GLOBAL EXISTENCE OF SOLUTIONS TO REACTION DIFFUSION SYSTEMS WITH MASS TRANSPORT TYPE BOUNDARY CONDITIONS

VANDANA SHARMA† AND JEFF MORGAN*

Abstract. We consider a reaction-diffusion system where some components react and diffuse on the boundary of a region, while other components diffuse in the interior and react with those on the boundary through mass transport. We establish local well-posedness and global existence of solutions for these systems using classical potential theory and linear estimates for initial boundary value problems.

Key words. reaction-diffusion equations, mass transport, conservation of mass, Laplace Beltrami operator, global existence, a priori estimates.

AMS subject classifications. 35K57, 35B45

1. Introduction. The idea that reaction-diffusion phenomena is essential to the growth of living organisms seems quite intuitive. Indeed, it would be rather hard to envision how any organism could grow and operate without moving its constituents around and using them in various bio-chemical reactions [16]. For example, bacterial cytokinesis is one process which can be modeled by reaction-diffusion systems. During the bacterial cytokinesis process, a proteinaceous contractile ring assembles in the middle of the cell. The ring tethers to the membrane and contracts to form daughter cells; that is, the "cell divides". One mechanism that centers the ring involves the pole-to-pole oscillation of proteins Min C, Min D and Min E. Oscillations cause the average concentration of Min C, an inhibitor of the ring assembly, to be lowest at the midcell and highest near the poles [35], [27]. This centering mechanism, relating molecular-level interactions to supra-molecular ring positioning can be modelled as a system of semilinear parabolic equations. The multi-dimensional version of the evolution of the Min concentrations can be described as a special case of the reaction-diffusion system

$$
\begin{aligned}
u_t &= D\Delta u + H(u) && x\in\Omega, \quad 0<t<T\\
v_t &= \tilde{D}\Delta_M v + F(u,v) && x\in M, \quad 0<t<T\\
D\frac{\partial u}{\partial \eta} &= G(u,v) && x\in M, \quad 0<t<T\\
u &= u_0 && x\in\Omega, \quad t=0\\
v &= v_0 && x\in M, \quad t=0
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 2$, with smooth boundary M, Δ and Δ_M denote the Laplace and Laplace Beltrami operators, η is the unit outward normal vector to Ω at points on M, and D and $\tilde{D}$ are $k\times k$ and $m\times m$ diagonal matrices with positive diagonal entries $\{d_j\}_{1\leq j\leq k}$ and $\{\tilde{d}_i\}_{1\leq i\leq m}$ respectively. $F:\mathbb{R}^k\times\mathbb{R}^m\to\mathbb{R}^m$, $G:\mathbb{R}^k\times\mathbb{R}^m\to\mathbb{R}^k$, $H:\mathbb{R}^k\to\mathbb{R}^k$, and u_0 and v_0 are componentwise nonnegative smooth functions that satisfy the compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0,v_0) \quad \text{on } M.$$

†Department of Mathematical and Statistical Sciences, Arizona State University, Tempe, USA, AZ 85281. Email: vandanas@asu.edu.

*Department of Mathematics, The University of Houston, Houston, USA, TX 77004. Email: jjmorgan@central.uh.edu. The authors acknowledge the generous support of NSF grant DMS-0714864.

For this model, Ω may represent the cell cytoplasm and M may represent its membrane. There are some components that are bound to the membrane, and other components that move freely in the cytoplasm. Also, the components on the membrane and cytoplasm react together on the membrane through mass action and boundary transport. In Section 7, we present two applications associated with (1.1), with one modeling the chemical reaction involving Min protiens for positioning of the ring, explained in [27]. We point out the study in [35] that also modeled these reactions.

In general, system (1.1) is somewhat reminiscent of two component systems where both of the unknowns react and diffuse inside Ω, with various homogeneous boundary conditions and nonnegative initial data. In that setting, global well-possedness and uniform boundedness has been studied by many researchers, and we refer the interested reader to the excellent survey of Pierre [23].

In the remainder of the introduction, we assume $H = 0$ and $k = m = 1$. A fundamental mathematical question concerning global existence for (1.1) asks, what conditions on F and G will guarantee that (1.1) has global solutions, and how are these conditions related to the results listed in [23]? The focus of this paper is to give a partial answer to this question and to apply our results to (1.1).

From a physical standpoint, it is natural to ask under what conditions the solutions of (1.1) are nonnegative, and the total mass is either conserved or reduced. It is also important to ask whether these conditions arise in problems similar to the above mentioned cell biology system. Conditions that are similar in spirit to those given in [20], [13] and [23] result in nonnegative solutions for system (1.1). More precisely, (1.1) has nonnegative solutions for all choices of nonnegative initial data u_0 and v_0 if and only if F, G, and H are quasi-positive. That is $F(a, 0), G(0, a) \geq 0$ whenever $a \geq 0$ (recall from above that $H = 0$ in the remainder of this introduction). Also, some control of total mass can be achieved by assuming there exists $\alpha > 0$ such that

$$F(u, v) + G(u, v) \leq \alpha(u + v + 1) \quad \text{for all } u, v \geq 0. \tag{1.2}$$

Assumption (1.2) (discussed later), generalizes mass conservation by implying that total mass, $\int_\Omega u(x, t)\, dx + \int_M v(\zeta, t)\, d\sigma$, grows at most exponentially in time t.

