Manipulating nonclassicality via quantum state engineering processes: Vacuum filtration and single photon addition

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Abstract

The effect of two quantum state engineering processes that can be used to burn hole at vacuum in the photon number distribution of quantum states of radiation field are compared using various witnesses of lower- and higher-order nonclassicality as well as a measure of nonclassicality. Specifically, the witnesses of nonclassical properties due to the effect of vacuum state filtration and a single photon addition on an even coherent state, binomial state and Kerr state are investigated using the criteria of lower- and higher-order antibunching, squeezing and sub-Poissonian photon statistics. Further, the amount of nonclassicality present in these engineered quantum states is quantified and analyzed by using an entanglement potential based on linear entropy. It is observed that all the quantum states studied here are highly nonclassical, and on many occasions the hole burning processes are found to introduce/enhance nonclassical features. However, it is not true in general. The investigation has further revealed that despite the fact that a hole at vacuum implies a maximally nonclassical state (as far as Lee's nonclassical depth is used as the quantitative measure of nonclassicality). However, any particular process of hole burning at vacuum does not ensure the existence of a particular nonclassical feature. Specifically, lower- and higher-order squeezing are not observed for photon added even coherent state and vacuum filtered even coherent state.

1 Introduction

The art of generating and manipulating quantum states as per need is referred to as the "quantum state engineering" [1–5]. This relatively new area of research has drawn much attention of the scientific community because of its success in experimentally producing various quantum states [6–10] having nonclassical properties and applications in realizing quantum information processing tasks, like quantum key distribution [11] and quantum teleportation [12, 13]. Engineered quantum states, such as cat states, Fock state and superposition of Fock states, are known to play a crucial role in performing fundamental tests of quantum mechanics and in establishing quantum supremacy in the context of quantum computation and communication ([14] and references therein).

In fact, to establish quantum supremacy or to perform a fundamental test of quantum mechanics, we would require a state having some features that would not be present in any classical state. Such a state having no classical analogue is referred to as the nonclassical state and is characterized by the negative values of Glauber-Sudarshan P function [15, 16]. Frequently used examples of nonclassical states include squeezed, antibunched, entangled, steered, and Bell nonlocal states; and the relevance of the states having these nonclassical features have already been established in various domains of physics. For example, squeezed state has been used in the successful experimental detection of gravitational wave in LIGO [17], antibunching is used in characterizing single photon sources [18] required in quantum cryptography, and entangled states are useful in almost all sub-fields of quantum information processing [19, 20]. In this article, we focus on two quantum state engineering operations (namely, hole burning at vacuum by filtration and single photon addition) to evaluate their role in inducing/enhancing the nonclassicality in the engineered output state.

To introduce the idea of these quantum state engineering operations, we can write the photon number distribution of an arbitrary quantum state in terms of Glauber-Sudarshan $P(\alpha)$ function as

$$p_n = \int P(\alpha) \left|\langle n|\alpha\rangle\right|^2 d^2\alpha. \tag{1}$$

If p_n vanishes for a particular value of Fock state parameter n, we refer to that as a "hole" or a hole in the photon number distribution at position n [21]. Notice that $p_n = 0$ reveals that $P(\alpha) < 0$ for some α, which is the signature of nonclassicality. Thus, the existence of a hole in the photon number distribution implies that the corresponding state is nonclassical, and corresponding technique of quantum state engineering to create hole is called hole burning [22]. Interestingly, this result also implies that qudits which are d-dimensional (finite dimensional) quantum states are always nonclassical as we can see that in such a state $p_d = p_{d+1} = \ldots = 0$. In principle, the hole can be created for an arbitrary n, but here for the sake of a comparative study, we restrict ourselves to the situation where the hole is created at $n = 0$, i.e., the desired engineered state has zero probability of getting vacuum state on measurement in Fock basis (in other words, $p_0 = 0$). In fact, Lee [23] had shown that a state with $p_n = 0$ is a maximally nonclassical as long as the nonclassicality is quantitatively measured using nonclassical depth. Such a state can be constructed in various ways. To elaborate on this, we describe an arbitrary pure quantum state as a superposition of the Fock states

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{2}$$

where c_n is the probability amplitude of state $|n\rangle$. A hole can be created at $n = 0$ by adding a single photon to obtain

$$|\psi_1\rangle = N_1 a^\dagger |\psi\rangle, \tag{3}$$

where $N_1 = \left(\langle\psi_1|aa^\dagger|\psi_1\rangle\right)^{-1/2}$ is the normalization constant. If we consider the initial quantum state $|\psi\rangle$ as a coherent state, the addition of a single photon would lead to a photon added coherent state which has been experimentally realized [6] and extensively studied [24] because of its interesting nonclassical properties and potential applications. Thus, in quantum state engineering, techniques for photon addition are known [25–27] and experimentally realized.

An alternative technique to create a hole at vacuum is vacuum filtration. The detailed procedure of this technique is recently discussed in [28]. Vacuum filtration implies removal of the coefficient of the vacuum state, c_0, in Eq. (2) and subsequent normalization. Clearly this procedure would yield

$$|\psi_2\rangle = N_2 \sum_{n=1}^{\infty} c'_n |n\rangle, \tag{4}$$

where the normalization constant $N_2 = \left(1 - |c_0|^2\right)^{-1/2}$. Both these states (i.e., $|\psi_1\rangle$, and $\psi_2\rangle$) are maximally nonclassical as far as Lee's result related to nonclassical depth is concerned [29]. However, recently lower-order nonclassical properties of $|\psi_1\rangle$ and $|\psi_2\rangle$ in (3)-(4) are reported to be different for $|\psi\rangle$ chosen as coherent state [28]. This led to several interesting questions, like- What happens if the initial state on which addition of photon or vacuum filtration process is to be applied is already nonclassical (specifically, pure state other than coherent state [30])? How do these processes affect the higher-order nonclassical properties of the quantum states? How does the depth of nonclassicality corresponding to a particular witness of nonclassicality changes with the parameters of the quantum state for these processes? The present article aims to answer these questions through a comparative study using a set of interesting quantum states $|\psi\rangle$ (and the corresponding single photon added $|\psi_1\rangle$ and vacuum filtered $|\psi_2\rangle$ states), each of which can be reduced to many more states. Specifically, in what follows, we would study the lower- and higher-order nonclassical properties of single photon addition and vacuum filtration of even coherent state (ECS), binomial state (BS) and Kerr state (KS). In fact, the quantum state engineering processes described mathematically in Eqs. (3)-(4) can be used to prepare a set of engineered quantum states, namely vacuum filtered ECS (VFECS), vacuum filtered BS (VFBS), vacuum filtered KS (VFKS), photon added ECS (PAECS), photon added BS (PABS) and photon added (PAKS). We aim to look at the nonclassical properties of these states with a focus on higher-order nonclassical properties and subsequently quantify the amount of nonclassicality in all these states. In what follows, the higher-order nonclassical properties are illustrated through the criteria of higher-order antibunching (HOA), higher-order squeezing (HOS) and higher-order sub-Poissonian photon statistics (HOSPS) with brief discussion of lower-order antibunching and squeezing.

