Isolated Singularities of Polyharmonic Operator in Even Dimension

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Abstract

We consider the equation $\Delta^2 u = g(x,u) \geq 0$ in the sense of distribution in $\Omega' = \Omega \setminus \{0\}$ where u and $-\Delta u \geq 0$. Then it is known that u solves $\Delta^2 u = g(x,u) + \alpha\delta_0 - \beta\Delta\delta_0$, for some non-negative constants α and β. In this paper we study the existence of singular solutions to $\Delta^2 u = a(x)f(u) + \alpha\delta_0 - \beta\Delta\delta_0$ in a domain $\Omega \subset I\!R^4$, a is a non-negative measurable function in some Lebesgue space. If $\Delta^2 u = a(x)f(u)$ in Ω', then we find the growth of the nonlinearity f that determines α and β to be 0. In case when $\alpha = \beta = 0$, we will establish regularity results when $f(t) \leq Ce^{\gamma t}$, for some $C, \gamma > 0$. This paper extends the work of Soranzo (1997) where the author finds the barrier function in higher dimensions ($N \geq 5$) with a specific weight function $a(x) = |x|^\sigma$. Later we discuss its analogous generalization for the polyharmonic operator.

Keywords: Elliptic system; polyharmonic operator; existence of solutions; singularity
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1. Introduction

Isolated singularities of elliptic operators are studied extensively, see for eg. [2],[10], [14], [15] and [16]. In this paper we wish to address the following problem and the questions related to it for the biharmonic(polyharmonic) operator in $I\!R^4 (I\!R^{2m})$:-

Question: If a non negative measurable function u is known to solve a PDE in the sense of distribution in a punctured domain, then what can one say about the differential equation satisfied by u in the entire domain?

In [2], Brezis and Lions answered this question for the Laplace operator with the assumption that

$$0 \leq -\Delta u = f(u) \text{ in } \Omega \setminus \{0\} \ , \ u \geq 0 \ , \ \liminf_{t \to \infty} \frac{f(t)}{t} > -\infty \ , \ \Omega \subset I\!R^N.$$

With the above hypotheses it was proved that both u and $f(u)$ belong to $L^1(\Omega)$, and satisfy $-\Delta u = f(u) + \alpha\delta_0$, for some $\alpha \geq 0$. For the dimension $N \geq 3$, P.L.Lions[10] found a sharp condition on f that determines whether α is zero or not in the previous expression. In [5], the authors further extended the result for dimension $N = 2$ by finding the minimal growth rate of the function f which guranteed α to be 0.

Taliaferro, in his series of papers (see for e.g. [15], [16], [8]) studied the isolated singularities of non-linear elliptic inequalities. In [16] the author studied the asymptotic behaviour of the positive solution of the differential inequality

$$0 \leq -\Delta u \leq f(u) \tag{1.1}$$

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in a punctured domain under various assumptions on f. If $N \geq 3$ and the function f has a "super-critical" growth as in Lions[10], (i.e. $\lim_{t\to\infty} \frac{f(t)}{t^{\frac{N}{N-2}}} = \infty$,) then there exists arbitrarily 'large solutions' of (1.1). When $N = 2$, it was proved that there exists a punctured neighborhood of the origin such that (1.1) admits arbitrarily large solutions near the origin, provided that $\log f(t)$ has a superlinear growth at infinity. Moreover author characterizes the singularity at the origin of all solutions u of (1.1) when $\log f(t)$ has a sublinear growth. Later Taliaferro, Ghergu and Moradifam in [8] generalized these results to polyharmonic inequalities.
The study of the polyharmonic equations of the type $(-\Delta)^m u = h(x, u)$ is associated to splitting the equation into a non-linear coupled system involving Laplace operator alone. Orsina and Ponce[12] proved the existence of solutions to

$$(1) \begin{cases} -\Delta u &= f(u,v) + \mu \quad \text{in } \Omega, \\ -\Delta v &= g(u,v) + \eta \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

with the assumption that the continuous functions f and g are non increasing in first and second variables respectively with $f(0,t) = g(s,0) = 0$. But here the authors assumed that μ and η are diffusive measures and Dirac distribution is not a diffusive measure. Considerable amount of existence/non-existence results have been proved for the problem (1) when f is a function of v alone and g depends only on u and μ, η are Radon measures. For eg. see [1] where the authors assumed $f(u,v) = v^p$, $g(u,v) = u^q$ and with non-homogenous boundary condition. In [6] authors dealt with sign changing functions f and g, with a polynomial type growth at infinity and the measure μ and η were assumed to be multiples of δ_0.