We suspect that the natural conditions, quasipositivity and conservation of mass, are not sufficient to obtain global existence in (1.1), and that it is possible to construct an example along the same lines as constructed in [24]. To this end, we impose a condition similar to Morgan's intermediate sums [21] and [22]. Namely, there exists a constant $K_g > 0$ such that

$$G(\zeta, \nu) \leq K_g(\zeta + \nu + 1) \quad \text{for all} \quad \nu, \zeta \geq 0.$$

In addition, we adopt a natural assumption of polynomial growth, which has been considered in the context of chemical and biological modeling (see Horn and Jackson [14]). That is, there exists $l \in \mathbb{N}$ and $K_f > 0$ such that

$$F(u, v) \leq K_f(u + v + 1)^l \quad \text{for all} \quad v \geq 0,\ u \geq 0.$$

In our analysis, we extend recent results of Huisken and Polden [25], Polden [15], and Sharples [31] associated with $W_2^{2,1}(M \times (0, T))$ results for solutions to linear Cauchy problems on a membrane. We also verify and make use of a remark of Brown [4] which states that if $d > 0$

and the Neumann data γ lies in $L_p(M \times (0,T))$ for $p > n+1$, then the solution to

$$
\begin{aligned}
\varphi_t &= d\Delta\varphi && x \in \Omega, \quad 0 < t < T \\
d\frac{\partial \varphi}{\partial \eta} &= \gamma && x \in M, \quad 0 < t < T \\
\varphi &= 0 && x \in \Omega, \quad t = 0
\end{aligned} \tag{1.3}
$$

is Hölder continuous on $\overline{\Omega} \times (0,T)$. We provide the proof of this result in section 5 for completeness of our arguments.

Note that the results of Amann [3] can be used to guarantee the local well posedness of (1.1) subject to appropriate conditions on initial data and on the functions F and G. However, those results do not provide the explicit estimates that are needed in our setting. Our approach keeps the analysis on comparatively simpler L_p spaces.

It is worth mentioning that some of the results in section 5 are valid for domains that are only C^1. Handling cases with weak smoothness conditions on curves or domain boundaries was one of the motivations for the results obtained in [4], [5], [9] and [10] , and these results may be of independent interest.

2. Notations, Definitions and Preliminary Estimates. Throughout this paper, $n \geq 2$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary M ($\partial\Omega$) belonging to the class $C^{2+\mu}$ with $\mu > 0$ such that Ω lies locally on one side of its boundary. η is the unit outward normal (from Ω) to M, and Δ and Δ_M are the Laplace and the Laplace Beltrami operators, respectively. For more details, see Rosenberg [30] and Taylor [34]. In addition, m, k, n, i and j are positive integers, D and $\tilde{D}$ are $k \times k$ and $m \times m$ diagonal matrices with positive diagonal entries $\{d_j\}_{1\leq j\leq k}$ and $\{\tilde{d}_i\}_{1\leq i\leq m}$, respectively.

2.1. Basic Function Spaces. Let $\mathcal{B}$ be a bounded domain on $\mathbb{R}^m$ with smooth boundary such that $\mathcal{B}$ lies locally on one side of $\partial\mathcal{B}$. We define all function spaces on $\mathcal{B}$ and $\mathcal{B}_T = \mathcal{B}\times(0,T)$. $L_q(\mathcal{B})$ is the Banach space consisting of all measurable functions on $\mathcal{B}$ that are $q^{th}(q \geq 1)$ power summable on $\mathcal{B}$. The norm is defined as

$$\|u\|_{q,\mathcal{B}} = \left(\int_{\mathcal{B}} |u(x)|^q dx\right)^{\frac{1}{q}}$$

Also,

$$\|u\|_{\infty,\mathcal{B}} = ess\sup\{|u(x)| : x \in \mathcal{B}\}.$$

Measurability and summability are to be understood everywhere in the sense of Lebesgue.

If $p \geq 1$, then $W_p^2(\mathcal{B})$ is the Sobolev space of functions $u : \mathcal{B} \to \mathbb{R}$ with generalized derivatives, $\partial_x^s u$ (in the sense of distributions) $|s| \leq 2$ belonging to $L_p(\mathcal{B})$. Here $s = (s_1, s_2,...,s_n)$, $|s| = s_1 + s_2 + .. + s_n$, $|s| \leq 2$, and $\partial_x^s = \partial_1^{s_1}\partial_2^{s_2}...\partial_n^{s_n}$ where $\partial_i = \frac{\partial}{\partial x_i}$. The norm in this space is

$$\|u\|_{p,\mathcal{B}}^{(2)} = \sum_{|s|=0}^{2} \|\partial_x^s u\|_{p,\mathcal{B}}$$

Similarly, $W_p^{2,1}(\mathcal{B}_T)$ is the Sobolev space of functions $u : \mathcal{B}_T \to \mathbb{R}$ with generalized derivatives, $\partial_x^s\partial_t^r u$ (in the sense of distributions) where $2r + |s| \leq 2$ and each derivative belonging to $L_p(\mathcal{B}_T)$. The norm in this space is

$$\|u\|_{p,\mathcal{B}_T}^{(2)} = \sum_{2r+|s|=0}^{2} \|\partial_x^s\partial_t^r u\|_{p,\mathcal{B}_T}$$

In addition to $W_p^{2,1}(\mathcal{B}_T)$, we will encounter other spaces with two different ratios of upper indices, $W_2^{1,0}(\mathcal{B}_T)$, $W_2^{1,1}(\mathcal{B}_T)$, $V_2(\mathcal{B}_T)$, $V_2^{1,0}(\mathcal{B}_T)$, and $V_2^{1,\frac{1}{2}}(\mathcal{B}_T)$ as defined in [19].