The rest of the paper is organized as follows. In Section 2, we have introduced the quantum states of our interest which include ECS, BS, KS, VFECS, VFBS, VFKS, PAECS, PABS, and PAKS. In Section 3, we have investigated the nonclassical properties of these states using various witnesses of lower- and higher-order noncassicality as well as a measure of nonclassicality. Specifically, in this section, we have compared nonclassicality features found in vacuum filtered and single photon added versions of the states of our interest using the witnesses of HOA, HOS and HOSPS. Finally, in Section 5, the results are analyzed, and the paper is concluded.

2 Quantum states of interest

In this paper, we have selected a set of three widely studied and important quantum states- (i) ECS, (ii) BS and (iii) KS. We subsequently noted that these states can further be engineered to generate corresponding vacuum filtered states and single photon added states. For example, one can generate VFBS and PABS from BS by using vacuum filtration [28] and photon addition [6] processes, respectively. In a similar manner, these processes can also generate VFECS and PAECS from ECS, and VFKS and PAKS from KS. In this section, we briefly describe ECS, BS, KS, VFBS, PABS, VFECS, PAECS, VFKS and PAKS. Specifically, we describe three parent states as Fock superposition states. Similarly, the six engineered states are also expressed as Fock superposition states for the convenience of identifying the corresponding photon number distributions (each of which essentially contains a hole at the vacuum). In the rest of the study, we wish to compare the impact of these two quantum state engineering processes (i.e., vacuum filtration and photon addition processes) on the nonclassical properties of the engineered states.

2.1 Even coherent state and states generated by holeburning on it

The analytical expression for ECS in number basis can be written as

$$|\phi(\alpha)\rangle = \frac{\exp\left[-\frac{|\alpha|^2}{2}\right]}{\sqrt{2(1+\exp[-2|\alpha|^2])}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)|n\rangle. \tag{5}$$

The parameter $\alpha = |\alpha| \exp(i\theta)$, in Eqs. (5), is complex in general and θ is phase angle in the complex plane.

Various schemes to generate ECS are reported in [31–33]. The nonclassical properties (witnessed through the antibunching and squeezing criteria, Q function, Wigner function, and photon number distribution, etc.) of ECS have been studied in the recent past [33]. In what follows, we study, both qualitatively and quantitatively, the role of vacuum filtration and photon addition on the nonclassical properties of VFECS and PAECS and compare them with the corresponding properties of ECS.

2.1.1 Vacuum filtered even coherent state

Experimentally, an ECS or a cat state can be generated in various ways [32], and the same can be further engineered to produce a hole at vacuum in its photon number distribution. Specifically, filtration of vacuum will burn a hole at $n = 0$ and produce VFECS, which can described in Fock basis as

$$|\phi_1(\alpha)\rangle = N_{\mathrm{VFECS}} \sum_{n=0,\, n\neq 0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)|n\rangle, \tag{6}$$

where

$$N_{\mathrm{VFECS}} = \left\{4\cosh\left(|\alpha|^2\right) - 1\right\}^{-1/2} \tag{7}$$

is the normalization constant. For simplicity, we may expand Eq. (6) as a superposition of a standard ECS and a vacuum state as follows

$$|\phi_1(\alpha)\rangle = N_{\mathrm{VFECS}} \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)|n\rangle - 2|0\rangle\right). \tag{8}$$

In what follows, Eq. (8) will be used to explore various nonclassical features that exist in VFECS. Specifically, we will compute a general expression for moment of annihilation and creation operators for this state and use a set of moment-based criteria of nonclassicality to identify the nonclassical properties of this engineered quantum state. Similar prescription will be followed for the other engineered quantum states of our interest.

2.1.2 Photon added even coherent state

One can define a single photon added ECS as

$$|\phi_2(\alpha)\rangle = N_{\mathrm{PAECS}}\hat{a}^{\dagger}|\phi(\alpha)\rangle = N_{\mathrm{PAECS}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)\sqrt{n+1}|n+1\rangle, \tag{9}$$

where

$$N_{\mathrm{PAECS}} = \left\{\cosh\left(|\alpha|^2\right) + |\alpha|^2 \sinh\left(|\alpha|^2\right)\right\}^{-1/2}/2 \tag{10}$$

is the normalization constant for PAECS.

2.2 Binomial state and the states generated by holeburning on it

BS is a finite superposition of Fock states having binomial photon number distribution. It is quite similar to the coherent state which is the linear combination of Fock states having the Poissonian photon number distribution [34]. BS can be defined as

$$|p,M\rangle = \sum_{n=0}^{M}\left[\frac{M!}{n!(M-n)!}p^{n}\left(1-p\right)^{M-n}\right]^{1/2}|n\rangle. \tag{11}$$

The binomial coefficient describes the presence of n photons with probability p in M number of ways. Recently, some of the present authors have extensively studied the nonclassical properties of BS, specifically, antibunching, squeezing, HOSPS [35–37], etc. However, no effort has yet been made to study the nonclassical properties of VFBS and PABS.

2.2.1 Vacuum filtered Binomial state

The vacuum filtration of BS can be obtained by simply eliminating vacuum state from the BS as

$$|p,M\rangle_{1} = N_{\rm VFBS}\sum_{n=0}^{M}\left[\frac{M!}{n!(M-n)!}p^{n}\left(1-p\right)^{M-n}\right]^{1/2}|n\rangle - N_{VFBS}\left[\left(1-p\right)^{M}\right]^{1/2}|0\rangle, \tag{12}$$

where

$$N_{\rm VFBS} = \{1-(1-p)^{M}\}^{-1/2} \tag{13}$$

is the normalization constant for the VFBS.

2.2.2 Photon added Binomial state

A hole at $n=0$ at a BS can also be introduced by the addition of a single photon on the BS. A few steps of computation yield the desired expression for PABS as

$$|p,M\rangle_{2} = N_{\rm PABS}\sum_{n=0}^{M}\left[\frac{M!(n+1)!}{(n!)^{2}(M-n)!}p^{n}\left(1-p\right)^{M-n}\right]^{1/2}|n+1\rangle, \tag{14}$$

where

$$N_{\rm PABS} = \left(1+Mp\right)^{-1/2} \tag{15}$$

is the normalization constant for single photon added BS.

