Topology and its Applications 157 (2010) 2052–2058

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Inducing sensitivity on hyperspaces

Puneet Sharma, Anima Nagar *

Department of Mathematics, I.I.T. Delhi, Hauz Khas, New Delhi 110016, India

ARTICLE INFO

Article history:
Received 22 June 2009
Received in revised form 14 April 2010
Accepted 11 May 2010

Keywords:
Hyperspace
Induced map
Sensitive
Strongly sensitive
Asymptotic sensitive
Li–Yorke sensitive

ABSTRACT

Let (X,d) be a compact metric space and $(\mathcal{K}(X), d_H)$ be the space of all non-empty compact subsets of X equipped with the Hausdorff metric d_H. The dynamical system (X, f) induces another dynamical system $(\mathcal{K}(X), \overline{f})$. We study the relations between the various forms of sensitivity of the systems (X, f) and $(\mathcal{K}(X), \overline{f})$. We prove that all forms of sensitivity of $(\mathcal{K}(X), \overline{f})$ partly imply the same for (X, f), and the converse holds in some cases.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

Let (X,d) be a metric space and let $f : X \to X$ be a continuous map. Then the pair (X, f) constitutes a dynamical system. The study here is concentrated on the orbit $\{f^n(x)\colon n \in \mathbb{N}\}$ of a given point $x \in X$, where f^n is the n fold composition of the map f with itself.

Let $\mathcal{K}(X)$ denote the collection of all non-empty compact subsets of X. The *Hausdorff metric* d_H, on $\mathcal{K}(X)$ is defined as,

$$d_H(A, B) = \max\bigl\{\rho(A, B), \rho(B, A)\bigr\}$$

where, $\rho(A, B) = \inf\{\epsilon > 0\colon\ d(b, A) < \epsilon,\ \text{for all } b \in B\}$.

It is well known that for compact X, the topology on $\mathcal{K}(X)$ given by the metric d_H is same as the *Vietoris or finite topology*, which is generated by a basis consisting of all sets of the form,

$$\langle U_1, U_2, \ldots, U_n\rangle = \left\{E \in \mathcal{K}(X)\colon\ E \subseteq \bigcup_{i=1}^{n} U_i \text{ and } E \cap U_i \neq \phi,\ 1 \leqslant i \leqslant n\right\},$$

where $U_1, U_2, \ldots, U_n$ are open subsets of X.

This topology is admissible in the sense that the map $i : X \to \mathcal{K}(X)$ given as $x \to \{x\}$ is continuous. Under this topology $\mathcal{F}(X)$, the set of all finite subsets of X, is dense in $\mathcal{K}(X)$. Also, $\mathcal{K}(X)$ is compact if and only if X is compact.

See [4,14] for details.

It can be seen that every continuous map f on X induces a continuous map $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined as $\overline{f}(K) = f(K) = \{f(k)\colon\ k \in K\}$. Thus, a dynamical system (X, f) induces another dynamical system $(\mathcal{K}(X), \overline{f})$. The original system (X, f) is a subsystem of the induced system $(\mathcal{K}(X), \overline{f})$, when a point $x \in X$ is identified as a subset $\{x\} \in \mathcal{K}(X)$. In recent times,

* Corresponding author.
E-mail addresses: puneet.iitd@yahoo.com (P. Sharma), anima@maths.iitd.ac.in (A. Nagar).

0166-8641/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2010.05.002

there have been many attempts to study relations between the dynamical properties of the map f and that of the induced map $\overline{f}$ [3,7,9–12]. While this article was being written [16] discussed sensitivity of $(\mathcal{K}(X), \overline{f})$. But, no condition on (X, f) implying this has been discussed there.

One of the most interesting characteristics of a dynamical system is when orbits of nearby points deviate after finite steps. This is also one of the important features depicting the chaotic behaviour of the system. This notion, also popularly called the "butterfly effect", has been widely studied and is termed as *sensitive dependence on initial conditions*. There are various forms of sensitivity, depending on how often the orbits of nearby points deviate.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. A self map f on a metric space (X, d) is,

- *sensitive* [8] or has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there exist a point $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$;
- *strongly sensitive* [1] if there exists $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, there is a $y \in X$ with $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$;
- *asymptotic sensitive* [6] if there exists $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there exists $y \in X$ such that $d(x, y) < \epsilon$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta$.
  In this case, the pair (x, y) is called an *asymptotic sensitive pair*.
- *Li–Yorke sensitive* [2] if there exists $\delta > 0$ such that for each $x \in X$ and $\epsilon > 0$ there exists $y \in X$ with $d(x, y) < \epsilon$ such that

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > \delta.$$

A pair $(x, y) \in X \times X$ is proximal (asymptotic) if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ ($\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$). A Li–Yorke (or scrambled) pair is a pair $(x, y) \in X \times X$ which is proximal but not asymptotic. A Li–Yorke pair (x, y) has modulus $\delta > 0$ if $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta$.