We also introduce $W_p^l(\mathcal{B})$, where $l > 0$ is not an integer, because initial data will be taken from these spaces. The space $W_p^l(\mathcal{B})$ with nonintegral l, is a Banach space consisting of elements of $W_p^{[l]}$ ($[l]$ is the largest integer less than l) with the finite norm

$$\|u\|_{p,\mathcal{B}}^{(l)} = \langle u\rangle_{p,\mathcal{B}}^{(l)} + \|u\|_{p,\mathcal{B}}^{([l])}$$

where

$$\|u\|_{p,\mathcal{B}}^{([l])} = \sum_{s=0}^{[l]} \|\partial_x^s u\|_{p,\mathcal{B}}$$

and

$$\langle u\rangle_{p,\mathcal{B}}^{(l)} = \sum_{s=[l]} \left(\int_{\mathcal{B}} dx \int_{\mathcal{B}} |\partial_x^s u(x) - \partial_y^s u(y)|^p . \frac{dy}{|x-y|^{n+p(l-[l])}}\right)^{\frac{1}{p}}$$

$W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$ spaces with non integral l also play an important role in the study of boundary value problems with nonhomogeneous boundary conditions, especially in the proof of exact estimates of their solutions. It is a Banach space when $p \geq 1$, which is defined by means of parametrization of the surface $\partial\mathcal{B}$. For a rigorous treatment of these spaces, we refer the reader to page 81 of Chapter 2 of [19].

The use of the spaces $W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$ is connected to the fact that the differential properties of the boundary values of functions from $W_p^{2,1}(\mathcal{B}_T)$ and of certain of its derivatives, $\partial_x^s\partial_t^r$, can be exactly described in terms of the spaces $W_p^{l,\frac{l}{2}}(\partial\mathcal{B}_T)$, where $l = 2 - 2r - s - \frac{1}{p}$.

For $0 < \alpha, \beta < 1$, $C^{\alpha,\beta}(\overline{\mathcal{B}_T})$ is the Banach space of Hölder continuous functions u with the finite norm

$$|u|_{\overline{\mathcal{B}}_T}^{(\alpha)} = \sup_{(x,t)\in\mathcal{B}_T} |u(x,t)| + [u]_{x,\mathcal{B}_T}^{(\alpha)} + [u]_{t,\mathcal{B}_T}^{(\beta)}$$

where

$$[u]_{x,\overline{\mathcal{B}}_T}^{(\alpha)} = \sup_{\substack{(x,t),(x',t)\in\mathcal{B}_T\\ x\neq x'}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha}$$

and

$$[u]_{t,\overline{\mathcal{B}}_T}^{(\beta)} = \sup_{\substack{(x,t),(x,t')\in\mathcal{B}_T\\ t\neq t'}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\beta}$$

We shall denote the space $C^{\frac{\alpha}{2},\frac{\alpha}{2}}(\overline{\mathcal{B}}_T)$ by $C^{\frac{\alpha}{2}}(\overline{\mathcal{B}}_T)$. $C(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{B}_T \to \mathbb{R}^n$, and $C^{1,0}(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{B}_T \to \mathbb{R}^n$ for which u_{x_i} is continuous for all $1 \leq i \leq n$. $C^{2,1}(\mathcal{B}_T,\mathbb{R}^n)$ is the set of all continuous functions $u : \mathcal{B}_T \to \mathbb{R}^n$ having continuous derivatives $u_{x_i}, u_{x_ix_j}$ and u_t in $\mathcal{B}_T$. Note that similar definitions can be given on $\overline{\mathcal{B}}_T$. Moreover notations and definitions for Hölder and Sobolev Spaces on manifolds are similar to the ones used in the Handbook of Global analysis [17]. More developments on Sobolev spaces, Sobolev inequalities, and the notion of best constants may be found in [6], [7], [12] and [34].

2.2. Preliminary Estimates. For completeness of our arguments, we state the following results, which will help us obtain a priori estimates for the Cauchy problem on the manifold M, and prove the existence of solutions in $W_p^{2,1}(M_T)$. Lemmas 2.1, 2.4 and 2.6 can be found on page 341, Chapter 4 in [19], as (2.24) and (2.25) on page 49 in [18], and [12] respectively. Lemma 2.2 is stated as Lemma 3.3 in Chapter 2 of [19].

Let $\mathcal{B}$ be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\mathcal{B}$ belonging to the class $C^{2+\mu}$ with $\mu > 0$ such that $\mathcal{B}$ lies locally on one side of the boundary $\partial\mathcal{B}$. Let $T > 0$ and $p > 1$. Suppose $\Theta \in L_p(\mathcal{B}_T)$, $w_0 \in W_p^2(\mathcal{B})$, $\gamma \in W_p{}^{2-\frac{1}{p},1-\frac{1}{2p}}(\partial\mathcal{B}_T)$. Also, let the coefficient matrix $(a_{i,j})$ be symmetric and continuous on $\overline{\mathcal{B}_T}$, and satisfy the uniform ellipticity condition. That is for some $\lambda > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } (x,t) \in \overline{\mathcal{B}_T} \text{ and for all } \xi \in \mathbb{R}^n$$

Finally, let the coefficients a_i be continuous on $\overline{\mathcal{B}_T}$. Consider the problem

$$\begin{aligned} \frac{\partial w}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial w}{\partial x_i} &= \Theta(x,t) \qquad (x,t) \in \mathcal{B}_T \\ w &= \gamma(x,t) \qquad (x,t) \in \partial\mathcal{B}_T \\ w\big|_{t=0} &= w_0(x) \qquad x \in \mathcal{B} \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $p > 1$ with $p \neq \frac{3}{2}$, and in the case $p > \frac{3}{2}$, assume the compatibility condition of zero order, $w_0|_{\partial\mathcal{B}} = \gamma|_{t=0}$. Then (2.1) has a unique solution $w \in W_p{}^{2,1}(\mathcal{B}_T)$, and there exists $C > 0$ depending on T, p and $\mathcal{B}$, and independent of Θ, w_0 and γ such that*

$$\|w\|_{p,\mathcal{B}_T}^{(2)} \leq C(\|\Theta\|_{p,\mathcal{B}_T} + \|w_0\|_{p,\mathcal{B}}^{(2-\frac{2}{p})} + \|\gamma\|_{p,\partial\mathcal{B}_T}^{(2-\frac{1}{p},1-\frac{1}{2p})})$$