2.3 Kerr state and the states generated by holeburning on it

A KS can be obtained when a coherent state of electromagnetic field interacts with nonlinear medium with Kerr type nonlinearity [38]. This interaction generates phase shifts which depend on the intensity. The Hamiltonian involved in this process is given as

$$H = \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar\chi\left(\hat{a}^{\dagger}\right)^{2}\left(\hat{a}\right)^{2}, \tag{16}$$

where χ depends on the third-order susceptibility of Kerr medium. Thus, the compact analytic form of the KS in the Fock basis can be given as

$$|\psi_{K}\left(n\right)\rangle = \sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\exp\left(-\frac{|\alpha|^{2}}{2}\right)\exp\left(-\iota\chi n\left(n-1\right)\right)|n\rangle. \tag{17}$$

2.3.1 Vacuum filtered Kerr state

Similarly, a VFKS, which can be obtained using the same quantum state engineering process that leads to VFECS and VFBS, is given by

$$|\psi_{K}\left(n\right)\rangle_{1} = N_{\rm VFKS}\left[\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\exp\left(-\iota\chi n\left(n-1\right)\right)|n\rangle - |0\rangle\right], \tag{18}$$

where

$$N_{\rm VFKS} = \left(\exp\left[|\alpha|^{2}\right]-1\right)^{-1/2} \tag{19}$$

is the normalization constant for the VFKS.

2.3.2 Photon added Kerr state

An addition of a photon to KS would yield PAKS which can be expanded in Fock basis as

$$|\psi_K(n)\rangle_2 \quad = N_{\rm PAKS} \quad \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\iota\chi n\left(n-1\right)\right) \sqrt{(n+1)}|n+1\rangle, \tag{20}$$

where

$$N_{\rm PAKS} \quad = \quad \left(\exp\left[\mid\alpha\mid^2\right]\left(1+\mid\alpha\mid^2\right)\right)^{-1/2} \tag{21}$$

is the normalization constant for the PAKS.

In the above, we have described six (three) quantum states of our interest as Fock superposition states having (without) holes at vacuum. In what follows, these expressions will be used to study the nonclassical properties of these states using a set of witnesses of nonclassicality. Specifically, we will use a set of witnesses of nonclassicality which are based on moments of annihilation and creation operators. Keeping this in mind, in the following subsection, we report the general form of such moments for all the six engineered states of our interest and the corresponding three parent states (thus overall nine states).

2.4 Expressions for moments of annihilation and creation operators

In 1992, Agarwal and Tara [39] introduced a criterion of nonclassicality in the form of a matrix of moments of creation and annihilation operators. This criterion was further modified to propose a moment-based criteria of entanglement [40] and nonclassicality [41, 42]. Therefore, it is convenient to find out the expectation value of the most general term describing higher-order moment $\langle\hat{a}^{\dagger j}\hat{a}^k\rangle$ for a given state to investigate the nonclassicality using the set of moment-based criteria.

2.4.1 Expectation values for even coherent states and the corresponding engineered states

The analytic expression of $\langle\hat{a}^{\dagger j}\hat{a}^k\rangle_i$ is obtained for the quantum states $i\in\{\rm ECS, VFECS, PAECS\}$ using Eqs. (8) and (9). For ECS and VFECS, expressions of the moments can be written in a compact form as

$$\langle\hat{a}^{\dagger j}\hat{a}^k\rangle_{\rm ECS} = \quad \frac{\exp\left[-|\alpha|^2\right]}{2(1+\exp[-2|\alpha|^2])} \sum_{n=0}^{\infty} \frac{\alpha^n(\alpha^\star)^{n-k+j}}{(n-k)!} \left(1+(-1)^n\right)\left(1+(-1)^{n-k+j}\right). \tag{22}$$

and

$$\langle\hat{a}^{\dagger j}\hat{a}^k\rangle_{\rm VFECS} \quad = \quad \begin{cases} N^2_{\rm VFECS} \sum\limits_{n=1}^{\infty} \frac{\alpha^n(\alpha^\star)^{n-k+j}}{(n-k)!} \left(1+(-1)^n\right)\left(1+(-1)^{n-k+j}\right) & \text{for } k\leq j, \\ N^2_{\rm VFECS} \sum\limits_{n=1}^{\infty} \frac{\alpha^{\star n}\alpha^{n+k-j}}{(n-j)!} \left(1+(-1)^n\right)\left(1+(-1)^{n+k-j}\right) & \text{for } k>j, \end{cases} \tag{23}$$

respectively. Similarly, we obtained analytic expression for $\langle\hat{a}^{\dagger j}\hat{a}^k\rangle_{\rm PAECS}$ for PAECS as

$$\langle\hat{a}^{\dagger j}\hat{a}^k\rangle_{\rm PAECS} = \quad N^2_{\rm PAECS} \sum_{n=0}^{\infty} \frac{\alpha^n(\alpha^\star)^{n-k+j}(n+1)(n-k+j+1)}{(n+1-k)!} \left(1+(-1)^n\right)\left(1+(-1)^{n-k+j}\right). \tag{24}$$

The above mentioned quantities are also functions of displacement parameter of ECS used to generate the engineered states, which will be used as a control parameter while discussion of nonclassicality induced due to engineering operations.

2.4.2 Expectation values for binomial state and the corresponding engineered states

Similarly, the compact analytic form of $\langle\hat{a}^{\dagger t}\hat{a}^r\rangle_{\rm BS}$ can be written as

$$\langle\hat{a}^{\dagger t}\hat{a}^r\rangle_{\rm BS} \quad = \quad \sum_{n=0}^{M} \left[\frac{p^{2n-r+t}(1-p)^{2M-2n+r-t}}{(M-n)!(M-n+r-t)!}\right]^{1/2} \frac{M!}{(n-r)!}. \tag{25}$$

In case of VFBS and PABS, the analytic form of $\langle\hat{a}^{\dagger t}\hat{a}^r\rangle_i$ is obtained as

$$\langle\hat{a}^{\dagger t}\hat{a}^r\rangle_{\rm VFBS} \quad = \quad \begin{cases} N^2_{\rm VFBS} \sum\limits_{n=1}^{M} \left[\frac{p^{2n-r+t}(1-p)^{2M-2n+r-t}}{(M-n)!(M-n+r-t)!}\right]^{1/2} \frac{M!}{(n-r)!} & \text{for r} \leq \text{t}, \\ N^2_{\rm VFBS} \sum\limits_{n=1}^{M} \left[\frac{p^{2n+r-t}(1-p)^{2M-2n-r+t}}{(M-n)!(M-n-r+t)!}\right]^{1/2} \frac{M!}{(n-t)!} & \text{for r} > \text{t}, \end{cases} \tag{26}$$

and

$$\langle\hat{a}^{\dagger t}\hat{a}^r\rangle_{\rm PABS} \quad = \quad N^2_{\rm PABS} \sum_{n=0}^{M} \left[\frac{p^{2n-r+t}(1-p)^{2M-2n+r-t}}{(M-n)!(M-n+r-t)!}\right]^{1/2} \frac{M!(n+1)!(n+1-r+t)!}{n!(n+1-r)!(n-r+t)!}, \tag{27}$$

respectively. Here, the obtained average values of moments are also dependent on BS parameters, which will be used to enhance/control the nonclassicality features in the generated states.