In plain words, sensitivity simply means that given any point, there exists another point arbitrarily close such that the orbits of these two points move apart by a fixed distance after some finite instants. The system is strongly sensitive if after a particular instant, for each successive instants, there are points arbitrarily close to any point, such that their orbits move apart by a fixed distance from the orbit of this particular point. If for any point there is a point arbitrarily close by, such that the orbits of these two points move apart infinitely often, then the system is asymptotically sensitive. If in addition of moving apart infinitely often, these orbits also come arbitrarily closer infinitely often, then the system is Li–Yorke sensitive. In general, these properties though distinct, satisfy the relations,

$$\begin{aligned} &\text{sensitive} \;\Leftarrow\; \text{strongly sensitive},\\ &\text{sensitive} \;\Leftarrow\; \text{asymptotic sensitive} \;\Leftarrow\; \text{Li–Yorke sensitive}. \end{aligned}$$

The proofs for these implications are straightforward, and are left to the reader.

There is as such no other known relation between these different forms of sensitivity as can be seen from the examples below.

Example 1.1. Let Σ_2 denote the space of all infinite sequences of 0's and 1's with the metric

$$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

Let X be the collection of all sequences which are eventually zero. Then, the shift map $\sigma : X \to X$ defined as $(\sigma(x))_n = x_{n+1}$ is strongly sensitive. However, as orbits of any two points eventually coincide, the map fails to be asymptotically sensitive.

Example 1.2. Let $A_1 = [1, 2] = [s_1, a_1]$. Define recursively, $A_{k+1} = [a_k + 1, a_k + k + 2] = [s_{k+1}, a_{k+1}]$. Similarly, let $B_1 = [-2, -1] = [b_1, t_1]$. Again, define $B_{k+1} = [b_k - 1 - \frac{1}{k}, b_k - 1] = [b_{k+1}, t_{k+1}]$. Also let $A = [-\frac{1}{2}, \frac{1}{2}]$. Let $X = A \cup \{A_k\colon k \in \mathbb{N}\} \cup \{B_k\colon k \in \mathbb{N}\}$.

Define $f : X \to X$ as

$$f(x) = \begin{cases} 2x + 2, & -\frac{1}{2} \leqslant x \leqslant 0;\\ 2 - 2x, & 0 \leqslant x \leqslant \frac{1}{2};\\ \frac{2(t_k - b_k)}{a_k - s_k} x + \frac{a_k b_k + s_k b_k - 2t_k s_k}{a_k - s_k}, & s_k \leqslant x \leqslant \frac{s_k + a_k}{2};\\ \frac{2(b_k - t_k)}{a_k - s_k} x + \frac{2a_k t_k - s_k b_k - a_k b_k}{a_k - s_k}, & \frac{s_k + a_k}{2} \leqslant x \leqslant a_k;\\ \frac{2(a_{k+1} - s_{k+1})}{t_k - b_k} x + \frac{-2a_{k+1} b_k + s_{k+1} b_k + t_k s_{k+1}}{t_k - b_k}, & b_k \leqslant x \leqslant \frac{t_k + b_k}{2};\\ \frac{-2(a_{k+1} - s_{k+1})}{t_k - b_k} x + \frac{2a_{k+1} t_k - s_{k+1} b_k - t_k s_{k+1}}{t_k - b_k}, & \frac{b_k + t_k}{2} \leqslant x \leqslant t_k. \end{cases}$$

The map defined above maps A to A_1, each A_k to B_k and each B_k to A_{k+1}.

This map is Li–Yorke sensitive and hence asymptotic sensitive, but fails to be strongly sensitive.

Example 1.3. Let $X = [1, \infty)$. Define a map $f : X \to X$ as

$$f(x) = x^2.$$

Then the map defined is asymptotic sensitive but fails to be Li–Yorke sensitive.

For a continuous map f on a compact metric space X, asymptotic sensitivity is equivalent to sensitivity (cf. [2]). However, even on compact metric spaces Li–Yorke sensitivity and strong sensitivity are different notions, as can be seen from the examples below.

Example 1.4. Consider the annulus region $S = \{(x, y){:}\ 1 \leqslant x^2 + y^2 \leqslant 4\}$ in $\mathbb{R}^2$. Then S can also be represented as $S = \{(r, \theta){:}\ 1 \leqslant r \leqslant 2;\ 0 \leqslant \theta \leqslant 2\pi\}$. Define $f : S \to S$ as, $f((r, \theta)) = (r, r + \theta)$. Then f is a continuous map defined on S which is asymptotically sensitive and strongly sensitive, but fails to be Li–Yorke sensitive.

For any neighborhood U of a point $x = (r, \theta)$, there exists $n_U \in \mathbb{N}$ such that $x_n = (r + \frac{\pi}{n}, \theta) \in U$ for all $n \geqslant n_U$. As $|f^n(x) - f^n(x_n)| > \pi$, for all $n \geqslant n_U$, the map f is strongly sensitive. Also for no two points $x = (r_1, \theta_1)$ and $y = (r_2, \theta_2)$ in the annulus, $\liminf_{n\to\infty} d(f^n(x), f^n(y))$ can be 0. Hence the map fails to be Li–Yorke sensitive.

Example 1.5. Let $\Sigma = \{0^r 10^{2^{n_1}} 10^{2^{n_2}} 10^{2^{n_3}} 1 \ldots{:}\ r \geqslant 0$, **and** $(n_k) \nearrow \infty$ (**a strictly increasing sequence**)$\} \subset \{0, 1\}^{\mathbb{N}}$. Let X be the closure of Σ and let σ be the shift operator on X.

