k) - \frac{1}{2\alpha}\langle E'(u_k),u_k\rangle = \frac{M(\|u_k\|^n)}{n} - \frac{m(\|u_k\|^n)\|u_k\|^n}{2\alpha}\\ &\quad -\frac{1}{2}\left(\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) F(x,u_k)\ dx - \frac{1}{\alpha}\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)u_k\ dx\right)\\ &\geq \frac{M(\|u_k\|^n)}{n} - \frac{m(\|u_k\|^n)\|u_k\|^n}{2\alpha} \geq \left(\frac{1}{2n}-\frac{1}{2\alpha}\right) m(\|u_k\|^n)\|u_k\|^n \geq \left(\frac{1}{2n}-\frac{1}{2\alpha}\right) m_0\|u_k\|^n.\end{aligned} \tag{3.5}$$

Also from (3.2) and (3.3) it follows that

$$E(u_k) - \frac{1}{2\alpha}\langle E'(u_k), u_k\rangle \leq C\left(1 + \epsilon_k \frac{\|u_k\|}{2\alpha}\right) \tag{3.6}$$

for some constant $C > 0$. Therefore from (3.5) and (3.6) we get that

$$\left(\frac{1}{2n} - \frac{1}{2\alpha}\right) m_0 \|u_k\|^n \leq C\left(1 + \epsilon_k \frac{\|u_k\|}{2\alpha}\right).$$

This implies that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$. ■

Let $\Gamma = \{\gamma \in C([0,1], W_0^{1,n}(\Omega)) : \ \gamma(0) = 0, \ E(\gamma(1)) < 0\}$ and define the Mountain Pass critical level as

$$l^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)). \tag{3.7}$$

Then we have the following result:

Lemma 3.3 *If* (2.4) *holds, then*

$$0 < l^* < \frac{1}{n} M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right).$$

Proof. Since for $u \not\equiv 0$, $E(tu) \to -\infty$ as $t \to \infty$ (as we proved in Lemma 3.1) and since by definition $l^* \leq \max_{t\in[0,1]} E(tu)$ for $u \in W_0^{1,n}(\Omega)\backslash\{0\}$ satisfying $E(u) < 0$, it is enough to show that there exists a $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$ and

$$\max_{t\in[0,\infty)} E(tw) < \frac{1}{n} M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right).$$

To prove this, we consider the sequence of Moser functions $\{w_k\}$ defined as

$$w_k(x) = \frac{1}{\omega_{n-1}^{\frac{1}{n}}} \begin{cases} (\log k)^{\frac{n-1}{n}}, \ 0 \leq |x| \leq \frac{\rho}{k}, \\ \frac{\log\left(\frac{\rho}{|x|}\right)}{(\log k)^{\frac{1}{n}}}, \ \frac{\rho}{k} \leq |x| \leq \rho, \\ 0, \ |x| \geq \rho \end{cases}$$

so that $\text{supp}(w_k) \subset B_\rho(0)$. It is easy to verify that $\|w_k\| = 1$ for all k. So we claim that there exists a $k \in \mathbb{N}$ such that

$$\max_{t\in[0,\infty)} E(tw_k) < \frac{1}{n} M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right).$$

Suppose this is not true then for all $k \in \mathbb{N}$ there exists a $t_k > 0$ such that

$$\begin{aligned} &\max_{t\in[0,\infty)} E(tw_k) = E(t_k w_k) \geq \frac{1}{n} M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right) \\ &\text{and } \frac{d}{dt}(E(tw_k))|_{t=t_k} = 0. \end{aligned} \tag{3.8}$$

From the proof of Lemma 3.2, $E(tw_k) \to -\infty$ as $t \to \infty$ uniformly in k. Then we infer that $\{t_k\}$ must be a bounded sequence in $\mathbb{R}$. From (3.8) and definition of $E(t_k w_k)$ we obtain

$$\frac{1}{n}M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right) < \frac{M(t_k^n)}{n}. \tag{3.9}$$

Since M is monotone increasing, from (3.9) we get that

$$t_k^n \geq \left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}. \tag{3.10}$$

From (3.10), we get

$$\frac{t_k}{\omega_{n-1}^{\frac{1}{n}}}(\log k)^{\frac{n-1}{n}} \to \infty \text{ as } k \to \infty. \tag{3.11}$$

Furthermore from (3.8), we have

$$\begin{aligned} m(t_k^n)t_k^n &= \int_\Omega \left(\int_\Omega \frac{F(y,t_k w_k)}{|x-y|^\mu}dy\right) f(x,t_k w_k)t_k w_k \; dx \\ &\geq \int_{B_{\rho/k}} f(x,t_k w_k)t_k w_k \int_{B_{\rho/k}} \frac{F(y,t_k w_k)}{|x-y|^\mu} \; dy \; dx. \end{aligned} \tag{3.12}$$

In addition, as in equation (2.11) p. 1943 in [5], it is easy to get that

$$\int_{B_{\rho/k}}\int_{B_{\rho/k}} \frac{dxdy}{|x-y|^\mu} \geq C_{\mu,n}\left(\frac{\rho}{k}\right)^{2n-\mu}$$

where $C_{\mu,n}$ is a positive constant depending on μ and n. From (2.4), we know that for each $d > 0$ there exists a s_d such that

$$sf(x,s)F(x,s) \geq d\exp\left(2|s|^{\frac{n}{n-1}}\right), \text{ whenever } s \geq s_d.$$

Since (3.11) holds, we can choose a $r_d \in \mathbb{N}$ such that

$$\frac{t_k}{\omega_{n-1}^{\frac{1}{n}}}(\log k)^{\frac{n-1}{n}} \geq s_d, \text{ for all } k \geq r_d.$$

Using these estimates in (3.12) and from (3.10), for d large enough we get that

$$m(t_k^n)t_k^n \geq d\exp\left((\log k)\left(\frac{2t_k^{\frac{n}{n-1}}}{\omega_{n-1}^{\frac{1}{n-1}}}\right)\right) C_{\mu,n}\left(\frac{\rho}{k}\right)^{2n-\mu} \geq dC_{\mu,n}\rho^{2n-\mu}.$$

Taking d large enough and since t_k^n is bounded, we arrive at a contradiction. This establishes our claim and we conclude the proof of the result. ■

Lemma 3.4 *If $\{u_k\}$ denotes a Palais Smale sequence then up to a subsequence, there exists $u \in W_0^{1,n}(\Omega)$ such that*

$$|\nabla u_k|^{n-2}\nabla u_k \rightharpoonup |\nabla u|^{n-2}\nabla u \textit{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n. \tag{3.13}$$

Proof. From Lemma 3.2, we know that the sequence $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$. Consequently, up to a subsequence, there exists $u \in W_0^{1,n}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$ and strongly in $L^q(\Omega)$ for any $q \in [1,\infty)$ as $k \to \infty$. Also still up to a subsequence we can assume $u_k(x) \to u(x)$ pointwise a.e. for $x \in \Omega$. Therefore the sequence $\{|\nabla u_k|^{n-2}\nabla u_k\}$ must be bounded in $(L^{\frac{n}{n-1}}(\Omega))^n$ whereas $|\nabla u|^n$ is bounded in $L^1(\Omega)$. So we use that there exists a non-negative radon measure ν such that up to a subsequence

$$|\nabla u_k|^n \to \nu \text{ in } (C(\overline{\Omega}))^* \text{ as } k \to \infty.$$

Moreover there exists $v \in (L^{\frac{n}{n-1}}(\Omega))^n$ such that,

$$|\nabla u_k|^{n-2}\nabla u_k \to v \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n \text{ as } k \to \infty.$$

Claim : $v = |\nabla u|^{n-2}\nabla u$.

To prove this, we set $\sigma > 0$ and $X_\sigma = \{x \in \overline{\Omega} : \ \nu(B_r(x) \cap \overline{\Omega}) \geq \sigma, \text{ for all } r > 0\}$. Then X_σ must be a finite set. Because if not, then there exists a sequence of distinct points $\{x_k\}$ in X_σ such that for all $r > 0$, $\nu(B_r(x_k) \cap \overline{\Omega}) \geq \sigma$ for all k. This implies that $\nu(\{x_k\}) \geq \sigma$ for all k, hence $\nu(X_\sigma) = +\infty$. But this is a contradiction to

$$\nu(X_\sigma) = \lim_{k\to\infty} \int_{X_\sigma} |\nabla u_k|^n \ dx \leq C.$$

So let $X_\sigma = \{x_1, x_2, \ldots, x_m\}$. Next, we claim that if we take $\sigma > 0$ such that $\sigma^{\frac{1}{n-1}} < \frac{2n-\mu}{2n}\alpha_n$, the for any K compact subset of $\overline{\Omega} \setminus X_\sigma$ we have

$$\lim_{k\to\infty} \int_K \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy\right) f(x,u_k)u_k \ dx = \int_K \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu} dy\right) f(x,u)u \ dx. \tag{3.14}$$

To show this, let $x_0 \in K$ and $r_0 > 0$ be such that $\nu(B_{r_0}(x_0) \cap \overline{\Omega}) < \sigma$ that is $x_0 \notin X_\sigma$. Also we consider a $\psi \in C^\infty(\Omega)$ satisfying $0 \leq \psi(x) \leq 1$ for $x \in \Omega$, $\psi \equiv 1$ in $B_{\frac{r_0}{2}}(x_0) \cap \overline{\Omega}$ and $\psi \equiv 0$ in $\overline{\Omega} \setminus (B_{r_0}(x_0) \cap \overline{\Omega})$. Then

$$\lim_{k\to\infty} \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_k|^n \leq \lim_{k\to\infty} \int_{B_{r_0}(x_0)\cap\overline{\Omega}} |\nabla u_k|^n \psi \leq \nu(B_{r_0}(x_0) \cap \overline{\Omega}) < \sigma.$$

Therefore for large enough $k \in \mathbb{N}$ and $\epsilon > 0$ small enough, it must be

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_k|^n \leq \sigma(1-\epsilon). \tag{3.15}$$

Now we estimate the following using (3.15) and Theorem 1.1

$$\begin{aligned}
&\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(x,u_k)|^q \ dx = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |h(x,u_k)|^q \exp\left(q|u_k|^{\frac{n}{n-1}}\right) \ dx \\
&\leq C_\delta \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \exp\left((1+\epsilon)q|u_k|^{\frac{n}{n-1}}\right) \ dx \\
&\leq C_\delta \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \exp\left((1+\epsilon)q\sigma^{\frac{1}{n-1}}(1-\epsilon)^{\frac{1}{n-1}} \left(\frac{|u_k|^n}{\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_k|^n}\right)^{\frac{1}{n-1}}\right) \ dx \leq C_0
\end{aligned} \tag{3.16}$$

for some constant $C_0 > 0$ while choosing $q > 1$ such that $(1+\epsilon)q\sigma^{\frac{1}{n-1}} \leq \alpha_n$. Consider

$$\begin{aligned}
&\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)u_k - \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u)u \right| \, dx \\
&\leq \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) (f(x,u_k)u_k - f(x,u)u) \right| \, dx \\
&\quad + \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,u_k)-F(y,u)}{|x-y|^\mu}dy\right) f(x,u_k)u_k \right| \, dx \\
&:= I_1 + I_2 \text{ (say)}.
\end{aligned}$$

From (2.2), we know that $F(u) \in L^r(\Omega)$ for any $r \in [1,\infty)$. Since $\mu \in (0,n)$, $y \to |x-y|^{-\mu} \in L^{r_0}(\Omega)$ for all $r_0 \in (1, \frac{n}{\mu})$ uniformly in $x \in \Omega$ (since Ω is bounded). So using Hölder's inequality we get that

$$\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy \in L^\infty(\Omega). \tag{3.17}$$

From the asymptotic growth of $f(x,t)$, it is easy to get that

$$\lim_{t\to\infty} \frac{f(x,t)t}{(f(x,t))^r} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } r > 1. \tag{3.18}$$

Using (3.17) we get

$$I_1 \leq C \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(x,u_k)u_k - f(x,u)u| \, dx$$

where $C > 0$ is a constant. Because of (3.18) and (3.16), the family $\{f(x,u_k)u_k\}$ is equi-integrable over $B_{\frac{r_0}{2}}(x_0) \cap \overline{\Omega}$. Also continuity of $f(x,t)$ gives that $f(x,u_k)u_k \to f(x,u)u$ pointwise a.e. in Ω as $k \to \infty$ and thus using Vitali's convergence theorem, it follows that $I_1 \to 0$ as $k \to \infty$. Next we show $I_2 \to 0$ as $k \to \infty$.