2.4.3 Expectation values for Kerr state and the corresponding engineered states

For KS, VFKS and PAKS, we use the same approach to obtain a compact generalized forms of $\langle\hat{a}^{\dagger q}\hat{a}^{s}\rangle_i$; and our computation yielded

$$\langle\hat{a}^{\dagger q}\hat{a}^{s}\rangle_{\rm KS} = \sum_{n=0}^{\infty}\frac{\alpha^{n}(\alpha^{\star})^{n-s+q}}{(n-s)!}\exp\left[-\mid\alpha\mid^{2}\right]\exp\left(\iota\chi\left[(n-s+q)\left(n-s+q-1\right)-n\left(n-1\right)\right]\right), \quad (28)$$

$$\langle\hat{a}^{\dagger q}\hat{a}^{s}\rangle_{\rm VFKS} = \begin{cases} N_{\rm VFKS}^{2}\sum\limits_{n=1}^{\infty}\frac{\alpha^{n}(\alpha^{\star})^{n-s+q}}{(n-s)!}\exp\left(\iota\chi\left[(n-s+q)\left(n-s+q-1\right)-n\left(n-1\right)\right]\right), \text{ for s} \leq \text{q}, \\ N_{\rm VFKS}^{2}\sum\limits_{n=1}^{\infty}\frac{\alpha^{\star n}\alpha^{n+s-q}}{(n-q)!}\exp\left(-\iota\chi\left[(n+s-q)\left(n+s-q-1\right)-n\left(n-1\right)\right]\right), \text{ for s} > \text{q}, \end{cases} \quad (29)$$

and

$$\langle\hat{a}^{\dagger q}\hat{a}^{s}\rangle_{\rm PAKS} = N_{\rm PAKS}^{2}\sum_{n=0}^{\infty}\frac{\alpha^{n}(\alpha^{\star})^{n-s+q}(n+1)!(n-s+q+1)!}{n!(n-s+q)!(n+1-s)!}\exp\left(\iota\chi\left[(n-s+q)\left(n-s+q-1\right)-n\left(n-1\right)\right]\right). \quad (30)$$

From the above expressions, it is clear that when $q = s$, there is no role of χ and the behavior of KS is similar to that of a coherent state. So the effect of this parameter (χ) can be observed only in HOS which also depends on the higher-order moments other than moments of number operator, i.e., $\langle\hat{a}^{\dagger q}\hat{a}^{s}\rangle_i : q \neq s$. In what follows, we use the expressions of moments given in Eqs. (23)-(30) to study various lower- and higher-order nonclassicality witnesses.

3 Nonclassicality witnesses

There are various criteria of nonclassicality, most of them are sufficient but not necessary in the sense that satisfaction of such a criterion can identify a nonclassical feature, but failure does not ensure that the state is classical. Further, most of the criteria (specially, all the criteria studied here) do not provide any quantitative measure of nonclassicality present in a state, and so they are referred to as witnesses of nonclassicality. These witnesses are based on either quasiprobability distribution or moments of annihilation and creation operators. In the present work, we have used a set of moment-based criteria to investigate nonclassical properties of our desired engineered quantum states. Specifically, we have investigated the possibilities of observing lower-order squeezing and antibunching as well as HOA, HOSPS, and HOS for all the states of our interest. To begin the investigation and the comparison process, let us start with the study of antibunching.

3.1 Lower- and higher-order antibunching

The phenomenon of lower-order antibunching is closely associated with the lower-order sub-Poissonian photon statistics [43]. However, they are not equivalent [44]. The concept of HOA (higher-order nonclassicality) also plays an important role in identifying the presence of weaker nonclassicality [45, 46]. It was first introduced in 1990 based on majorization technique [47] followed by some of its modifications [48, 49]. In this section, we study the generalized HOA criterion introduced by Pathak and Garcia [49] to investigate lower-order antibunching and HOA. To do so, we use the following criterion for $\xi^{\rm th}$ order antibunching [49, 50]

$$\mathscr{A}(\xi) = \langle\hat{a}^{\dagger(\xi+1)}\hat{a}^{(\xi+1)}\rangle - \langle\hat{a}^{\dagger}\hat{a}\rangle^{\xi+1} < 0, \quad (31)$$

where ξ is a positive integer. Depending upon the values of ξ, Eq. (31) reduces to lower- and higher-order criteria of antibunching for $\xi = 1$ and $\xi \geq 2$, respectively. The analytic expressions of moments (23)-(30) can be used to investigate the nonclassicality using inequality (31) for the set of states. The obtained results are illustrated in Fig. 1 where we have compared the results between the vacuum filtered and single photon added states. During this attempt, we also discuss the nonclassicality present in the quantum states used for the preparation of the engineered quantum states (cf. Figs. 1 (a)-(c)). In Figs. 1 (b)-(c), we have shown the result for photon added and vacuum filtered BS and KS, where it can be observed that the depths of both lower- and higher-order witnesses in the negative region are larger for photon added BS and KS in comparison with the vacuum filtered BS and KS, respectively. However, an opposite nature is observed for ECS where the depth of lower- and higher-order witnesses is more for the vacuum filtration in comparison with the photon addition if the values of α remain below certain values; whereas for the photon addition the depth of lower- and higher-order antibunching witnesses is found to be greater than that for vacuum filtration for the higher values of α (cf. Fig. 1 (a)). However, HOA is not observed for the ECS and KS and thus both operations can be ascribed as nonclassicality inducing operations as far as this nonclassical feature is concerned.

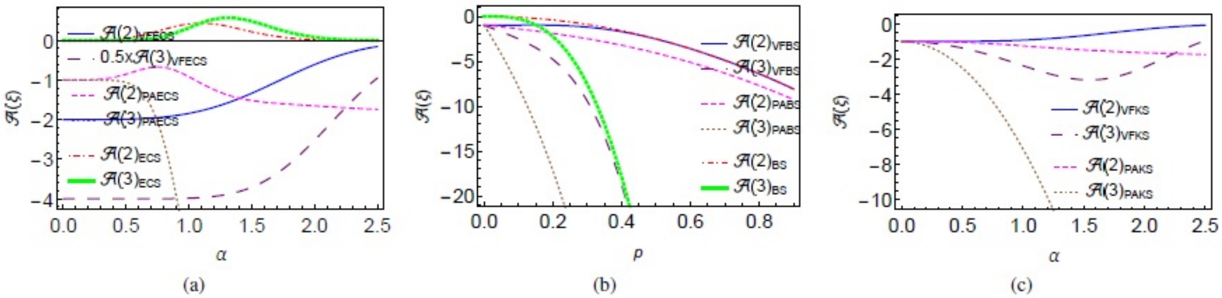


Figure 1: (Color online) Lower- and higher-order antibunching witnesses as functions of displacement parameter α (for ECS and KS) and probability p (for BS with parameter $M = 10$) for (a) ECS, PAECS and VFECS, (b) BS, PABS and VFBS, and (c) KS, PAKS and VFKS. The quantities shown in all the plots are dimensionless.