Our paper is closely related to the work of Soranzo [14] where author considers the equation:

$$\Delta^2 u = |x|^\sigma u^p \ \text{ with } u > 0,\ -\Delta u > 0 \text{ in } \Omega \subset I\!R^N, N \geq 4 \text{ and } \sigma \in (-4, 0).$$

A complete description of the singularity was provided when $1 < p < \frac{N+\sigma}{N-4}$ for $N \geq 5$, or $1 < p < \infty$ when $N = 4$. In this work we prove that the results of Soranzo can be improved for the dimension $N = 4$ by replacing u^p by more general exponential type function.

2. Preliminaries

We assume that Ω is a bounded open set in $I\!R^N, N \geq 4$ with smooth boundary and $0 \in \Omega$. We denote Ω' to be $\Omega \setminus \{0\}$. In this section we discuss some of the well known results for biharmonic operator.

Theorem 2.1. *(Brezis - Lions [2]) Let $u \in L^1_{loc}(\Omega')$ be such that $\Delta u \in L^1_{loc}(\Omega')$ in the sense of distributions in Ω', $u \geq 0$ in Ω such that*

$$-\Delta u + au \geq g \ \text{ a.e in } \Omega,$$

where a is a positive constant and $g \in L^1_{loc}(\Omega)$. Then there exist $\varphi \in L^1_{loc}(\Omega)$ and $\alpha \geq 0$ such that

$$-\Delta u = \varphi + \alpha\delta_0 \text{ in } \mathcal{D}'(B_R) \tag{2.1}$$

where δ_0 is the Dirac mass at origin.In particular, $u \in M^p_{loc}(B_R)$ [1] where $p = N/N-2$ when $N \geq 3$ and $1 \leq p < \infty$ is arbitrary when $N = 2$.

Theorem 2.2. *(Weyl Lemma, Simader[13]) Suppose $G \subset I\!R^N$ be open and let $u \in L^1_{loc}(G)$ satisfies*

$$\int_G u\Delta^2\varphi dx = 0 \text{ for all } \varphi \in C_c^\infty(G), \text{ i.e. } \Delta^2 u = 0 \text{ in } \mathcal{D}'(G).$$

Then there exists $\tilde{u} \in C^\infty(G)$ with $\Delta^2\tilde{u} = 0$ and $u = \tilde{u}$ a.e in G.

[1] $M^p_{loc}(B_R)$ denotes the *Marcinkeiwicz space*

Theorem 2.3. *(Weak maximum principle:) Let $u \in W^{4,r}(\Omega)$ be a solution of*

$$\begin{cases} \Delta^2 u = f(x) \geq 0 & in\ \Omega \\ u \geq 0, -\Delta u \geq 0 & on\ \partial\Omega \end{cases}$$

Then we have $u \geq 0$ and $-\Delta u \geq 0$ in Ω.

Proof of maximum principle easily follows by splitting the equation into a (coupled) system of second order PDE's say: $w = -\Delta u$ and $-\Delta w = f$ with the corresponding boundary conditions. Using similar ideas we can infact prove a maximum principle with weaker assumptions on the the smoothness of u, which is stated below:

Theorem 2.4. *Let $u, \Delta u \in L^1(\Omega)$ and $\Delta^2 u \geq 0$ in the sense of distributions. Also assume that $u, \Delta u$ are continuous near $\partial\Omega$ and $u > 0, -\Delta u > 0$ near $\partial\Omega$. Then $u(x) \geq 0$ in Ω.*

Definition 2.1. *Fundamental solution of Δ^2 is defined as a locally integrable function Φ in $I\!R^N$ for which $\Delta^2 \Phi = \delta_0$ and precisely expressed as*

$$\Phi(x) = a_N \begin{cases} |x|^{4-N} & if \quad N \geq 5 \\ \log \frac{5}{|x|} & if \quad N = 4 \\ |x| & if \quad N = 3 \\ |x|^2 \log \frac{5}{|x|} & if \quad N = 2 \end{cases}$$

for some constant $a_N > 0$.