First by using the semigroup property of the Riesz potential we get that for some constant $C > 0$ independent of k

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{F(y,u_k)-F(y,u)}{|x-y|^\mu}dy\right) \chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}}(x) f(x,u_k)u_k \, dx \\
&\leq \left(\int_\Omega \left(\int_\Omega \frac{|F(y,u_k)-F(y,u)|dy}{|x-y|^\mu}\right) |F(x,u_k)-F(x,u)| \, dx\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_\Omega \left(\int_\Omega \chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}}(y) \frac{f(y,u_k)u_k}{|x-y|^\mu}dy\right) \chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}}(x) f(x,u_k)u_k \, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

From (3.16) and since $\sigma^{\frac{1}{n-1}} < \frac{2n-\mu}{2n}\alpha_n$ we obtain

$$\begin{aligned}
&\left(\int_\Omega \left(\int_\Omega \chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}}(y) \frac{f(y,u_k)u_k}{|x-y|^\mu}dy\right) \chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}}(x) f(x,u_k)u_k \, dx\right)^{\frac{1}{2}} \leq \\
&\|\chi_{B_{\frac{r_0}{2}}\cap\overline{\Omega}} f(x,u_k)u_k\|_{L^{\frac{2n}{2n-\mu}}(\Omega)} \leq C.
\end{aligned}$$

Now we claim that

$$\lim_{k\to\infty}\int_\Omega\left(\int_\Omega \frac{|F(y,u_k)-F(y,u)|}{|x-y|^\mu}dy\right)|F(x,u_k)-F(x,u)|\ dx=0. \tag{3.19}$$

From (3.2), (3.3) and (3.4) we get that there exists a constant $C>0$ such that

$$\begin{aligned}\int_\Omega\left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right)F(x,u_k)\ dx\le C,\\ \int_\Omega\left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right)f(x,u_k)u_k\ dx\le C.\end{aligned} \tag{3.20}$$

We argue as along equation (2.20) in Lemma 2.4 in [5]. Now using (3.20), (h4) and the semigroup property of the Riesz Potential we obtain,

$$\int_\Omega\int_{|u|\ge M}\frac{F(y,u)}{|x-y|^\mu}F(x,u)dy\ dx=o(M),\ \int_\Omega\int_{|u_k|\ge M}\frac{F(y,u_k)}{|x-y|^\mu}F(x,u_k)dy\ dx=o(M), \tag{3.21}$$

$$\int_\Omega\int_{|u|\ge M}\frac{F(y,u_k)}{|x-y|^\mu}F(x,u)dy\ dx=o(M), \tag{3.22}$$

and

$$\int_\Omega\int_{|u_k|\ge M}\frac{F(y,u_k)}{|x-y|^\mu}F(x,u)dy\ dx=o(M)\text{ as }M\to\infty. \tag{3.23}$$

So,

$$\begin{aligned}&\int_\Omega\left(\int_\Omega \frac{|F(y,u_k)-F(y,u)|}{|x-y|^\mu}dy\right)|F(x,u_k)-F(x,u)|\ dx\le\\
&2\int_\Omega\left(\int_\Omega \frac{\chi_{u_k\ge M}(y)F(y,u_k)}{|x-y|^\mu}dy\right)F(x,u_k)\ dx\\
&+4\int_\Omega\left(\int_\Omega \frac{F(y,u_k)\chi_{u\ge M}(x)F(x,u)}{|x-y|^\mu}dy\right)\ dx+4\int_\Omega\left(\int_\Omega \frac{\chi_{u_k\ge M}(y)F(y,u_k)F(x,u)}{|x-y|^\mu}dy\right)\ dx\\
&+2\int_\Omega\left(\int_\Omega \frac{\chi_{u\ge M}(y)F(y,u)}{|x-y|^\mu}dy\right)F(x,u)\ dx\\
&+\int_\Omega\left(\int_\Omega \frac{|F(y,u_k)\chi_{u_k\le M}-F(y,u)\chi_{u\le M}|}{|x-y|^\mu}dy\right)|F(x,u_k)\chi_{u_k\le M}-F(x,u)\chi_{u\le M}|\ dx.\end{aligned}$$

Then from Lebesgue dominated convergence theorem the above integrand tends to 0 as $k\to\infty$. Hence using (3.21), (3.22) and (3.23), it is easy to conclude (3.19) and $I_2\to 0$ as $k\to\infty$. This implies that

$$\lim_{k\to\infty}\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}}\left|\left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right)f(x,u_k)u_k-\left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right)f(x,u)u\right|\ dx=0.$$

Now to conclude (3.14), we repeat this procedure over a finite covering of balls using the fact that K is compact. Lastly, the proof of (3.13) can be achieved by classical arguments as in the proof of Lemma 4 in [28]. ■

Lemma 3.5 *Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence for E at level l^* then there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that as $k \to \infty$ (up to a subsequence)*

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\phi\ dx \to \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}dy\right) f(x,u_0)\phi\ dx,\ \textit{for all}\ \phi \in C_c^\infty(\Omega).$$

Proof. If $\{u_k\}$ is a Palais Smale sequence at l^* for E then it must satisfy (3.2) and (3.3). We remark that $E(u^+) \le E(u)$ for each $u \in W_0^{1,n}(\Omega)$, then we can assume $u_k \ge 0$ for each $k \in \mathbb{N}$. From Lemma 3.2 we know that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$ so there exists a $C_0 > 0$ such that $\|u_k\| \le C_0$. Also there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ in $W_0^{1,n}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1,\infty)$ and pointwise a.e. in Ω as $k \to \infty$. Let $\Omega' \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in Ω'. With easy computations, we get that

$$\begin{aligned}\left\|\frac{\varphi}{1+u_k}\right\|^n &= \int_\Omega \left|\frac{\nabla\varphi}{1+u_k} - \varphi\frac{\nabla u_k}{(1+u_k)^2}\right|^n\ dx\\ &\le 2^{n-1}(\|\varphi\|^n + \|u_k\|^n).\end{aligned}$$

This implies that $\frac{\varphi}{1+u_k} \in W_0^{1,n}(\Omega)$. So using $\frac{\varphi}{1+u_k}$ as a test function (3.2), we get the following estimate

$$\begin{aligned}&\int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) \frac{f(x,u_k)}{1+u_k}\ dx \le \int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) \frac{f(x,u_k)\varphi}{1+u_k}\ dx\\ &\le \epsilon_k \left\|\frac{\varphi}{1+u_k}\right\| + \int_\Omega m(\|u_k\|^n)|\nabla u_k|^{n-2}\nabla u_k \nabla\left(\frac{\varphi}{1+u_k}\right)\ dx\\ &\le \epsilon_k 2^{\frac{n-1}{n}}(\|\varphi\| + \|u_k\|) + m(\|u_k\|^n)\int_\Omega |\nabla u_k|^{n-2}\nabla u_k\left(\frac{\nabla\varphi}{1+u_k} - \varphi\frac{\nabla u_k}{(1+u_k)^2}\right)\ dx\\ &\le \epsilon_k 2^{\frac{n-1}{n}}(\|\varphi\| + \|u_k\|) + m(\|u_k\|^n)\int_\Omega |\nabla u_k|^{n-1}\left(|\nabla\varphi| + |\nabla u_k|\right)\ dx\\ &\le \epsilon_k 2^{\frac{n-1}{n}}(\|\varphi\| + \|u_k\|) + m(\|u_k\|^n)[\|\varphi\|\|u_k\|^{n-1} + \|u_k\|^n].\end{aligned}$$

But using $\|u_k\| \le C_0$ for all k and (m2), we infer that there must exists a $C_1 > 0$ such that

$$\int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) \frac{f(x,u_k)}{1+u_k}\ dx \le C_1. \tag{3.24}$$

Also for the same reason, (3.3) gives that

$$\int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)u_k\ dx \le C_2 \tag{3.25}$$

for some $C_2 > 0$. Gathering (3.24) and (3.25) we obtain

$$\begin{aligned}&\int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\ dx\\ &\le 2\int_{\Omega'\cap\{u_k<1\}} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) \frac{f(x,u_k)}{1+u_k}\ dx + \int_{\Omega'\cap\{u_k\ge 1\}} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) u_k f(x,u_k)\ dx\\ &\le 2\int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) \frac{f(x,u_k)}{1+u_k}\ dx + \int_{\Omega'} \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) u_k f(x,u_k)\ dx\\ &\le 2C_1 + C_2 := C_3.\end{aligned}$$

Thus the sequence $\{w_k\} := \left\{\left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\right\}$ is bounded in $L^1_{\text{loc}}(\Omega)$ which implies that up to a subsequence, $w_k \rightharpoonup w$ in the *weak**-topology as $k \to \infty$, where w denotes a Radon measure. So for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k\to\infty} \int_\Omega \int_\Omega \left(\frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\phi \; dx = \int_\Omega \phi \; dw, \; \forall \phi \in C_c^\infty(\Omega).$$

Since u_k satisfies (3.2), we get that

$$\int_E \phi dw = \lim_{k\to\infty} m(\|u_k\|) \int_E |\nabla u_k|^{n-2}\nabla u_k \nabla\phi \; dx, \;\; \forall E \subset \Omega.$$

Together with Lemma 3.4, this implies that w is absolutely continuous with respect to the Lebesgue measure. Thus, Radon-Nikodym theorem asserts that there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that for any $\phi \in C_c^\infty(\Omega)$, $\int_\Omega \phi \; dw = \int_\Omega \phi g \; dx$. Therefore for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k\to\infty} \int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)\phi \;\; dx = \int_\Omega \phi g \; dx = \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}dy\right) f(x,u_0)\phi \;\; dx$$

which completes the proof. ■

In the next Lemma, we show that weak limit of any $(PS)_c$ sequence is a weak solution of (KC).

Lemma 3.6 *Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence of E. Then there exists a $u \in W_0^{1,n}(\Omega)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$ and*

$$\left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) F(x,u_k) \to \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) F(x,u) \text{ in } L^1(\Omega) \tag{3.26}$$

as $k \to \infty$. Moreover, u forms a weak solution of (KC).

Proof. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence of E at level c. From Lemma 3.2 we know that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$. Thus there exists a $u \in W_0^{1,n}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$, $u_k \to u$ pointwise a.e. in $\mathbb{R}^n$ and $u_k \to u$ strongly in $L^q(\Omega)$, $q \in [1,\infty)$ as $k \to \infty$. Also from (3.2), (3.3) and (3.4) we get that there exists a constant $C > 0$ such that (3.20) holds. Now the proof of (3.26) follows similarly the proof of (3.19) (see also equation (2.20) of Lemma 2.4 in [5]). Also, from this we get u forms a weak solution of (KC) using Lemma 3.5 and Lemma 3.4. ■

Now we define the associated Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{1,n}(\Omega) \setminus \{0\} : \; \langle E'(u), u\rangle = 0\}$$

and $l^{**} = \inf_{u\in\mathcal{N}} E(u)$.

Lemma 3.7 *If (m3) holds then $l^* \leq l^{**}$.*

Proof. Let $u \in \mathcal{N}$ and $h : (0, +\infty) \to \mathbb{R}$ be defined as $h(t) = E(tu)$. Then

$$h'(t) = m(\|tu\|^n)\|u\|^n t^{n-1} - \int_\Omega \left(\int_\Omega \frac{F(y, tu)}{|x-y|^\mu} dy \right) f(x, tu)u \, dx.$$

Since u satisfies $\langle E'(u), u \rangle = 0$, we get

$$\begin{aligned} h'(t) = {} & \|u\|^{2n} t^{2n-1} \left(\frac{m(\|tu\|^n)}{t^n \|u\|^n} - \frac{m(\|u\|^n)}{\|u\|^n} \right) \\ & + t^{2n-1} \left[\int_\Omega \left(\int_\Omega \frac{\frac{F(y,u)f(x,u)}{u^{n-1}(x)}}{|x-y|^\mu} dy - \int_\Omega \frac{\frac{F(y,tu)f(x,tu)}{(tu(x))^{n-1} t^n}}{|x-y|^\mu} dy \right) u^n(x) \, dx \right]. \end{aligned}$$

Claim: For any $x \in \Omega, t \to tf(x,t) - nF(x,t)$ is increasing on $\mathbb{R}^+$. (3.27)

indeed, from (h3), for $0 < t_1 < t_2$, we have

$$t_1 f(x, t_1) - nF(x, t_1) \leq t_1 f(x, t_1) - nF(x, t_2) + \frac{f(x, t_2)}{t_2^{n-1}} (t_2^n - t_1^n) \leq t_2 f(x, t_2) - nF(x, t_2).$$

Using this we get that $tf(x,t) - nF(x,t) \geq 0$ for $t \geq 0$ which implies that $t \to \frac{F(x,tu)}{t^n}$ is non-decreasing for $t > 0$. Therefore for $0 < t < 1$ and $x \in \Omega$, we get $\frac{F(x,tu)}{t^n} \leq F(x, u)$ and this implies

$$\begin{aligned} h'(t) \geq {} & \|u\|^{2n} t^{2n-1} \left(\frac{m(\|tu\|^n)}{\|tu\|^n} - \frac{m(\|u\|^n)}{\|u\|^n} \right) \\ & + t^{2n-1} \left[\int_\Omega \left(\int_\Omega \left(F(y,u) - \frac{F(y,tu)}{t^n} \right) \frac{dy}{|x-y|^\mu} \right) \frac{f(x,tu)}{(tu(x))^{n-1}} u^n(x) \, dx \right]. \end{aligned}$$

This gives that $h'(t) \geq 0$ for $0 < t \leq 1$ and $h'(t) < 0$ for $t > 1$. Hence $E(u) = \max_{t \geq 0} E(tu)$. Now we define $g : [0,1] \to W_0^{1,n}(\Omega)$ as $g(t) = (t_0 u)t$ where $t_0 > 1$ is such that $E(t_0 u) < 0$. So $g \in \Gamma$, where Γ is as defined in the definition of l^*. Therefore we obtain

$$l^* \leq \max_{t \in [0,1]} E(g(t)) \leq \max_{t \geq 0} E(tu) = E(u).$$

Since $u \in \mathcal{N}$ is arbitrary, we get $l^* \leq l^{**}$. This completes the proof. ■

We recall the following Lemma from [26] which is known as the higher integrability Lemma.

Lemma 3.8 *Let $\{v_k \in W_0^{1,n}(\Omega) : \|v_k\| = 1\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly to a non zero $v \in W_0^{1,n}(\Omega)$. Then for every $p \in \left(1, (1 - \|v\|)^{-\frac{1}{n-1}}\right)$,*

$$\sup_k \int_\Omega \exp\left(p\alpha_n |v_k|^{\frac{n}{n-1}} \right) < +\infty.$$

Proof of Theorem 2.3: Let $\{u_k\}$ denotes a Palais Smale sequence at the level l^*. Then $(u_k)_{k \in I\!N}$ can be obtained as a minimizing sequence associated to the variational problem (3.7).

