Kirchhoff equations with Choquard exponential type nonlinearity involving the fractional Laplacian

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Abstract

In this article, we deal with the existence of non-negative solutions of the class of following non local problem

$$\begin{cases} -M\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,dxdy\right)(-\Delta)^s_{n/s}u = \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}\,dy\right) g(x,u) \text{ in } \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(-\Delta)^s_{n/s}$ is the n/s-fractional Laplace operator, $n \geq 1$, $s \in (0,1)$ such that $n/s \geq 2$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $M : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, where g behaves like $\exp(|u|^{\frac{n}{n-s}})$ as $|u| \to \infty$.

Key words: Doubly non local problems, Kirchhoff equation, Choquard nonlinearity, Trudinger-Moser nonlinearity.

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1 Introduction

Let $n \geq 1$, $s \in (0,1)$ such that $n/s \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary then we intend to study the existence of a non negative solutions of following fractional Kirchhoff type problem with Trudinger-Moser type Choquard nonlinearity

$$(\mathcal{M}) \quad \begin{cases} -M\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,dxdy\right)(-\Delta)^s_{n/s}u = \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}\,dy\right) g(x,u) \text{ in } \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega, \end{cases}$$

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where $(-\Delta)^s_{n/s}$ is the n/s-fractional Laplace operator which, up to a normalizing constant, is defined as

$$(-\Delta)^s_{n/s}u(x) = 2\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))}{|x-y|^{2n}}dy, \quad x\in\mathbb{R}^n,\ u\in C_0^\infty(\mathbb{R}^n).$$

The functions $M:\mathbb{R}^+\to\mathbb{R}^+$ and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ are continuous satisfying some appropriate conditions which will be stated later on.

Our problem $(\mathcal{M})$ is basically driven by the Hardy-Littlewood-Sobolev inequality and the Trudinger-Moser inequality. Let us first recall the following well known Hardy-Littlewood-Sobolev inequality [Theorem 4.3, p.106] [13].

Proposition 1.1 ***(Hardy-Littlewood-Sobolev inequality)*** *Let $t,\ r>1$ and $0<\mu<n$ with $1/t+\mu/n+1/r=2$, $g\in L^t(\mathbb{R}^n)$ and $h\in L^r(\mathbb{R}^n)$. Then there exists a sharp constant $C(t,n,\mu,r)$, independent of g, h such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{g(x)h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le C(t,n,\mu,r)\|g\|_{L^t(\mathbb{R}^n)}\|h\|_{L^r(\mathbb{R}^n)}. \tag{1.1}$$

If $t=r=\frac{2n}{2n-\mu}$ then

$$C(t,n,\mu,r)=C(n,\mu)=\pi^{\frac{\mu}{2}}\frac{\Gamma\left(\frac{n}{2}-\frac{\mu}{2}\right)}{\Gamma\left(n-\frac{\mu}{2}\right)}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right\}^{-1+\frac{\mu}{n}}.$$

In this case there is equality in (1.1) if and only if $g\equiv(constant)h$ and

$$h(x)=A(\gamma^2+|x-a|^2)^{\frac{-(2n-\mu)}{2}}$$

for some $A\in\mathbb{C}$, $0\neq\gamma\in\mathbb{R}$ and $a\in\mathbb{R}^n$.

The study of Choquard equations originates from the work of S. Pekar [19] and P. Choquard [12] where they used elliptic equations with Hardy-Littlewood-Sobolev type nonlinearity to describe the quantum theory of a polaron at rest and to model an electron trapped in its own hole in the Hartree-Fock theory, respectively. For more details on the application of Choquard equations, we refer [17]. On the other hand, the boundary value problems involving Kirchhoff equations arise in several physical and biological systems. These type of non-local problems were initially observed by Kirchhoff in 1883 in the study of string or membrane vibrations to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Lü [14] in 2015 studied the following Kirchhoff problem with Choquard nonlinearity

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+(1+\mu g(x))u=(|x|^{-\alpha}*|u|^p)u|^{p-2}u \text{ in } \mathbb{R}^3$$

for $a>0,\ b\ge 0,\ \alpha\in(0,3),\ p\in(2,6-\alpha),\ \mu>0$ is a parameter and g is a nonnegative continuous potential with some growth assumptions. He proved the existence of solution to the above problem for μ sufficiently large and also showed their concentration behavior when μ approaches $+\infty$. In [11], authors discuss the existence and concentration of sign-changing solutions to a class of Kirchhoff-type systems with Hartree-type nonlinearity in $\mathbb{R}^3$ by the minimization argument on the sign-changing Nehari manifold and a quantitative deformation lemma. In the nonlocal case that is problems involving the fractional Laplace operator, Kirchhoff problem with Choquard nonlinearity has been studied by Pucci et al. in [21] via variational techniques.

The study of elliptic equations involving nonlinearity with exponential growth are motivated by the following Trudinger-Moser inequality in [15], namely

Theorem 1.2 *let Ω be a open bounded domain then we define $\tilde{W}_0^{s,n/s}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|^{\frac{n}{s}} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}dxdy$. Then there exists a positive constant $\alpha_{n,s}$ given by*

$$\alpha_{n,s} = \frac{n}{\omega_{n-1}}\left(\frac{\Gamma(\frac{n-s}{2})}{\Gamma(s/2)2^s\pi^{n/2}}\right)^{-\frac{n}{n-s}},$$

where ω_{n-1} be the surface area of the unit sphere in $\mathbb{R}^n$ and $C_{n,s}$ depending only on n and s such that

$$\sup_{u\in\tilde{W}_0^{s,n/s}(\Omega),\ \|u\|\leq 1}\int_\Omega \exp\left(\alpha|u|^{\frac{n}{n-s}}\right)\ dx \leq C_{n,s}|\Omega| \tag{1.2}$$

for each $\alpha\in[0,\alpha_{n,s}]$. Moreover there exists a $\alpha^*_{n,s}\geq\alpha_{n,s}$ such that the right hand side of (1.2) is $+\infty$ for $\alpha>\alpha^*_{n,s}$.

It is proved in [18] (see Proposition 5.2) that

$$\alpha^*_{n,s} = n\left(\frac{2(n\mathcal{W}_n)^2\Gamma(\frac{n}{s}+1)}{n!}\sum_{i=0}^{\infty}\frac{(n+i-1)!}{i!(n+2i)^{\frac{n}{s}}}\right)^{\frac{s}{n-s}},$$

where $\mathcal{W}_n = \frac{w_{n-1}}{n}$ is the volume of the unit sphere in $\mathbb{R}^n$. It is still unknown whether $\alpha^*_{n,s}=\alpha_{n,s}$ or not.

The p-fractional Kirchhoff problems involving the Trudinger-Moser type nonlinearity has been recently addressed in [16, 23]. We also refer [6, 7] to the readers, in the linear case i.e. when $p=2$. The Choquard equations with exponential type nonlinearities has been comparatively less attended. In this regard, we cite [1] where authors studied a singularly perturbed nonlocal Schrödinger equation via variational techniques. We also refer [2] for reference. On a similar note, there is no literature available on Kirchhoff problems involving the Choquard exponential nonlinearity except the very recent article [3] where authors studied the existence of positive solutions to the following problem

$$-m\left(\int_\Omega|\nabla u|^n\ dx\right)\Delta_n u = \left(\int_\Omega\frac{F(y,u)}{|x-y|^\mu}dy\right)f(x,u),\ u>0 \text{ in } \Omega,\ u=0 \text{ in } \partial\Omega$$

where $-\Delta_n = \nabla.(|\nabla u|^{n-2}\nabla u)$, $\mu \in (0,n)$, $n \geq 2$, m and f are continuous functions satisfying some additional assumptions, using the concentration compactness arguments. They also established multiplicity result corresponding to a perturbed problem via minimization over suitable subsets of Nehari manifold. Whereas in the p-fractional laplacian case, motivated by above research, our paper represents the first article to consider the Kirchhoff problem with Choquard exponential nonlinearity.

The problem of the type $(\mathcal{M})$ are categorized under doubly nonlocal problems because of the presence of the term $M\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}dxdy\right)$ and $\left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}\, dy\right) g(x,u)$ which does not allow the problem $(\mathcal{M})$ to be a pointwise identity. Additionally, we also deal with the degenerate case of Kirchhoff problem which is a new result even in the case of $s=1$. This phenomenon arises mathematical difficulties which makes the study of such a class of problem interesting. Generally, the main difficulty encountered in Kirchhoff problems is the competition between the growths of M and g. Precisely, mere weak limit of a Palais Smale (PS) sequence is not enough to claim that it is a weak solution to $(\mathcal{M})$ because of presence of the function M, which holds in the case of $M \equiv 1$. Next technical hardship emerge while proving convergence of the Choquard term with respect to (PS) sequence. We use delicate ideas in Lemma 3.4 and Lemma 3.5 to establish it. Following a variational approach, we prove that the corresponding energy functional to $(\mathcal{M})$ satisfies the Mountain pass geometry and the Mountain pass critical level stays below a threshold (see Lemma 3.3) using the Moser type functions established by Parini and Ruf in [18]. Then we perform a convergence analysis of the Choquard term with respect to the (PS)-sequences in Lemma 3.4. This along with the higher integrability Lemma 2.5 benefited us to get the weak limit of (PS)-sequence as a weak solution of $(\mathcal{M})$ leading to build the proof of our main result. The approach although may not be completely new but the problem is comprehensively afresh.