Theorem 2.5. *Suppose $g : \Omega' \times [0, \infty) \to I\!R^+$ be a measurable function and let u, Δu and $\Delta^2 u \in L^1_{loc}(\Omega')$. Let $\Delta^2 u = g(x, u)$ in $\mathcal{D}'(\Omega')$ with $u \geq 0$ and $-\Delta u \geq 0$ a.e in Ω'. Then $u, g(x, u) \in L^1_{loc}(\Omega)$ and there exist a non-negative constants α , β such that $\Delta^2 u = g(x, u) + \alpha\delta_0 - \beta\Delta\delta_0$ in $\mathcal{D}'(\Omega)$.*

Proof: Let us write $w = -\Delta u$. Then $-\Delta w = g(x, u) \geq 0$ in $\mathcal{D}'(\Omega')$ and also given that $w, g(x, u) \in L^1_{loc}(\Omega')$. Now as a direct application of Brezis-Lions Theorem 4.4, we obtain

$$-\Delta w = g(x, u) + \alpha\delta_0 \text{ for some } \alpha \geq 0 \tag{2.2}$$

and $w, g(x, u) \in L^1_{loc}(\Omega)$. Since $-\Delta u = w \geq 0$ in Ω' again by Theorem 4.4 $u \in L^1_{loc}(\Omega)$ and

$$-\Delta u = w + \beta\delta_0 \text{ for some } \beta \geq 0.$$

Now substituting $w = -\Delta u - \beta\delta_0$ in (2.2) we get

$$\Delta^2 u = g(x, u) + \alpha\delta_0 - \beta\Delta\delta_0. \tag{2.3}$$

Extending $g(x, u)$ to be zero outside Ω we get $\Delta^2(u - f(u) * \Phi - \alpha\Phi - \beta\Gamma) = 0$ in $\mathcal{D}'(\Omega)$. By Weyl's lemma for biharmonic operators, there exists a biharmonic function $h \in C^\infty(\Omega)$ and

$$u = g(x, u) * \Phi + \alpha\Phi + \beta\Gamma + h \ a.e \text{ in } \Omega.$$

Note that $\Gamma(x)$ belongs to Marcinkeiwicz space $M^{\frac{N}{N-2}}(\Omega)$ when $N \geq 2$. By the property of the convolution of an L^1 function with the functions in $M^{\frac{N}{N-2}}(I\!R^N)$ we obtain $u \in M^{\frac{N}{N-2}}_{loc}(\Omega)$. $\square$.
The above result has been proved in [14](see Theorem 2) as an application of their main result on the system of equations. Proof is essentially based on the idea of Brezis-Lions type estimates. We have instead given a direct alternative proof for the same result. Theorem 2.5 can be extended for polyharmonic operator in a standard way, for details see Theorem 4.1 .

3. Biharmonic operator in $I\!R^4$

In this section we will restrict ourselves to the dimension $N = 4$ and $g(x, u)$ to take a specific form $g(x, u) = a(x)f(u)$. Let Ω be a bounded open set in $I\!R^4$, $0 \in \Omega$ and denote $\Omega^{'} = \Omega \setminus \{0\}$. We assume

(H1) $f : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function which is nondecreasing in $I\!R^+$ and $f(0) = 0$.

(H2) $a(x)$ is a non-negative measurable function in $L^k(\Omega)$ for some $k > \frac{4}{3}$.

(H3) There exists $r_0 > 0$ such that $\operatorname{essinf}_{B_{r_0}} a(x) > 0$.

Let u be a measurable function which solves the following problem:

$$(P) \qquad \begin{cases} \Delta^2 u = a(x)f(u) & \text{in } \Omega' \\ u \geq 0 \ , \quad -\Delta u \geq 0 & \text{in } \Omega' \end{cases}$$

From Theorem 2.5 we know that u is a distributional solution of $(P_{\alpha,\beta})$

$$(P_{\alpha,\beta}) \qquad \begin{cases} \left.\begin{array}{l} \Delta^2 u \quad = a(x)f(u) + \alpha\delta_0 - \beta\Delta\delta_0 \\ u \geq 0 \quad -\Delta u \geq 0 \end{array}\right\} \text{ in } \Omega, \\ \alpha, \beta \geq 0,\ u \text{ and } a(x)f(u) \in L^1(\Omega). \end{cases}$$

The assumption $(H3)$ suggests that the presence of such a weight function does not reduce the singularity of $a(x)f(u)$ at origin. In particular, if $a(x) = |x|^\sigma$ for $\sigma \in (-3, 0)$, then $a(x)$ satisfies $(H2)$ and $(H3)$.