Then by Lemma 3.6 we know that there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ weakly in $W_0^{1,n}(\Omega)$ as $k \to \infty$. So if $u_0 \equiv 0$ then using Lemma 3.6, we infer that

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy \right) F(x,u_k)\ dx \to 0 \text{ as } k \to \infty.$$

This gives that $\lim_{k\to\infty} E(u_k) = \frac{1}{n} \lim_{k\to\infty} M(\|u_k\|^n) = l^*$ which implies in the light of Lemma 3.3 that for large enough k

$$M(\|u_k\|^n) < M\left(\left(\frac{2n-\mu}{2n} \alpha_n \right)^{n-1} \right).$$

Therefore since M is non decreasing, we get

$$\frac{2n}{2n-\mu} \|u_k\|^{\frac{n}{n-1}} < \alpha_n.$$

Now, this implies that $\sup_k \int_\Omega f(x,u_k)^q\ dx < +\infty$ for some $q > \frac{2n}{2n-\mu}$ and along with Proposition 2.2, Theorem 1.1 and the Vitali's convergence theorem,

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy \right) f(x,u_k) u_k\ dx \to 0 \text{ as } k \to \infty.$$

Hence $\lim_{k\to\infty} \langle E'(u_k), u_k \rangle = 0$ gives $\lim_{k\to\infty} m(\|u_k\|^n)\|u_k\|^n = 0$. From (m1) we then obtain $\lim_{k\to\infty} \|u_k\|^n = 0$. Thus using Lemma 3.6, it must be that $\lim_{k\to\infty} E(u_k) = 0 = l^*$ which contradicts $l^* > 0$. Thus $u_0 \not\equiv 0$.

Claim (1): u_0 is a weak solution of (KC).

Before proving this, we show that $u_0 > 0$ in Ω. From Lemma 3.2 we know that $\{u_k\}$ must be bounded. Therefore there exists a constant $\tau > 0$ such that up to a subsequence $\|u_k\| \to \tau$ as $k \to \infty$. Since $E'(u_k) \to 0$, again up to a subsequence $|\nabla u_k|^{n-2}\nabla u_k \rightharpoonup |\nabla u_0|^{n-2}\nabla u_0$ weakly in $(L^{\frac{n}{n-1}}(\Omega))^n$. Furthermore, by Lemma 3.4 and by Lemma 3.5, we get as $k \to \infty$,

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy \right) f(x,u_k)\varphi\ dx \to \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu} dy \right) f(x,u_0)\varphi\ dx$$

and

$$m(\tau^n) \int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \nabla \varphi\ dx = \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu} dy \right) f(x,u_0)\varphi\ dx, \text{ for all } \varphi \in W_0^{1,n}(\Omega).$$

In particular, taking $\varphi = u_0^-$ in the above equation we get $m(\tau^n)\|u_0^-\| = 0$ which implies together with assumption (m1) that $u_0^- = 0$ a.e. in Ω. Therefore $u_0 \geq 0$ a.e. in Ω.

From Theorem 1.1, we have $f(\cdot, u_0) \in L^q(\Omega)$ for $1 \leq q < \infty$. Also as in (3.17), we can similarly get that $\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}\ dy \in L^\infty(\Omega)$. Hence $\left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}\ dy \right) f(x,u_0) \in L^q(\Omega)$ for $1 \leq q < \infty$. By elliptic regularity results, we finally get that $u_0 \in L^\infty(\Omega)$ and $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. Therefore, $u_0 > 0$ in Ω follows from the strong maximum principle and $u_0 \not\equiv 0$.

Now we claim that

$$m(\|u_0\|^n)\|u_0\|^n \geq \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}\ dy \right) f(x,u_0) u_0\ dx. \tag{3.28}$$

Arguing by contradiction, suppose that

$$m(\|u_0\|^n)\|u_0\|^n < \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}\ dy \right) f(x,u_0)u_0\ dx$$

which implies that $\langle E'(u_0), u_0\rangle < 0$. For $t > 0$, using (3.27) we have that

$$\begin{aligned} \langle E'(tu_0), u_0\rangle &\geq m(t^n\|u_0\|^n)t^{n-1}\|u_0\|^n - \frac{1}{n}\int_\Omega \left(\int_\Omega \frac{f(y,tu_0)tu_0(y)}{|x-y|^\mu}\ dy \right) f(x,tu_0)u_0\ dx \\ &\geq m_0 t^{n-1}\|u_0\|^n - \frac{1}{n}\int_\Omega \left(\int_\Omega \frac{f(y,tu_0)tu_0(y)}{|x-y|^\mu}\ dy \right) f(x,tu_0)u_0\ dx. \end{aligned}$$

Since (h3) gives that

$$\lim_{t\to 0^+} \frac{f(x,t)}{t^\gamma} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } \gamma \in [0, n-1],$$

we can choose $t > 0$ sufficiently small so that $\langle E'(tu_0), u_0\rangle > 0$. Thus there exists a $t_* \in (0,1)$ such that $\langle E'(t_*u_0), u_0\rangle = 0$ that is $t_*u_0 \in \mathcal{N}$. So using Lemma 3.7, $(m3)'$ and (3.27) we get

$$\begin{aligned} l^* \leq l^{**} \leq E(t_*u_0) &= E(t_*u_0) - \frac{1}{2n}\langle E'(t_*u_0), u_0\rangle \\ &= \frac{M(\|t_*u_0\|^n)}{n} - \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{F(y,t_*u_0)}{|x-y|^\mu} dy \right) F(x,t_*u_0)\ dx - \frac{1}{2n} m(\|t_*u_0\|^n)\|t_*u_0\|^n \\ &\quad + \frac{1}{2n}\int_\Omega \left(\int_\Omega \frac{F(y,t_*u_0)}{|x-y|^\mu} dy \right) f(x,t_*u_0)t_*u_0\ dx \\ &< \frac{M(\|u_0\|^n)}{n} - \frac{1}{2n} m(\|u_0\|^n)\|u_0\|^n \\ &\quad + \frac{1}{2n}\int_\Omega \left(\int_\Omega \frac{F(y,t_*u_0)}{|x-y|^\mu} dy \right) (f(x,t_*u_0)t_*u_0 - nF(x,t_*u_0))\ dx \\ &\leq \frac{M(\|u_0\|^n)}{n} - \frac{1}{2n} m(\|u_0\|^n)\|u_0\|^n + \frac{1}{2n}\int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu} dy \right) (f(x,u_0)u_0 - nF(x,u_0)) \\ &\leq \liminf_{k\to\infty} \frac{M(\|u_k\|^n)}{n} - \frac{1}{2n} m(\|u_k\|^n)\|u_k\|^n \\ &\quad + \frac{1}{2n}\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy \right) (f(x,u_k)u_k - nF(x,u_k))\ dx \\ &= \liminf_{k\to\infty} \left(E(u_k) - \frac{1}{2n}\langle E'(u_k), u_k\rangle \right) = l^*. \end{aligned}$$

This gives a contradiction, that is (3.28) holds true.

Claim (2): $E(u_0) = l^*$.

From Lemma 3.6 we know that

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu} dy \right) F(x,u_k)\ dx \to \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu} dy \right) F(x,u_0)\ dx.$$

Using this and the weakly lower semicontinuity of norms in $\lim_{k\to\infty} E(u_k) = l^*$, we obtain $E(u_0) \leq l^*$. If $E(u_0) < l^*$ then it must be

$$\lim_{k\to\infty} M(\|u_k\|^n) > M(\|u_0\|^n)$$

which implies that $\lim_{k\to\infty}\|u_k\|^n > \|u_0\|^n$, since M is continuous and increasing. From this we get

$$\tau^n > \|u_0\|^n.$$

Moreover we get

$$M(\tau^n) = n\left(l^* + \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}dy\right) F(x,u_0)\ dx\right). \tag{3.29}$$

Now we define the sequence $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\tau}$ then $v_k \rightharpoonup v_0$ weakly in $W_0^{1,n}(\Omega)$ and $\|v_0\| < 1$. From Lemma 3.8 we have that

$$\sup_{k\in\mathbb{N}}\int_\Omega \exp\left(p|v_k|^{\frac{n}{n-1}}\right) < +\infty, \text{ for } 1 < p < \frac{\alpha_n}{(1-\|v_0\|)^{\frac{1}{n-1}}}. \tag{3.30}$$

Also from $(m3)'$, Claim (1) and Lemma 3.7 we obtain

$$E(u_0) = \frac{M(\|u_0\|^n)}{n} - \frac{m(\|u_0\|^n)\|u_0\|^n}{2n} + \frac{1}{2n}\int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}dy\right)(f(x,u_0)u_0 - nF(x,u_0))\ dx \geq 0.$$

Using this with (3.29) we get that

$$M(\tau^n) = nl^* - nE(u_0) + M(\|u_0\|^n) < M\left(\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\right) + M(\|u_0\|^n)$$

which implies together with (m1) that

$$\tau^n < \frac{\alpha_n^{n-1}\left(\frac{2n-\mu}{2n}\right)^{n-1}}{1-\|v_0\|^n}.$$

Thus it is possible to find a $\tau_* > 0$ such that for $k \in \mathbb{N}$ large enough

$$\|u_k\|^{\frac{n}{n-1}} < \tau_* < \frac{\alpha_n\left(\frac{2n-\mu}{2n}\right)}{(1-\|v_0\|^n)^{\frac{1}{n-1}}}.$$

Then we choose a $q > 1$ but close to 1 such that

$$\frac{2n}{2n-\mu}q\|u_k\|^{\frac{n}{n-1}} \leq \frac{2n}{2n-\mu}\tau_* < \frac{\alpha_n}{(1-\|v_0\|^n)^{\frac{1}{n-1}}}.$$

Therefore from (3.30) we conclude that

$$\int_\Omega \exp\left(\frac{2n}{2n-\mu}q|u_k|^{\frac{n}{n-1}}\right) \leq C \tag{3.31}$$

for some constant $C > 0$. Using (3.31)

$$\int_\Omega \left(\int_\Omega \frac{F(y,u_k)}{|x-y|^\mu}dy\right) f(x,u_k)u_k\ dx \to \int_\Omega \left(\int_\Omega \frac{F(y,u_0)}{|x-y|^\mu}dy\right) f(x,u_0)u_0\ dx.$$

We conclude that $\|u_k\| \to \|u_0\|$ and we get a contradiction and claim (2) is proved. Now, from claims (1) and (2), the proof of Theorem 2.3 follows. ■

4 The Nehari Manifold method for $(\mathcal{P}_{\lambda,M})$

The energy functional $\mathcal{J}_{\lambda,M} : W_0^{1,n}(\Omega) \longrightarrow \mathbb{R}$ associated to the problem $\mathcal{P}_{\lambda,M}$ is defined as

$$\mathcal{J}_{\lambda,M}(u) = \frac{1}{n}M(\|u\|^n) - \frac{\lambda}{q+1}\int_\Omega h(x)|u|^{q+1}\ dx - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx$$

where $|x|^{-\mu} * F(u)$ denotes $\int_\Omega \frac{F(u(y))}{|x-y|^\mu}\ dy$, F, M are anti-derivatives of f, m (vanishing at 0) respectively and $f(s) = s|s|^p exp(|s|^\beta)$.

Definition 4.1 *A function $u \in W_0^{1,n}(\Omega)$ is said to be weak solution of $\mathcal{P}_{\lambda,M}$ if $\forall\ \phi \in W_0^{1,n}(\Omega)$ we have*

$$m(\|u\|^n)\int_\Omega |\nabla u|^{n-2}\nabla u.\nabla\phi\ dx = \lambda\int_\Omega h(x)u^{q-1}u\phi\ dx + \int_\Omega (|x|^{-\mu} * F(u))f(u)\phi\ dx.$$

We observe that $\mathcal{J}_{\lambda,M}$ is only bounded below on suitable subsets of $W_0^{1,n}(\Omega)$. In order to prove the existence of weak solutions to $(\mathcal{P}_{\lambda,M})$, we establish the existence of minimizers of $\mathcal{J}_{\lambda,M}$ under the natural constraint of the Nehari Manifold:

$$N_{\lambda,M} := \{u \in W_0^{1,n}(\Omega)|\ \langle \mathcal{J}_{\lambda,M}'(u), u\rangle = 0\}$$

where $\langle .\ ,.\rangle$ denotes the duality between $W_0^{1,n}(\Omega)$ and $W^{-1,n}(\Omega)$. Therefore, $u \in N_{\lambda,M}$ if and only if

$$\|u\|^n\ m(\|u\|^n) - \lambda\int_\Omega h(x)u^{q+1}\ dx - \int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx = 0.$$

Remark 4.2 *We notice that $N_{\lambda,M}$ contains every solution of $(\mathcal{P}_{\lambda,M})$.*

For $u \in W_0^{1,n}(\Omega)$, we define the fiber map $\Phi_{u,M} : \mathbb{R}^+ \to \mathbb{R}$ as

$$\Phi_{u,M}(t) = \mathcal{J}_{\lambda,M}(tu) = \frac{M(\|tu\|^n)}{n} - \frac{\lambda}{q+1}\int_\Omega h(x)|tu|^{q+1}\ dx - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(tu))F(tu)\ dx,$$

$$\Phi_{u,M}'(t) = t^{n-1}\|u\|^n m(\|tu\|^n) - \lambda t^q\int_\Omega h(x)|u|^{q+1}\ dx - \int_\Omega (|x|^{-\mu} * F(tu))f(tu)u\ dx$$

and

$$\Phi_{u,M}''(t) = nt^{2n-2}\|u\|^{2n}m'(\|tu\|^n) + (n-1)t^{n-2}\|u\|^n m(\|tu\|^n) - \lambda qt^{q-1}\int_\Omega h(x)|u|^{q+1}\ dx$$
$$- \int_\Omega (|x|^{-\mu} * f(tu).u)f(tu)u\ dx - \int_\Omega (|x|^{-\mu} * F(tu))f'(tu)u^2\ dx.$$