3.2 Lower- and higher-order squeezing

The concept of squeezing originates from the uncertainty relation. There is a minimum value of an uncertainty relation involving quadrature operators where the variance of two non-commuting quadratures (say position and momentum) are equal and their product satisfies minimum uncertainty relation. Such a situation is closest to the classical scenario, in the sense that there is no uncertainty in the classical picture and this is the closest point that one can approach remaining within the framework of quantum mechanics. Coherent state satisfies this minimum uncertainty relation and is referred to as a classical (or more precisely closest to classical state). If any of the quadrature variances reduces below the corresponding value for a minimum uncertainty (coherent) state (at the cost of increase in the fluctuations in the other quadrature) then the corresponding state is called squeezed state.

The higher-order nonclassical properties can be investigated by studying HOS. There are two different criteria for HOS [51–53]: Hong-Mandel criterion [52] and Hillery criterion [53]. The concept of the HOS was first introduced by Hong and Mandel using higher-order moments of the quadrature operators [52]. According to this criterion, it is observed if the higher-order moment for a quadrature operator for a quantum state is observed to be less than the corresponding coherent state value. Another type of HOS was introduced by Hillery who introduced amplitude powered quadrature and used variance of this quadrature to define HOS [53]. Here, we aim to analyze the possibility of HOS using Hong-Mandel criterion for lth order squeezing defined as [52]

$$S(l) = \frac{\langle(\Delta \mathrm{X})^l\rangle - \left(\frac{1}{2}\right)_{\frac{l}{2}}}{\left(\frac{1}{2}\right)_{\frac{l}{2}}} < 0, \tag{32}$$

where $X = \left(a + a^\dagger\right)/\sqrt{2}$ is the dimensionless quadrature, the symbol $(x)_n$ is the conventional Pochhammer symbol, and l is an even integer >2 for HOS, which represents the order of the squeezing. Computation of $\langle(\Delta \mathrm{X})^l\rangle$ was a bit tedious and HOS of Hong-Mandel type was reported only in a few states until the recent past when Verma and Pathak [36] reduced the complexity associated with the operator ordering involved in the computation of $\langle(\Delta \mathrm{X})^l\rangle$ and provided the following c-number-based criterion for Hong-Mandel type HOS which can be computed easily from the expressions of moments provided in the previous section

$$\langle(\Delta \mathrm{X})^l\rangle = \sum_{r=0}^{l}\sum_{i=0}^{\frac{r}{2}}\sum_{k=0}^{r-2i}(-1)^r\,\frac{1}{2^{\frac{l}{2}}}\,(2i-1)!^{2i}C_k{}^{l}C_r{}^{r}C_{2i}\langle\hat{a}^\dagger + \hat{a}\rangle^{l-r}\langle\hat{a}^{\dagger k}\hat{a}^{r-2i-k}\rangle < \left(\frac{1}{2}\right)_{\frac{l}{2}} = \frac{1}{2^{\frac{l}{2}}}\,(l-1)!!. \tag{33}$$

Note that $l = 2$ corresponds to lower-order squeezing. We have investigated the possibility of observing HOS analytically using Eqs. (23)-(30) and inequality (33) for all engineered quantum states of our interest and have shown the corresponding results in Figs. 2 (a)-(c) where we have compared the HOS in the set of quantum states and the states obtained by photon addition and vacuum filtration. These operations fail to induce this nonclassical feature in the engineered states prepared from ECS, which also did not show signatures of squeezing. In Fig. 2 (a), we illustrate Hong-Mandel type HOS with respect to parameter p where we have shown the existence HOS for BS, VFBS and PABS. It can be observed that the state engineering operations fail to increase this particular feature of nonclassicality in BS. Additionally, higher-order nonclassicality is absent for higher values of p when corresponding lower-order squeezing is present. In case of KS, PAKS and VFKS, we have observed that HOS is observed when the values of α are greater than certain values for the individual curves of the corresponding states (cf. Fig. 2 (b)). Note that photon addition may provide some advantage in this case, but vacuum filtration would not as for the same value of displacement parameter KS and PAKS (VFKS) have (has not) shown squeezing. Interestingly, the

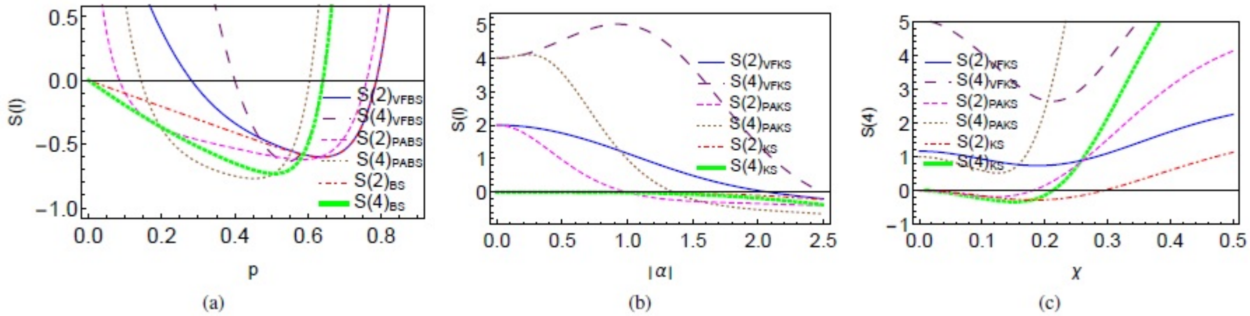


Figure 2: (Color online) Illustration of lower- and higher-order squeezing for (a) BS, PABS and VFBS; (b) KS, PAKS and VFKS at the fixed value of $\chi = 0.02$; (c) KS, VFKS and PAKS as a function of χ with $\alpha = 1$. The negative regions of the curves illustrate the presence of squeezing.