Now assume that

$$\lim_{t\to\infty} \frac{f(t)}{t^2} = c \in (0, \infty]. \tag{3.1}$$

i.e. $f(t)$ grows atleast at a rate of t^2 near infinity. Then for some t_0 large enough, we have $f(t) \geq \frac{c}{2}t^2$ for all $t \geq t_0$. Suppose u is a solution of $(P_{\alpha,\beta})$ and f satisfies 3.1. Then we know that for some biharmonic function h

$$u(x) = a(x)f(u) * \Phi + \alpha\Phi + \beta\Gamma + h \ a.e \ \text{ in } \ \Omega$$

where Φ is the fundamental solution of biharmonic operator in $I\!R^4$ and Γ is the fundamental solution of $-\Delta$ in $I\!R^4$. Since α and $a(x)f(u)$ are non-negative, we have $u(x) \geq \beta\,\Gamma(x) + h(x)$. If $\beta \neq 0$, fix an $\tilde{r} \in (0, r_0)$ such that $u(x) \geq \dfrac{\beta}{2\pi^2|x|^2} \geq t_0$ whenever $|x| < \tilde{r}$. Now,

$$\int_{B_{\tilde{r}}} a(x)f(u) \geq C\int_{B_{\tilde{r}}} |x|^{-4} = \infty$$

which is a contradiction since $a(x)f(u) \in L^1(\Omega)$. Thus $\beta = 0$ if $f(t)$ grows at a rate faster than t^2 near infinity. We state this result in the next lemma.

Lemma 3.1. *Let f satisfies the condition (3.1) and u solves (P). Then for some α non-negative $\Delta^2 u = a(x)f(u) + \alpha\delta_0$ in $\mathcal{D}'(\Omega)$.*

Now onwards we assume that f satisfies (3.1). We would like to address following questions in this paper:

1. Can we find a sharp condition on f that determines whether $\alpha = 0$ or not in $(P_{\alpha,0})$?
2. If $\alpha = 0$, is it true that u is regular in Ω?

Definition 3.1. *We call f a sub-exponential type function if*

$$\lim_{t\to\infty} f(t)e^{-\gamma t} \leq C \quad \text{for} \quad \text{some} \quad \gamma, \mathrm{C} > 0.$$

We call f to be of super-exponential type if it is not a sub-exponential type function.

We will show that the above two questions can be answered based on the non-linearity being a sub-exponential type function or not.

Theorem 3.1. *(Removable Singularity) Let f be a super-exponential type function and u is a distributional solution of (P). Then u extends as a distributional solution of $(P_{0,0})$.*

Proof: Given u solves (P), we know that $\Delta^2 u = a(x)f(u) + \alpha\delta_0 - \beta\Delta\delta_0$ for some $\alpha, \beta \geq 0$. To show the extendability of the distributional solution we need to prove $\alpha = \beta = 0$. Since f is of super exponential type function, from Lemma 3.1 it is clear that $\beta = 0$. Let us assume that $\alpha > 0$ and derive a contradiction. Note that we can find an r small enough such that $u(x) \geq -\dfrac{\alpha}{16\pi^2}\log|x|$ whenever $|x| < r$. Since f is not a sub-exponential type function, for a given $\gamma > 0$ there exists $t_0 > 0$ such that $f(t) \geq e^{\gamma t}$ for all $t \geq t_0$. Thus,

$$f(u(x)) \,\geq\, f\left(-\frac{\alpha}{16\pi^2}\log|x|\right) \,\geq\, e^{-\frac{\gamma\alpha}{16\pi^2}\log|x|}, \text{ for } |x| << 1.$$

Now if we choose $\gamma = \frac{64\pi^2}{\alpha}$ in the above inequality, it contradicts the fact $a(x)f(u) \in L^1(\Omega)$. Thus $\alpha = \beta = 0$ in $(P_{\alpha,\beta})$. □

Theorem 3.2. *If $f(t) = t^p$ where $1 \leq p < \frac{4+\sigma}{2}$ and $a(x) = |x|^\sigma$, for $\sigma \in (-2, 0)$ then $(P_{\alpha,\beta})$ is solvable for α, β small enough.*

Proof follows from Theorem 4(ii)) of Soranzo[14]. The idea was to split the equation into a coupled system and find a sub and super solution for the system. In the next theorem when f satisfies 3.1, we find a super solution for $(P_{\alpha,0})$ directly without splitting the equation into a coupled system and then use the idea of monotone iteration to show the existence of a non-negative solution for α small enough. When $\beta \neq 0$, such a direct monotone iteration technique is not applicable as $\Delta\delta_0$ is not a positive or a negative distribution, ie $\phi \geq 0$, does not imply $\langle\Delta\delta_0, \phi\rangle \geq 0$ or ≤ 0.