The Nehari Manifold is closely related to the the maps $\Phi_{u,M}$ by the relation $tu \in N_{\lambda,M}$ iff $\Phi_{u,M}'(t) = 0$. In particular, $u \in N_{\lambda,M}$ iff $\Phi_{u,M}'(1) = 0$. So we study the geometry of the energy functional on the following components of the Nehari Manifold:

$$N_{\lambda,M}^+ := \{u \in N_{\lambda,M} : \Phi_{u,M}''(1) > 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi_{u,M}'(t) = 0, \Phi_{u,M}''(t) > 0\},$$

$$N_{\lambda,M}^{-} := \{u \in N_{\lambda,M} : \Phi_{u,M}^{''}(1) < 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi_{u,M}^{'}(t) = 0, \Phi_{u,M}^{''}(t) < 0\},$$
$$N_{\lambda,M}^{0} := \{u \in N_{\lambda,M} : \Phi_{u,M}^{''}(1) = 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi_{u,M}^{'}(t) = 0, \Phi_{u,M}^{''}(t) = 0\}.$$

We also define $H(u) = \int_\Omega h|u|^{q+1}\,dx$ and study the behaviour of fibering maps $\Phi_{u,M}$ according to the sign of $H(u)$. Let

$$H^{+} := \{u \in W_0^{1,n}(\Omega) : H(u) > 0\},$$
$$H_0^{-} := \{u \in W_0^{1,n}(\Omega) : H(u) \le 0\}.$$

4.1 Analysis of Fiber Maps

Here we analyze accurately the geometry of the energy functional on the Nehari manifold. We split the study in different cases.
Case 1: $u \in H_0^{-}$
Define $\psi : \mathbb{R}^{+} \to \mathbb{R}$ such that

$$\psi_u(t) = t^{n-1-q} m(\|tu\|^n)\|u\|^n - t^{-q}\int_\Omega (|x|^{-\mu} * F(tu))f(tu)u\,dx.$$

Since

$$\begin{aligned}\Phi_{u,M}^{'}(t) &= t^{n-1}\|u\|^n m(\|tu\|^n) - \lambda t^q \int_\Omega h(x)|u|^{q+1}\,dx - \int_\Omega (|x|^{-\mu} * F(tu))f(tu)u\,dx\\ &= t^q(\psi_u(t) - \lambda\int_\Omega h(x)|u|^{q+1}\,dx),\end{aligned}$$

$tu \in N_{\lambda,M}$ iff $t > 0$ is a solution of $\psi_u(t) = \lambda\int_\Omega h(x)|u|^{q+1}$.

$$\begin{aligned}\psi_u^{'}(t) &= (n-1-q)t^{n-2-q}m(\|tu\|^n)\|u\|^n + nt^{2n-2-q}m'(\|tu\|^n)\|u\|^{2n}\\ &\quad + \frac{q}{t^{q+1}}\int_\Omega (|x|^{-\mu} * F(tu))f(tu).u\,dx - t^{-q}\Big[\int_\Omega (|x|^{-\mu} * f(tu).u)f(tu).u\,dx\\ &\quad + \int_\Omega (|x|^{-\mu} * F(tu))f^{'}(tu).u^2\,dx\Big].\end{aligned} \tag{4.1}$$

Due to the exponential growth of f, for large t we have $\psi_u^{'}(t) < 0$ and since $u \in H_0^{-}$, there exists $t^* > 0$ such that $\psi_u(t^*) = \lambda\int_\Omega h(x)|u|^{q+1}$, *i.e.* $t^*u \in N_{\lambda,M}$.
If there exists an another point t_1 such that $t^* < t_1$ and $\psi_u(t_1) = \lambda\int_\Omega h(x)|u|^{q+1} \le 0$, *i.e.*

$$t_1^{n-1-q}(at_1^n\|u\|^n + b)\|u\|^n \le t_1^{-q}\int_\Omega (|x|^{-\mu} * F(t_1u))f(t_1u)u\,dx \tag{4.2}$$

and $\psi_u'(t_1) \ge 0$. Then by using $f'(t_1u)t_1u > (p+1)f(t_1u)$ and $p > 2n-2-q$ we obtain from (4.2),

$$\psi_u'(t_1) < (2n-1-q)\left[t_1^{n-2-q}(at_1^n\|u\|^n + b)\|u\|^n - t_1^{-q-1}\int_\Omega (|x|^{-\mu} * F(t_1u))f(t_1u)u\,dx\right] \le 0.$$

Therefore $\psi_u'(t_1) < 0$ which yields a contradiction. Therefore there exists a unique t^* such that $\psi_u(t^*) = \lambda \int_\Omega h(x)|u|^{q+1}\,dx$. Also for $0 < t < t^*$, $\Phi_{u,M}'(t) = t^q(\psi_u(t) - \lambda \int_\Omega h(x)|u|^{q+1}\,dx) > 0$. Consequently, $\Phi_{u,M}$ is increasing in $(0, t^*)$ and decreasing on (t^*, ∞). Therefore there exists a unique critical point of $\Phi_{u,M}$ which is also a global maximum point. Furthermore, since $\psi_u'(t) = \dfrac{\left(t\Phi_{u,M}''(t) - q\Phi_{u,M}'(t)\right)}{t^q}$, we get $t^*u \in N_{\lambda,M}^-$.

Case 2: $u \in H^+$

In this case, we establish that there exists $\lambda_0 > 0$ and a t_* such that for $\lambda \in (0, \lambda_0)$, $\Phi_{u,M}$ has exactly two critical points $t_1(u)$ and $t_2(u)$ such that $t_1(u) < t_*(u) < t_2(u)$ where $t_1(u)$ is local minimum point and $t_2(u)$ is local maximum point. To prove this case, we need the analysis performed in the next subsection.

4.2 Preliminary Results for Case-2

For $0 \not\equiv u \in H^+$, we have that $\psi_u(t) \to -\infty$ as $t \to \infty$ and for small $t > 0$, $\psi_u(t) > 0$. Then there exists at least a point of maximum of $\psi_u(t)$, say t_*, and $\psi_u'(t_*) = 0$, *i.e.*

$$(2n-1-q)t_*^{2n-2-q}a\|u\|^{2n} + (n-1-q)t_*^{n-2-q}b\|u\|^n + \frac{q}{t_*^{q+1}}\int_\Omega (|x|^{-\mu} * F(t_*u))f(t_*u)u\,dx$$
$$= t_*^{-q}\left[\int_\Omega (|x|^{-\mu} * F(t_*u))f'(t_*u)u^2\,dx + \int_\Omega (|x|^{-\mu} * f(t_*u)u)f(t_*u).u\,dx\right].$$

This implies that

$$(2n-1-q)a\|t_*u\|^{2n} + (n-1-q)b\|t_*u\|^n + q\int_\Omega (|x|^{-\mu} * F(t_*u))f(t_*u)t_*u\,dx$$
$$= \int_\Omega (|x|^{-\mu} * F(t_*u))f'(t_*u)(t_*u)^2\,dx + \int_\Omega (|x|^{-\mu} * f(t_*u)t_*u)f(t_*u)t_*u\,dx.$$

Then we have

$$2\sqrt{(2n-1-q)a\|t_*u\|^{2n}b(n-1-q)\|t_*u\|^n} \leq B(t_*u)$$

from which it follows

$$\|t_*u\|^{3n/2} \leq \frac{B(t_*u)}{2\sqrt{(2n-1-q)(n-1-q)ab}}$$

where $B(u) = \int_\Omega (|x|^{-\mu} * F(u))f'(u)u^2 + \int_\Omega (|x|^{-\mu} * f(u)u)f(u)u\,dx$. Using $\psi_u'(t_*) = 0$, we replace the value of $a\|tu\|^{2n}$ in the definition of $\psi_u(t)$ to obtain

$$\psi_u(t_*) = \frac{1}{(2n-1-q)t_*^{q+1}}\left[B(t_*u) - (2n-1)\int_\Omega (|x|^{-\mu} * F(t_*u))f(t_*u)t_*u\,dx + nb\|t_*u\|^n\right]. \tag{4.3}$$

Now we prove the following result and establish the proof in various steps.

Lemma 4.3 *Let*

$$\Gamma := \left\{ u \in W_0^{1,n}(\Omega) : \|u\|^{3n/2} \leq \frac{B(u)}{2\sqrt{(2n-1-q)(n-1-q)ab}} \right\}$$

where $B(u) = \int_\Omega (|x|^{-\mu} * F(u)) f'(u)(u)^2 + \int_\Omega (|x|^{-\mu} * f(u)u) f(u)u \; dx$. *Then there exists a* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$, $\Gamma_0 > 0$ *holds where*

$$\Gamma_0 := \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left[B(u) - (2n-1) \int_\Omega (|x|^{-\mu} * F(u)) f(u).u \; dx + nb\|u\|^n - \lambda(2n-1-q)H(u) \right]. \tag{4.4}$$

Proof. Step 1: Claim: $\inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\| > 0$.
Let us suppose that it doesn't hold then there exists a sequence $\{u_k\} \subset \Gamma \setminus \{0\} \cap H^+$ such that $\|u_k\| \to 0$ and $\|u_k\|^{3n/2} \leq \frac{B(u_k)}{2\sqrt{(2n-1-q)(n-1-q)ab}}, \forall \; k$. Then by the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} B(u_k) &= \int_\Omega (|x|^{-\mu} * F(u_k)) f'(u_k) u_k^2 \; dx + \int_\Omega (|x|^{-\mu} * f(u_k)u_k) f(u_k) u_k \; dx \\ &\leq C(n,\mu) \left(\|f(u_k)u_k\|^2_{L^{2n/(2n-\mu)}(\Omega)} + \|F(u_k)\|_{L^{2n/(2n-\mu)}(\Omega)} \|f'(u_k)(u_k)^2\|_{L^{2n/(2n-\mu)}(\Omega)} \right). \end{aligned}$$

Since $f(u) = u|u|^p exp(|u|^\beta)$ and $f'(u) = ((p+1) + \beta|u|^\beta)|u|^p exp(|u|^\beta)$, then we have

$$\begin{aligned} |B(u_k)| \leq C(n,\mu) \bigg(\int_\Omega (|u_k|^{p+2} exp(|u_k|^\beta))^{\frac{2n}{2n-\mu}} \; dx \bigg)^{\frac{2n-\mu}{n}} + C(n,\mu) \left(\int_\Omega (F(u_k))^{\frac{2n}{2n-\mu}} \; dx \right)^{\frac{2n-\mu}{2n}} \\ \times \left(\int_\Omega (((p+1) + \beta|u_k|^\beta)|u_k|^{p+2} exp(|u_k|^\beta))^{\frac{2n}{2n-\mu}} \; dx \right)^{\frac{2n-\mu}{2n}}. \end{aligned}$$

Then using $F(t) \leq t f(t)$ and by the Hölder's inequality, we obtain

$$\begin{aligned} |B(u_k)| \leq C_1 \bigg(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} \; dx \bigg)^{\frac{2n-\mu}{n\alpha'}} . \bigg(\int_\Omega exp\bigg(|u_k|^\beta \frac{2n\alpha}{2n-\mu} \bigg) \; dx \bigg)^{\frac{2n-\mu}{n\alpha}} \\ + C_2 \bigg(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} \; dx \bigg)^{\frac{2n-\mu}{2n\alpha'}} . \bigg(\int_\Omega exp\bigg(|u_k|^\beta \frac{2n\alpha}{2n-\mu} \bigg) \; dx \bigg)^{\frac{2n-\mu}{2n\alpha}} \times \\ \bigg[\bigg(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} \; dx \bigg)^{\frac{2n-\mu}{2n\alpha'}} . \bigg(\int_\Omega exp\bigg(|u_k|^\beta \frac{2n\alpha}{2n-\mu} \bigg) \; dx \bigg)^{\frac{2n-\mu}{2n\alpha}} \\ + \bigg(\int_\Omega |u_k|^{\frac{2n\alpha'(p+\beta+2)}{2n-\mu}} \; dx \bigg)^{\frac{2n-\mu}{2n\alpha'}} . \bigg(\int_\Omega exp\bigg(|u_k|^\beta \frac{2n\alpha}{2n-\mu} \bigg) \; dx \bigg)^{\frac{2n-\mu}{2n\alpha}} \bigg]. \end{aligned}$$

Let α be such that $2n\alpha/(2n-\mu))\|u_k\|^\beta \le \alpha_n$ and $v_k = \frac{u_k}{\|u_k\|}$, then by the Trudinger-Moser inequality we obtain

$$\begin{aligned}|B(u_k)| &\le C_1\left(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{n\alpha'}}.\left(\sup_{\|v_k\|\le 1}\int_\Omega exp(|v_k|^\beta\alpha_n)\,dx\right)^{\frac{2n-\mu}{n\alpha}}\\ &\quad + C_2\left(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{2n\alpha'}}.\left(\sup_{\|v_k\|\le 1}\int_\Omega exp(|v_k|^\beta\alpha_n)\,dx\right)^{\frac{2n-\mu}{n\alpha}}\times\\ &\quad \left[\left(\int_\Omega |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{2n\alpha'}} + \left(\int_\Omega |u_k|^{\frac{2n\alpha'(p+\beta+2)}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{2n\alpha'}}\right].\end{aligned}$$

Using the Sobolev embedding, it implies that

$$\begin{aligned}|B(u_k)| &\le C_1(n,k,\beta,\mu)(\|u_k\|^{2(p+2)} + \|u_k\|^{(p+2)}(\|u_k\|^{(p+2)} + \|u_k\|^{(p+\beta+2)}))\\ &\le C\|u_k\|^{(2p+4)} + \|u_k\|^{(2p+\beta+4)}.\end{aligned}$$

Hence using $u_k \in \Gamma\backslash\{0\}$ and by the Sobolev embedding theorem, we get $1 \le C(\|u_k\|^{(2p+4-\frac{3n}{2})} + \|u_k\|^{(2p+\beta+4-\frac{3n}{2})}$ and $2p+4-\frac{3n}{2} > 0$ which is a contradiction as $\|u_k\| \to 0$ as $k \to \infty$. Therefore we have $\inf_{u\in\Gamma\backslash\{0\}\cap H^+} \|u\| > 0$.