Our article is divided into 3 sections- Section 2 illustrates the functional set up to study $(\mathcal{M})$ and contains the main result that we intend to establish. Section 3 contains the proof of our main result.

2 Functional Setting and Main result

Let us consider the usual fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega); \frac{(u(x)-u(y))}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega\times\Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}}$$

where $\Omega \subset \mathbb{R}^n$ is an open set. We denote $W_0^{s,p}(\Omega)$ as the completion of the space $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. To study fractional Sobolev spaces in details we refer to [5]. Now we define

$$X_0 = \{u \in W^{s,n/s}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

with respect to the norm

$$\|u\|_{X_0} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} dxdy\right)^{\frac{s}{n}} = \left(\int_Q \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} dxdy\right)^{\frac{s}{n}},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Then X_0 is a reflexive Banach space and continuously embedded in $W_0^{s,p}(\Omega)$. Also $X_0 \hookrightarrow\hookrightarrow L^q(\Omega)$ compactly for each $q \in [1,\infty)$. Note that the norm $\|.\|_{X_0}$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$. We denote $\|.\|_{X_0}$ by $\|.\|$ in future, for notational convenience. This type of functional setting was introduced by Servadei and Valdinoci for $p = 2$ in [22] and for $p \neq 2$ in [8].
Moreover, we define the space

$$\tilde{W}_0^{s,p}(\Omega) = \overline{C_0(\Omega)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}}.$$

The space $\tilde{W}_0^{s,p}(\Omega)$ is equivalent to the completion of $C_0^\infty(\Omega)$ with respect to the semi norm $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} dxdy$ (see for example [[9], Remark 2.5]). If $\partial\Omega$ is Lipschitz, then $\tilde{W}_0^{s,p}(\Omega) = X_0$, (see[[10], Proposition B.1]). The embedding $W_0^{s,\frac{n}{s}}(\Omega) \ni u \longmapsto \exp(|u|^\beta) \in L^1(\Omega)$ is compact for all $\beta \in \left(1, \frac{n}{n-s}\right)$ and is continuous when $\beta = \frac{n}{n-s}$.
We now state our assumptions on M and g. The function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which satisfies the following assumptions:

$(M1)$ For all t, $s \geq 0$, it holds

$$\hat{M}(t+s) \geq \hat{M}(t) + \hat{M}(s),$$

where $\hat{M}(t) = \int_0^t M(s)ds$, the primitive of M.

$(M2)$ There exists a $\gamma > 1$ such that $t \mapsto \frac{M(t)}{t^{\gamma-1}}$ is non increasing for each $t > 0$.

$(M3)$ For each $b > 0$, there exists a $\kappa := \kappa(b) > 0$ such that $M(t) \geq \kappa$ whenever $t \geq b$.

The condition $(M3)$ asserts that the function M has possibly a zero only when $t = 0$.

Remark 2.1 *From $(M2)$, we can easily deduce that $\gamma\hat{M}(t) - M(t)t$ is non decreasing for $t > 0$ and*

$$\gamma\hat{M}(t) - M(t)t \geq 0 \quad \forall\, t \geq 0. \tag{2.1}$$

We also have the following remark as a consequence of (2.1).

Remark 2.2 *For each $t \geq 0$, by using (2.1) we have*

$$\frac{d}{dt}\left(\frac{\hat{M}(t)}{t^\gamma}\right) = \frac{M(t)}{t^\gamma} - \frac{\gamma\hat{M}(t)}{t^{\gamma+1}} \leq 0.$$

So the map $t \mapsto \frac{\hat{M}(t)}{t^\gamma}$ is non increasing for $t > 0$. Hence

$$\hat{M}(t) \geq \hat{M}(1)t^\gamma \text{ for all } t \in [0,1], \tag{2.2}$$

and

$$\hat{M}(t) \leq \hat{M}(1)t^\gamma \text{ for all } t \geq 1. \tag{2.3}$$

We note that the condition $(M1)$ is valid whenever M is non decreasing.

Example 1 *Let $M(t) = m_0 + at^{\gamma-1}$, where $m_0, a \geq 0$ and $\gamma > 1$ such that $m_0 + a > 0$ then M satisfies the conditions $(M1) - (M3)$. If $m_0 = 0$, this forms an example of the degenerate case whereas of the non degenerate case if $m_0 > 0$.*

The nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(x,t) = h(x,t)\exp(|t|^{\frac{n}{n-s}})$, where $h(x,t)$ satisfies the following assumptions:

$(g1)$ $h \in C^1(\overline{\Omega} \times \mathbb{R})$, $h(x,t) = 0$, for all $t \leq 0$, $h(x,t) > 0$, for all $t > 0$.

$(g2)$ For any $\epsilon > 0$, $\lim\limits_{t\to\infty} \sup\limits_{x\in\overline{\Omega}} h(x,t)\exp(-\epsilon|t|^{\frac{n}{n-s}}) = 0$, $\lim\limits_{t\to\infty} \inf\limits_{x\in\overline{\Omega}} h(x,t)\exp(\epsilon|t|^{\frac{n}{n-s}}) = \infty$.

$(g3)$ There exist positive constants T, T_0 and γ_0 such that

$$0 < t^{\gamma_0}G(x,t) \leq T_0 g(x,t) \text{ for all } (x,t) \in \Omega \times [t_0, +\infty).$$

$(g4)$ For $\gamma > 1$ (defined in (M2)), there exists a $l > \frac{\gamma n}{2s} - 1$ such that the map $t \mapsto \frac{g(x,t)}{t^l}$ is increasing on $\mathbb{R}^+ \setminus \{0\}$, uniformly in $x \in \Omega$.

Remark 2.3 *Condition $(g4)$ implies that for each $x \in \Omega$,*

$$t \mapsto \frac{g(x,t)}{t^{\frac{\gamma n}{2s}-1}} \text{ is increasing for } t > 0 \text{ and } \lim_{t\to 0^+} \frac{g(x,t)}{t^{\frac{\gamma n}{2s}-1}} = 0,$$

uniformly in $x \in \Omega$. Also, for each $(x,t) \in \Omega \times \mathbb{R}$ we have

$$(l+1)G(x,t) \leq tg(x,t).$$

Example 2 *Let $g(x,t) = h(x,t)e^{|t|^{\frac{n}{n-s}}}$, where $h(x,t) = \begin{cases} 0 \text{ if } t \leq 0 \\ t^{\alpha+(\frac{\gamma n}{2s}-1)}\exp(dt^\beta) \text{ if } t > 0. \end{cases}$ for some $\alpha > 0$, $0 < d \leq \alpha_{n,s}$ and $1 \leq \beta < \frac{n}{n-s}$. Then g satisfies all the conditions from $(g1) - (g4)$.*

Definition 2.4 *We say that $u \in X_0$ is a weak solution of $(\mathcal{M})$ if, for all $\phi \in X_0$, it satisfies*

$$M(\|u\|^{\frac{n}{s}}) \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{2n}} dxdy = \int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) g(x,u)\phi \; dx.$$

Before stating our main Theorem, we recall a result of [18] which will be used to find an upper bound for the Mountain Pass critical level. Assume that $0 \in \Omega$ and $B_1(0) \subset \Omega$. Then we consider the following Moser type functions which is given by equation (5.2) of [18]. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$\tilde{w_k}(x) = \begin{cases} |\log k|^{\frac{n-s}{n}}, & \text{if } 0 \le |x| \le \frac{1}{k}, \\ \frac{|\log(|x|)|}{|\log(1/k)|^{s/n}}, & \text{if } \frac{1}{k} \le |x| \le 1, \\ 0, & \text{if } |x| \ge 1, \end{cases} \tag{2.4}$$

then $\operatorname{supp}(\tilde{w}_k) \subset B_1(0) \subset \Omega$ and $\tilde{w}_k|_{B_1(0)} \in W_0^{s,\frac{n}{s}}(B_1(0))$.
Now by Proposition 5.1 of [18] we know that

$$\lim_{k\to\infty} \|\tilde{w}_k\|^{\frac{n}{s}} = \lim_{k\to\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{w}_k(x)-\tilde{w}_k(y)|^{\frac{n}{s}}}{|x-y|^{2n}} dxdy = \gamma_{n,s}, \tag{2.5}$$

where

$$\gamma_{n,s} := \frac{2(n\mathcal{W}_n)^2\Gamma(\frac{n}{s}+1)}{n!} \sum_{i=0}^{\infty} \frac{(n+i-1)!}{i!(n+2i)^{\frac{n}{s}}}.$$

where $\mathcal{W}_n$ denotes the volume of n-dimensional unit sphere. We also recall the following result of Lions known as higher integrability Lemma in case of fractional Laplacian, proved in [20].