Theorem 3.3. *Let f and a satisfy the hypotheses $(H1)-(H3)$. Additionally assume $\lim_{t\to\infty}\frac{f(t)}{t^2} = c \in (0, \infty]$. Then there exists an $\alpha_* > 0$ such that for all $\alpha \leq \alpha_*$ the problem $(P_{\alpha,0})$ admits a solution in $B_r(0)$.*

Proof: We use the idea of sub and super solution to construct a distributional solution for $(P_{\alpha,0})$ for α small enough. Clearly $u_0 = 0$ is a subsolution for $(P_{\alpha,0})$. Given that f is a sub-exponential type nonlinearity, there exists a $\gamma > 0$ and a $C > 0$, such that $f(t) \leq Ce^{\gamma t}$ for all $t \in I\!R^+$.

Now define

$$\overline{u}(x) = \frac{-\log|x| + C\phi}{\gamma} \text{ in } B_1(0). \tag{3.2}$$

where ϕ is the unique solution of the following Navier boundary value problem,

$$\begin{cases} \Delta^2\phi &= -\dfrac{a(x)}{|x|}\log|x| \text{ in } B_1(0) \\ \phi = & 0 \quad = \Delta\phi \text{ on } \partial B_1(0). \end{cases} \tag{3.3}$$

We notice that since $a(x) \in L^k(\Omega)$, for some $k > \frac{4}{3}$, the term $a(x)|x|^{-1}\log|x| \in L^p(B_1)$ for some $p > 1$. Hence the existence of a unique weak solution $\phi \in W^{4,p}(B_1)$ is guarenteed by Gazzolla [7], Theorem 2.20. Now by maximum principle we have $\phi \geq 0, -\Delta\phi \geq 0$.
Therefore,

$$\overline{u} \geq 0 \text{ in } B_1(0), \tag{3.4}$$

$$-\Delta\overline{u} = \frac{2}{\gamma|x|^2} - \frac{C}{\gamma}\Delta\phi \,\geq\, 0. \tag{3.5}$$

and

$$\Delta^2\overline{u} = \frac{\delta_0}{8\pi^2\gamma} + \frac{C}{\gamma|x|}a(x)\left|\log|x|\right|. \tag{3.6}$$

Note that $a(x)f(\overline{u}) \leq \dfrac{C}{|x|}a(x)e^{C\phi}$. By Sobolev embedding, we know $W^{4,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, and hence $e^{C\phi}$ is bounded in $B_1(0)$. Now we fix an $r > 0$ where $e^{C\phi} \leq \dfrac{|\log|x||}{\gamma}$ in $B_r(0)$. We let $\Omega = B_r(0)$ (where r depends only on γ and C) be a strict subdomian of $B_1(0)$ where $\dfrac{C}{\gamma|x|}a(x)|\log|x|| \geq a(x)f(\overline{u})$. Now from the choice of r and equations 3.4) , (3.5) and (3.6) it is obvious that $\overline{u}$ is a super solution of $(P_{\alpha,0})$ where $\alpha = \dfrac{1}{8\pi^2\gamma}$. Now let us define inductively with $u_0 = 0$

$$(P^n_{\alpha,0}) \begin{cases} \Delta^2 u_n = a(x)f(u_{n-1}) + \alpha\delta_0 & \text{in } \mathcal{D}'(\Omega) \\ u_n > 0, -\Delta u_n > 0 & \text{in } \Omega \\ u_n = \Delta u_n = 0 & \text{on } \partial\Omega \end{cases}$$

Existence of such a sequence $\{u_n\}$ can be obtained by writing $u_n = w_n + \alpha\Phi$ where

$$\begin{cases} \Delta^2 w_n = a(x)f(u_{n-1}) \text{ in } \Omega, \\ w_n = -\alpha\Phi, \Delta w_n = -\alpha\Delta\Phi \text{ on } \partial\Omega, \\ w_n \in W^{4,r}(\Omega) \text{ for some } r > 1. \end{cases}$$

Existence of w_1 is clear since $f(0) = 0$ and from Theorem 2.2 of [7]. First let us show the positivity of u_1 and $-\Delta u_1$ in Ω. Since w_1 is bounded, we can choose ϵ small enough so that $u_1 = w_1 + \alpha\Phi > 0$ and $-\Delta u_1 > 0$ in B_ϵ. In $\Omega \setminus B_\epsilon$ by weak comparison principle we can show that $u_1 > 0$ and $-\Delta u_1 > 0$. Next we need to show that $u_1 \leq \overline{u}$. Note that by construction, $\overline{u} > 0$ and $-\Delta\overline{u} > 0$ in $\overline{B_r} \setminus \{0\}$. Then, $\overline{u} - u_1$ satisfies the set of equations

$$\begin{cases} \Delta^2(\overline{u} - u_1) \geq 0 & \text{in } \mathcal{D}(\Omega), \\ \overline{u} - u_1 > 0, -\Delta(\overline{u} - u_1) > 0 & \text{near } \partial\Omega. \end{cases}$$

Now using the maximum principle for distributional solutions (Theorem 2.4) we find $u_1 \leq \overline{u}$.