Lemma 2.2. *Suppose $q \geq p$, $2 - 2r - s - \left(\frac{1}{p} - \frac{1}{q}\right)(m+2) \geq 0$ and $0 < \delta \leq \min\{d; \sqrt{T}\}$. Then there exists $c_1, c_2 > 0$ depending on r, s, m, p and $\mathcal{B}$ such that*

$$\|D_t^r D_x^s u\|_{q,\mathcal{B}_T} \leq c_1\delta^{2-2r-s-\left(\frac{1}{p}-\frac{1}{q}\right)(m+2)}\|u\|_{p,\mathcal{B}_T}^{(2)} + c_2\delta^{-(2r+s+\left(\frac{1}{p}-\frac{1}{q}\right)(m+2))}\|u\|_{p,\mathcal{B}_T}$$

for all $u \in W_p^{2,1}(\mathcal{B}_\mathcal{T})$. Moreover, if $2 - 2r - s - \frac{(m+2)}{p} > 0$, then for $0 \leq \alpha < 2 - 2r - s - \frac{(m+2)}{p}$ there exist constants c_3, c_4 depending on r, s, m, p and $\mathcal{B}$ such that

$$|D_t^r D_x^s u|_{\mathcal{B}_T}^{(\alpha)} \leq c_3\delta^{2-2r-s-\frac{m+2}{p}-\alpha}\|u\|_{p,\mathcal{B}_T}^{(2)} + c_4\delta^{-(2r+s+\frac{(m+2)}{p}+\alpha)}\|u\|_{p,\mathcal{B}_T}$$

for all $u \in W_p^{2,1}(\mathcal{B}_\mathcal{T})$.

Corollary 2.3. *Suppose the conditions of Lemma 2.1 are fulfilled and $p > \frac{m+2}{2}$. Then there exists $\hat{c} > 0$ depending on m, p and $\mathcal{B}$ such that the solution of problem (2.1) is Hölder continuous, and*

$$|w|_{\mathcal{B}_T}^{(2-\frac{m+2}{p})} \leq \hat{c}\|w\|_{p,\mathcal{B}_T}^{(2)}$$

Lemma 2.4. *Suppose $1 < p < \infty$. If $p < m$ then $W_p^1(\mathcal{B})$ embedds continuously into $W_p^{(1-\frac{1}{p})}(\partial\mathcal{B})$ and $L_q(\mathcal{B})$ for $p \le q \le p^* = \frac{mp}{m-p}$. Furthermore, if $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\|v\|_{q,\mathcal{B}}^p \le \epsilon \|v_x\|_{p,\mathcal{B}}^p + C_\epsilon \|v\|_{1,\mathcal{B}}^p$$

for all $v \in W_p^1(\mathcal{B})$, and

$$\|v\|_{2,\partial\mathcal{B}}^2 \le \epsilon \|v_x\|_{2,\mathcal{B}}^2 + C_\epsilon \|v\|_{2,\mathcal{B}}^2$$

for all $v \in W_2^1(\mathcal{B})$.

Lemma 2.5. *Let $p > m$ and $0 < \alpha < 1 - \frac{m}{p}$. Then $W_p^1(\mathcal{B})$ embedds compactly in $C^\alpha(\overline{\mathcal{B}})$.*

Lemma 2.6. *Let M be a compact Riemannian manifold of dimension $m \ge 1$ and $p > m$. Then the embedding $W_p^1(M) \subset C^\alpha(M)$ is compact for all $0 < \alpha < 1 - \frac{m}{p}$.*

The following result follows from the Gagliardo Nirenberg inequality in [8] on bounded C^1 domains, and Young's inequality on page 40 in [18].

Lemma 2.7. *Let $\epsilon > 0$ and $1 < p < \infty$. Then there exists $C_{\epsilon,p} > 0$ such that*

$$\|v_x\|_{p,\mathcal{B}} \le \epsilon \|v_{xx}\|_{p,\mathcal{B}} + C_{\epsilon,p} \|v\|_{p,\mathcal{B}}$$

for all $v \in W_p^2(\mathcal{B})$.

3. Statements of Main Results. The primary concern of this work is the system

$$\begin{aligned}
u_t &= D\Delta u + H(u) && x \in \Omega, \quad 0 < t < T \\
v_t &= \tilde{D}\Delta_M v + F(u,v) && x \in M, \quad 0 < t < T \\
D\frac{\partial u}{\partial \eta} &= G(u,v) && x \in M, \quad 0 < t < T \\
u &= u_0 && x \in \Omega, \quad t = 0 \\
v &= v_0 && x \in M, \quad t = 0
\end{aligned} \tag{3.1}$$

where D and $\tilde{D}$ are $k \times k$ and $m \times m$ diagonal matrices with positive diagonal entries, $F = (F_i) : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m, G = (G_j) : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ and $H = (H_j) : \mathbb{R}^k \to \mathbb{R}^k$, and $u_0 = (u_{0j}) \in W_p^2(\Omega)$, $v_0 = (v_{0i}) \in W_p^2(M)$ with $p > n$. Also, u_0 and v_0 satisfy the compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

Remark 1. *Since $p > n$, u_0 and v_0 are Hölder continuous functions on $\overline{\Omega}$ and M respectively (see [1], [8]).*

Definition 3.1. *A function (u, v) is said to be a solution of (3.1) if and only if*

$$u \in C(\overline{\Omega} \times [0,T), \mathbb{R}^k) \cap C^{1,0}(\overline{\Omega} \times (0,T), \mathbb{R}^k) \cap C^{2,1}(\Omega \times (0,T), \mathbb{R}^k)$$

and

$$v \in C(M \times [0,T), \mathbb{R}^m) \cap C^{2,1}(M \times (0,T), \mathbb{R}^m)$$

such that (u,v) *satisfies* (3.1). *If* $T = \infty$, *the solution is said to be a global solution.* Moreover, a solution (u,v) defined for $0 \le t < b$ is a *maximal solution* of (3.1) if and only if (u,v) solves (3.1) with $T = b$, and if $d > b$ and $(\tilde{u}, \tilde{v})$ solves (3.1) for $T = d$ then there exists $0 < c < b$ such that $(u(\cdot,c), v(\cdot,c)) \neq (\tilde{u}(\cdot,c), \tilde{v}(\cdot,c))$.