For any point $x = 0^r 10^{2^{n_1}} 10^{2^{n_2}} 10^{2^{n_3}} 1 \ldots \in X$ and a basic open set $U = [0^r 10^{2^{n_1}} 10^{2^{n_2}} 10^{2^{n_3}} 1 \ldots 10^{2^{n_k}} 1]$ containing x, consider $y = 0^r 10^{2^{n_1}} 10^{2^{n_2}} 10^{2^{n_3}} 1 \ldots 10^{2^{n_k}} 10^{\infty} \in U$. Then (x, y) forms a Li–Yorke pair. Since any general point in the space X is either of the form x or y, the system (X, σ) is Li–Yorke sensitive.

Further, for any point in any neighborhood of points of the form y, as the number of zeros between two consecutive 1's increases to infinity (or the tail of sequence becomes eventually zero), the map σ cannot be strongly sensitive here.

There are some other notions of sensitivity, which primarily depend on the points of the space. A system (X, f) is *Lyapunov unstable* at a point x if for each $x \in X$ there exists $\delta_x > 0$, such that for each $\epsilon > 0$, there exist a point $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta_x$. A system is called *pointwise sensitive* if it is Lyapunov unstable at each point. It is possible that a system is pointwise sensitive but fails to be sensitive on X, as shown by the example below.

Example 1.6. Let $X = [-1, 1]$. Define $f : X \to X$ as

$$f(x) = \begin{cases} 2 - 2x, & -1 \leqslant x \leqslant -\frac{1}{2}; \\ 2x, & -\frac{1}{2} \leqslant x \leqslant 0; \\ 3x - \frac{1}{2^m}, & \frac{1}{2^{m+1}} \leqslant x \leqslant \frac{4}{3(2^{m+1})}; \\ \frac{3}{2^m} - 3x, & \frac{4}{3(2^{m+1})} \leqslant x \leqslant \frac{5}{3(2^{m+1})}; \\ 3x - \frac{4}{2^{m+1}}, & \frac{5}{3(2^{m+1})} \leqslant x \leqslant \frac{1}{2^m}. \end{cases}$$

Each of the intervals $[\frac{1}{2^{m+1}}, \frac{1}{2^m}]$ remain invariant under f and all the points of the form $\frac{1}{2^k}$ are fixed under the action of f. Also, it can be verified that the map defined is pointwise sensitive (sensitive on each interval $[\frac{1}{2^{m+1}}, \frac{1}{2^m}]$ and $[-1, 0]$) but fails to be sensitive on X.

Some other definitions based on Li–Yorke pairs but having some common features with sensitivity have been described in [5]. We again recall that if for some $x \in X$, there exists $y \in X$ with $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$, then the pair (x, y) is called a *Li–Yorke pair*. A dynamical system (X, f) has *chaotic dependence on initial conditions* if for any $x \in X$ and $\epsilon > 0$ there exists $y \in X$ such that $d(x, y) < \epsilon$ and the pair (x, y) is Li–Yorke. Such sensitivity is in general weaker than Li–Yorke sensitivity. Again, such systems may fail to be even sensitive. The map f (in the example above) has chaotic dependence on initial conditions but fails to be Li–Yorke sensitive.

In this article, we study the relationship between various forms of sensitivity of the dynamical system (X, f), for compact (X, d), and the induced system $(\mathcal{K}(X), \overline{f})$. We show that sensitivity, asymptotic sensitivity, strong sensitivity and Li–Yorke sensitivity of the system $(\mathcal{K}(X), \overline{f})$ partly imply the same for (X, f). As regards the converse, we contradict the counterexample in [7] which says that f sensitive need not imply $\overline{f}$ sensitive on $\mathcal{K}(X)$ by showing the error therein, and provide a counterexample for the same. We prove that some forms of sensitivity for (X, f) imply the same for $(\mathcal{K}(X), \overline{f})$, whereas for some forms this cannot be guaranteed.

We restrict ourselves to comparison of only the various properties related to sensitivity in this article, and will not discuss its implications in presence of any other dynamical property. For example, (X, f) is weakly mixing (topological mixing) if and only if $(\mathcal{K}(X), \overline{f})$ is so. Since weakly mixing maps are Li–Yorke sensitive, in this case Li–Yorke sensitivity of the maps f and $\overline{f}$ are equivalent. Also we prove in Propositions 2.3 and 2.8 that strong sensitivity of f and $\overline{f}$ are equivalent in general. Hence, in such a case, all kinds of sensitivity are equivalent for (X, f) and the induced system $(\mathcal{K}(X), \overline{f})$.

Throughout, $\overline{C}$ denotes the closure of C in the metric space (X, d), and $S_d(x, \epsilon) = \{y \in X\colon\ d(x, y) < \epsilon\}$.

2. Main results

Henceforth, (X, d) is a compact metric space and f a continuous self map on X.

Proposition 2.1. *If $(\mathcal{K}(X), \overline{f})$ is sensitive, then (X, f) is sensitive.*

Proof. Let $(\mathcal{K}(X), \overline{f})$ be sensitive with sensitivity constant δ. Let $\epsilon > 0$ be arbitrary. Let $x \in X$ and U be the ϵ-neighborhood of x in X. Then, as $\mathcal{U} = S_{d_H}(\{x\}, \epsilon)$ is an ϵ-neighborhood of $\{x\} \in \mathcal{K}(X)$ and $\overline{f}$ is sensitive, there exist $A \in \mathcal{U}$ and $n \in \mathbb{N}$ such that $d_H(\overline{f}^n(\{x\}), \overline{f}^n(A)) > \delta$. Thus, there exists $y \in A \subset U$ such that $d(f^n(x), f^n(y)) > \delta$ and hence the proposition holds. □

Remark 2.2. The above result is proved in both [7,10], but we have included the proof here for the sake of completion.