Assume that there exists a function u_k solving $(P^k_{\alpha,0})$ for $k = 1, 2\cdots n$ and

$$0 \leq u_1 \leq u_2 \ldots \leq u_n \leq \overline{u} \text{ in } \Omega.$$

Since f is non-decreasing we have $a(x)f(u_n) \in L^p(\Omega)$, for some $p > 1$. Thus by Sobolev embedding there exists a $w_{n+1} \in C(\overline{\Omega}) \cap W^{4,p}(\Omega)$. Also,

$$\begin{cases} \Delta^2(u_{n+1} - u_n) = a(x)f(u_n) - a(x)f(u_{n-1}) \geq 0 & \text{in } \Omega \\ u_{n+1} = u_n, \Delta u_{n+1} = \Delta u_n & \text{on } \partial\Omega. \end{cases}$$

Again from weak comparison principle $0 < u_n \leq u_{n+1}$ and $0 \leq -\Delta u_n \leq -\Delta u_{n+1}$. As before one can show that $u_{n+1} \leq \overline{u}$. Now if we define $u(x) = \lim_{n\to\infty} u_n(x)$ one can easily verify that u is a solution of $(P_{\alpha,0})$ for $\alpha = \frac{1}{8\pi^2\gamma}$. For a given f sub-exponential type function we have found a ball of radius r such that $(P_{\alpha,0})$ posed on $B_r(0)$ has a solution u_α for $\alpha = \frac{1}{8\pi^2\gamma}$. This solution u_α is a supersolution for $(P_{\alpha',0})$ posed in $B_r(0)$ and for $\alpha' \in (0, \alpha)$. Thus one can repeat the previous iteration and show that for all $\alpha' \in (0, \alpha)$ there exists a weak solution for $(P_{\alpha',0})$ in $B_r(0)$. □

Corollary 3.1. *Suppose for a given $\gamma > 0$ there exists a C_γ such that $f(t) \leq C_\gamma e^{\gamma t}$ for all $t \in \mathbb{R}^+$. Then $(P_{\alpha,0})$ has a solution in $B_{r_\alpha}(0)$ for all $\alpha \in (0, \infty)$. In particular if $f(t) = t^p, p > 2$ or $e^{t^\delta}, \delta < 1$ then $(P_{\alpha,0})$ is solvable for all $\alpha > 0$.*

Next we recall a Brezis-Merle [3] type of estimate for Biharmonic operator in $I\!R^4$. Let h be a distributional solution of

$$(2)\quad \begin{cases} \Delta^2 h = f & \text{in } \Omega \\ h = \Delta h = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded domain in $I\!R^4$.

Theorem 3.4. *(C.S Lin [9]) Let $f \in L^1(\Omega)$ and h is a distributional solution of (2). For a given $\delta \in (0, 32\pi^2)$ there exists a constant $C_\delta > 0$ such that the following inequality holds:*

$$\int_\Omega \exp\left(\frac{\delta h}{\|f\|_1}\right) dx \leq C_\delta (diam\Omega)^4$$

where diam Ω denote the diameter of Ω.

Theorem 3.5. *Let f be a sub-exponential type function. Let u be a solution of $(P_{0,0})$ with $u = \Delta u = 0$ on $\partial\Omega$. Then u is regular in Ω.*

Proof: Let u be a solution of $\Delta^2 u = a(x)f(u)$ in Ω with Navier boundary conditions. Write $g(x) = a(x)f(u)$, then $g \in L^1(\Omega)$. Fix a $l > 0$ and split $g = g_1 + g_2$ where $\|g_1\|_1 < \frac{1}{l}$ and $g_2 \in L^\infty(\Omega)$. Let u_2 be the unique solution of

$$\begin{cases} \Delta^2 u_2 = g_2 \text{ in } \Omega, \\ u_2 = 0, \Delta u_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Then

$$\begin{cases} \Delta^2 u_1 = g_1 \text{ in } \Omega, \\ u_1 = 0, \Delta u_1 = 0 \text{ on } \partial\Omega. \end{cases}$$

Choosing $\delta = 1$ in Theorem 3.4, we find $\int_\Omega exp(\frac{|u_1|}{\|g_1\|_1}) < C_1(diam\,\Omega)^4$. Thus $e^{l|u_1|} \in L^1(\Omega)$. Since $u_2 \in L^\infty(\Omega)$, we have $e^{l|u|} \in L^1(\Omega)$ for all $l > 0$. We use this higher intergrability property of u in establishing its regularity.