We say F, G and H are *quasipositive* if and only if $F_i(\zeta,\xi) \ge 0$ whenever $\xi \in \mathbb{R}^m_+$ and $\zeta \in \mathbb{R}^k_+$ with $\xi_i = 0$ for $i = 1, ..., m$, and $G_j(\zeta,\xi) \ge 0$, $H_j(\zeta) \ge 0$ whenever $\xi \in \mathbb{R}^m_+$ and $\zeta \in \mathbb{R}^k_+$ with $\zeta_j = 0$, for $j = 1, ..., k$.

The purpose of this study is to give sufficient conditions guaranteeing that (3.1) has a global solution. The following Theorems comprise local and global existence of the solution.

THEOREM 3.2. *Suppose* F, G *and* H *are locally Lipschitz. Then there exists* $T_{\max} > 0$ *such that* (3.1) *has a unique, maximal solution* (u,v) *with* $T = T_{\max}$. *Moreover, if* $T_{\max} < \infty$ *then*

$$\limsup_{t \to T_{\max}^-} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t \to T_{\max}^-} \|v(\cdot,t)\|_{\infty,M} = \infty$$

In addition to the assumptions stated above, we say condition $V_{i,j}$ holds for $1 \le j \le k$ and $1 \le i \le m$ if and only if

($V_{i,j}1$) There exist $\alpha, \beta, \sigma > 0$ such that

$$\sigma F_i(\zeta,\nu) + G_j(\zeta,\nu) \le \alpha(\zeta_j + \nu_i + 1) \quad \text{and} \quad H_j(\zeta) \le \beta(\zeta_j + 1) \quad \text{for all} \quad \nu \in \mathbb{R}^m_{\ge 0},\ \zeta \in \mathbb{R}^k_{\ge 0}$$

($V_{i,j}2$) There exists $K_g > 0$ such that

$$G_j(\zeta,\nu) \le K_g(\zeta_j + \nu_i + 1) \quad \text{for all} \quad \nu \in \mathbb{R}^m_{\ge 0},\ \zeta \in \mathbb{R}^k_{\ge 0}$$

($V_{i,j}3$) There exists $l \in \mathbb{N}$ and $K_f > 0$ such that

$$F_i(\zeta,\nu) \le K_f(|\zeta| + |\nu| + 1)^l \quad \text{for all} \quad \nu \in \mathbb{R}^m_{\ge 0},\ \zeta \in \mathbb{R}^k_{\ge 0}$$

REMARK 2. ($V_{i,j}2$) *is related to the so - called linear "intermediate sums" condition used by Morgan in* [21], [22] *in the special case when the system has only two equations. This condition in* [21], [22], *as well as* [23] *pertains to interactions between the first m-1 equations in an m component system. Again, see* [21], [22] *and* [23]. ($V_{i,j}1$) *helps control mass, and allows higher order nonlinearities in* F, *but requires cancellation of high-order positive terms by* G. ($V_{i,j}3$) *implies* F *is polynomially bounded above.*

REMARK 3. *We will show that* ($V_{i,j}1$) *provides* L_1 *estimates for* u_j *on* Ω *and* M, *and* v_i *on* M. ($V_{i,j}2$) *helps us bootstrap* L_p *estimates for* u_j *on* $M \times (0, T_{max})$ *and* $\Omega \times (0, T_{max})$, *and* v_i *on* $M \times (0, T_{max})$. *Finally,* ($V_{i,j}2$) *and* ($V_{i,j}3$) *allow us to use* L_p *estimates to obtain sup norm estimates on* u_j *and* v_i.

THEOREM 3.3. *Suppose* F, G *and* H *are locally Lipschitz, quasi positive, and* u_0, v_0 *are componentwise nonnegative functions. Also, assume that for each* $1 \le j \le k$ *and* $1 \le i \le m$, *there exists* $l_i \in \{1, ..., k\}$ *and* $k_j \in \{1, ..., m\}$ *so that both* V_{i,l_i} *and* $V_{k_j,j}$ *are satisfied. Then* (3.1) *has a unique component-wise nonegative global solution.*

COROLLARY 3.4. *Suppose* $k = m = 1$, F, G *and* H *are locally Lipschitz and quasipositive, and* u_0, v_0 *are nonnegative functions. If* $V_{1,1}$ *is satisfied, then* (3.1) *has a unique nonnegative* *global solution.*

In the process of obtaining our results, we will derive $W_p^{2,1}(M_T)$ estimates of the Cauchy problem on M_T, and Hölder estimates of the solution to the Neumann problem on Ω_T. The Hölder estimates for the solution to the Neumann problem are given as a comment in Brown [4]. We give the statement as Theorem 3.6 below, and supply a proof in section 5. Let $\tilde{d}, d > 0$. Consider the systems

$$\begin{aligned} &\Psi_t = \tilde{d}\Delta_M \Psi + f && (\xi, t) \in M \times (0,T) \\ &\Psi\big|_{t=0} = \Psi_0 && \xi \in M \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\varphi_t = d\Delta\varphi + \theta && x \in \Omega, \quad 0 < t < T \\ &d\frac{\partial \varphi}{\partial \eta} = \gamma && x \in M, \quad 0 < t < T \\ &\varphi = \varphi_0 && x \in \Omega, \quad t = 0 \end{aligned} \tag{3.3}$$