Proposition 2.3. *If $(\mathcal{K}(X), \overline{f})$ is strongly sensitive, then (X, f) is strongly sensitive.*

Proof. The proof is similar. □

Proposition 2.4. *If $(\mathcal{K}(X), \overline{f})$ is asymptotic sensitive, then (X, f) is asymptotic sensitive.*

Proof. Let $(\mathcal{K}(X), \overline{f})$ be asymptotic sensitive with sensitivity constant δ. Let $x \in X$ and let $\epsilon > 0$ be given. We show that there exists $a \in S_d(x, \epsilon)$ such that,

$$\limsup_{n\to\infty} d\big(f^n(x), f^n(a)\big) > \frac{\delta}{2}.$$

As $\{x\} \in \mathcal{K}(X)$ and $S_{d_H}(\{x\}, \epsilon)$ is a neighborhood of $\{x\}$ in $\mathcal{K}(X)$, there exists $A_1 \in S_{d_H}(\{x\}, \epsilon)$ such that

$$\limsup_{n\to\infty} d_H\big(\overline{f}^n(\{x\}), \overline{f}^n(A_1)\big) > \delta.$$

Therefore there exist an integer $n_1 \in \mathbb{N}$ and $a_1 \in A_1$ such that $d(f^{n_1}(x), f^{n_1}(a_1)) > \delta$.

If (x, a_1) forms an asymptotic sensitive pair with $\limsup_{n\to\infty} d(f^n(x), f^n(a_1)) > \frac{\delta}{2}$, then we are done. If not, then there exists an integer m_1, $m_1 > n_1$, such that $d(f^n(x), f^n(a_1)) < \frac{\delta}{2}$ for all $n \geqslant m_1$. Now, there exists a neighborhood $U_1 \subset S_d(x, \epsilon)$ of a_1 such that $d(f^{n_1}(x), f^{n_1}(y)) > \delta$ for all $y \in U_1$. As U_1 is a neighborhood of a_1, there exists $\epsilon_1 > 0$ with

$$\overline{S_d(a_1, \epsilon_1)} \subset U_1.$$

Again, as $\overline{f}$ is asymptotic sensitive, there exists a compact set $A_2 \in S_{d_H}(\{a_1\}, \epsilon_1)$ such that $(\{a_1\}, A_2)$ is an asymptotic sensitive pair for $\overline{f}$. Thus, there exist integer n_2, $n_2 > m_1$, and $a_2 \in A_2$ such that $d(f^{n_2}(a_1), f^{n_2}(a_2)) > \delta$. Then, $d(f^{n_2}(x), f^{n_2}(a_2)) > \frac{\delta}{2}$.

If (x, a_2) forms an asymptotic sensitive pair with $\limsup_{n\to\infty} d(f^n(x), f^n(a_2)) > \frac{\delta}{2}$, then we are done. If not, then there exists an integer m_2, $m_2 > n_2$, such that $d(f^n(x), f^n(a_2)) < \frac{\delta}{2}$ for all $n \geqslant m_2$. Again, there exists a neighborhood $U_2 \subset \overline{S_d(a_1, \epsilon_1)}$ of a_2 such that $d(f^{n_2}(x), f^{n_2}(y)) > \frac{\delta}{2}$ for all $y \in U_2$. As U_2 is a neighborhood of a_2, there exists $\epsilon_2 > 0$ with $\overline{S_d(a_2, \epsilon_2)} \subset U_2$.

Proceeding inductively, we either get the asymptotic sensitive pair (x, a_k) with $\limsup_{n\to\infty} d(f^n(x), f^n(a_k)) > \frac{\delta}{2}$ or we get a sequence $\{a_n\}$ in $S_d(x, \epsilon)$. Let a be a limit point of this sequence. As $a \in U_i$, $d(f^{n_i}(x), f^{n_i}(a)) > \frac{\delta}{2}$ for each i. Thus, $\limsup_{n\to\infty} d(f^n(x), f^n(a)) > \frac{\delta}{2}$ and f is asymptotic sensitive at x. □

Remark 2.5. The above proposition directly follows from Proposition 2.1 and the fact that for compact spaces sensitivity and asymptotic sensitivity are equivalent. But we give a direct proof here.

Proposition 2.6. *If $(\mathcal{K}(X), \overline{f})$ is Li–Yorke sensitive, then (X, f) has chaotic dependence on initial conditions. Further, if $(\mathcal{F}(X), \overline{f})$ is Li–Yorke sensitive then (X, f) is Li–Yorke sensitive.*

Proof. Let $(\mathcal{K}(X), \overline{f})$ be Li–Yorke sensitive with sensitivity constant δ. Let $x \in X$ and let $\epsilon > 0$ be given. We now show that there exists $a \in S_d(x, \epsilon)$ such that the pair (x, a) forms a Li–Yorke pair.