Step 2: Claim: $0 < \inf_{u\in\Gamma\backslash\{0\}\cap H^+}\left\{\int_\Omega (|x|^{-\mu} * f(u)u)(p+2-2n+\beta|u|^\beta)exp(|u|^\beta)|u|^{p+2}\,dx\right\}$.

Since $F(s) \le \frac{f(s)s}{p+2}$ then by the definition of Γ and from Step 1, we obtain $0 < \inf_{u\in\Gamma\backslash\{0\}\cap H^+} B(u)$ *i.e.*

$$\begin{aligned}0 &< \inf_{u\in\Gamma\backslash\{0\}\cap H^+}\left\{\int_\Omega (|x|^{-\mu} * F(u))f'(u)u^2 + \int_\Omega (|x|^{-\mu} * f(u)u)f(u)u\right\}\\ &\le \inf_{u\in\Gamma\backslash\{0\}\cap H^+}\left\{\int_\Omega (|x|^{-\mu} * f(u)u)\left(f(u).u + f'(u)\frac{u^2}{p+2}\right)\right\}\\ &= \inf_{u\in\Gamma\backslash\{0\}\cap H^+}\left\{\int_\Omega (|x|^{-\mu} * f(u)u)|u|^{p+2}exp(|u|^\beta)\left(1 + \frac{(p+1)+\beta|u|^\beta}{p+2}\right)\right\}.\end{aligned}$$

Since $p+2-2n > 0$, we infer

$$0 < \inf_{u\in\Gamma\backslash\{0\}\cap H^+}\{\int_\Omega (|x|^{-\mu} * f(u)u)(p+2-2n+\beta|u|^\beta)exp(|u|^\beta)|u|^{p+2}\,dx\}.$$

Step 3: Claim: $\Gamma_0 > 0$. First,

$$\int_\Omega h(x)|u|^{q+1} \le \left(\int_\Omega |h(x)|^\gamma\right)^{1/\gamma}\left(|u|^{(1+q)\gamma'}\right)^{1/\gamma'} \le l\|u\|^{q+1}$$

where $l = \|h\|_{L^\gamma(\Omega)}$. Choosing

$$\lambda < \frac{bn}{(2n-1-q)l}M_0 := \lambda_0$$

where $M_0 = \inf_{u\in\Gamma\backslash\{0\}\cap H^+} \|u\|^{n-1-q} > 0$, we get that $\lambda l(2n-1-q)\|u\|^{1+q} < nb\|u\|^n$ for any $u \in \Gamma\backslash\{0\}\cap H^+$. Then for $u \in \Gamma\backslash\{0\}\cap H^+$ and $p+1 > 2n-1$,

$$\begin{aligned} B(u) + nb\|u\|^n - (2n-1)\int_\Omega (|x|^{-\mu} * F(u))f(u).u - \lambda(2n-1-q)H(u) \geq \\ \int_\Omega (|x|^{-\mu} * F(u))(f'(u)u^2 - (2n-1)f(u).u) + \int_\Omega (|x|^{-\mu} * f(u).u)f(u).u\ dx \\ + nb\|u\|^n - (2n-1-q)\lambda H(u) > 0. \end{aligned}$$

Therefore $\Gamma_0 > 0$. ■

Now we establish the claim made in Case-2. We notice from Lemma 4.3 and Equation (4.3) that for $u \in H^+\backslash\{0\}$, there exists a $t_* > 0$, local maximum of ψ_u verifying $\psi_u(t_*) - \lambda H(u) > 0$ since $t_*u \in \Gamma \setminus \{0\} \cap H^+$. From $\psi_u(0) = 0$, $\psi_u(t_*) > \lambda H(u) > 0$ and $\lim_{t\to\infty}\psi_u(t) = -\infty$, there exists $t_1 = t_1(u) < t_* < t_2(u) = t_2$ such that $\psi_u(t_1) = \lambda\int_\Omega h(x)|u|^{q+1}\ dx = \psi_u(t_2)$ with $\psi_u'(t_1) > 0, \psi_u'(t_2) < 0$. Therefore, $t_1u \in N^+_{\lambda,M}$ and $t_2u \in N^-_{\lambda,M}$. We now prove that $t_1u \in N^+_{\lambda,M}$ and $t_2u \in N^-_{\lambda,M}$ are unique. If not then there exists $t_3u \in N^+_{\lambda,M}$ and t_{**} such that $t_2 < t_{**} < t_3$ and $\psi_u'(t_{**}) = 0$ and $\psi_u(t_{**}) < \lambda H(u)$. But our Lemma 4.3 induces that if $\psi_u'(t_{**}) = 0$ then $\psi_u(t_{**}) > \lambda H(u)$ which is a contradiction.

In the sequel, we will denote t_* the smallest critical point of ψ_u.

Lemma 4.4 *If* $\lambda \in (0,\lambda_0)$ *then* $N^0_{\lambda,M} = \{0\}$.

Proof. Suppose $u \not\equiv 0$ and $u \in N^0_{\lambda,M}$. Then $\Phi'_{u,M}(1) = 0$ and $\Phi''_{u,M}(1) = 0$, *i.e.*

$$a\|u\|^{2n} + b\|u\|^n = \lambda H(u) + \int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx \text{ and} \tag{4.5}$$

$$(2n-1)a\|u\|^{2n} + (n-1)b\|u\|^n = \lambda qH(u) + B(u). \tag{4.6}$$

Let $u \in H^+ \cap N^0_{\lambda,M}$, then from (4.5) and (4.6) (by replacing the value $\lambda H(u)$), we obtain

$$2\sqrt{(2n-1-q)(n-1-q)ab\|u\|^{3n}} \leq B(u)$$

which implies $u \in \Gamma\backslash\{0\}\cap H^+$. Again from (4.5), (4.6) and substituting the value of $a\|u\|^{2n}$, we obtain

$$B(u) - (2n-1)\int_\Omega (|x|^{\mu} * F(u))f(u).u + nb\|u\|^n - \lambda(2n-1-q)H(u) = 0$$

which contradicts Lemma 4.3. If $u \in H_0^- \cap N^0_{\lambda,M}$ and from Case-1, "1" is the only critical point of $\Phi_{u,M}$ and $\Phi''_{u,M}(1) = 0$. But $u \in H_0^-$ implies that $\psi_u'(1) < 0$ and then $\phi''_{u,M}(1) < 0$ which is a contradiction and the lemma is proved. ■

4.3 Existence of weak solutions to $(\mathcal{P}_{\lambda,M})$

In this section we prove that $\mathcal{J}_{\lambda,M}$ is bounded below on $N_{\lambda,M}$ and achieves its minimum. Define $\theta = \inf_{u\in N_{\lambda,M}} \mathcal{J}_{\lambda,M}(u)$.

Theorem 4.5 *$\mathcal{J}_{\lambda,M}(u)$ is bounded below and coercive on $N_{\lambda,M}$ such that $\theta \geq -C(q,n,b)\lambda^{\frac{n}{n-q-1}}$.*

Proof. Let $u \in N_{\lambda,M}$. Then,

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(u) &= \frac{1}{n}\left[\frac{a}{2}\|u\|^{2n} + b\|u\|^n\right] - \frac{\lambda}{q+1}H(u) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx \\
&= \frac{1}{n}\left[\frac{a}{2}\|u\|^{2n} + b\|u\|^n\right] - \frac{\lambda}{q+1}H(u) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx \\
&- \frac{1}{p+2}\left[a\|u\|^{2n} + b\|u\|^n - \lambda H(u) - \int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx\right] \\
&= a\|u\|^{2n}\left(\frac{p+2-2n}{2n(p+2)}\right) + b\|u\|^n\left(\frac{p+2-n}{n(p+2)}\right) - \lambda\left(\frac{p+1-q}{(1+q)(p+2)}\right)H(u) \\
&- \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))\left(F(u) - \frac{2f(u)u}{p+2}\right)\ dx.
\end{aligned}$$

Since $0 \leq F(u) \leq \frac{2}{p+2}f(u).u$ and $H(u) \leq l\|u\|^{q+1}$. Then by the Sobolev inequality we obtain

$$\mathcal{J}_{\lambda,M}(u) \geq a\|u\|^{2n}\left(\frac{p+2-2n}{2n(p+2)}\right) + b\|u\|^n\left(\frac{p+2-n}{n(p+2)}\right) - \lambda l\left(\frac{p+1-q}{(1+q)(p+2)}\right)\|u\|^{q+1}.$$

Therefore since $q < n-1$, $\mathcal{J}_{\lambda,M}$ is coercive on $N_{\lambda,M}$, *i.e.* $\mathcal{J}_{\lambda,M}(u) \to \infty$ as $\|u\| \to \infty$.
For $u \in N_{\lambda,M}$ we have also,

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(u) &= \frac{b}{n}\|u\|^n - \frac{\lambda}{q+1}H(u) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx \\
&\quad + \frac{1}{2n}\left(\lambda H(u) + \int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx - b\|u\|^n\right) \\
&= \frac{1}{2n}b\|u\|^n - \lambda\left(\frac{1}{q+1} - \frac{1}{2n}\right)H(u) + \frac{1}{2}\left(\int_\Omega (|x|^{-\mu} * F(u))\left(\frac{f(u)u}{n} - F(u)\right)\ dx\right) \\
&\geq \frac{1}{2n}b\|u\|^n - \lambda\left(\frac{1}{q+1} - \frac{1}{2n}\right)H(u)
\end{aligned}$$

since $\left(\frac{f(u)u}{n} - F(u)\right) \geq 0$. Then for $u \in H_0^-$, we get $\mathcal{J}_{\lambda,M}(u) \geq 0$.
Now for $u \in H^+$, setting $r = \frac{n}{1+q}$ and by the Sobolev embedding we obtain

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(u) &\geq \frac{b}{2n}\|u\|^n - \frac{\lambda(2n-1-q)}{2n(q+1)}H(u) \geq \frac{b}{2n}\|u\|^n - \frac{\lambda(2n-1-q)}{2n(q+1)}l\left(\int_\Omega |u|^n\ dx\right)^{1/r} \\
&= c_1\|u\|^n - c_2\|u\|^{q+1}
\end{aligned}$$

where $c_1 = \frac{b}{2n}$ and $c_2 = c_2(\lambda)$.
We observe that the minimum of the function $g(x) = c_1x^n - c_2x^{q+1}$ is achieved at $x =$

$\left(\frac{c_2(q+1)}{c_1 n}\right)^{\frac{1}{n-q-1}}$. Therefore,

$$\inf_{u\in N_{\lambda,M}} \mathcal{J}_{\lambda,M}(u) \geq g\left(\frac{c_2(q+1)}{c_1 n}\right)^{\frac{1}{n-q-1}} = \left(\frac{c_2^n}{c_1^{q+1}}\right)^{\frac{1}{n-1-q}} \left(\left(\frac{q+1}{n}\right)^{\frac{n}{n-1-q}} - \left(\frac{q+1}{n}\right)^{\frac{q+1}{n-1-q}}\right).$$

From this it follows that

$$\theta \geq -C(q,n,b)\lambda^{\frac{n}{n-q-1}}$$

where $C(q,n,b) > 0$. This completes the proof of Theorem 4.5. ■

Now since $\mathcal{J}_{\lambda,M}$ is bounded below on $N_{\lambda,M}$, by the Ekeland variational principle we get a sequence $\{u_k\}_{k\in\mathbb{N}} \subset N_{\lambda,M}\backslash\{0\}$ such that

$$\begin{cases} \mathcal{J}_{\lambda,M}(u_k) \leq \theta + \dfrac{1}{k}; \\ \mathcal{J}_{\lambda,M}(v) \geq \mathcal{J}_{\lambda,M}(u_k) - \dfrac{1}{k}\|u_k - v\|, \quad \forall v \in N_{\lambda,M}. \end{cases} \tag{4.7}$$

Lemma 4.6 *There exists a constant $C_0 > 0$ such that $\theta \leq -C_0$.*

Proof. Let $u \in H^+$, then $\exists\ t_1(u) > 0$ such that $t_1 u \in N^+_{\lambda,M}$ and $\psi_{u,M}(t_1) = \lambda H(u)$. In that case,

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(t_1 u) &= \frac{1}{n}\left(\frac{a}{2}\|t_1 u\|^{2n} + b\|t_1 u\|^n\right) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(t_1 u))F(t_1 u)\ dx - \frac{\lambda}{q+1}\int_\Omega h(x)|t_1 u|^{q+1}\ dx \\
&= \frac{1}{n}\left(\frac{a}{2}\|t_1 u\|^{2n} + b\|t_1 u\|^n\right) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(t_1 u))F(t_1 u)\ dx \\
&\quad - \frac{1}{q+1}\left(a\|t_1 u\|^{2n} + b\|t_1 u\|^n - \int_\Omega (|x|^{-\mu} * F(t_1 u))f(t_1 u)t_1 u\ dx\right).
\end{aligned}$$