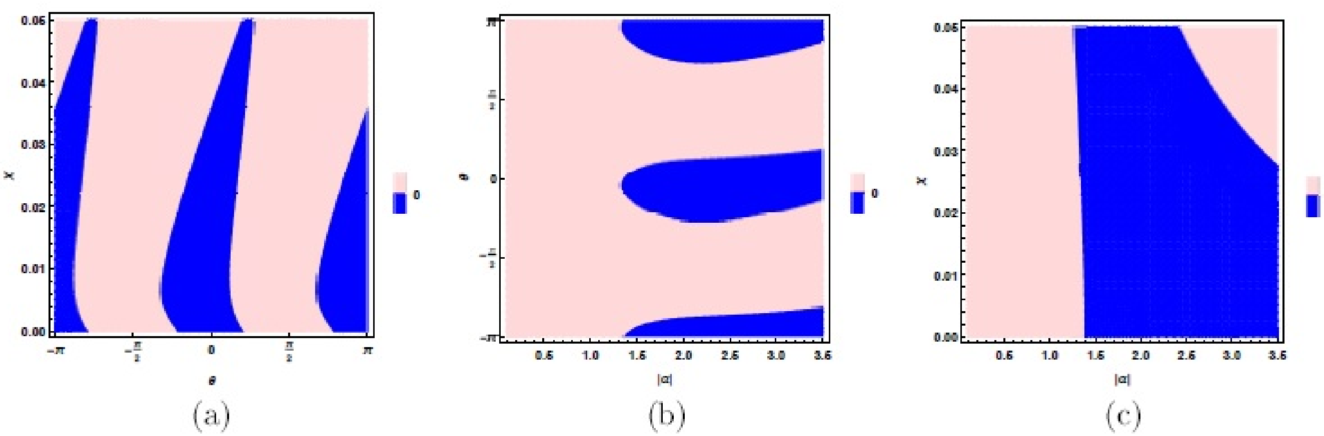


Figure 3: (Color online) The dependence of HOS witness ($l = 4$) on Kerr parameter χ and displacement parameters $|\alpha|$ and θ for PAKS with (a) $|\alpha| = 3$, (b) $\chi = 0.02$, (c) $\theta = 0$.

presence of squeezing also depends upon the Kerr nonlinearity parameter χ, which is shown in Fig. 2 (c). Similar to Fig. 2 (b) photon addition shows advantage over KS which disappears for larger values of χ, while vacuum filtering is not beneficial.

In Fig. 3, we have shown using the dark (blue) color in the contour plots of the HOS witness for PAKS that squeezing can be observed for higher values of $|\alpha|$ and smaller values of χ. Additionally, the phase parameter θ of α is also relevant for observing the nonclassicality as squeezing occurs in the vicinity of $\theta = m\pi$, while disappears for $\theta = \frac{m\pi}{2}$ with integer m. Similar behavior is observed in KS and VFKS (not shown here).

3.3 Higher-order sub-Poissonian photon statistics

Quantum statistical properties of a radiation field can be investigated through HOSPS. Mathematically, it is observed if the higher-order variance of the photon number is less than its Poissonian level ($\langle(\Delta N)^l\rangle < \langle(\Delta N)^l\rangle_{\text{Poissonian}}$). Moreover, HOSPS is found useful in various aspects of higher-order nonclassical phenomena [54]. The condition for HOSPS is defined as

$$D(l) \quad = \quad \sum_{u=0}^{l}\sum_{v=0}^{u} S_2(u,\,v)\,{}^lC_u\,(-1)^u\,\mathscr{A}(v)\langle N\rangle^{l-u} \quad < \quad 0. \tag{34}$$

where $S_2(u,\,v)$ stands for the Stirling numbers of second kind, N is the usual number operator. The inequality in Eq. (34) is the condition for the lth order sub-Poissonian photon statistics representing corresponding higher-order counterparts for $l > 2$. The higher-order moments in Eqs. (23)-(30) are used to calculate the above inequality (34) with the help of (31) for states obtained after vacuum filtration and photon addition in ECS, BS and KS as well as the parent states, and the corresponding results are depicted in Fig. 4. Nonclassicality is not revealed by HOSPS criteria of even orders in case of ECS, while corresponding engineered states show nonclassicality. Additionally, nonclassicality is induced by vacuum filtration for odd

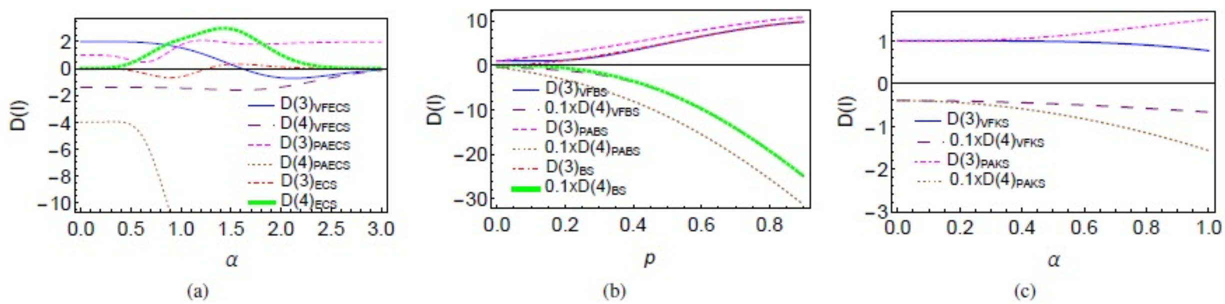


Figure 4: (Color online) Illustration of HOSPS as a function of displacement parameter α (for ECS and KS) and probability p (for BS) for (a) ECS, (b) BS, and (c) KS and corresponding engineered states. HOSPS is not observed in KS.