As $\{x\} \in \mathcal{K}(X)$ and $\overline{f}$ is Li–Yorke sensitive, there exists $A_1 \in S_{d_H}(\{x\}, \epsilon)$ such that, $\liminf_{n\to\infty} d_H(\overline{f}^n(\{x\}), \overline{f}^n(A_1)) = 0$ and

$$\limsup_{n\to\infty} d_H\big(\overline{f}^n(\{x\}), \overline{f}^n(A_1)\big) > \delta.$$

Also, for any $a_1 \in A_1$, $d(f^n(x), f^n(a_1)) \leqslant d_H(\overline{f}^n(\{x\}), \overline{f}^n(A_1))$. Hence for any $a_1 \in A_1$, $\liminf_{n\to\infty} d(f^n(x), f^n(a_1)) = 0$ and the pair (x, a_1) is proximal.

We observe that, $\limsup_{n\to\infty} d_H(\overline{f}^n(\{x\}), \overline{f}^n(A_1)) > \delta$ and so there exist an integer n_1 and $a_1 \in A_1$ such that $d(f^{n_1}(x), f^{n_1}(a_1)) > \delta$. If (x, a_1) forms a Li–Yorke pair, we are done. If not, (x, a_1) is an asymptotic pair. In particular, there exists an integer m_1 such that $d(f^n(x), f^n(a_1)) < \frac{1}{2}$ for all $n \geqslant m_1$. Thus, there exists a neighborhood $U_1 \subset S_d(x, \epsilon)$ of a_1 such that $d(f^{n_1}(x), f^{n_1}(y)) > \delta$ and $d(f^{m_1}(x), f^{m_1}(y)) < \frac{1}{2}$ for all $y \in U_1$. As U_1 is a neighborhood of a_1, there exists $\epsilon_1 > 0$ such that $\overline{S_d(a_1, \epsilon_1)} \subset U_1$.

Again, as $\overline{f}$ is Li–Yorke sensitive, there exists a compact set $A_2 \in S_{d_H}(\{a_1\}, \epsilon_1)$ such that $(\{a_1\}, A_2)$ is a Li–Yorke pair for $\overline{f}$. Thus for any $a_2 \in A_2$, $\liminf_{n\to\infty} d(f^n(a_1), f^n(a_2)) = 0$. As $\lim_{n\to\infty} d(f^n(x), f^n(a_1)) = 0$, we have, $\liminf_{n\to\infty} d(f^n(x), f^n(a_2)) = 0$, and there exists an integer n_2, $n_2 > n_1$, such that $d(f^{n_2}(x), f^{n_2}(a_2)) > \delta$ for some $a_2 \in A_2$.

If (x, a_2) forms a Li–Yorke pair, we are done. If not, then there exists $m_2 \in \mathbb{N}$ such that $d(f^n(x), f^n(a_2)) < \frac{1}{4}$ for all $n \geqslant m_2$. Thus, there exists a neighborhood $U_2 \subset S_d(a_1, \epsilon_1)$ of a_2, such that $d(f^{n_2}(x), f^{n_2}(y)) > \delta$ and $d(f^{m_2}(x), f^{m_2}(y)) < \frac{1}{4}$ for all $y \in U_2$. As U_2 is a neighborhood of a_2, there exists $\epsilon_2 > 0$ such that $\overline{S_d(a_2, \epsilon_2)} \subset U_2$.

Proceeding inductively, we either get a Li–Yorke pair or we get a sequence $\{a_n\}$ in $S_d(x, \epsilon)$. Let a be a limit point of this sequence. As $a \in S_d(a_i, \epsilon_i) \subset \overline{S_d(a_i, \epsilon_i)} \subset U_i \subset S_d(x, \epsilon)$, $d(f^{n_i}(x), f^{n_i}(a)) > \delta$ and $d(f^{m_i}(x), f^{m_i}(a)) < \frac{1}{2^i}$.

Thus, $\liminf_{n\to\infty} d(f^n(x), f^n(a)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(a)) > 0$.

Now, if $(\mathcal{F}(X), \overline{f})$ is Li–Yorke sensitive then $(\{x\}, A)$ is a Li–Yorke pair in $\mathcal{F}(X)$ with modulus (some) $\delta > 0$. Since A is finite, there exists $y \in A$ such that (x, y) is a Li–Yorke pair with modulus δ. Hence, (X, f) is Li–Yorke sensitive. □

In [7] Rongbao Gu has claimed that f sensitive need not imply $\overline{f}$ be sensitive in $\mathcal{K}(X)$, by giving a counterexample (Example 3.7 mentioned therein). We contradict his observation by showing the error in his example and then prove that $\overline{f}$ in this example is sensitive with the aid of the next proposition.

Example 2.7. Let I be the unit interval and let f be the tent map, $f(x) = 1 - 2|x - \frac{1}{2}|$, defined on I. Let S^1 be the unit circle in the complex plane and let T, defined as $T(e^{i\theta}) = e^{i(\theta+1)}$, be the rotation defined on S^1. Then the product $\mathcal{C} = I \times S^1$ is a cylinder and the metric

$$\rho\big((x, e^{i\alpha}), (y, e^{i\beta})\big) = \max\big\{|x - y|, \big|e^{i\alpha} - e^{i\beta}\big|\big\}$$

gives the product topology on it.

The product map $h : \mathcal{C} \to \mathcal{C}$ defined as,

$$h\big((x, e^{i\theta})\big) = \big(f(x), T\big(e^{i\theta}\big)\big)$$

is sensitive since the tent map f is sensitive.