Since $\Phi'_{u,M}(t_1) = 0$, $\Phi''_{u,M}(t_1) > 0$ and from (4.1) we obtain

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(t_1 u) &= \frac{-(n-1-q)}{2n(q+1)} b\|t_1 u\|^n + \int_\Omega (|x|^{-\mu} * F(t_1 u))\left(\frac{2n+q}{2n(q+1)} f(t_1 u)t_1 u \right. \\
&\quad \left. - \frac{1}{2}F(t_1 u) - \frac{f'(t_1 u)(tu)^2}{2n(q+1)}\right)\ dx - \frac{1}{2n(q+1)}\int_\Omega (|x|^{-\mu} * f(t_1 u)t_1 u) f(t_1 u)t_1 u\ dx \\
&\leq \frac{-(n-1-q)}{2n(q+1)} b\|t_1 u\|^n + \int_\Omega (|x|^{-\mu} * F(t_1 u))\left(\frac{2n+q}{2n(q+1)} - \frac{p+2}{2n(q+1)}\right. \\
&\quad \left. - \frac{p+1}{2n(q+1)}\right) f(t_1 u)t_1 u\ dx - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(t_1 u))F(t_1 u)\ dx.
\end{aligned}$$

Since $q < n-1$ and $p+1 > 2n-1$ we set $2n + q - (2p+3) \leq 3n - 1 - (4n-1) < 0$ and then $\theta \leq \inf_{u\in N^+_{\lambda,M}\cap H^+} \mathcal{J}_{\lambda,M}(u) \leq -C_0 < 0$. ■

Then by (4.7) and Lemma 4.6, we have for large k,

$$\mathcal{J}_{\lambda,M}(u_k) \leq -\frac{C_0}{2}. \tag{4.8}$$

Also since $u_k \in N_{\lambda,M}\backslash\{0\}$ we have

$$\mathcal{J}_{\lambda,M}(u_k) = a\|u_k\|^{2n}\left(\frac{p+2-2n}{2n(p+2)}\right) + b\|u_k\|^n\left(\frac{p+2-n}{n(p+2)}\right) - \lambda\left(\frac{p+1-q}{(1+q)(p+2)}\right)H(u_k) - \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u_k))\left(F(u_k) - \frac{2f(u_k)u_k}{p+2}\right)\ dx.$$

then together with (4.8), we have

$$-\lambda\left(\frac{p+1-q}{(1+q)(p+2)}\right)H(u_k) \leq -\frac{C_0}{2} \Longrightarrow H(u_k) \geq \frac{C_0(p+2)(1+q)}{2\lambda(p+1-q)}C_0 > 0$$

i.e.

$$H(u_k) > C > 0 \ \text{ and } \ u_k \in N_{\lambda,M} \cap H^+ \ \text{ for } \ k \ \text{ large enough.} \tag{4.9}$$

The following result shows that minimizers for $\mathcal{J}_{\lambda,M}$ in any of the subsets of $N_{\lambda,M}$ are critical points for $\mathcal{J}_{\lambda,M}$.

Lemma 4.7 *Let u be a local minimizer for $J_{\lambda,M}$ on any subsets of $N_{\lambda,M}$ such that $u \notin N^0_{\lambda,M}$. Then u is a critical point of $\mathcal{J}_{\lambda,M}$.*

Proof. Let u be a local minimizer for $\mathcal{J}_{\lambda,M}$. Then, in any case u is a minimizer for $\mathcal{J}_{\lambda,M}$ under the constraint $I_{\lambda,M}(u) := \langle \mathcal{J}'_{\lambda,M}(u), u\rangle = 0$. Hence , by the theory of Lagrange multipliers , there exists a $\mu \in \mathbb{R}$ such that $\mathcal{J}'_{\lambda,M} = \mu I'_{\lambda,M}(u)$. Thus $\langle \mathcal{J}'_{\lambda,M}(u), u\rangle = \mu\langle I'_{\lambda,M}(u), u\rangle = \mu\Phi''_{\lambda,M}(1) = 0$, but $u \notin N^0_{\lambda,M}$ and so $\Phi''_{\lambda,M}(1) \neq 0$. Hence $\mu = 0$. ■

Lemma 4.8 *Let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l}M_0$. Then given any $u \in N_{\lambda,M}\backslash\{0\}$, then there exists $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W^{1,n}_0(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$, and $\xi(w)(u-w) \in N_{\lambda,M}$ and for all $w \in W^{1,n}_0(\Omega)$*

$$\langle \xi'(0), w\rangle = \frac{n(2a\|u\|^n + b)\int_\Omega |\nabla(u)|^{n-2}\nabla u.\nabla w\ dx - \lambda(q+1)\int_\Omega h(x)|u|^{q-1}uw\ dx - \langle S(u), w\rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)} \tag{4.10}$$

where

$$R(u) = \int_\Omega (|x|^{-\mu} * F(u))(qf(u) - f'(u).u).u\ dx - \int_\Omega (|x|^{-\mu} * f(u).u)f(u)u\ dx$$

and

$$\langle S(u), w\rangle = \int_\Omega (|x|^{-\mu} * F(u))(f'(u)u + f(u))w\ dx + \int_\Omega (|x|^{-\mu} * f(u)u)f(u)w\ dx.$$

Proof. Fix $u \in N_{\lambda,M}\backslash\{0\}$, define a function $G_u : \mathbb{R} \times W^{1,n}_0(\Omega) \to \mathbb{R}$ as

$$G_u(t,v) = at^{2n-1-q}\|u-v\|^{2n} + bt^{n-1-q}\|u-v\|^n - \frac{1}{t^q}\int_\Omega (|x|^{-\mu} * F(t(u-v)))f(t(u-v)).(u-v)\ dx - \lambda\int_\Omega h|u-v|^{q+1}\ dx.$$

Then $G_u \in C^1(\mathbb{R}\times W_0^{1,n}(\Omega),\mathbb{R})$ and

$$G_u(1,0) = a\|u\|^{2n} + b\|u\|^n - \int_\Omega (|x|^{-\mu} * F(u))f(u).u\ dx - \lambda\int_\Omega h|u|^{q+1}\ dx = \Phi'_{u,M}(1) = 0$$

and

$$\frac{\partial}{\partial t}G_u(1,0) = a(2n-1-q)\|u\|^{2n}+b(n-1-q)\|u\|^n+q\int_\Omega(|x|^{-\mu}*F(u))f(u).u-B(u) = \Phi''_{u,M}(1) \neq 0.$$

Then by the implicit function theorem, there exists $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W_0^{1,n}(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$ and $G_u(\xi(w), w) = 0\ \ \forall w \in B(0,\epsilon)$ which is equivalent to $\langle \mathcal{J}'_{\lambda,M}(\xi(w)(u-w), \xi(w)(u-w))\rangle = 0\ \ \forall w \in B(0,\epsilon)$. Thus, $\xi(w)(u-w) \in N_{\lambda,M}$ and differentiating

$$\begin{aligned}
&G_u(\xi(w), w) = a(\xi(w))^{2n-1-q}\|u-w\|^{2n} + b(\xi(w))^{n-1-q}\|u-w\|^n\\
&-\frac{1}{(\xi(w))^q}\int_\Omega(|x|^{-\mu} * F(\xi(w))(u-w))f(\xi(w)(u-w))(u-w) - \lambda\int_\Omega h(x)|u-w|^{q+1} = 0
\end{aligned}$$

with respect to w, we obtain (4.10). ■

Similarly we have:

Lemma 4.9 *Let $\lambda \in (0,\lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l}M_0$. Then there exists $u \in N^-_{\lambda,M}\backslash\{0\}$, then there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0,\epsilon) \subset W_0^{1,n}(\Omega) \to \mathbb{R}$ such that $\xi^-(0) = 1$, and $\xi^-(w)(u-w) \in N^-_{\lambda,M}$ and for all $w \in W_0^{1,n}(\Omega)$*

$$\langle(\xi^-)'(0), w\rangle = \frac{n(2a\|u\|^n + b)\int_\Omega |\nabla(u)|^{n-2}\nabla u.\nabla w\ dx - \lambda(q+1)\int_\Omega h(x)|u|^{q-1}uw\ dx - \langle S(u), w\rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)}$$

where $R(u)$ and $S(u)$ are as in lemma 4.8.

Proof. For any $u \in N^-_{\lambda,M}$, $\Phi'_{u,M}(1) = 0$ and $\Phi''_{u,M}(1) < 0$. This implies $u \in \Gamma\backslash\{0\}$. Then by Lemma 4.8 there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0,\epsilon) \subset W_0^{1,n}(\Omega) \to \mathbb{R}$ such that $\xi^-(0) = 1$, and $\xi^-(w)(u-w) \in N_{\lambda,M}$ for all $w \in B(0,\epsilon)$. Then by the continuity of $\mathcal{J}'_{\lambda,M}$ and ξ^- and by choosing ϵ small enough we have

$$\begin{aligned}
\Phi''_{\xi^-(u)(u-w),M}(1) &= n\|\xi^-(u)(u-w)\|^{2n}m(\|\xi^-(u)(u-w)\|^n) + (n-1)\|\xi^-(u)(u-w)\|^n m(\|tu\|^n)\\
&- \lambda q\int_\Omega h(x)|\xi^-(u)(u-w)|^{q+1}\ dx\\
&- \int_\Omega(|x|^{-\mu} * f(\xi^-(u)(u-w)).\xi^-(u)(u-w))f(\xi^-(u)(u-w))\xi^-(u)(u-w)\ dx\\
&- \int_\Omega(|x|^{-\mu} * F(\xi^-(u)(u-w)))f'(\xi^-(u)(u-w))(\xi^-(u)(u-w))^2\ dx < 0
\end{aligned}$$

that implies $\xi^-(w)(u-w) \in N^-_{\lambda,M}$. ■

Now we prove the following result:

Proposition 4.10 *Let* $\lambda \in (0, \lambda_0)$ *where* $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. *Assume* $u_k \in N_{\lambda,M}$ *is satisfying* (4.7). *Then* $\|\mathcal{J}_{\lambda,M}^{'}(u_k)\|_* \to 0$ *as* $k \to \infty$.

Proof. Step 1: $\liminf_{k\to\infty} \|u_k\| > 0$.
We know that from (4.9) that for large k, $H(u_k) \geq C > 0$, so by using Hölder inequality we obtain $C < \ H(u_k) \leq C_1 \|u_k\|^{q+1}$.
Step 2: We claim that

$$\liminf_{k\to\infty} (2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n + q\int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\ dx - B(u_k) > 0.$$

Without loss of generality, we can assume that $u_k \in N_{\lambda,M}^+$ (if not replace u_k by $t_1(u_k)u_k$). Arguing by contradiction, suppose that there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$0 \leq (2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n + q\int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\ dx - B(u_k) = o_k(1).$$

From Step 1 and the above equation we obtain that $\liminf_{k\to\infty} B(u_k) > 0$ and $(2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n \leq B(u_k)$ *i.e.* $u_k \in \Gamma\backslash\{0\}$ for all large k.
Since $u_k \in N_{\lambda,M}^+ \backslash \{0\}$

$$-nb\|u_k\|^n + \lambda(2n-1-q)H(u_k) + (2n-1)\int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\ dx - B(u_k) = o_k(1)$$

which is a contradiction since $\Gamma_0 > 0$.
Step 3: $\|\mathcal{J}_{\lambda,M}^{'}(u_k)\|_* \to 0$ as $k \to \infty$.
By using Lemma 4.8, there exists a differentiable function $\xi_k : B(0, \epsilon_k) \to \mathbb{R}$ for some $\epsilon_k > 0$ such that $\xi_k(0) = 1$ and $\xi_k(w)(u_k - w) \in N_{\lambda,M} \quad \forall w \in B(0, \epsilon_k)$. Choose $0 < \rho < \epsilon_k$ and $f \in W_0^{1,n}(\Omega)$ such that $\|f\| = 1$. Let $w_\rho = \rho f$. Then $\|w\|_\rho = \rho < \epsilon_k$ and define $\eta_\rho = \xi_k(w_\rho)(u_k - w_\rho)$. Then from the Taylor expansion and (4.7), we obtain

$$\begin{aligned} \frac{1}{k}\|\eta_\rho - u_k\| &\geq \mathcal{J}_{\lambda,M}(u_k) - \mathcal{J}_{\lambda,M}(\eta_\rho) = \langle \mathcal{J}_{\lambda,M}^{'}(\eta_\rho), u_k - \eta_\rho\rangle + o(\|u_k - \eta_\rho\|) \\ &= (1 - \xi_k(w_\rho))\langle \mathcal{J}_{\lambda,M}^{'}(\eta_\rho), u_k\rangle + \rho\xi_k(w_\rho)\langle \mathcal{J}_{\lambda,M}^{'}(\eta_\rho), f\rangle + o(\|u_k - \eta_\rho\|). \end{aligned} \tag{4.11}$$

We also infer

$$\frac{1}{\rho}\|\eta_\rho - u_k\| = \|\frac{(\xi_k(w_\rho) - 1)}{\rho}u_k - \xi_k(w_\rho)f\| \to \|u_k\langle \xi_k^{'}(0), f\rangle - f\| \ \text{ as } \ \rho \to 0.$$

Since $u_k \in N_{\lambda,M}$, we have also $\frac{1-\xi_k(w_\rho)}{\rho}\langle \mathcal{J}_{\lambda,M}^{'}(\eta_\rho), u_k\rangle \to 0$ as $\rho \to 0$.
Thus, dividing the expression in (4.11) by ρ and doing $\rho \to 0^+$, we get

$$\langle \mathcal{J}_{\lambda,M}^{'}(u_k), f\rangle \leq \frac{\|f\|}{k}(\|u_k\|\|\xi_k^{'}(0)\|_* + O(1))$$

which implies that

$$\|\mathcal{J}_{\lambda,M}^{'}(u_k)\|_* \to 0 \ \text{ as } \ k \to \infty$$

if $\|\xi_k'(0)\|_*$ is bounded uniformly in k. To prove that, using (4.4) and the boundedness of the sequence $\{u_k\}$ in $W_0^{1,n}(\Omega)$, we only need to show that for any $f \in W_0^{1,n}(\Omega)$, $\langle S(u_k), f\rangle$ is uniformly bounded in k. For the subcritical case, *i.e.* $\beta \in (0, \frac{n}{n-1})$, it holds since for any $\epsilon > 0$ and $q > 1$, there exists $C_{\epsilon,q,\beta} > 0$ such that

$$\exp(q|t|^\beta) \leq C_{\epsilon,q,\beta} \exp(\epsilon|t|^{\frac{n}{n-1}}), \quad \forall t \in \mathbb{R}.$$

Then by Theorem 1.1 we obtain $\langle S(u_k), f\rangle \leq C\|f\|$ with $C > 0$ independent of k. Consider now the critical case, *i.e.* $\beta = \frac{n}{n-1}$. From the boundedness of $R(u_k)$ (see the statement of Lemma 4.8), it follows that

$$\sup_k \int_\Omega (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx < \infty,$$

$$\sup_k \int_\Omega (|x|^{-\mu} * F(u_k)) f'(u_k) u_k^2 \, dx < \infty$$

and

$$\sup_k \int_\Omega (|x|^{-\mu} * f(u_k) u_k) f(u_k) u_k \, dx < \infty.$$

Then for any $\phi \in C_c^\infty(\Omega)$, we have by Vitali's convergence theorem and up to a subsequence

$$\langle S(u_k), \phi\rangle \to \langle S(u_0), \phi\rangle \tag{4.12}$$

where u_0 is the weak limit of $(u_k)_{k\in\mathbb{N}}$ in $W_0^{1,n}(\Omega)$. From (4.12), we have that there exists $C > 0$ independent of k such that

$$|\langle S(u_k), \phi\rangle| \leq C\|\phi\|. \tag{4.13}$$

Using a density argument, we conclude that (4.13) holds for any $\phi \in W_0^{1,n}(\Omega)$. This completes the proof in the critical case.