Lemma 2.5 *Let $\{v_k : \|v_k\| = 1\}$ be a sequence in $W_0^{s,n/s}(\Omega)$ converging weakly to a non-zero function v. Then for every p such that $p < \alpha_{n,s}(1-\|v\|^{\frac{n}{s}})^{\frac{-s}{n-s}}$,*

$$\sup_k \int_\Omega \exp(p|v_k|^{\frac{n}{n-s}}) < +\infty.$$

Now we state our main result:

Theorem 2.6 *Suppose $(M1)-(M3)$ and $(g1)-(g4)$ hold. Assume in addition that for $\beta > \frac{2\alpha^*_{n,s}}{\alpha_{n,s}}$,*

$$\lim_{t\to+\infty} \frac{tg(x,t)G(x,t)}{\exp\left(\beta t^{\frac{n}{n-s}}\right)} = \infty \text{ uniformly in } x \in \overline{\Omega}. \tag{2.6}$$

Then, problem $(\mathcal{M})$ admit a non negative non trivial solution.

3 Proof of Main result

We begin this section with the study of mountain pass structure and Palais-Smale sequences corresponding to the energy functional $J : X_0 \to \mathbb{R}$ associated to the problem $(\mathcal{M})$ which is defined as

$$J(u) = \frac{s}{n}\hat{M}(\|u\|^{\frac{n}{s}}) - \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}dy\right) G(x,u)\ dx.$$

From the assumptions, $(g1) - (g4)$, we obtain that for any $\epsilon > 0$, $r \geq 1$, $1 \leq \alpha < l+1$ there exists $C(\epsilon) > 0$ such that

$$|G(x,t)| \leq \epsilon|t|^\alpha + C(\epsilon)|t|^r \exp((1+\epsilon)|t|^{\frac{n}{n-s}}), \text{ for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Now by Proposition 1.1, for any $u \in X_0$ we obtain

$$\int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}dy\right) G(x,u)\ dx \leq C(n,\mu)\|G(\cdot,u)\|^2_{L^{\frac{2n}{2n-\mu}}(\Omega)}. \tag{3.2}$$

This implies that J is well defined using Theorem 1.2. Also one can easily see that J is Fréchet differentiable and the critical points of J are the weak solutions of $(\mathcal{M})$.

Lemma 3.1 *Assume that the conditions $(M1)$ and $(g1) - (g4)$ hold. Then J satisfies the Mountain Pass geometry around* 0.

Proof. From (3.1), (3.2), Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}dy\right) G(x,u)\ dx \\
&\leq C(n,\mu)2^2 \left(\epsilon^{\frac{2n}{2n-\mu}}\int_\Omega |u|^{\frac{2n\alpha}{2n-\mu}} + (C(\epsilon))^{\frac{2n}{2n-\mu}}\int_\Omega |u|^{\frac{2rn}{2n-\mu}} \exp\left(\frac{2n(1+\epsilon)}{2n-\mu}|u|^{\frac{n}{n-s}}\right)\right)^{\frac{2n-\mu}{n}} \\
&\leq C\left(\epsilon^{\frac{2n}{2n-\mu}}\int_\Omega |u|^{\frac{2n\alpha}{2n-\mu}} + C_1(\epsilon)\|u\|^{\frac{2rn}{2n-\mu}}\left(\int_\Omega \exp\left(\frac{4n(1+\epsilon)\|u\|^{\frac{n}{n-s}}}{2n-\mu}\left(\frac{|u|}{\|u\|}\right)^{\frac{n}{n-s}}\right)\right)^{\frac{1}{2}}\right)^{\frac{2n-\mu}{n}}.
\end{aligned} \tag{3.3}$$

So if we choose $\epsilon > 0$ small enough and u such that $\dfrac{4n(1+\epsilon)\|u\|^{\frac{n}{n-s}}}{2n-\mu} \leq \alpha_{n,s}$ then using the fractional Trudinger-Moser inequality (1.2) in (3.3), we obtain

$$\begin{aligned}
\int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}dy\right) G(x,u)\ dx &\leq C_2(\epsilon)\left(\|u\|^{\frac{2n\alpha}{2n-\mu}} + \|u\|^{\frac{2rn}{2n-\mu}}\right)^{\frac{2n-\mu}{n}} \\
&\leq C_3(\epsilon)\left(\|u\|^{2\alpha} + \|u\|^{2r}\right).
\end{aligned}$$

Using (2.2) and above estimate, we have

$$J(u) \geq \frac{s}{n}\hat{M}(1)\|u\|^{\frac{\gamma n}{s}} - C_3(\epsilon)\left(\|u\|^{2\alpha} + \|u\|^{2r}\right),$$

when $\|u\| \leq 1$. Choosing $\alpha > \frac{\gamma n}{2s}$, $r > \frac{\gamma n}{2s}$ and $\rho > 0$ such that $\rho < \min\left\{1, \left(\frac{\alpha_{n,s}(2n-\mu)}{4n(1+\epsilon)}\right)^{\frac{n-s}{n}}\right\}$ we obtain $J(u) \geq \sigma > 0$ for all $u \in X_0$ with $\|u\| = \rho$ and for some $\sigma > 0$ depending on ρ.

The condition $(g4)$ implies that there exist some positive constants C_1 and C_2 such that

$$G(x,t) \geq C_1 t^{l+1} - C_2 \text{ for all } (x,t) \in \Omega \times [0,\infty). \tag{3.4}$$

Let $\phi \in X_0$ such that $\phi \geq 0$ and $\|\phi\| = 1$ then by (3.4) we obtain

$$\begin{aligned}\int_\Omega \left(\int_\Omega \frac{G(y,t\phi)}{|x-y|^\mu}dy\right) G(x,t\phi)\ dx &\geq \int_\Omega\int_\Omega \frac{(C_1(t\phi)^{l+1}(y) - C_2)(C_1(t\phi)^{l+1}(x) - C_2)}{|x-y|^\mu}\ dxdy \\ &= C_1^2 t^{2(l+1)} \int_\Omega\int_\Omega \frac{\phi^{l+1}(y)\phi^{l+1}(x)}{|x-y|^\mu}\ dxdy \\ &\quad - 2C_1C_2t^{l+1}\int_\Omega\int_\Omega \frac{\phi^{l+1}(y)}{|x-y|^\mu}\ dxdy + C_2^2\int_\Omega\int_\Omega \frac{1}{|x-y|^\mu}\ dxdy.\end{aligned}$$

This together with (2.3), we obtain

$$\begin{aligned}J(t\phi) &\leq \frac{s}{n}M(1)\|t\phi\|^{\frac{\gamma n}{s}} - \frac{1}{2}\int_\Omega\left(\int_\Omega \frac{G(y,t\phi)}{|x-y|^\mu}dy\right) G(x,t\phi)\ dx \\ &\leq C_3 + C_4 t^{\frac{\gamma n}{s}} - C_5 t^{2(l+1)} + C_6 t^{l+1},\end{aligned}$$

where $C_i's$ are positive constants for $i = 3,4,5,6$. This implies that $J(t\phi) \to -\infty$ as $t \to \infty$, since $l+1 > \frac{\gamma n}{2s}$. Thus there exists a $v_0 \in X_0$ with $\|v_0\| > \rho$ such that $J(v_0) < 0$. Therefore, J satisfies Mountain Pass geometry near 0. □

Let $\Gamma = \{\gamma \in C([0,1], X_0) : \gamma(0) = 0, J(\gamma(1)) < 0\}$ and define the Mountain Pass critical level $c_* = \inf\limits_{\gamma\in\Gamma}\max\limits_{t\in[0,1]} J(\gamma(t))$. Then by Lemma 3.1 and the Mountain pass theorem we know that there exists a Palais Smale sequence $\{u_k\} \subset X_0$ for J at c_* that is

$$J(u_k) \to c_* \text{ and } J'(u_k) \to 0 \text{ as } k \to \infty.$$

Lemma 3.2 *Every Palais-Smale sequence of J is bounded in X_0.*

Proof. Let $\{u_k\} \subset X_0$ denotes a $(PS)_c$ sequence of J that is

$$J(u_k) \to c \text{ and } J'(u_k) \to 0 \text{ as } k \to \infty$$

for some $c \in \mathbb{R}$. This implies

$$\begin{aligned}&\frac{s\hat{M}(\|u_k\|^{\frac{n}{s}})}{n} - \frac{1}{2}\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) G(x,u_k)\ dx \to c \text{ as } k \to \infty, \\ &\left| M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^{\frac{n}{s}-2}(u_k(x)-u_k(y))(\phi(x)-\phi(y))}{|x-y|^{2n}}dxdy \right. \\ &\qquad \left. - \int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\phi\ dx \right| \leq \epsilon_k\|\phi\| \end{aligned} \tag{3.5}$$

where $\epsilon_k \to 0$ as $k \to \infty$. In particular, taking $\phi = u_k$ we get

$$\left| M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}} - \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k)u_k \; dx \right| \le \epsilon_k \|u_k\|. \tag{3.6}$$