As mentioned in [7], the induced map $\bar{h}$ is not sensitive at the point $I \times S^1 \in \mathcal{K}(\mathcal{C})$, since $\forall F \in B(I \times S^1, \frac{\epsilon}{4})$, $d_H(\bar{h}^n(I \times S^1), \bar{h}^n(F)) < \epsilon$. We show that this is not true.

Choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \frac{\epsilon}{4}$. Consider the set $E = \{\{\frac{\mu}{2^k}\} \times S^1\colon 0 \leqslant \mu \leqslant 2^k\}$. Then $E \in B(I \times S^1, \frac{\epsilon}{4}) \subset \mathcal{K}(\mathcal{C})$ and $\bar{h}^{2^k}(E) = \{0\} \times S^1$. Since $I \times S^1$ is fixed by $\bar{h}$, and so $d_H(\bar{h}^{2^k}(E), \bar{h}^{2^k}(I \times S^1)) = d_H(\{0\} \times S^1, (I \times S^1)) = 1$. Thus $\bar{h}$ is sensitive at $I \times S^1$.

We note that here, the map h above is strongly sensitive, and for strongly sensitive maps we have

Proposition 2.8. *If (X, f) is strongly sensitive, then $(\mathcal{K}(X), \overline{f})$ is also strongly sensitive.*

Proof. Let f be strongly sensitive with sensitivity constant δ. To prove the strong sensitivity of $\overline{f}$ on $\mathcal{K}(X)$, it is enough to prove the same on $\mathcal{F}(X)$, as $\mathcal{F}(X)$ is dense in $\mathcal{K}(X)$.

Let $A = \{x_1, x_2, \ldots, x_k\} \in \mathcal{F}(X)$ and let $S_{d_H}(A, \epsilon)$ be the ϵ-neighborhood of A. As f is strongly sensitive, for each $i = 1, 2, \ldots, k$, there exists $n_i \in \mathbb{N}$ such that $\sup_{y \in S(x_i, \epsilon)} d(f^n(x_i), f^n(y)) > \delta$ for all $n \geqslant n_i$. Let $N = \max\{n_i\colon 1 \leqslant i \leqslant k\}$.

We shall show that $\sup_{B \in S_{d_H}(A, \epsilon)} d_H(\overline{f}^n(A), \overline{f}^n(B)) > \frac{\delta}{2}$ for all $n \geqslant N$.

Let $r \geqslant N$. For each x_i, there exists $y_i \in S_d(x_i, \epsilon)$ such that $d(f^r(x_i), f^r(y_i)) > \delta$. Let $C = \{z_1, z_2, \ldots, z_k\}$ where,

$$z_i = \begin{cases} y_i, & d(f^r(x_1), f^r(x_i)) \leqslant \frac{\delta}{2}; \\ x_i, & \text{otherwise}. \end{cases}$$

Then, $d(f^r(x_1), f^r(z_i)) > \frac{\delta}{2}$ for each i and hence $d_H(\overline{f}^r(A), \overline{f}^r(C)) > \frac{\delta}{2}$. Consequently, $\sup_{B \in S_{d_H}(A,\epsilon)} d_H(\overline{f}^r(A), \overline{f}^r(B)) > \frac{\delta}{2}$ holds for all $r \geqslant N$, thus establishing strong sensitivity on $\mathcal{F}(X)$. □

Remark 2.9. Recently in [15] it has been shown that most of the important sensitive dynamical systems are all strongly sensitive (the author here calls them cofinitely sensitive). Hence, we can say that for most cases, sensitivity is equivalent for both (X, f) and $(\mathcal{K}(X), \overline{f})$. Also, by [15], all sensitive interval maps are strongly sensitive, and hence the main result in [13] follows as a corollary to Proposition 2.8.

However, for sensitivity of f, we can at best have

Proposition 2.10. *If X is locally connected, and (X, f) is sensitive, then $(\mathcal{F}(X), \overline{f})$ is pointwise sensitive.*

Proof. Let (X, f) be sensitive and let $A = \{x_1, x_2, \ldots, x_k\} \in \mathcal{F}(X)$. Without loss of generality, let $\mathcal{U} = \langle U_1, U_2, \ldots, U_k\rangle$ be an arbitrary neighborhood of the point A in the hyperspace where $x_i \in U_i$, $i = 1, 2, \ldots, k$. As X is locally connected, let U_1^* be the connected neighborhood of x_1 contained in U_1.

As (X, f) is sensitive, corresponding to x_1, there exist $y_1 \in U_1^*$ and $n \in \mathbb{N}$ such that $d(f^n(x_1), f^n(y_1)) > \delta$. It can be seen that $f^n(x_1)$ and $f^n(y_1)$ are more than δ apart and as U_1^* is connected, every possible distance less than $d(f^n(x_1), f^n(y_1))$ is attained. Since there are k distinct points $x_1, x_2, \ldots, x_k$ and $d(f^n(x_1), f^n(y_1)) > \delta$, there exists $z_1 \in U_1^*$ such that $d(f^n(z_1), f^n(x_r)) > \frac{\delta}{2k}$ for all $r = 2, \ldots, k$ (by triangle inequality).