THEOREM 3.5. *If $1 < p < \infty$ and $T > 0$, then there exists $\hat{C}_{p,T} > 0$ such that whenever $\Psi_0 \in W_p^{2-\frac{2}{p}}(M)$ and $f \in L_p(M_T)$, there exists a unique solution $\Psi \in W_p^{2,1}(M_T)$ of (3.2), and*

$$\|\Psi\|_{p,M_T}^{(2)} \leq \hat{C}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})$$

THEOREM 3.6. *Suppose $p > n+1$ and $T > 0$ and $\theta \in L_p(\Omega \times (0,T))$, $\gamma \in L_p(M \times (0,T))$ and $\varphi_0 \in W_p^2(\Omega)$ such that*

$$d\frac{\partial \varphi_0}{\partial \eta} = \gamma(x, 0) \quad \text{on } M.$$

Then there exists $C_{p,T} > 0$ independent of θ, γ and φ_0 and a unique weak solution $\varphi \in V_2^{1,\frac{1}{2}}(\Omega_T)$ of (3.3), such that if $0 < \beta < 1 - \frac{n+1}{p}$ then

$$|\varphi|_{\Omega_{\hat{T}}}^{(\beta)} \leq C_{p,T}(\|\theta\|_{p,\Omega_T} + \|\gamma\|_{p,M_T} + \|\varphi_0\|_{p,\Omega}^{(2)})$$

The proofs of Theorems 3.5 and 3.6 are given in sections 4 and 5. The remaining results are proved in section 6, and examples are given in section 7.

4. $W_p^{2,1}$ estimates for the Cauchy problem on a manifold. Let $n \geq 2$ and M be a compact $n-1$ dimensional Riemannian manifold without boundary. Consider (3.2) where $\tilde{d} > 0$, $f \in L_p(M_T)$ and $\Psi_0 \in W_p^{2-\frac{2}{p}}(M)$. Searching the literature, we surprisingly could not find $W_p^{2,1}(M_T)$ estimates for the solutions to (3.2). Tracing through the work in this direction, we found that Huisken and Polden [15] and [25], and J.J Sharples [31] give a result in the setting where $p = 2$. Using their $W_2^{2,1}(M_T)$ estimate, we obtain $W_p^{2,1}(M_T)$ a priori estimates for solutions of (3.2) for all $p > 1$. For $a > 0$ and smooth functions $f, g : M \times [0, \infty) \to \mathbb{R}$, Polden considered weighted inner products:

$$\langle f, g \rangle_{LL_a} = \int_0^\infty e^{-2at} \langle f(\cdot, t), g(\cdot, t) \rangle_{L^2(M)} dt$$

$$\langle f,g\rangle_{LW_a^1}=\int_0^\infty e^{-2at}\langle f(\cdot,t),g(\cdot,t)\rangle_{W_2^1(M)}dt$$

$$\langle f,g\rangle_{LW_a^2}=\int_0^\infty e^{-2at}\langle f(\cdot,t),g(\cdot,t)\rangle_{W_2^2(M)}dt$$

$$\langle f,g\rangle_{WW_a}=\langle f(\cdot,t),g(\cdot,t)\rangle_{LW_a^1}+\langle D_tf,D_tg\rangle_{LL_a}$$

Where LL_a, LW_a and WW_a are the Hilbert spaces formed by the completion of $C^\infty(M\times[0,\infty))$ in the corresponding norms, and WW_a^0 is the completion of subspace of $C^\infty(M\times[0,\infty))$ with compact support in WW_a. See [31] for the proof of the following result.

THEOREM 4.1. *Suppose Ψ_0 lies in $W_2^1(M)$ and $f\in LL_a(M\times[0,\infty))$. Then for sufficiently large a, the system (3.2) has a unique weak solution in WW_a^0.*

Furthermore using a priori estimates in [31], they showed that the solution belongs to $W_2^{2,1}(M\times[0,\infty))$.

THEOREM 4.2. *Let $\Psi\in WW_a$ be the unique solution of (3.2) with $\Psi_0\in W_2^1(M)$ and $f\in LL_a(M_T)$. Then $\Psi\in LW_a^2$, and there exists $C>0$ independent of Ψ_0 and f such that*

$$\|\Psi\|^2_{LW_a^2}\le C(\|\Psi_0\|^2_{W_2^1(M)}+\|f\|^2_{LL_a})$$

Proof. See Lemma 4.3 in [31]. □

The result below is an immediate consequence.

COROLLARY 4.3. *Let $0<T<\infty$. Suppose $\Psi_0\in W_2^1(M)$ and $f\in L_2(M_T)$. Then there exists a unique weak solution to (3.2) in $W_2^{2,1}(M_T)$, and there exists $C>0$ independent of Ψ_0 and f such that*

$$\|\Psi\|^2_{W_2^{2,1}(M_T)}\le C(\|\Psi_0\|^2_{W_2^1(M)}+\|f\|^2_{L_2(M_T)})$$

We will use the $W_2^{2,1}(M_T)$ result to derive $W_p^{2,1}(M_T)$ a priori estimates for solutions to (3.2) for all $p>1$. To obtain these estimates, we transform the Cauchy problem defined locally on M to a bounded domain on $\mathbb{R}^{n-1}$ and obtain the estimates over this bounded domain. Then we pull the resulting estimates back to the manifold. Repeating this process over every neighborhood on the manifold, and using compactness of the manifold, we get estimates over the entire manifold.