Now Remark (2.3) gives us that

$$(l+1) \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k) \; dx \le \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k)u_k \; dx. \tag{3.7}$$

Then using (3.5), (3.6) along with (3.7) and (2.1), we get

$$\begin{aligned} J(u_k) - \frac{1}{2(l+1)}\langle J'(u_k), u_k\rangle &= \frac{s}{n}\hat{M}(\|u_k\|^{\frac{n}{s}}) - \frac{1}{2(l+1)} M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}} \\ &\quad - \frac{1}{2}\left[\int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k) \; dx - \frac{1}{(l+1)} \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k)u_k \; dx \right] \\ &\ge \frac{s\hat{M}(\|u_k\|^{\frac{n}{s}})}{n} - \frac{M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}}}{2(l+1)} \\ &\ge \left(\frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}}. \end{aligned} \tag{3.8}$$

To prove the Lemma, we assume by contradiction that $\{\|u_k\|\}$ is an unbounded sequence. Then without loss of generality, we can assume that, up to a subsequence, $\|u_k\| \to \infty$ and $\|u_k\| \ge \alpha > 0$ for some α and for all k. This along with (3.8) and $(M3)$ gives us

$$J(u_k) - \frac{1}{2(l+1)}\langle J'(u_k), u_k\rangle \ge \left(\frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) \kappa \|u_k\|^{\frac{n}{s}} \tag{3.9}$$

where κ depends on α. Also from (3.5) and (3.6) it follows that

$$J(u_k) - \frac{1}{2(l+1)}\langle J'(u_k), u_k\rangle \le C\left(1 + \epsilon_k \frac{\|u_k\|}{2(l+1)} \right) \tag{3.10}$$

for some constant $C > 0$. Therefore from (3.9) and (3.10) we get that

$$\left(\frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) \kappa \|u_k\|^{\frac{n}{s}} \le C\left(1 + \epsilon_k \frac{\|u_k\|}{2(l+1)} \right)$$

which gives a contradiction because $l+1 > \frac{\gamma n}{2s}$ and $\frac{n}{s} > 1$. This implies that $\{u_k\}$ must be bounded in X_0. □

Assume that $0 \in \Omega$ and $\rho > 0$ be such that $B_\rho(0) \subset \Omega$. Then for $x \in \mathbb{R}^n$, we define $w_k(x) := \tilde{w}_k\left(\frac{x}{\rho}\right)$, where $\tilde{w}_k$ is same as (2.4) then $\text{supp}(w_k) \in B_\rho(0) \subset \Omega$. We note that $w_k \in W_0^{s,\frac{n}{s}}(\mathbb{R}^n)$ and by (2.5), we have

$$\lim_{k\to\infty} \|w_k\|^{\frac{n}{s}} = \lim_{k\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tilde{w}_k(x) - \tilde{w}_k(y)|^{\frac{n}{s}}}{|x-y|^{2n}} dxdy = \gamma_{n,s}. \tag{3.11}$$

Next, we use w_k's efficiently to obtain the following bound on c_*.

Lemma 3.3 *It holds that*

$$0 < c_* < \frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right).$$

Proof. Using Lemma 3.1, we deduce that $c_* > 0$ and $J(t\phi) \to -\infty$ as $t \to \infty$ if $0 \leq \phi \in X_0 \setminus \{0\}$ with $\|\phi\| = 1$. Also by definition of c_*, we have $c_* \leq \max_{t\in[0,1]} J(t\phi)$ for each non negative $\phi \in X_0 \setminus \{0\}$ with $J(\phi) < 0$ which assures that it is enough to prove that there exists a non negative $w \in X_0 \setminus \{0\}$ such that

$$\max_{t\in[0,\infty)} J(tw) < \frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right).$$

To prove this, we consider the sequence of non negative functions $\{w_k\}$(defined before this Lemma) and claim that there exists a $k \in \mathbb{N}$ such that

$$\max_{t\in[0,\infty)} J(tw_k) < \frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right).$$

Suppose this is not true, then for all $k \in \mathbb{N}$ there exists a $t_k > 0$ such that

$$\begin{aligned} &\max_{t\in[0,\infty)} J(tw_k) = J(t_k w_k) \geq \frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right) \\ &\text{and } \frac{d}{dt}(J(tw_k))|_{t=t_k} = 0. \end{aligned} \tag{3.12}$$

From the proof of Lemma 3.1, $J(tw_k) \to -\infty$ as $t \to \infty$ for each k. Then we infer that $\{t_k\}$ must be a bounded sequence in $\mathbb{R}$ which implies that there exists a t_0 such that, up to a subsequence which we still denote by $\{t_k\}$, $t_k \to t_0$ as $k \to \infty$. From (3.12) and definition of $J(t_k w_k)$ we obtain

$$\frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right) < \frac{s}{n}\hat{M}(\|t_k w_k\|^{\frac{n}{s}}). \tag{3.13}$$

Since $\hat{M}$ is monotone increasing, from (3.13) we get that

$$\|t_k w_k\|^{\frac{n}{s}} \geq \left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}. \tag{3.14}$$

From (3.14) and since (3.11) holds, we infer that

$$t_k(\log k)^{\frac{n-s}{n}} \to \infty \text{ as } k \to \infty. \tag{3.15}$$

Furthermore from (3.12), we have

$$\begin{aligned} M(\|t_k w_k\|^{\frac{n}{s}})\|t_k w_k\|^{\frac{n}{s}} &= \int_\Omega \left(\int_\Omega \frac{G(y, t_k w_k)}{|x-y|^\mu} dy\right) g(x, t_k w_k) t_k w_k \, dx \\ &\geq \int_{B_{\rho/k}} g(x, t_k w_k) t_k w_k \int_{B_{\rho/k}} \frac{G(y, t_k w_k)}{|x-y|^\mu} \, dy \, dx. \end{aligned} \tag{3.16}$$

In addition, as in equation (2.11) p. 1943 in [1], it is easy to get that

$$\int_{B_{\rho/k}}\int_{B_{\rho/k}}\frac{dxdy}{|x-y|^{\mu}}\geq C_{\mu,n}\left(\frac{\rho}{k}\right)^{2n-\mu},$$

where $C_{\mu,n}$ is a positive constant depending on μ and n. From (2.6), it is easy to deduce that for $\beta>\frac{2\alpha^*_{n,s}}{\alpha_{n,s}}$ and for each $d>0$ there exists a $r_d\in\mathbb{N}$ such that

$$rg(x,r)G(x,r)\geq d\exp\left(\beta|r|^{\frac{n}{n-s}}\right)\text{ whenever } r\geq r_d.$$

Since (3.15) holds, we can choose a $N_d\in\mathbb{N}$ such that

$$t_k(\log k)^{\frac{n-s}{n}}\geq r_d \text{ for all } k\geq N_d.$$

Using these estimates in (3.16) and from (3.14), for d large enough we get that

$$\begin{aligned}M(\|t_kw_k\|^{\frac{n}{s}})\|t_kw_k\|^{\frac{n}{s}}&\geq d\exp\left(\beta t_k^{\frac{n}{n-s}}|\log k|\right)C_{\mu,n}\left(\frac{\rho}{k}\right)^{2n-\mu}\\&=dC_{\mu,n}\rho^{2n-\mu}\exp\left(\left(\beta t_k^{\frac{n}{n-s}}-(2n-\mu)\right)\log k\right)\\&\geq dC_{\mu,n}\rho^{2n-\mu}\exp\left(\log k\left(\frac{(2n-\mu)\beta\alpha_{n,s}}{2n\|w_k\|^{\frac{n}{n-s}}}-(2n-\mu)\right)\right)\end{aligned}\tag{3.17}$$

Since $\beta>\frac{2\alpha^*_{n,s}}{\alpha_{n,s}}=\frac{2n\gamma_{n,s}^{\frac{s}{n-s}}}{\alpha_{n,s}}$ and (3.11) hold, the R.H.S. of (3.17) tends to $+\infty$ as $k\to\infty$. Whereas from continuity of M it follows that

$$\lim_{k\to\infty}M\left(\|t_kw_k\|^{\frac{n}{s}}\right)\|t_kw_k\|^{\frac{n}{s}}=M\left(t_0^{\frac{n}{s}}\gamma_{n,s}\right)(t_0^{\frac{n}{s}}\gamma_{n,s}),$$

which is a contradiction. This establishes our claim and we conclude the proof of Lemma. □

In order to prove that a Palais-Smale sequence converges to a weak solution of problem $(\mathcal{M})$, we need the following convergence Lemma. The idea of proof is borrowed from Lemma 2.4 in [1].