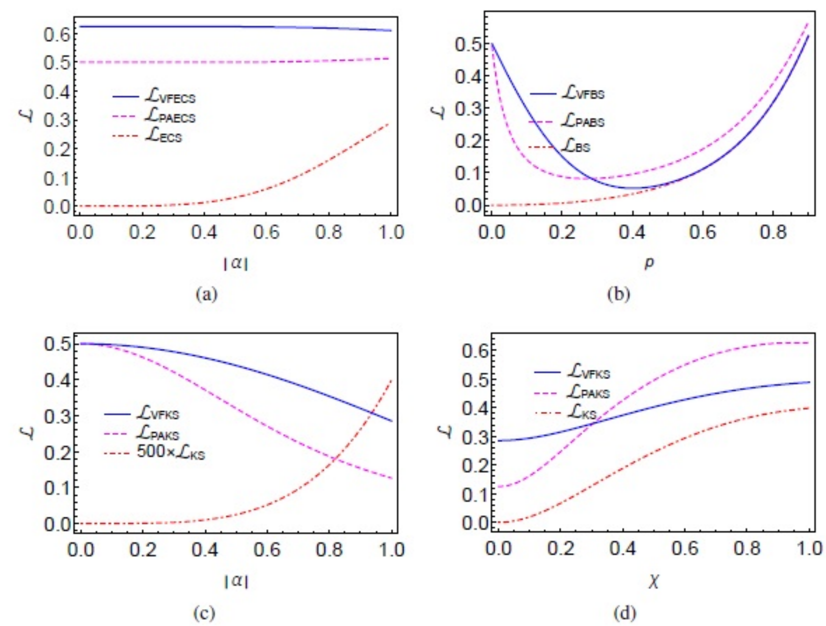
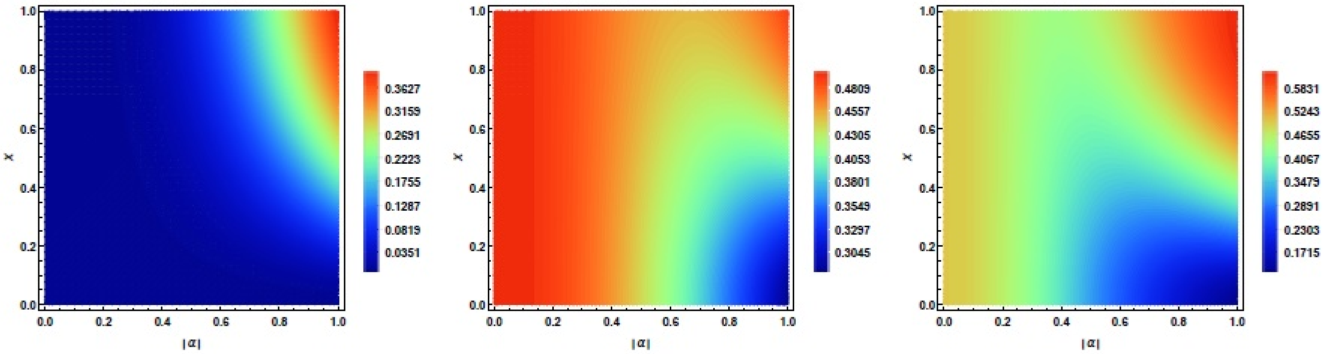


Figure 5: (Color online) Illustration of linear entropy for (a) ECS, PAECS and VFECS, (b) BS, PABS and VFBS, (c) KS, PAKS and VFKS with α or p for $\chi = 0.02$. (d) Dependence of nonclassicality in KS, PAKS and VFKS on χ for $\alpha = 1$.

orders while it was not observed in the parent state (cf. Fig. 4 (a)). This clearly shows the role of hole burning operations in inducing nonclassicality for odd orders. However, in case of even orders, the same operations are also observed to destroy the nonclassicality in the parent state. From Figs. 4 (b) and (c), it is observed that BS and KS do not show HOSPS for the odd values of l even after application of state engineering operations. Additionally, HOSPS is not observed for the KS for even values of l, too. Consequently, the nonclassical feature witnessed through the HOSPS criterion in PAKS can be attributed solely to the hole burning process.

4 Nonclassicality measure

In 2005, a measure of nonclassicality was proposed as entanglement potential, which is the amount of entanglement in two output ports of a beam splitter with the quantum state ρ_{in} and vacuum $|0\rangle\langle 0|$ sent through two input ports [55]. The amount of entanglement quantifies the amount of nonclassicality in the input quantum state as classical state can not generate entanglement in the output. The post beam splitter state can be obtained as $\rho_{out} = U\left(\rho_{in} \otimes |0\rangle\langle 0|\right) U^{\dagger}$ with $U = \exp[-iH\theta]$, where $H = (a^{\dagger}b + ab^{\dagger})/2$, and $a^{\dagger}$ (a), $b^{\dagger}$ (b) are the creation (annihilation) operators of the input modes. For example, considering



(a) (b) (c)

Figure 6: (Color online) Illustration of linear entropy for (a) KS (b) VFKS (c) PAKS.

quantum state (2) and a vacuum state $|0\rangle$ as input states, we can write the analytic expression of the two-mode output state as

$$|\phi\rangle = U\left(|\psi\rangle \otimes |0\rangle\right) \equiv U|\psi, 0\rangle = \sum_{n=0}^{\infty} \frac{c_n}{2^{n/2}} \sum_{j=0}^{n} \sqrt{{}^{n}C_j}\ |j,\, n-j\rangle. \tag{35}$$

We can measure the amount of entanglement in the output state to quantify the amount of input nonclassicality in $|\psi\rangle$. Here, we use linear entropy of single mode subsystem (obtained after tracing over the other subsystem) as entanglement potential. The linear entropy for an arbitrary bipartite state ρ_{AB} is defined as [56]

$$\mathcal{L} = 1 - \mathrm{Tr}\left(\rho_B^2\right), \tag{36}$$

where ρ_B is obtained by tracing over subsystem A. We have obtained the analytic expressions of linear entropy for ECS, KS, BS and corresponding engineered states and have reported them as Appendix A Eqs. (A-1)-(A-9). In general, significance of hole burning operations can be clearly established through corresponding results shown in Fig. 5. Specifically, one can clearly see the amount of nonclassciality (revealed through the amount of entanglement it can generate at a beam splitter) increases due to these operations.

From Figs. 5 (a) and (c), it can be observed that vacuum filtered ECS and KS are more nonclassical than corresponding photon added counterparts. However, in case of BS and its engineered states, it is observed that only up to a certain value of p VFBS is more nonclassical than PABS (cf. Fig. 5 (a)). In fact, the amount of additional nonclassicality induced due to filtration decreases with p and eventually becomes zero (i.e., the amount of nonclassicality of VFBS becomes equal to that of BS as far as linear entropy is considered as a measure of nonclassicality). It is interesting to observe the effect of Kerr coupling parameter χ on the amount of nonclassicality induced due to nonlinearity. It is observed that for small (relatively large) values of χ nonclassicality present in VFKS (PAKS) is more than that in PAKS (VFKS) (cf. Fig. 5 (d)). This dependence is more clearly visible in Fig. 6, where one can observe strong nonclassicality in PAKS and VFKS (KS) favor (favors) smaller (higher) values of α and large χ.

5 Conclusion

In summary, this article is focused on the comparison of the effects of two processes (vacuum state filtration and single photon addition) used in quantum state engineering to burn hole at vacuum as far as the higher-order nonclassical properties of the quantum states prepared using these two processes are concerned. Specifically, various quantum state engineering processes for burning holes at vacuum lead to different $\sum_{m=1} c_m|m\rangle$ as far as the values of c_ms are concerned (even when the parent state is the same). To study its significance in nonclassical properties of the engineered states, we considered a small set of finite and infinite dimensional quantum states (namely, ECS, BS, and KS). This provided us a set of six engineered quantum states, namely VFECS, PAECS, VFBS, PABS, VFKS, and PAKS and three parent states for our analysis. This set of engineered quantum states can have a great importance in quantum information processing and quantum optics as they are found to be highly nonclassical. Especially when some exciting applications of their parent states are already investigated in the context of continuous variable quantum information processing and/or quantum optics. The present study also addresses the significance of these hole burning processes in inducing (enhancing) particular nonclassical features in the large set of engineered and parent quantum states.