■

Theorem 4.11 *Let $\beta < \frac{n}{n-1}$ and let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. Then there exists a weak solution to $(\mathcal{P}_{\lambda,M})$ $u_\lambda \in N_{\lambda,M}^+ \cap H^+$ such that $\mathcal{J}_{\lambda,M}(u_\lambda) = \inf_{u\in N_{\lambda,M}\setminus\{0\}} \mathcal{J}_{\lambda,M}(u)$.*

Proof. Let u_k be a minimizing sequence satisfying $\mathcal{J}_{\lambda,M}(u_k) \to \theta$ as $k \to \infty$ and $\mathcal{J}_{\lambda,M}(v) \geq \mathcal{J}_{\lambda,M}(u_k) - \frac{1}{k}\|u_k - v\|$, $\forall v \in N_\lambda$. Using $\mathcal{J}_{\lambda,M}(|u|) \leq \mathcal{J}_{\lambda,M}(u)$ for any $u \in W_0^{1,n}(\Omega)$ and from the proof of the Ekeland principle (see [33, p. 51-53]), we can assume that u_k is non-negative. By using Proposition 4.10 we obtain $\{u_k\}$ is (PS)$_\theta$ sequence. Then from Lemma 3.2 we get $\{u_k\}$ is a bounded sequence in $W_0^{1,n}(\Omega)$. Also there exists a subsequence of $\{u_k\}$ (denoted by same sequence) and a non-negative u_λ such that $u_k \rightharpoonup u_\lambda$ weakly in $W_0^{1,n}(\Omega)$ and $u_k \to u_\lambda$ strongly in $L^r(\Omega)$ for $r \geq 1$ and $u_k \to u_\lambda$ a.e. in Ω. Then using $f(t) \leq C_{\epsilon,\beta} \exp(\epsilon t^{\frac{n}{n-1}})$ for $\epsilon > 0$ small enough and from Theorem 1.1, we obtain that $f(u_k)$ and $(|x|^{-\mu} * F(u_k))$ are uniformly bounded in $L^q(\Omega)$ for all $q > 1$. Then again by Vitali's convergence theorem, we obtain

$$\left| \int_\Omega (|x|^{-\mu} * F(u_k)) f(u_k)(u_k - u_\lambda)\ dx \right| \to 0 \text{ as } \ k \to \infty.$$

and by Proposition 4.10, we have $\langle \mathcal{J}'_{\lambda,M}(u_k), u_k - u_\lambda \rangle \to 0$. Then we conclude that

$$m(\|u_k\|^n) \int_\Omega |\nabla u_k|^{n-2} \nabla u_k . \nabla(u_k - u_\lambda)\ dx \to 0. \tag{4.14}$$

On the other hand, using $u_k \to u_\lambda$ weakly and by boundedness of $m(\|u_k\|^n)$ we have

$$m(\|u_k\|^n) \int_\Omega |\nabla u_\lambda|^{n-2} \nabla u_\lambda . \nabla(u_k - u_\lambda)\ dx \to 0. \tag{4.15}$$

Substracting (4.15) from (4.14), we get,

$$m(\|u_k\|^n) \int_\Omega (|\nabla u_k|^{n-2} \nabla u_k - |\nabla u_\lambda|^{n-2} \nabla u_\lambda) . \nabla(u_k - u_\lambda)\ dx \to 0.$$

Now by using this and following inequality,

$$|a_1 - a_2|^n \leq 2^{n-2}(|a|_1^{n-2} a_1 - |a_2|^{n-2} a_2)(a_1 - a_2) \text{ for all } \ a_1, a_2 \in \mathbb{R}^n$$

with $a_1 = \nabla u_k$ and $a_2 = \nabla u_\lambda$, we obtain

$$m(\|u_k\|^n) \int_\Omega |\nabla u_k - \nabla u_\lambda|^n\ dx \to 0 \ \text{ as } k \to \infty.$$

Since $m(t) \geq b$, then we obtain $u_k \to u_\lambda$ strongly in $W_0^{1,n}(\Omega)$ and by Lemma 3.5

$$\int_\Omega (|x|^{-\mu} * F(u_k)) f(u_k) \phi\ dx \to \int_\Omega (|x|^{\mu} * F(u_\lambda)) f(u_\lambda) \phi\ dx$$

and also

$$\int_\Omega h(x) u_k^{q-1} u_k \phi\ dx \to \int_\Omega h(x) u_\lambda^{q-1} u_\lambda \phi\ dx$$

for all $\phi \in W_0^{1,n}(\Omega)$. Therefore, u_λ satisfies $(\mathcal{P}_{\lambda,M})$ in weak sense and hence $u_\lambda \in N_{\lambda,M}$. Moreover, $\theta \leq \mathcal{J}_{\lambda,M}(u_\lambda) \leq \liminf_{k\to\infty} \mathcal{J}_{\lambda,M}(u_k) = \theta$. Hence u_λ is a minimizer for $\mathcal{J}_{\lambda,M}$ in $N_{\lambda,M}$.

Using (4.9), we have $\int_\Omega h(x)|u_\lambda|^{q+1} > 0$, then there exists $t_1(u_\lambda)$ such that $t_1(u_\lambda)u_\lambda \in N^+_{\lambda,M}$. We now claim that $t_1(u_\lambda) = 1$ *i.e.* $u_\lambda \in N^+_{\lambda,M}$. Suppose that $t_1(u_\lambda) < 1$ and then $t_2(u_\lambda) = 1$ and $u_\lambda \in N^-_{\lambda,M}$. Now $\mathcal{J}_{\lambda,M}(t_1(u_\lambda)u_\lambda) < \mathcal{J}_{\lambda,M}(u_\lambda) \leq \theta$ which yields a contradiction, since $t_1(u_\lambda)u_\lambda \in N_{\lambda,M}$. Thus, u_λ is non-negative and nontrivial. From the strong comparison principle (see for instance [34]), we get $u_\lambda > 0$ in Ω. ■

Theorem 4.12 *Let* $\beta < \frac{n}{n-1}$ *and let* $\lambda \in (0, \lambda_0)$ *where* $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. *Then* $u_\lambda \in N^+_{\lambda,M} \cap H^+$ *is a non-negative local minimum for* $\mathcal{J}_{\lambda,M}$ *in* $W_0^{1,n}(\Omega)$.

Proof. Since $u_\lambda \in N^+_{\lambda,M} \cap H^+$ then we have a $t_*(u_\lambda)$ such that $1 = t_1(u_\lambda) < t_*(u_\lambda)$. Hence by the continuity of $u \to t_*(u)$, given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$(1+\epsilon) < t_*(u_\lambda - w) \quad \text{for all } \|w\| < \delta_\epsilon$$

and from Lemma 4.8 we have, for $\delta > 0$ small enough, a continuously differentiable map $t : B(0,\delta) \to \mathbb{R}^+$ such that $t(w)(u_\lambda - w) \in N_{\lambda,M}, t(0) = 1$. Then we have

$$t_1(u_\lambda - w) = t(w) < 1+\epsilon < t_*(u_\lambda - w)$$

for δ small enough. Since $t_*(u_\lambda - w) > 1$ for all $\|w\| < \delta$, we obtain

$$\mathcal{J}_{\lambda,M}(u_\lambda) \leq \mathcal{J}_{\lambda,M}(t_1(u_\lambda - w)(u_\lambda - w)) \leq \mathcal{J}_{\lambda,M}(u_\lambda - w), \text{ if } \|w\| < \delta$$

which implies that u_λ is a local minimizer for $\mathcal{J}_{\lambda,M}$.

■

Theorem 4.13 *Let $\beta < \frac{n}{n-1}$ and let $\lambda \in (0,\lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. Then $\mathcal{J}_{\lambda,M}$ achieve its minimizers on $N^-_{\lambda,M}$.*

Proof. Let $u \in N^-_{\lambda,M}$. Then

$$(2n-1)a\|u\|^{2n} + (n-1)b\|u\|^n - \lambda q H(u) - \int_\Omega (|x|^{-\mu} * f(u)u) f(u).u - \int_\Omega (|x|^{-\mu} * F(u)) f'(u)u^2 < 0.$$

Then (4.5) implies that

$$(2n-1-q)a\|u\|^{2n} + (n-1-q)b\|u\|^n + q\int_\Omega (|x|^{-\mu} * F(u)) f(u).u - \left(\int_\Omega (|x|^{-\mu} * f(u)u) f(u).u + \int_\Omega (|x|^{-\mu} * F(u)) f'(u)u^2\right) < 0. \tag{4.16}$$

Using $p+1 > 2n$ it is easy to deduce from (4.16) that $\exists\ c > 0,\ \|u\| \geq c > 0$ for any $u \in N^-_{\lambda,M}$ from which it follows that $N^-_{\lambda,M}$ is a closed set. Also as in Lemma 4.3 we can prove that $N^-_{\lambda,M} \subset \Gamma$ and then $\inf_{u \in N^-_{\lambda,M}} B(u) \geq \tilde{c} > 0$. Therefore, for $\lambda < \lambda_0$ small enough,

$$\inf_{u \in N^-_{\lambda,M}\setminus\{0\}} B(u) + nb\|u\|^n - (2n-1-q)\lambda H(u) - (2n-1)\int_\Omega |x|^{-\mu} * F(u) f(u) u > 0. \tag{4.17}$$

Now taking $\theta^- = \min_{u \in N^-_{\lambda,M}} \mathcal{J}_{\lambda,M}(u) > -\infty$. From Ekeland variational principle, there exist $\{v_k\}_{k\in\mathbb{N}}$ a non-negative minimizing sequence such that

$$\mathcal{J}_{\lambda,M}(v_k) \leq \inf_{u \in N^-_{\lambda,M}} \mathcal{J}_{\lambda,M}(u) + \frac{1}{k} \text{ and } \mathcal{J}_{\lambda,M}(u) \geq \mathcal{J}_{\lambda,M}(v_k) - \frac{1}{k}\|v_k - u\| \ \ \forall\ u \in N^-_{\lambda,M}.$$

From $\mathcal{J}_{\lambda,M}(v_k) \to \theta^-$ as $k \to \infty$ and $v_k \in N_{\lambda,M}$, it is easy to prove that $\|v_k\| \leq C$ (as in Lemma 3.2). Indeed,

$$\left| a\|v_k\|^{2n} + b\|v_k\|^n - \lambda H(v_k) - \int_\Omega (|x|^{-\mu} * F(v_k))f(v_k)v_k \, dx \right| = o(\|v_k\|)$$

and

$$C + o(\|v_k\|) \geq \mathcal{J}_{\lambda,M}(v_k) - \frac{1}{2n}\langle \mathcal{J}'_{\lambda,M}(v_k), v_k\rangle \geq \frac{b}{2n}\|v_k\|^{2n} - C(\lambda)\|v_k\|^{q+1}$$

imply $\|v_k\| \leq C$. Thus we get $\|S(v_k)\|_* \leq C_1$ and from (4.17) we have $\|\xi_k^-(0)\|_* \leq C_2$. Now the rest of the proof can be done as in the proof of Theorem 4.11 with the help of Lemma 4.9. ■

Proof of Theorem 2.4 for $\beta \in \left(1, \frac{n}{n-1}\right)$: The proof follows from Lemma 4.11 and Theorem 4.12. ■

Now we establish the following compactness result in the critical case.

Lemma 4.14 *There exists* $C = C(p,q,n) > 0$ *such that for any* $\{u_k\} \subset W_0^{1,n}(\Omega)$ *satisfying*

$$\mathcal{J}'_{\lambda,M}(u_k) \to 0 \quad \text{and} \quad \mathcal{J}_{\lambda,M}(u_k) \to c \leq \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} - C\lambda^{\frac{2(p+2)}{2p+3-q}} \quad \text{as } k \to \infty$$

is relatively compact in $W_0^{1,n}(\Omega)$.