Lemma 3.4 *If $\{u_k\}$ is a Palais Smale sequence for J at c then there exists a $u\in X_0$ such that, up to a subsequence.*

$$\left(\int_\Omega\frac{G(y,u_k)}{|x-y|^{\mu}}\,dy\right)G(x,u_k)\to\left(\int_\Omega\frac{G(y,u)}{|x-y|^{\mu}}\,dy\right)G(x,u)\text{ in } L^1(\Omega)\tag{3.18}$$

Proof. From Lemma 3.2, we know that the sequence $\{u_k\}$ must be bounded in X_0. Consequently, up to a subsequence, there exists a $u\in X_0$ such that $u_k\rightharpoonup u$ weakly in X_0 and strongly in $L^q(\Omega)$ for any $q\in[1,\infty)$ as $k\to\infty$. Also, still up to a subsequence, we can assume that $u_k(x)\to u(x)$ pointwise a.e. for $x\in\Omega$.

From (3.5), (3.6) and (3.7) we get that there exists a constant $C>0$ such that

$$\begin{aligned}&\int_\Omega\left(\int_\Omega\frac{G(y,u_k)}{|x-y|^{\mu}}dy\right)G(x,u_k)\,dx\leq C,\\&\int_\Omega\left(\int_\Omega\frac{G(y,u_k)}{|x-y|^{\mu}}dy\right)g(x,u_k)u_k\,dx\leq C.\end{aligned}\tag{3.19}$$

Now, it is well known that if $f \in L^1(\Omega)$ then for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_U f(x)\ dx \right| < \epsilon,$$

for any measurable set $U \subset \Omega$ with $|U| \leq \delta$. Also $f \in L^1(\Omega)$ implies that for any fixed $\delta > 0$ there exists $M > 0$ such that

$$|\{x \in \Omega : |f(x)| \geq M\}| \leq \delta.$$

Now using (3.19), we have

$$\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(\cdot, u_k) \in L^1(\Omega)$$

and also by (3.2)

$$\left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) G(\cdot, u) \in L^1(\Omega).$$

Now we fix $\delta > 0$ and choose $M > \max\left\{ \left(\frac{CT_0}{\delta}\right)^{\frac{1}{\gamma_0+1}}, t_0 \right\}$. Then we use $(g3)$ to obtain

$$\begin{aligned}\int_{\Omega \cap \{u_k \geq M\}} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx &\leq T_0 \int_{\Omega \cap \{u_k \geq M\}} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) \frac{g(x,u_k)}{u_k^{\gamma_0}}\ dx \\ &\leq \frac{T_0}{M^{\gamma_0+1}} \int_{\Omega \cap \{u_k \geq M\}} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k) u_k\ dx < \delta.\end{aligned}$$

Next we consider

$$\begin{aligned}&\left| \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx - \int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \right| \\ &\leq 2\delta + \left| \int_{\Omega \cap \{u_k \leq M\}} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx - \int_{\Omega \cap \{u \leq M\}} \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \right|\end{aligned}$$

To prove the result, it is enough to establish that as $k \to \infty$

$$\int_{\Omega \cap \{u_k \leq M\}} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx \to \int_{\Omega \cap \{u \leq M\}} \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx. \quad (3.20)$$

Since $\left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy \right) G(\cdot, u) \in L^1(\Omega)$, so by Fubini's theorem we get

$$\begin{aligned}&\lim_{K \to \infty} \int_{\Omega \cap \{u \leq M\}} \left(\int_{\Omega \cap \{u \geq K\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \\ &= \lim_{K \to \infty} \int_{\Omega \cap \{u \geq K\}} \left(\int_{\Omega \cap \{u \leq M\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx = 0.\end{aligned}$$

Thus we can fix a $K > \max\left\{ \left(\frac{CT_0}{\delta}\right)^{\frac{1}{\gamma_0+1}}, t_0 \right\}$ such that

$$\int_{\Omega\cap\{u\le M\}} \left(\int_{\Omega\cap\{u\ge K\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \le \delta.$$

From $(g3)$, we get

$$\begin{aligned}
&\int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\ge K\}} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx \\
&\le \frac{1}{K^{\gamma_0+1}} \int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\ge K\}} \frac{u_k^{\gamma_0+1}(y) G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx \\
&\le \frac{T_0}{K^{\gamma_0+1}} \int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\ge K\}} \frac{u_k(y) g(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx \\
&\le \frac{T_0}{K^{\gamma_0+1}} \int_{\Omega} \left(\int_{\Omega} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k) u_k\ dx \le \delta.
\end{aligned}$$

Thus we have proved that

$$\begin{aligned}
&\left| \int_{\Omega\cap\{u\le M\}} \left(\int_{\Omega\cap\{u\ge K\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \right. \\
&\qquad \left. - \int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\ge K\}} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx \right| \le 2\delta
\end{aligned}$$

Finally, to complete the proof of Lemma, we need to verify that as $k \to \infty$

$$\begin{aligned}
&\left| \int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\le K\}} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k)\ dx - \right. \\
&\qquad \left. \int_{\Omega\cap\{u\le M\}} \left(\int_{\Omega\cap\{u\le K\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u)\ dx \right| \to 0
\end{aligned} \tag{3.21}$$

for fixed positive K and M. It is easy to see that

$$\left(\int_{\Omega\cap\{u_k\le K\}} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k) \chi_{\Omega\cap\{u_k\le M\}} \to \left(\int_{\Omega\cap\{u\le K\}} \frac{G(y,u)}{|x-y|^\mu} dy \right) G(x,u) \chi_{\Omega\cap\{u\le M\}}$$

pointwise a.e. as $k \to \infty$. Now choose $r = \alpha$ in (3.1), which gives us that there exist a constant $C_{M,K} > 0$ depending on M and K such that

$$\begin{aligned}
&\int_{\Omega\cap\{u_k\le M\}} \left(\int_{\Omega\cap\{u_k\le K\}} \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k) dx \\
&\le C_{M,K} \int_{\Omega\cap\{u_k\le M\}} \left(\int_{\{u_k\le K\}} \frac{|u_k(y)|^r}{|x-y|^\mu} dy \right) |u_k(x)|^r dx \\
&\le C_{M,K} \int_{\Omega} \int_{\Omega} \left(\frac{|u_k(y)|^r}{|x-y|^\mu}\ dy \right) |u_k(x)|^r\ dx \\
&\le C_{M,K} C(n,\mu) \|u_k\|^{2r}_{L^{\frac{2nr}{2n-\mu}}(\Omega)} \to C_{M,K} C(n,\mu) \|u\|^{2r}_{L^{\frac{2nr}{2n-\mu}}(\Omega)} \text{ as } k \to \infty,
\end{aligned}$$

where we used the Hardy-Littlewood-Sobolev inequality in the last inequality and then used the fact that $u_k \to u$ strongly in $L^q(\Omega)$ for each $q \in [1,\infty)$. This implies that, using Theorem 4.9 of [4], there exists a constant $h \in L^1(\Omega)$ such that, up to a subsequence, for each k

$$\left|\left(\int_{\Omega\cap\{u_k\leq K\}} \frac{G(y,u_k)}{|x-y|^\mu}dy\right) G(x,u_k)\chi_{\Omega\cap\{u_k\leq M\}}\right| \leq |h(x)|$$

This helps us to employ the Lebesgue dominated convergence theorem and conclude (3.21). □

Lemma 3.5 *Let $\{u_k\} \subset X_0$ be a Palais Smale sequence of J. Then there exists a $u \in X_0$ such that, up to a subsequence, for all $\phi \in X_0$*

$$\int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\phi\ dx \to \int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu}dy\right) g(x,u)\phi\ dx \ as\ k\to\infty\ . \quad (3.22)$$

Proof. As we argued in previous Lemma, we have that there exists a $u \in X_0$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in X_0, $u_k \to u$ pointwise a.e. in $\mathbb{R}^n$, $\|u_k\| \to \tau$ as $k\to\infty$, for some $\tau \geq 0$ and $u_k \to u$ strongly in $L^q(\Omega)$, $q\in[1,\infty)$ as $k\to\infty$.