We can show that $a(x)f(u) \in L^r(\Omega)$ for some $r > 1$. In fact,

$$\begin{aligned} \int_\Omega (a(x)f(u))^r &\leq \tilde{C} \int_\Omega a(x)^r e^{\gamma r u} \\ &\leq C_2 \left(\int_\Omega a(x)^{pr}\right)^{1/p} \left(\int_\Omega e^{p'\gamma r u}\right)^{1/p'} < \infty \end{aligned}$$

if we choose $p, r > 1$ close enough to 1 so that $1 < p.r \leq k$, where $a(x) \in L^k(\Omega)$. Now let v be the unique weak solution of

$$\begin{cases} \Delta^2 v = a(x)f(u) \text{ in } \Omega, \\ v = 0, \Delta v = 0 \text{ on } \partial\Omega. \end{cases}$$

We have $v \in C^{3,\gamma'}(\overline{\Omega})$ for all $\gamma' \in (0,1)$. Now $u = v + h$ for some biharmonic function h. Therefore $u \in C^{3,\gamma'}(\Omega)$. □.

Remark 3.1. *The previous theorem is true even if $a(x) \in L^k(\Omega)$ for some $k > 1$.*

When f is super exponential in nature an arbitrary solution of $\Delta^2 u = a(x)f(u)$ in $\mathcal{D}'(\Omega)$ need not be bounded. We consider the following example.

Example 3.1. *Let $w(x) = (-4\log|x|)^{\frac{1}{\mu}}$ for some $\mu > 1$. Then one can verifty that whenever $x \neq 0$,*

$$\Delta^2 w = b_1 e^{w^\mu} w^{1-4\mu}[b_2 w^{2\mu} - b_3]$$

for some positive constants b_i. Since $f(w) = b_1 e^{w^\mu} w^{1-4\mu}[b_2 w^{2\mu} - b_3]$ is super exponential in nature, w extends as an unbounded distributional solution of $\Delta^2 w = f(w)$ in $B_r(0)$ for r small enough.

4. Polyharmonic Operator in $\mathbb{R}^{2m}$

We suppose Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2m$ with smooth boundary and $0 \in \Omega$. We denote Ω' as $\Omega \setminus \{0\}$.

Theorem 4.1. *Suppose $g : \Omega' \times [0, \infty) \to I\!R^+$ is a measurable function and $\Delta^k u \in L^1_{loc}(\Omega')$ for $k = 0, 1, ..m$. Let $(-\Delta)^m u = g(x, u)$ in $\mathcal{D}'(\Omega')$ with $(-\Delta)^k u \geq 0$ for $k = 0, 1, .., m-1$ a.e in Ω'. Then $u, g(x, u) \in L^1_{loc}(\Omega)$ and there exist non-negative constants $\alpha_0, ..., \alpha_{m-1}$ such that* $(-\Delta)^m u = g(x, u) + \sum_{i=0}^{m-1} \alpha_i (-\Delta)^i \delta_0$ *in* $\mathcal{D}'(\Omega)$.

Now we restrict ourselves to dimension $N = 2m$ and $g(x, u)$ to take a specific form $g(x, u) = a(x) f(u)$. Throughout this section we make the following assumption:

$(H1')$ $f : [0, \infty) \mapsto [0, \infty)$ is a continuous function which is non-decreasing in $\mathbb{R}^+$ and $f(0) = 0$.

$(H2')$ $a(x)$ is non negative measurable function in $L^k(\Omega)$ for some $k > \frac{2m}{2m-1}$.

$(H3')$ There exists $r_0 > 0$ such that $\text{essinf}_{B_{r_0}} a(x) > 0$.

Let u be a measurable function which satisfies the problem below,

$$(P^1) \qquad \begin{cases} (-\Delta)^m u = a(x) f(u) & \text{in } \Omega' \\ (-\Delta)^k u \geq 0 \text{ in } \Omega', \quad k = 0, .., m-1 \\ u \in C^{2m}(\overline{\Omega} \setminus \{0\}). \end{cases}$$

Then by 4.1 we know that u is a distribution solution of $(P^1_{\alpha_0,..,\alpha_{m-1}})$

$$(P^1_{\alpha_0,..,\alpha_{m-1}}) \qquad \begin{cases} (-\Delta)^m u \;= a(x) f(u) + \sum_{i=0}^{m-1} \alpha_i (-\Delta)^i \delta_0 \text{ in } \Omega \\ (-\Delta)^k u \geq 0, \quad k = 0, .., m-1 \text{ in } \Omega' \\ \alpha_i \geq 0, \text{ for } i = 0, .., m-1 \text{ and } u, a(x) f(u) \in L^1(\Omega). \end{cases}$$