Proof. As in Lemma 3.2 we can prove that $\{u_k\}$ is bounded in $W_0^{1,n}(\Omega)$ and up to a subsequence $u_k \to u$ in $L^\alpha(\Omega)$ for all $\alpha \geq 1$, $u_k(x) \to u$ a.e in Ω and $\nabla u_k \to \nabla u$ a.e. in Ω and weakly in $L^n(\Omega)$. Also still up to a subsequence, there exist radon measures ν_1 and ν_2 such that $|\nabla u_k|^n \to \nu_1$ and $(|x|^{-\mu} * F(u_k))f(u_k)u_k \to \nu_2$ weakly as $k \to \infty$.
Let $B = \{x \in \overline{\Omega} : \exists\ r = r_x > 0, \nu_1(B_r \cap \Omega) < \left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}\}$ and let $A = \overline{\Omega}\backslash B$. Then by Lemma 3.4 we can infer that A is a finite set, say $\{x_1, x_2, \ldots, x_t\}$. Since $\mathcal{J}'_{\lambda,M}(u_k) \to 0$ and since $\{u_k\}$ is bounded in $W_0^{1,n}(\Omega)$, we have that $\forall\ \phi \in C_c^\infty(\Omega)$,

$$\begin{aligned} 0 = \lim_{k\to\infty} \langle \mathcal{J}'_{\lambda,M}(u_k), \phi\rangle = \lim_{k\to\infty} \Big[& m(\|u_k\|^n) \int_\Omega |\nabla u_k|^{n-2}\nabla u_k.\nabla\phi \, dx - \lambda \int_\Omega h(x)|u_k|^{q-1}u_k\phi \, dx \\ & - \int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)\phi \, dx \Big], \end{aligned} \tag{4.18}$$

$$\begin{aligned} 0 = \lim_{k\to\infty} \langle \mathcal{J}'_{\lambda,M}(u_k), u_k\phi\rangle = \lim_{k\to\infty} \Big[& m(\|u_k\|^n) \int_\Omega (|\nabla u_k|^{n-2}\nabla u_k.\nabla\phi u_k \, dx + |\nabla u_k|^n\phi) \\ & - \lambda \int_\Omega h(x)|u_k|^{q+1}\phi \, dx - \int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\phi \, dx \Big], \end{aligned} \tag{4.19}$$

$$\begin{aligned} 0 = \lim_{k\to\infty} \langle \mathcal{J}'_{\lambda,M}(u_k), u\phi\rangle = \lim_{k\to\infty} \Big[& m(\|u_k\|^n) \int_\Omega (|\nabla u_k|^{n-2}\nabla u_k.\nabla\phi u + |\nabla u_k|^{n-2}\nabla u_k.\nabla u\phi) \, dx \\ & - \int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u\phi \, dx \Big] - \lambda \int_\Omega h(x)|u|^q\phi \, dx. \end{aligned} \tag{4.20}$$

Substituting (4.20) in (4.19) and taking into account (4.18), we get $\forall\ \phi \in C_c^\infty(\Omega)$

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\phi = \lim_{k\to\infty} m(\|u_k\|^n)\int_\Omega |\nabla u_k|^n\phi - |\nabla u_k|^{n-2}\nabla u_k.\nabla u\phi\ dx \\ + \int_\Omega (|x|^{-\mu} * F(u))f(u)u\phi\ dx + o_k(1).\end{aligned} \tag{4.21}$$

Now we take the cut-off function $\psi_\delta \in C_c^\infty(\Omega)$ such that $\psi_\delta = 1$ in $B_\delta(x_j)\ \forall\ j = \{1,\ldots,t\}$ and $\psi_\delta(x) = 0$ in $B_{2\delta}^c(x_j)$ with $|\psi_\delta| \leq 1$. Then by taking $\phi = \psi_\delta$ in (4.21) and since as $\delta \to 0$

$$\begin{aligned}0 &\leq \left|\int_\Omega (|\nabla u_k|^{n-2}\nabla u_k.\nabla u)\psi_\delta\ dx\right| \leq \int_\Omega |\nabla u_k|^{n-1}|\nabla u||\psi_\delta|\ dx \\ &\leq \int_{\cup_j B_{2\delta(x_j)}} |\nabla u_k|^{n-1}|\nabla u|\ dx \leq \left(\int_\Omega |\nabla u_k|^n\ dx\right)^{n/(n-1)}\left(\int_{\cup_j B_{2\delta}(x_j)} |\nabla u|^n\ dx\right)^{1/n} \to 0,\end{aligned}$$

we deduce after letting $\delta \to 0$ that

$$\nu_2(A) \geq m_0\nu_1(A) \geq m_0\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1}. \tag{4.22}$$

On the other hand, by using the same argument as in Lemma 3.4 (in particular see (3.13)) we can prove that for any compact set $K \subset \Omega_\delta = \Omega\setminus \cup_{i=1}^t B_{2\delta}(x_i)$

$$\lim_{k\to\infty}\int_K (|x|^{-\mu} * F(u_k))f(u_k)u_k\ dx = \int_K (|x|^{-\mu} * F(u))f(u)u\ dx.$$

Thus, we obtain

$$\begin{aligned}nc = \lim_{k\to\infty} n\ \mathcal{J}_{\lambda,M}(u_k) - \frac{1}{2}\langle \mathcal{J}_{\lambda,M}^{'}(u_k), u_k\rangle = \lim_{k\to\infty}\left(M(\|u_k\|^n) - \frac{1}{2}m(\|u_k\|^n)\|u_k\|^n\right) \\ + \lim_{k\to\infty}\frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u_k))(f(u_k)u_k - nF(u_k))\ dx + \lambda\left(\frac{1}{2} - \frac{n}{q+1}\right)\int_\Omega h(x)|u_k|^{q+1}\ dx.\end{aligned}$$

Since

$$\int_\Omega (|x|^{-\mu} * F(u_k))F(u_k)\ dx \to \int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx,$$

$$\frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u_k))f(u_k)u_k\ dx \to \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx + \frac{\nu_2(A)}{2},$$

together with (4.22) it follows that

$$\begin{aligned}nc \geq \frac{m_0}{2}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} + \lambda\left(\frac{1}{2} - \frac{n}{q+1}\right)\int_\Omega h(x)u^{q+1}\ dx - \frac{n}{2}\int_\Omega (|x|^{-\mu} * F(u))F(u)\ dx \\ + \frac{1}{2}\int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx.\end{aligned}$$

Consequently,

$$\begin{aligned} c &\geq \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} + \lambda\left(\frac{1}{2n} - \frac{1}{(q+1)}\right)\int_\Omega hu^{q+1}\ dx \\ &+ \left(\frac{1}{2n} - \frac{1}{2(p+1)}\right)\int_\Omega (|x|^{-\mu} * F(u))f(u)u\ dx \\ &\geq \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} - \|h\|_{L^{r'}(\Omega)}\lambda\left(\frac{2n-1-q}{2n(q+1)}\right)\left(\int_\Omega u^{p+2}\ dx\right)^{\frac{q+1}{p+2}} \\ &+ c_1\frac{2p+2-2n}{2n(2p+2)(p+2)}\left(\int_\Omega u^{p+2}\ dx\right)^2 \geq \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} - \inf_{t\in\mathbb{R}^+}\rho(t) \end{aligned}$$

with $r' = \left(1 - \frac{q+1}{p+2}\right)^{-1}$, $c_1 = c_1(\Omega) > 0$ and $\rho(t) = \|h\|_{L^{r'}(\Omega)}\lambda\left(\frac{2n-1-q}{2n(q+1)}\right)t^{\frac{q+1}{2(p+2)}} - \frac{(2p+2-2n)c_1}{2n(2p+2)(p+2)}t$.
Thus $c \geq \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} - \tilde{C}\lambda^{\frac{2(p+2)}{2p+3-q}}$ which completes the proof. ■

Now we prove Theorem 2.5 which concerns the critical case $\beta = \frac{n}{n-1}$.

Proof of Theorem 2.5 Let u_k be a nonnegative minimizing sequence for $\mathcal{J}_{\lambda,M}$ on $N_{\lambda,M}\backslash\{0\}$ satisfying (4.7) then u_k is bounded in $W_0^{1,n}(\Omega)$. Using Proposition 4.10 we get u_k is a Palais Smale sequence at level $\theta < \frac{m_0}{2n}\left(\frac{2n-\mu}{2n}\alpha_n\right)^{n-1} - \tilde{C}\lambda^{\frac{2(p+2)}{2p+3-q}}$. Taking λ small enough, using Lemma 4.6 and Lemma 4.14, $\{u_k\}$ admits a strongly convergent subsequence. Let $u \in W_0^{1,n}(\Omega)$ be the limit of this subsequence. Then arguing as in the proof of Theorems 4.12 and 4.13, we prove that u is a non-trivial weak solution and $\mathcal{J}_{\lambda,M}(u) = \theta$. By elliptic regularity and strong maximum principle, we infer that $u > 0$ in Ω. This completes the proof of Theorem 2.5.
■

Acknowledgements. The authors would like to thank the anonymous referee for valuable comments which have improved the paper.

References

[1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 17 (1990), 393-413.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., 122 (1994), 519-543.

[3] C. O. Alves, F. J. S. A. Corrêa and T. F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl., 49 (2005), 85-93.

[4] C. O. Alves and F. J. S. A. Corrêa, *On existence of solutions for a class of problem involving a nonlinear operator*, Comm. Appl. Nonlinear Anal., 8 (2001), 43-56.

[5] C. O. Alves, D. Cassani, C. Tarsi and M. Yang, *Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^n$*, J. Differential Equations, 261 (2016), 1933-1972.

[6] C. O. Alves and A. El Hamidi, *Nehari manifold and existence of positive solutions to a class of quasilinear problem*, Nonlinear Anal., 60(4) (2005), 611-624.

[7] C. O. Alves and M. Yang, *Existence of Solutions for a Nonlocal Variational Problem in* $\mathbb{R}^2$ *with Exponential Critical Growth*, Journal of Convex Analysis, 24(4) (2017), 1197-1215.

[8] L. Bergé and A. Couairon, *Nonlinear propagation of self-guided ultra-short pulses in ionized gases*, Phys. Plasmas, 7 (2000), 210-230.

[9] K. J. Brown and T. F. Wu, *A fibering map approach to a semilinear elliptic boundary value problem*, Electron. J. Differ. Equ., 69 (2007), 1-9.

[10] B. Cheng, *New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems*, J. Math. Anal. Appl., 394 (2012), 488-495.

[11] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., 71 (1999), 463-512.

[12] P. Drabek and S. I. Pohozaev, *Pohozaev Positive solutions for the p-Laplacian: application of the fibering method*, Proc. Royal Soc. Edinburgh Sect. A, 127 (1997), 703-726.

[13] A. El Hamidi, *Multiple solutions with changing sign energy to a nonlinear elliptic equation*, Commun. Pure Appl. Anal., 3 (2004), 253-265.

[14] G. M. Figueiredo, *Existence of positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl., 401 (2013), 706-713.

[15] G. M. Figueiredo and J. R. Santos Júnior, *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, Diff. Integral Equ., 25 (2012), 853-868.

[16] G. M. Figueiredo and U. B. Severo, *Ground state solution for a Kirchhoff problem with exponential critical growth*, Milan J. Math., 84 (2016), 23-39.

[17] S. Goyal, P.K. Mishra and K. Sreenadh, *n- Kirchhoff type equations with exponential nonlinearities*, RACSAM, 116 (2016), 219-245.

[18] S. Goyal and K. Sreenadh, *Lack of coercivity for n-laplace equation with critical exponential nonlinearities in a bounded domain*, Electron. J. Differ. Equ., 15 (2014), 1-22.

[19] S. Goyal and K. Sreenadh, *The Nehari manifold approach for N-Laplace equation with singular and exponential nonlinearities in* $\mathbb{R}^n$, Commun. Contemp. Math., 17(3) (2015), 1450011 (22 pages).

[20] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[21] C. Y. Lei, G. S. Liu and L. T. Guo, *Multiple positive solutions for a Kirchhoff type problem with a critical nonlinearity*, Nonlinear Analysis: Real World Applications, 31 (2016), 343-355.

[22] F. Li, C. Gao and X. Zhu, *Existence and concentration of sign-changing solutions to Kirchhoff-type system with Hartree-type nonlinearity*, J. Math. Anal. Appl., 448 (2017), 60-80.

[23] Y. Li, F. Li and J. Shi, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations, 253 (2012), 2285-2294.

[24] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard nonlinear equation*, Studies in Appl. Math., 57 (1976/77), 93-105.

[25] E. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, AMS, Providence, Rhode island, 2001.

[26] P. L. Lions, *The concentration compactness principle in the calculus of variations part-I*, Rev. Mat. Iberoamericana, 1 (1985), 185-201.

[27] D. Lü, *A note on Kirchhoff-type equations with Hartree-type nonlinearities*, Nonlinear Anal., 99 (2014), 35-48.

[28] J. Marcos do Ò, *Semilinear Dirichlet problems for the N-laplacian in* $\mathbb{R}^n$ *with nonlinearities in critical growth range*, Diff. Integral Equ., 5 (1996), 967-979.

[29] V. Moroz, and J. V. Schaftingen, *A guide to the Choquard equation*, Journal of Fixed Point Theory and Applications, 19(1) (2017), 773-813.

[30] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., 20 (1971), 1077-1092.

[31] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin 1954.

[32] P. Pucci, M. Xiang and B. Zhang, *Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian*, to appear Adv. Calc. Var., https://doi.org/10.1515/acv-2016-0049.

[33] M. Struwe, *Variational Methods*, Springer-Verlag Berlin Heidelberg, 2008.

[34] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12(3) (1984), 191-202.

[35] J. Wang, L. Tian, J. Xu and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differential Equations, 253 (2012), 2314-2351.

[36] T. F. Wu, *On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function*, J. Math. Anal. Appl., 318 (2006), 253-270.

[37] T. F. Wu, *Multiplicity results for a semilinear elliptic equation involving sign-changing weight function*, Rocky Mountain J. Math., 39(3) (2009), 995-1011.

[38] T. F. Wu, *Multiple positive solutions for a class of concave-convex elliptic problems in* Ω *involving sign-changing weight*, J. Funct. Anal., 258(1) (2010), 99-131.