Let $\Omega' \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega)$ such that $0\leq\varphi\leq 1$ and $\varphi\equiv 1$ in Ω'. Then by taking φ as a test function in (3.5), we obtain the following estimate

$$\begin{aligned}&\int_{\Omega'} \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\ dx \leq \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\varphi\ dx\\ &\leq \epsilon_k\|\varphi\| + M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^{2n}} \frac{|u_k(x)-u_k(y)|^{\frac{n}{s}-2}(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{2n}}\ dxdy\\ &\leq \epsilon_k\|\varphi\| + C\|u_k\|\|\varphi\| \leq C,\end{aligned}$$

since $\|u_k\|\leq C_0$ for all k. This implies that the sequence $\{\mu_k\} := \left\{\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\right\}$ is bounded in $L^1_{\text{loc}}(\Omega)$ which implies that up to a subsequence, $\mu_k \to \mu$ in the *weak**-topology as $k\to\infty$, where μ denotes a Radon measure. So for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k\to\infty}\int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\phi\ dx = \int_\Omega \phi\ d\mu,\ \forall\ \phi\in C_c^\infty(\Omega).$$

Since u_k satisfies (3.5), for any measurable set $E\subset\Omega$, taking $\phi \in C_c^\infty(\Omega)$ such that supp$\phi \subset E$, we get that

$$\begin{aligned}\mu(E) &= \int_E \phi\ d\mu = \lim_{k\to\infty}\int_E\int_\Omega \left(\frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\phi(x)\ dx\\ &= \lim_{k\to\infty}\int_\Omega\int_\Omega \left(\frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)\phi(x)\ dx\\ &= \lim_{k\to\infty} M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^{\frac{n}{s}-2}(u_k(x)-u_k(y))(\phi(x)-\phi(y))}{|x-y|^{2n}}\ dxdy\\ &= M(\tau^{\frac{n}{s}})\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{2n}}\ dxdy,\end{aligned}$$

where we used the continuity of M and weak convergence of u_k to u in X_0. This implies that μ is absolutely continuous with respect to the Lebesgue measure. Thus, Radon-Nikodym theorem establishes that there exists a function $h \in L^1_{\text{loc}}(\Omega)$ such that for any $\phi \in C_c^\infty(\Omega)$, $\int_\Omega \phi \ d\mu = \int_\Omega \phi h \ dx$. Therefore for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k\to\infty} \int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy\right) g(x,u_k)\phi \ \ dx = \int_\Omega \phi h \ dx = \int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy\right) g(x,u)\phi \ \ dx$$

and the above holds for any $\phi \in X_0$ using the density argument. This completes the proof.□

Now we define the Nehari manifold associated to the functional J, as

$$\mathcal{N} := \{0 \not\equiv u \in X_0 : \langle J'(u), u\rangle = 0\}$$

and let $b := \inf_{u\in\mathcal{N}} J(u)$. Then we need the following Lemma to compare c_* and b.

Lemma 3.6 *If condition $(g4)$ holds, then for each $x \in \Omega$, $tg(x,t) - \frac{\gamma n}{2s}G(x,t)$ is increasing for $t \geq 0$. In particular $tg(x,t) - \frac{\gamma n}{2s}G(x,t) \geq 0$ for all $(x,t) \in \Omega \times [0,\infty)$ which implies $\frac{G(x,t)}{t^{\frac{\gamma n}{2s}}}$ is non-decreasing for $t > 0$.*

Proof. Suppose $0 < t < r$. Then for each $x \in \Omega$, we obtain

$$\begin{aligned} tg(x,t) - \frac{\gamma n}{2s}G(x,t) &= \frac{g(x,t)}{t^l}t^{l+1} - \frac{\gamma n}{2s}G(x,r) + \frac{\gamma n}{2s}\int_t^r g(x,\tau)d\tau \\ &< \frac{g(x,t)}{t^l}t^{l+1} - \frac{\gamma n}{2s}G(x,r) + \frac{\gamma n}{2s(l+1)}\frac{g(x,r)}{r^l}(r^{l+1} - t^{l+1}) \\ &\leq rg(x,r) - \frac{\gamma n}{2s}G(x,r), \end{aligned}$$

using $(g4)$. This completes the proof. □

Lemma 3.7 *Under the assumptions $(M2)$ and $(g4)$, it holds $c_* \leq b$.*

Proof. Let $u \in \mathcal{N}$ be non negative and we define $h : (0,\infty) \to \mathbb{R}$ by $h(t) = J(tu)$. Then for all $t > 0$

$$h'(t) = \langle J'(tu), u\rangle = M(t^{\frac{n}{s}}\|u\|^{\frac{n}{s}})t^{\frac{n}{s}-1}\|u\|^{\frac{n}{s}} - \int_\Omega \left(\int_\Omega \frac{G(y,tu)}{|x-y|^\mu}dy\right) g(x,tu)u \ dx.$$

Since $\langle J'(u), u\rangle = 0$ and $t \mapsto \frac{g(x,t)}{t^{\frac{\gamma n}{2s}-1}}$ is increasing for $t > 0$, we have

$$\begin{aligned} h'(t) =& \|u\|^{\frac{\gamma n}{s}} t^{\frac{\gamma n}{s}-1}\left(\frac{M(t^{\frac{n}{s}}\|u\|^{\frac{n}{s}})}{t^{(\gamma-1)\frac{n}{s}}\|u\|^{(\gamma-1)\frac{n}{s}}} - \frac{M(\|u\|^{\frac{n}{s}})}{\|u\|^{(\gamma-1)\frac{n}{s}}}\right) \\ &+ t^{\frac{\gamma n}{s}-1}\int_\Omega\left(\int_\Omega \frac{\frac{G(y,u)g(x,u)}{u^{\frac{\gamma n}{2s}-1}(x)}}{|x-y|^\mu}dy - \int_\Omega \frac{\frac{G(y,tu)g(x,tu)}{(tu)^{\frac{\gamma n}{2s}-1}(x)t^{\frac{\gamma n}{2s}}}}{|x-y|^\mu}dy\right)u^{\frac{\gamma n}{2s}}(x)dx \\ \geq& \|u\|^{\frac{\gamma n}{s}}t^{\frac{\gamma n}{s}-1}\left(\frac{M(t^{\frac{n}{s}}\|u\|^{\frac{n}{s}})}{t^{\frac{(\gamma-1)n}{s}}\|u\|^{\frac{(\gamma-1)n}{s}}} - \frac{M(\|u\|^{\frac{n}{s}})}{\|u\|^{\frac{(\gamma-1)n}{s}}}\right) \\ &+ t^{\frac{\gamma n}{s}-1}\int_\Omega\left(\int_\Omega\left(G(y,u) - \frac{G(y,tu)}{t^{\frac{\gamma n}{2s}}}\right)\frac{1}{|x-y|^\mu}dy\right)\frac{g(x,tu)}{(tu)^{\frac{\gamma n}{2s}-1}(x)}u^{\frac{\gamma n}{2s}}(x)dx. \end{aligned}$$

when $0 < t < 1$. So using Lemma 3.6 and $(M2)$ we have $h'(1) = 0$, $h'(t) \geq 0$ for $0 < t < 1$ and $h'(t) < 0$ for $t > 1$. Hence $J(u) = \max_{t\geq 0} J(tu)$. Now define $f : [0,1] \to X_0$ as $f(t) = (t_0 u)t$, where $t_0 > 1$ is such that $J(t_0 u) < 0$. Then we have $f \in \Gamma$ and therefore

$$c_* \leq \max_{t\in[0,1]} J(f(t)) \leq \max_{t\geq 0} J(tu) = J(u) \leq \inf_{u\in\mathcal{N}} J(u) = b.$$

Hence the proof is complete. □

Definition 3.8 *A solution u_0 of $(\mathcal{M})$ is a ground state if u_0 is a weak solution of $(\mathcal{M})$ and satisfies $J(u_0) = \inf_{u\in\mathcal{N}} J(u)$.*

Since $c_* \leq b$ in order to obtain a ground state solution u_0 for $(\mathcal{M})$, it is enough to show that there exists a weak solution of $(\mathcal{M})$ such that $J(u_0) = c_*$.

Lemma 3.9 *Any nontrivial solution of problem $(\mathcal{M})$ is nonnegative.*

Proof. Let $u \in X_0 \setminus \{0\}$ be a critical point of functional J. Clearly $u^- = \max\{-u, 0\} \in X_0$. Then $\langle J'(u), u^-\rangle = 0$, i.e.

$$\begin{aligned} M(\|u\|^{\frac{n}{s}}) \int_{\mathbb{R}^{2n}} & \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{2n}} dxdy \\ &= \int_\Omega \left(\int_\Omega \frac{G(y,u)}{|x-y|^\mu} dy\right) g(x,u)u^- dx. \end{aligned}$$

For a.e. $x, y \in \mathbb{R}^n$, using $|u^-(x) - u^-(y)| \leq |u(x) - u(y)|$, we have

$$\begin{aligned} &|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(u^-(x)-u^-(y)) \\ &= -|u(x)-u(y)|^{\frac{n}{s}-2}(u^+(x)u^-(y) + u^-(x)u^+(y) + |u^-(x)-u^-(y)|^2) \\ &\leq -|u^-(x)-u^-(y)|^{\frac{n}{s}} \end{aligned}$$

and $g(x,u)u^- = 0$ a.e. $x \in \Omega$ by assumption. Hence,

$$0 \leq -M(\|u\|^{\frac{n}{s}})\|u^-\|^{\frac{n}{s}} \leq 0.$$

So, $u^- \equiv 0$ since $\|u\| > 0$ and $(M3)$ holds. Hence $u \geq 0$ a.e. in Ω. □

Proof of Theorem 2.6: Since J satisfies the Mountain Pass geometry (refer Lemma 3.1), by Mountain Pass Lemma we know that there exists a Palais Smale $\{u_k\}$ sequence for J at c_*. Then by Lemma 3.2, $\{u_k\}$ must be bounded in X_0 so that, up to a subsequence, $u_k \rightharpoonup u_0$ weakly in X_0, strongly in $L^q(\Omega)$ for $q \in [1,\infty)$, pointwise a.e. in Ω, for some $u_0 \in X_0$ and $\|u_k\| \to \rho_0 \geq 0$ as $k \to \infty$.