Let $\mathcal{F}$ be a subset of $\mathbb{R}_+$ with following property:
$p>1$ belongs to $\mathcal{F}$ if and only if there exists $C_{p,T}>0$ such that whenever $\Psi_0\in W_p^{2-\frac{2}{p}}(M)$ and $f\in L_p(M_T)$, then there exists a unique $\Psi\in W_p^{2,1}(M_T)$, such that Ψ solves (3.2) and

$$\|\Psi\|^{(2)}_{p,M_T}\le C_{p,T}(\|f\|_{p,M_T}+\|\Psi_0\|^{(2-\frac{2}{p})}_{p,M})$$

Note: From Corollary 4.3, $2\in\mathcal{F}$. Also note that we can prove Theorem 3.5 by showing $\mathcal{F}=(1,\infty)$.

LEMMA 4.4. $[2,\infty)\subset\mathcal{F}$.

Proof. We will show that if $p\in\mathcal{F}$ then $[p,p+\frac{1}{n-1}]\subset\mathcal{F}$. To this end, let $p\in\mathcal{F}$ and $q\in[p,p+\frac{1}{n-1}]$ such that $\Psi_0\in W_q^{2-\frac{2}{q}}(M)$ and $f\in L_q(M_T)$. Then $f\in L_p(M_T)$ and $\Psi_0\in W_p^{2-\frac{2}{p}}(M)$. Since $p\in\mathcal{F}$, there exists $C_{p,T}>0$ independent of Ψ_0 and f, and a unique $\Psi\in W_p^{2,1}(M_T)$ solving (3.2) such that

$$\|\Psi\|^{(2)}_{p,M_T}\le C_{p,T}(\|f\|_{p,M_T}+\|\Psi_0\|^{(2-\frac{2}{p})}_{p,M}) \tag{4.1}$$

Let $B(0,1)$ be the open ball in $\mathbb{R}^{n-1}$ of radius 1 centered at the origin. Now, M is a C^2 manifold. Therefore, for each point $\xi\in M$ there exists an open set V_ξ of M containing ξ and a C^2 diffeomorphism $\phi_\xi: B(0,1)\xrightarrow{\text{onto}} V_\xi$. Let $\Phi=\Psi\circ\phi_\xi$, $\tilde{f}=f\circ\phi_\xi$ and $\Phi_0=\Psi_0\circ\phi_\xi$. Using the Laplace Beltrami operator (defined in [30]), (3.2) takes the form

$$\begin{aligned}&\Phi_t=\frac{\tilde{d}}{\sqrt{det\ g}}\partial_j(g^{ij}\sqrt{det\ g}\ \partial_i\Phi)+\tilde{f}(x,t) && x\in B(0,1),\quad 0<t<T\\ &\Phi=\Phi_0 && x\in B(0,1),\quad t=0\end{aligned} \tag{4.2}$$

where g is the metric on M and g^{ij} is the i,j^{th} entry of the inverse of the matrix corresponding to metric g. That is, in the bounded region $B(0,1)\times(0,T)$, we have

$$\mathcal{L}(\Phi)=\Phi_t-\sum_{i,j=1}^{n-1}a_{ij}\Phi_{x_ix_j}+\sum_{i=1}^{n-1}a_i\Phi_{x_i}=\tilde{f} \tag{4.3}$$

$$\Phi\big|_{t=0}=\Phi_0 \tag{4.4}$$

where,

$$a_{ij}=\tilde{d}\ g^{ij}$$

$$a_i=\frac{-\tilde{d}}{\sqrt{det\ g}}\partial_j(g^{ij}\sqrt{det\ g})$$

Note $\Psi\in W_p^{2,1}(M_T)$ implies $\Phi\in W_p^{2,1}(B(0,1)\times(0,T))$. Take $0<2r<1$ and define a cut off function $\psi\in C_0^\infty(\mathbb{R}^{n-1},[0,1])$ such that,

$$\psi(x)=\begin{cases}1 & \forall x\in B(0,r)\\ 0 & \forall x\in\mathbb{R}^{n-1}\backslash B(0,2r)\end{cases} \tag{4.5}$$

In $Q=B(0,2r)$, $Q_T=B(0,2r)\times(0,T)$ and $S_T=\partial B(0,r)\times(0,T)$, $w=\psi\Phi$ satisfies the equation

$$\begin{aligned}\frac{\partial w}{\partial t}-\sum_{i,j=1}^{n-1}a_{ij}\frac{\partial^2 w}{\partial x_i\partial x_j}+\sum_{i=1}^{n-1}a_i\frac{\partial w}{\partial x_i}&=\theta && (x,t)\in Q_T\\ w&=0 && (x,t)\in S_T\\ w\big|_{t=0}&=\psi\Phi_0 && t=0,x\in Q\end{aligned}$$

where,

$$\theta = \tilde{f}\psi - 2\sum_{i=1}^{n-1} a_{ij}\frac{\partial \Phi}{\partial x_i}\frac{\partial \psi}{\partial x_j} - \Phi\sum_{i,j=1}^{n-1} a_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \Phi\sum_{i=1}^{n-1} a_i\frac{\partial \psi}{\partial x_i}$$

Since $\psi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$ and $\Phi \in W_p^{2,1}(B(0,1)\times(0,T))$, therefore $\theta - \tilde{f}\psi \in W_p^{1,1}(Q_T)$.