In [4], Soranzo et.al considered a specific equation $(-\Delta)^m u = |x|^\sigma u^p$ in Ω', with $\sigma \in (-2m, 0)$ and $(-\Delta)^k u \geq 0$, for $k = 0, 1, \ldots m$. By Corollary 1 of [4], if $N = 2m$ and $p > \max\{1, \frac{N+\sigma}{2}\}$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$ in $(P^1_{\alpha_0, \ldots \alpha_{m-1}})$. This result can be sharpened for any weight function $a(x)$ satisfying $(H3)$ in a standard way and we skip the details of the proof.

Remark 4.1. *Let u satisfy (P^1) and $\lim_{t \to \infty} \frac{f(t)}{t^m} = c \in (0, \infty]$. Then we have $\alpha_1 = \alpha_2 = .. = \alpha_{m-1} = 0$ in $(P^1_{\alpha_0, \ldots, \alpha_{m-1}})$ and hence u is a distributional solution of $(-\Delta)^m u = a(x) f(u) + \alpha_0 \delta_0$ in Ω.*

Now the following theorem gives us a sharp condition on f which determines $\alpha_0 = 0$ in $(P^1_{\alpha_0, 0, \ldots, 0})$ and the proof is as similar to $Theorem$ 3.1.

Theorem 4.2. *Let f be a super-exponential type function and u is distribution solution of (P^1). Then u extends as a distributional solution of $(P^1_{0,0,..,0})$.*

Theorem 4.3. *Let f and a satisfy the hypotheses $(H1') - (H3')$. Additionally assume $\lim_{t \to \infty} \frac{f(t)}{t^m} = c \in (0, \infty]$. Then there exists an $\alpha_0 > 0$ such that for all $\alpha \leq \alpha_0$ the problem $(P^1_{\alpha, 0, \ldots 0})$ admits a solution in $B_r(0)$, where the radius of the ball depends on the nonlinearity f.*

Proof: We proceed as in Theorem 3.3, by constructing sub and super distributional solution for $(P^1_{\alpha,0,..,0})$ for all α small enough. We note that $u_0 = 0$ is a sub-solution, and let

$$\overline{u}(x) = \frac{-\log|x| + C\phi}{\gamma} \text{ in } B_1(0) \tag{4.1}$$

where ϕ is the unique solution of the following Navier boundary value problem,

$$\begin{cases} (-\Delta)^m \phi &= -\dfrac{a(x)}{|x|} \log|x| \text{ in } B_1(0) \\ \phi = \Delta\phi &= .. = (\Delta)^{m-1}\phi \text{ on } \partial B_1(0). \end{cases} \tag{4.2}$$

Then $\overline{u}$ is a supersolution of $(P^1_{\alpha,0...0})$ in a small ball $B_r(0)$. Rest of the proof follows exactly as in the case of biharmonic operator. $\square$

Next we state a Brezis-Merle type of type of estimates for poly-harmonic operator in $\mathbb{R}^{2m}$.

Theorem 4.4. *(Martinazzi [11]) Let $f \in L^1(B_R(x_0)), B_R(x_0) \subset \mathbb{R}^{2m}$, and let v solve*

$$\begin{cases} (-\Delta)^m v = f \text{ in } B_R(x_0), \\ v = \Delta^2 v = ... = \Delta^{m-1} v = 0 \text{ on } \partial B_R(x_0) \end{cases}$$

Then, for any $p \in (0, \dfrac{\gamma_m}{\|f\|_{L^1(B_R(x_0))}})$, we have $e^{2mp|v|} \in L^1(B_R(x_0))$ and

$$\int_{B_R(x_0)} e^{2mp|v|} dx \leq C(p) R^{2m},$$

where $\gamma_m = \dfrac{(2m-1)!}{2} \left|S^{2m}\right|$.

Finally with the help of above theorem we prove a regularity result for the polyharmonic operator.

Theorem 4.5. *Let $a(x)$ and f satisfies the properties as in $(H1') - (H3')$ and also assume that f be a sub-exponential type function. Let u be a solution $(P^1_{0,0,..,0})$ with $u = \Delta u = ... = \Delta^{m-1} u = 0$ on $\partial\Omega$. Then $u \in C^{2m-1,\gamma'}(\Omega)$, for all $\gamma' \in (0,1)$.*

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