Claim 1: $u_0 \not\equiv 0$ in Ω.

Proof. We argue by contradiction. Suppose that $u_0 \equiv 0$. Then using Lemma 3.4, we obtain

$$\int_\Omega \left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy\right) G(x,u_k)\, dx \to 0 \text{ as } k \to \infty. \tag{3.23}$$

This together with $\lim\limits_{k\to\infty} J(u_k) = c_*$ gives that

$$\lim_{k\to\infty} \frac{s}{n}\hat{M}(\|u_k\|^{\frac{n}{s}}) = c_* < \frac{s}{n}\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right).$$

Thus $\hat{M}$ being increasing function gives that there exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|^{\frac{n}{s}} \leq \left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}$ for all $k \geq k_0$. We fix $k \geq k_0$ and choose $p > 1$ close to 1 and $\epsilon > 0$ small enough such that

$$\frac{2np(1+\epsilon)}{2n-\mu}\|u_k\|^{\frac{n}{n-s}} < \alpha_{n,s}.$$

Using the growth assumptions on g and Theorem 1.2 we have

$$\begin{aligned}\|g(\cdot,u_k)u_k\|^{\frac{2n-\mu}{2n}}_{L^{\frac{2n}{2n-\mu}}(\Omega)} &\leq C(\epsilon)\left(\int_\Omega |u_k|^{\frac{2n\alpha}{2n-\mu}}dx + \int_\Omega |u_k|^{\frac{2nr}{2n-\mu}}\exp\left(\frac{2n(1+\epsilon)}{2n-\mu}|u_k|^{\frac{n}{n-s}}\right)dx\right)\\ &\leq C(\epsilon)\left(\int_\Omega |u_k|^{\frac{2n\alpha}{2n-\mu}}dx + \left(\int_\Omega |u_k|^{\frac{2nrp'}{2n-\mu}}dx\right)^{\frac{1}{p'}}\right.\\ &\qquad\left.\left(\int_\Omega \exp\left(\frac{2np(1+\epsilon)}{2n-\mu}\|u_k\|^{\frac{n}{n-s}}\left(\frac{|u_k|}{\|u_k\|}\right)^{\frac{n}{n-s}}\right)dx\right)^{\frac{1}{p}}\right)\end{aligned}$$

where $1 < \alpha < l+1$ and $1 < r$. Thus,

$$\|g(\cdot,u_k)u_k\|_{L^{\frac{2n}{2n-\mu}}(\Omega)} \leq C(\epsilon)\left(\|u_k\|^{\frac{2n-\mu}{2n\alpha}}_{L^{\frac{2n\alpha}{2n-\mu}}(\Omega)} + \|u_k\|^{\frac{2n-\mu}{2nr}}_{L^{\frac{2nrp'}{2n-\mu}}(\Omega)}\right) \to 0 \text{ as } k\to\infty, \tag{3.24}$$

where p' denotes the Hölder conjugate of p and $C(\epsilon) > 0$ is a constant depending on ϵ which may change value at each step. From the semigroup property of the Riesz potential and Hardy-Littlewood-Sobolev inequality we get that

$$\begin{aligned}&\left|\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right)g(x,u_k)u_k dx\right|\\ &\leq \left(\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right)G(x,u_k)\ dx\right)^{\frac{1}{2}}\left(\int_\Omega\left(\int_\Omega \frac{g(y,u_k)u_k}{|x-y|^\mu}dy\right)g(x,u_k)u_k\ dx\right)^{\frac{1}{2}}\\ &\leq \left(\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right)G(x,u_k)\ dx\right)^{\frac{1}{2}} C_{n,\mu}\|g(\cdot,u_k)u_k\|_{L^{\frac{2n}{2n-\mu}}(\Omega)} \to 0\end{aligned}$$

as $k \to \infty$ using (3.23) and (3.24). This together with $\langle J'(u_k), u_k\rangle = 0$ implies that $M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}} \to 0$. From $(M3)$, we deduce that $\|u_k\| \to 0$. Furthermore, we obtain $\lim_{k\to\infty} J(u_k) = 0 = c_*$, which is a contradiction to the fact that $c_* > 0$. Hence, we must have $u_0 \not\equiv 0$.

Claim 2: $M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} \geq \int_\Omega\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right)g(x,u_0)u_0 dx.$

Proof. Suppose by contradiction that $M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} < \int_\Omega\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right)g(x,u_0)u_0\ dx$. That is, $\langle J'(u_0), u_0\rangle < 0$.

It is easy to see, using $(M2)$, that $M(t)t \geq M(1)t^\gamma$ when $t \in [0,1]$. So for $0 < t < \frac{1}{\|u_0\|}$, using Lemma 3.6 and Hardy-Littlewood-Sobolev inequality we have that

$$\begin{aligned}\langle J'(tu_0), u_0\rangle &\geq M(t^{\frac{n}{s}}\|u_0\|^{\frac{n}{s}})t^{\frac{n}{s}-1}\|u_0\|^{\frac{n}{s}} - \frac{2s}{\gamma n}\int_\Omega\left(\int_\Omega \frac{g(y,tu_0)tu_0(y)}{|x-y|^\mu}\,dy\right)g(x,tu_0)u_0(x)\,dx\\ &\geq M(1)t^{\frac{\gamma n}{s}-1}\|u_0\|^{\frac{\gamma n}{s}} - \frac{C}{t}\left(\int_\Omega |g(x,tu_0)tu_0|^{\frac{2n}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{n}}.\end{aligned}$$

But from the growth assumptions on g we already know that for $\epsilon > 0$, $\alpha > \frac{\gamma n}{2s}$ and $r > \frac{\gamma n}{2s}$,

$$\begin{aligned}&\left(\int_\Omega |g(x,tu_0)tu_0|^{\frac{2n}{2n-\mu}}\,dx\right)^{\frac{2n-\mu}{n}}\\ &\leq C(\epsilon)\left(\int_\Omega |tu_0|^{\frac{2n\alpha}{2n-\mu}} + \|tu_0\|^{\frac{2rn}{2n-\mu}}\left(\int_\Omega \exp\left(\frac{4n(1+\epsilon)\|tu_0\|^{\frac{n}{n-s}}}{2n-\mu}\left(\frac{|tu_0|}{\|tu_0\|}\right)^{\frac{n}{n-s}}\right)\right)^{\frac{1}{2}}\right)^{\frac{2n-\mu}{n}}\\ &\leq C(\epsilon)\left(\|tu_0\|^{2\alpha} + \|tu_0\|^{2r}\right)\end{aligned}$$

by choosing $t < \left(\frac{(2n-\mu)\alpha_{n,s}}{4n(1+\epsilon)\|u_0\|^{\frac{n}{n-s}}}\right)^{\frac{n-s}{n}}$ and using Trudinger-Moser inequality. Therefore for $t > 0$ small enough as above, we obtain

$$\langle J'(tu_0), u_0\rangle \geq M(1)t^{\frac{\gamma n}{s}-1}\|u_0\|^{\frac{\gamma n}{s}} - C(\epsilon)\left(t^{2\alpha-1}\|u_0\|^{2\alpha} + t^{2r-1}\|u_0\|^{2r}\right)$$

which suggests that $\langle J'(tu_0), u_0\rangle > 0$ when t is sufficiently small. Thus there exists a $\sigma \in (0,1)$ such that $\langle J'(\sigma u_0), u_0\rangle = 0$ that is, $\sigma u_0 \in \mathcal{N}$. Thus from Lemmas 3.6, 3.7 and Remark 2.1, it follows that

$$\begin{aligned}&c_* \leq b \leq J(\sigma u_0) = J(\sigma u_0) - \frac{s}{n\gamma}\langle J'(\sigma u_0), \sigma u_0\rangle\\ &= \frac{s}{n}\hat{M}(\|\sigma u_0\|^{\frac{n}{s}}) - \frac{sM(\|\sigma u_0\|^{\frac{n}{s}})\|\sigma u_0\|^{\frac{n}{s}}}{n\gamma} + \frac{s}{n\gamma}\int_\Omega\left(\int_\Omega \frac{G(y,\sigma u_0)}{|x-y|^\mu}dy\right)\left(g(x,\sigma u_0)\sigma u_0 - \frac{n\gamma}{2s}G(x,\sigma u_0)\right)\\ &< \frac{s}{n}\hat{M}(\|u_0\|^{\frac{n}{s}}) - \frac{s}{n\gamma}M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} + \frac{s}{n\gamma}\int_\Omega\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right)\left(g(x,u_0)u_0 - \frac{n\gamma}{2s}G(x,u_0)\right)dx.\end{aligned}$$

Also by lower semicontinuity of norm and Fatou's Lemma, we obtain

$$\begin{aligned}c_* \leq b &< \liminf_{k\to\infty}\left(\frac{s}{n}\hat{M}(\|u_k\|^{\frac{n}{s}}) - \frac{s}{n\gamma}M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}}\right)\\ &\quad + \liminf_{k\to\infty}\frac{s}{n\gamma}\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right)\left[g(x,u_k)u_k - \frac{n\gamma}{2s}G(x,u_k)\right]dx\\ &\leq \lim_{k\to\infty}\left[J(u_k) - \frac{s}{n\gamma}\langle J'(u_k), u_k\rangle\right] = c_*,\end{aligned}$$

which is a contradiction. Hence Claim 2 is proved.