The general expressions for moments of the set of states are reported in the compact analytic form, which are used here to investigate nonclassical features of these states using a set of criteria of higher-order nonclassicality (e.g., criteria of HOA, HOS and HOSPS). The obtained expressions can be further used to study other moment-based criteria of nonclassicality. The hole burning operations are found to be extremely relevant as the states studied here are found to be highly nonclassical when quantified through a measure of nonclassicality (entanglement potential). In brief, both the vacuum filtration and photon addition operations can be ascribed as antibunching inducing operations in KS and ECS while antibunching enhancing operations for BS. As far as HOS is concerned no such advantage of these operations is visible as these operations fail to induce squeezing in ECS and often decrease the amount of squeezing present in the parent state. Additionally, the operations are successful in inducing HOSPS in KS and enhances this feature in the rest of the parent states. The relevance of higher-order nonclassicality in the context of the present study can be understood from the fact that these hole burning operations show an increase in the depth of HOA witness and decrease in the amount of HOS with order. While in case of HOSPS even orders show nonclassicality whereas odd orders fail to detect it. Finally, the measure of nonclassicality reveals vacuum filtration as a more powerful tool than photon addition for enhancing nonclassicality in the parent state, but photon addition is observed to be advantageous in some specific cases.

Based on our present investigation and the discussion above, we conclude the paper with the hope that the methods used here will be helpful in further theoretical studies on nonclassical properties of engineered quantum states (both finite and infinite dimensional), and the results reported here will be used in the experimental quantum information processing and/or quantum optics.

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Appendix: A

Analytic expression for linear entropy

Analytical expression for linear entropy of ECS

$$\mathcal{L}_{\rm ECS} = 1-\frac{\exp\left[-2|\alpha|^2\right]}{4(1+\exp[-2|\alpha|^2])^2}\sum_{n,m,r=0}^{\infty}\frac{|\alpha|^{2n+2r}(1+(-1)^n)(1+(-1)^m)(1+(-1)^r)\left(1+(-1)^{n+r-m}\right)}{n!r!}\sum_{k_1=0}^{n}\binom{n}{k_1}\binom{r}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r}, \tag{A-1}$$

VFECS

$$\mathcal{L}_{\rm VFECS} = 1-(N_{\rm VFECS})^4\sum_{n,m,r=1}^{\infty}\frac{|\alpha|^{2n+2r}(1+(-1)^n)(1+(-1)^m)(1+(-1)^r)\left(1+(-1)^{n+r-m}\right)}{n!r!}\sum_{k_1=0}^{n}\binom{n}{k_1}\binom{r}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r}, \tag{A-2}$$

and PAECS

$$\begin{aligned}\mathcal{L}_{\rm PAECS} &= 1-(N_{PAECS})^4\sum_{n,m,r=0}^{\infty}\frac{|\alpha|^{2n+2r}(1+(-1)^n)(1+(-1)^m)(1+(-1)^r)\left(1+(-1)^{n+r-m}\right)}{n!r!}\\ &\times \sum_{k_1=0}^{n+1}\binom{n+1}{k_1}\binom{r+1}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r+2}(m+1)\,(n-m+r+1)\,.\end{aligned} \tag{A-3}$$

Similarly, analytical expression for linear entropy of BS

$$\mathcal{L}_{\rm BS} = 1-\sum_{n,m,r=0}^{M}\frac{1}{n!r!}\left[\frac{(M!)^4p^{2(n+r)}(1-p)^{4M-2n-2r}}{(M-n)!(M-m)!(M-r)!(M-n-r+m)!}\right]^{1/2}\sum_{k_1=0}^{n}\binom{n}{k_1}\binom{r}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r}, \tag{A-4}$$

VFBS

$$\mathcal{L}_{\rm VFBS} = 1-(N_{\rm VFBS})^4\sum_{n,m,r=1}^{M}\frac{1}{n!r!}\left[\frac{(M!)^4p^{2(n+r)}(1-p)^{4M-2n-2r}}{(M-n)!(M-m)!(M-r)!(M-n-r+m)!}\right]^{1/2}\sum_{k_1=0}^{n}\binom{n}{k_1}\binom{r}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r}, \tag{A-5}$$

and PABS

$$\begin{aligned}\mathcal{L}_{\rm PABS} &= 1-(N_{\rm PABS})^4\sum_{n,m,r=0}^{M}\frac{1}{n!r!}\left[\frac{(M!)^4p^{2(n+r)}(1-p)^{4M-2n-2r}}{(M-n)!(M-m)!(M-r)!(M-n-r+m)!}\right]^{1/2}\\ &\times \sum_{k_1=0}^{n+1}\binom{n+1}{k_1}\binom{r+1}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r+2}(m+1)\,(n-m+r+1)\end{aligned} \tag{A-6}$$

are obtained. Finally, analytical expression for linear entropy of KS, VFKS, PAKS can be given as

$$\begin{aligned}\mathcal{L}_{\rm KS} &= 1-\sum_{n,m,r=0}^{\infty}\frac{|\alpha|^{2n+2r}\exp\left[-2|\alpha|^2\right]}{n!r!}\exp[\iota\chi(m(m-1)-n(n-1)-r(r-1)\\ &+ (n-m+r)(n-m+r-1))]\sum_{k_1=0}^{n}\binom{n}{k_1}\binom{r}{r+k_1-m}\left(\frac{1}{2}\right)^{n+r},\end{aligned} \tag{A-7}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{VFKS}} &= 1-(N_{\mathrm{VFKS}})^4 \sum_{n,m,r=1}^{\infty} \tfrac{|\alpha|^{2n+2r}}{n!r!} \exp[\iota\chi(m(m-1)-n(n-1)-r(r-1) \\ &+ (n-m+r)(n-m+r-1))] \sum_{k_1=0}^{n} \tbinom{n}{k_1} \tbinom{r}{r+k_1-m} \left(\tfrac{1}{2}\right)^{n+r}, \end{aligned} \tag{A-8}$$

and

$$\begin{aligned} \mathcal{L}_{\mathrm{PAKS}} &= 1-(N_{\mathrm{PAKS}})^4 \sum_{n,m,r=0}^{\infty} \tfrac{|\alpha|^{2n+2r}}{n!r!} \exp[\iota\chi(m(m-1)-n(n-1)-r(r-1) \\ &+ (n-m+r)(n-m+r-1))] \sum_{k_1=0}^{n+1} \tbinom{n+1}{k_1} \tbinom{r+1}{r+k_1-m} \left(\tfrac{1}{2}\right)^{n+r+2} (m+1)\,(n-m+r+1)\,, \end{aligned} \tag{A-9}$$

respectively.