Thus, $B = \{z_1, x_2, x_3, \ldots, x_k\} \in \langle U_1, U_2, \ldots, U_k\rangle$ such that

$$d_H\big(\overline{f}^n(A), \overline{f}^n(B)\big) > \frac{\delta}{2k}.$$

Thus, $(\mathcal{F}(X), \overline{f})$ is pointwise sensitive. □

Even when X is not locally connected, it seems that (X, f) is sensitive implies $(\mathcal{F}(X), \overline{f})$ is pointwise sensitive. Although, $(\mathcal{K}(X), \overline{f})$ may fail to be sensitive as in shown in the example below.

Example 2.11. Let $\Sigma = \{0, 1\}^{\mathbb{N}}$ be the shift space with the shift operator σ defined as $(\sigma(\bar{x}))^n = x^{n+1}$ where $\bar{x} = (x^n)$. The product topology on Σ can be generated by the metric $D(\bar{a}, \bar{b}) = \sum_{n=1}^{\infty} \frac{|a^n - b^n|}{2^{i-1}}$, where $\bar{a} = (a^n)$, $\bar{b} = (b^n) \in \Sigma$.

Let T be the irrational rotation on the circle S^1 given by $T(\theta) = \theta + \alpha$ where α is a very small irrational multiple of 2π. By dividing S^1 into two hemispheres, define a sequence $\bar{x} = (x^n) \in \Sigma$ as

$$x^n = \begin{cases} 0, & 0 \leqslant T^n(0) < \pi; \\ 1, & \pi \leqslant T^n(0) < 2\pi. \end{cases}$$

The sequence generated above, codes the trajectory of the point $\theta = 0$.

This sequence $\bar{x}$ generates a subshift (X, σ) of the shift space (Σ, σ), where $X = \overline{\{\sigma^n(\bar{x})\colon n \geqslant 0\}}$.

Since $\{T^n(0)\}$ is dense in S^1, and $\sigma^n(\bar{x})$ is a coding of the trajectory of this point, it can be seen that each point in X corresponds to a point in S^1. However, it can be noted that the points in X do not code the orbit of any $\theta \in S^1$ under T other than those of the form $T^n(0)$.

As no point of X is isolated, (X, σ) is sensitive [15]. However, we claim that $(\mathcal{K}(X), \overline{\sigma})$ is not sensitive.

Let $k \in \mathbb{N}$ be an odd integer and $\{\omega_i\colon 1 \leqslant i \leqslant k\}$ be the distinct k-th roots of unity. Let $\bar{x}_i \in X$ be the sequence corresponding to ω_i and $A = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$. Let $\mathcal{U} = \langle U_1, U_2, \ldots, U_k\rangle$ be a neighborhood of $A \in \mathcal{K}(X)$ where U_i are disjoint neighborhoods of $\bar{x}_i$.

For each U_i there exists $n_i \in \mathbb{N}$ such that the cylinder $[x_i^1 x_i^2 \ldots x_i^{n_i}] \subseteq U_i$, and there exists $y_i \in [x_i^1 x_i^2 \ldots x_i^{n_i}]$ such that it corresponds to the point $\beta_i = T^{k_i}(0)$ on S^1.

We now show that there exists an arc J_i, around β_i, such that the sequences corresponding to any point on this arc are in $[x_i^1 x_i^2 \ldots x_i^{n_i}]$.

If an arc J_i, containing β_i, stays completely in a single hemisphere for n_i iterates under T, then every sequence generated by points of the form $T^{k_i}(0) \in J_i$ is contained in the cylinder $[x_i^1 x_i^2 \ldots x_i^{n_i}]$. If the arc J_i intersects both the hemispheres at some k-th iterate, $k < n_i$, then we reduce the arc from J_i to J_i' such that its k-th image is also fully contained in the hemisphere containing $T^k(\beta_i)$. We can always find such an arc, since $T^l(0)$ can never be equal to π, for any $l \in \mathbb{N}$. Continuing in this way iteratively, we obtain an arc containing β_i which lies completely in one of the hemispheres for first n_i iterates.

Thus, corresponding to any cylinder $C = [x_i^1 x_i^2 \ldots x_i^{n_i}]$ we obtain $\overline{y_i} \in C$ such that $\overline{y_i}$ corresponds to some $T^{s_i}(0)$ and there exists an arc J_i containing $T^{s_i}(0)$ such that the coding of all the points in J_i is contained in C.

We now show that orbits of any pair of points $\{y_1, y_2, \ldots, y_k\}$, $\{z_1, z_2, \ldots, z_k\} \in \mathcal{U}$, $y_i, z_i \in U_i$ under $\overline{\sigma}$ get $\frac{1}{2^{m_k}}$ apart, where m_k depends on k and increases with k. Without loss of generality, we can assume the points y_i, z_i to be of the form $\sigma^{p_i}(\bar{x})$ and $\sigma^{q_i}(\bar{x})$ and hence generated by the points $T^{p_i}(0)$ and $T^{q_i}(0)$ on the circle respectively.

The points $\overline{y_i}$ and $\overline{z_i}$ will differ at r-th place if the r-th iterate of the corresponding arc J_i intersects both the hemispheres. As each $\overline{x_i}$ corresponds to the distinct k-th roots of unity and k is odd, for $j \neq i$, $\overline{y_j}$ and $\overline{z_j}$ will not differ at the r-th place since the r-th iterate of the corresponding arc will always lie in a single hemisphere.