Claim 3: $J(u_0) = c_*$.

Proof. Using $\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right)G(x,u_k)dx \to \int_\Omega\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right)G(x,u_0)dx$ and lower semicontinuity of norm we have $J(u_0) \leq c_*$. Now we are going to show that the case $J(u_0) < c_*$ can not occur. Indeed, if $J(u_0) < c_*$ then $\|u_0\|^{\frac{n}{s}} < \rho_0^{\frac{n}{s}}$. Moreover,

$$\frac{s}{n}\hat{M}(\rho_0^{\frac{n}{s}}) = \lim_{k\to\infty} \frac{s}{n}\hat{M}(\|u_k\|^{\frac{n}{s}}) = c_* + \frac{1}{2}\int_\Omega \left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right) G(x,u_0)dx, \tag{3.25}$$

This gives that

$$\rho_0^{\frac{n}{s}} = \hat{M}^{-1}\left(\frac{n}{s}c_* + \frac{n}{2s}\int_\Omega \left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right) G(x,u_0)dx\right).$$

Next defining $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\rho_0}$, we have $v_k \rightharpoonup v_0$ in X_0 and $\|v_0\| < 1$. Thus by Lemma 2.5,

$$\sup_{k\in\mathbb{N}} \int_\Omega \exp(p|v_k|^{\frac{n}{n-s}})\ dx < \infty \ \text{ for all } 1 < p < \frac{\alpha_{n,s}}{(1-\|v_0\|^{\frac{n}{s}})^{\frac{s}{n-s}}}. \tag{3.26}$$

On the other hand, by Claim 2, (2.1) and Lemma 3.6, we have

$$\begin{aligned} J(u_0) \geq\ & \frac{s}{n}\hat{M}(\|u_0\|^{\frac{n}{s}}) - \frac{s}{n\gamma}M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} \\ & + \frac{s}{n\gamma}\int_\Omega \left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right)\left(g(x,u_0)u_0 - \frac{n\gamma}{2s}G(x,u_0)\right)dx \geq 0. \end{aligned}$$

Using this together with Lemma 3.3 and the equality, $\frac{n}{s}(c_* - J(u_0)) = \hat{M}\left(\rho_0^{\frac{n}{s}}\right) - \hat{M}\left(\|u_0\|^{\frac{n}{s}}\right)$ we obtain

$$\hat{M}\left(\rho_0^{\frac{n}{s}}\right) \leq \frac{n}{s}c_* + \hat{M}(\|u_0\|^{\frac{n}{s}}) < \hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right) + \hat{M}(\|u_0\|^{\frac{n}{s}})$$

and therefore by $(M1)$

$$\rho_0^{\frac{n}{s}} < \hat{M}^{-1}\left(\hat{M}\left(\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}\right) + \hat{M}(\|u_0\|^{\frac{n}{s}})\right) \leq \left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}} + \|u_0\|^{\frac{n}{s}}. \tag{3.27}$$

Since $\rho_0^{\frac{n}{s}}(1-\|v_0\|^{\frac{n}{s}}) = \rho_0^{\frac{n}{s}} - \|u_0\|^{\frac{n}{s}}$, from (3.27) it follows that

$$\rho_0^{\frac{n}{s}} < \frac{\left(\frac{2n-\mu}{2n}\alpha_{n,s}\right)^{\frac{n-s}{s}}}{1-\|v_0\|^{\frac{n}{s}}}.$$

Thus, there exists $\beta > 0$ such that $\|u_k\|^{\frac{n}{n-s}} < \beta < \frac{\alpha_{n,s}(2n-\mu)}{2n(1-\|v_0\|^{\frac{n}{s}})^{\frac{s}{n-s}}}$ for k large. We can choose $q > 1$ close to 1 such that $q\|u_k\|^{\frac{n}{n-s}} \leq \beta < \frac{(2n-\mu)\alpha_{n,s}}{2n(1-\|v_0\|^{\frac{n}{s}})^{\frac{s}{n-s}}}$ and using (3.26), we conclude that for k large

$$\int_\Omega \exp\left(\frac{2nq|u_k|^{n/n-s}}{2n-\mu}\right)dx \leq \int_\Omega \exp\left(\frac{2n\beta|v_k|^{n/n-s}}{2n-\mu}\right)dx \leq C.$$

Let us recall (2.3) and (3.24) to get that

$$\left|\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)u_k\ dx\right| \leq C\left(\|u_k\|^{\frac{2n-\mu}{2n\alpha}}_{L^{\frac{2n\alpha}{2n-\mu}}(\Omega)} + \|u_k\|^{\frac{2n-\mu}{2nr}}_{L^{\frac{2nrq'}{2n-\mu}}(\Omega)}\right)$$
$$\to C\left(\|u_0\|^{\frac{2n-\mu}{2n\alpha}}_{L^{\frac{2n\alpha}{2n-\mu}}(\Omega)} + \|u_0\|^{\frac{2n-\mu}{2nr}}_{L^{\frac{2nrq'}{2n-\mu}}(\Omega)}\right)$$

as $k\to\infty$. Then the pointwise convergence of $\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)u_k$ to $\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right) g(x,u_0)u_0$ as $k\to\infty$ asserts that

$$\lim_{k\to\infty}\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)u_k\ dx = \int_\Omega\left(\int_\Omega \frac{G(y,u_0)}{|x-y|^\mu}dy\right) g(x,u_0)u_0\ dx$$

while using the Lebesgue dominated convergence theorem. Now Lemma 3.5, we get

$$\int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_k)(u_k-u_0)dx \to 0 \text{ as } k\to\infty.$$

Since $\langle J'(u_k), u_k-u_0\rangle \to 0$, it follows that

$$M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^{2n}} \frac{|u_k(x)-u_k(y)|^{\frac{n}{s}-2}(u_k(x)-u_k(y))((u_k-u_0)(x)-(u_k-u_0)(y))}{|x-y|^{2n}}dxdy \to 0. \quad (3.28)$$

We define $U_k(x,y)=u_k(x)-u_k(y)$ and $U_0(x,y)=u_0(x)-u_0(y)$ then using $u_k\rightharpoonup u_0$ weakly in X_0 and boundedness of $M(\|u_k\|^{\frac{n}{s}})$, we have

$$M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^{2n}} \frac{|U_0(x,y)|^{\frac{n}{s}-2}U_0(x,y)(U_k(x,y)-U_0(x,y))}{|x-y|^{2n}}dxdy \to 0 \text{ as } k\to\infty. \quad (3.29)$$

Subtracting (3.29) from (3.28), we get

$$M(\|u_k\|^{\frac{n}{s}})\int_{\mathbb{R}^{2n}} \frac{(|U_k(x,y)|^{\frac{n}{s}-2}U_k(x,y)-|U_0(x,y)|^{\frac{n}{s}-2}U_0(x,y))(U_k(x,y)-U_0(x,y))}{|x-y|^{2n}}dxdy \to 0$$

as $k\to\infty$. Now using this and the following inequality

$$|a-b|^p \leq 2^{p-2}(|a|^{p-2}a-|b|^{p-2}b)(a-b) \text{ for all } a,b\in\mathbb{R} \text{ and } p\geq 2, \quad (3.30)$$

with $a=u_k(x)-u_k(y)$ and $b=u_0(x)-u_0(y)$, we obtain

$$M(\rho_0^{\frac{n}{s}})\int_{\mathbb{R}^{2n}} \frac{|U_k(x)-U_0(x)|^{\frac{n}{s}}}{|x-y|^{2n}}dxdy \to 0 \text{ as } k\to\infty.$$

This implies that $u_k\to u$ strongly in X_0 and hence $J(u)=c_*$ which is a contradiction. Therefore, claim 3 holds true. Hence $J(u)=c_*=\lim_{k\to\infty} J(u_k)$ and $\|u_k\|\to\rho_0$ gives that $\rho_0=\|u_0\|$. Finally we have

$$M(\|u_0\|^{\frac{n}{s}})\int_\Omega \frac{|u_0(x)-u_0(y)|^{\frac{n}{s}-2}(u_0(x)-u_0(y))(\phi(x)-\phi(y))}{|x-y|^{2n}}dxdy$$
$$= \int_\Omega\left(\int_\Omega \frac{G(y,u_k)}{|x-y|^\mu}dy\right) g(x,u_0)\phi\ dx,$$

for all $\phi\in X_0$. Thus, u_0 is a non trivial solution of $(\mathcal{M})$. By Lemma 3.9 we obtain that u_0 is the required nonnegative solution of $(\mathcal{M})$ which completes the proof. □

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