Hence, if y_i and z_i differ at the r-th place, there exists an $m_k \in \mathbb{N}$, depending on k and the angle α, for which the predecessor and successor arcs of J_i remain in the same hemisphere as J_i for the next m_k iterates, i.e.

$$z_{i-1}[r, r+m_k] = y_i[r, r+m_k]; \qquad z_i[r, r+m_k] = y_{i+1}[r, r+m_k].$$

Thus, the Hausdorff distance in $\mathcal{K}(X)$ between $\overline{\sigma}^r\{y_1, y_2, \dots, y_k\}$ and $\overline{\sigma}^r\{z_1, z_2, \dots, z_k\}$ will be at most $\frac{1}{2^{m_k}}$.

As k increases, the points ω_i get closer to each other, and hence the predecessor and successor arcs of the arc J_i, will lie in the same hemisphere as J_i for a longer period, thus increasing m_k subsequently.

Hence, for any k-point set, a point in its neighbourhood for any subsequent iterate will get closer and closer to it, as k increases, and so in particular $\overline{\sigma}$ will not be sensitive on X.

Remark 2.12. We recall that for (X, f), a pair (x, y) is a Li–Yorke pair (with modulus δ) if it is proximal but not asymptotic (δ-asymptotic). Hence, Li–Yorke sensitivity (resp. chaotic dependence on initial conditions) implies that for every $x \in X$, and every neighbourhood U of x, the pair (x, y) cannot be asymptotic for every $y(\neq x) \in U$. We note that if (X, f) has the property that for every $x \in X$, there exists a neighbourhood U of x such that the pair (x, y) is asymptotic for every $y \in U$, then the property is also satisfied by $(\mathcal{K}(X), \overline{f})$, and vice versa. Thus, the property of not being asymptotic in the neighbourhood of the diagonal is equivalent in both (X, f) and $(\mathcal{K}(X), \overline{f})$. Also, if (X, f) has the property that for every $x \in X$, and every neighbourhood U of x, there exists $y \in U$ such that the pair (x, y) is proximal, then the property is also satisfied by $(\mathcal{K}(X), \overline{f})$. For any $A \in \mathcal{K}(X)$, we take $b \notin A$ such that the pair (a, b) is proximal for some $a \in A$. Then the pair $(A, A \cup \{b\})$ is proximal in $\mathcal{K}(X)$. The converse holds vacuously. Thus, the property of proximity in the neighbourhood of the diagonal is equivalent in both (X, f) and $(\mathcal{K}(X), \overline{f})$.

This observation strengthens the belief that Li–Yorke sensitivity (resp. chaotic dependence on initial conditions) of f should imply that for $\overline{f}$. But, those points that form an asymptotic pair, need not necessarily form a Li–Yorke pair. The instances when proximity is achieved, for even a finite set of points in X, need not overlap. Again, all that we can say is that $\overline{f}$ when Li–Yorke sensitive guarantees that f is chaotically dependent on initial conditions. We leave strengthening this implication, as well as discussing the converse implication, in any form, open here. Our belief here is that, we may not be able to say anything in this regard, without taking into account the other dynamical properties of the system.

Acknowledgements

The authors thank Philip Boyland and the referees for many useful suggestions and remarks. The first author thanks CSIR and the second author thanks DST for financial support.

References

[1] Christophe Abraham, Gerard Biau, Benoit Cadre, Chaotic properties of mappings on a probability space, J. Math. Anal. Appl. 266 (2002) 420–431.
[2] Ethan Akin, Sergii Kolyada, Li–Yorke sensitivity, Nonlinearity 16 (2003) 1421–1433.
[3] John Banks, Chaos for induced hyperspace maps, Chaos Solitons Fractals 25 (2005) 681–685.
[4] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[5] Francois Blanchard, Eli Glasner, Sergii Kolyada, Alejandro Maass, On Li–Yorke pairs, J. Reine Angew. Math. 547 (2002) 51–68.
[6] Bau-Sen Du, A dense orbit almost implies sensitivity to initial conditions, Bull. Inst. Math. Acad. Sin. 26 (2) (1998) 85–94.
[7] Rongbao Gu, Kato's chaos in set valued discrete systems, Chaos Solitons Fractals 31 (2007) 765–771.
[8] J. Guckenheimer, Sensitive dependence to initial conditions for one dimensional maps, Comm. Math. Phys. 70 (1979) 133–160.
[9] Roman-Flores Heriberto, Robinson's chaos in set valued discrete systems, Chaos Solitons Fractals 125 (2005) 33–42.
[10] Roman-Flores Heriberto, A note on transitivity in set valued discrete systems, Chaos Solitons Fractals 17 (2003) 99–104.
[11] Dominik Kwietniak, Piotr Oprocha, Topological entropy and chaos for maps induced on hyperspaces, Chaos Solitons Fractals 33 (2007) 76–86.
[12] Gongfu Liao, Xianfeng Ma, Lidong Wang, Individual chaos implies collective chaos for weakly mixing discrete dynamical systems, Chaos Solitons Fractals 32 (2007) 604–608.
[13] Heng Liu, Enhui Shi, Gongfu Liao, Sensitivity for set-valued discrete systems, Nonlinear Anal. 71 (12) (2009) 6122–6125.
[14] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.
[15] T.K. Subrahmonian Moothathu, Stronger forms of sensitivity for dynamical systems, Nonlinearity 20 (9) (2007) 2115–2126.
[16] Yangeng Wang, Guo Wei, William H. Campbell, Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems, Topology Appl. 156 (2009) 803–811.