Case 1. Suppose $p < n$. From Lemma 2.4, $\theta - \tilde{f}\psi \in L_{\min\{q,\ p+\frac{p^2}{n-p}\}}(Q_T)$. In particular since $p + \frac{1}{n-1} < p + \frac{p^2}{n-p}$, and $\tilde{f}\psi \in L_q(Q_T)$, we have $\theta \in L_q(Q_T)$. As a result

$$\begin{aligned}\|\theta\|_{q,Q_T} &\le \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_2\|\Phi_x\|_{q,Q_T}\\ &\le \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_2\|\Phi_x\|_{p,Q_T}^{(1)}\end{aligned}$$

where $C_1, C_2 > 0$ are independent of f. Now in order to estimate $\|\Phi_x\|_{p,Q_T}^{(1)}$, apply the change of variable

$$\|\Phi_x\|_{p,Q_T}^{(1)} = \|\Psi_x|\det((\phi_\xi^{-1})')|\|_{p,(\phi_\xi(Q))_T}^{(1)}$$

and using (4.1), we get

$$\|\theta\|_{q,Q_T} \le \|\tilde{f}\psi\|_{q,Q_T} + C_1\|\Phi\|_{q,Q_T} + C_{2p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})})$$

where $C_{2p,T} > 0$ is independent of f and Ψ_0. At this point, we need an estimate on $\|\Phi\|_{q,Q_T}$. Again $\|\Phi\|_{q,Q_T} = \|\Psi|\det(({\phi_\xi}^{-1})')|\|_{q,(\phi_\xi(Q))_T}$ and from Lemma 2.4,

$$\|\Psi|\det(({\phi_\xi}^{-1})')|\|_{q,(\phi_\xi(Q))_T} \le \tilde{C}\|\Psi|\det(({\phi_\xi}^{-1})')|\|_{p,(\phi_\xi(Q))_T}^{(1)}$$

Thus

$$\|\theta\|_{q,Q_T} \le K_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})}) \tag{4.6}$$

where $K_{p,T} > 0$ is independent of f and Ψ_0.

Since $g_{i,j}$ are C^1 functions on the compact manifold M, $a_{i,j}$ and a_i satisfy the hypothesis (bounded continuous function in $\overline{Q_T}$) of Lemma 2.1. Therefore using Lemma 2.1,

$$\|w\|_{q,Q_T}^{(2)} \le C_{q,T}(\|\theta\|_{q,Q_T} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})}) \tag{4.7}$$

where $C_{q,T} > 0$ is independent of θ and $\psi\Phi_0$. Combining (4.6) and (4.7) we get,

$$\begin{aligned}\|w\|_{q,Q_T}^{(2)} &\le C_{q,T}(\|\theta\|_{q,Q_T} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})})\\ &\le \tilde{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})})\end{aligned}$$

where $\tilde{K}_{p,T} > 0$ is independent of f, θ and $\psi\Phi_0$. Note that $w = \Phi$ on $W_T = B(0,r)\times(0,T)$. Thus

$$\|\Phi\|_{q,W_T}^{(2)} \le \tilde{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|_{p,M}^{(2-\frac{2}{p})} + \|\psi\Phi_0\|_{q,Q}^{(2-\frac{2}{q})}) \tag{4.8}$$

Observe (4.8) is over $B(0,r) \times (0,T) \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. To get the estimate back on the manifold, apply the change of variable, $\|\Phi\|^{(2)}_{q,W_T} = \|\Psi|\det((\phi^{-1})')|\|^{(2)}_{q,\phi(W_T)}$ and using first mean value theorem of integration there exist $\hat{\xi} \in \phi(W_T)$, and $\tilde{K}_{p,T,\hat{\xi}}$ such that

$$\|\Psi\|^{(2)}_{q,\phi(W_T)} \leq \tilde{K}_{p,T,\hat{\xi}}(\|f\|_{p,M_T} + \|\Psi_0\|^{(2-\frac{2}{p})}_{p,M} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,\phi(Q)}) \tag{4.9}$$

So far, an estimate in one open neighborhood of some point $\xi \in M$ is obtained. As one varies the point ξ on M, there exist corresponding open neighborhoods V_ξ and a smooth diffemorphisms $\phi_\xi : B(0,r) \longrightarrow V_\xi$, which results in different $\tilde{K}_{p,T,\hat{\xi}}$ for every V_ξ. Consider an open cover of M such that $M = \bigcup_{\xi \in M} V_\xi$. Since M is compact, there exists $\{\xi_1, \xi_2, ..., \xi_N\}$ such that $M \subset \bigcup_{\substack{\xi_j \in M \\ 1 \leq j \leq N}} V_{\xi_j}$ and $\tilde{K}_{p,T,\hat{\xi}_j}$ corrresponding to each V_{ξ_j}. Let, $C_{p,M,T} = \sum_{1 \leq j \leq N} \tilde{K}_{p,T,\hat{\xi}_j}$. Inequality (4.9) implies

$$\|\Psi\|^{(2)}_{q,M_T} \leq C_{p,M,T}(\|f\|_{q,M_T} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,M})$$

Thus $[p, p + \frac{1}{n-1}] \subset \mathcal{F}$.

Case 2. Suppose $p \geq n$.

By Lemma 2.4 and Theorem 4.12 in [1], if $q \in [p, \infty)$, $\Psi_0 \in W_q^{2-\frac{2}{q}}(M)$, and $f \in L_q(M_T)$ then $\theta \in L_q(Q_T)$, and proceeding similarly to Case 1, we get

$$\|\Psi\|^{(2)}_{q,M_T} \leq C_{q,M,T}(\|f\|_{q,M_T} + \|\Psi_0\|^{(2-\frac{2}{q})}_{q,M})$$

where $C_{q,M,T} > 0$ is independent of f, θ and $\psi\Phi_0$. Hence $[2, \infty) \subset \mathcal{F}$.□

Proof of Theorem 3.5: From Lemma 4.4, we have $[2, \infty) \subset \mathcal{F}$. It remains to show that $(1, 2) \subset \mathcal{F}$. Let $1 < p < 2$, $f \in L_p(M_T)$ and $\Psi_0 \in W^{2-\frac{2}{p}}(M)$. Since $C^\infty(\overline{M_T})$ is dense in $L_p(M_T)$ and $C^\infty(\overline{M})$ is dense in $W_p^{2-\frac{2}{p}}(M)$, there exist a sequences of functions $\{f_k\} \subseteq C^\infty(\overline{M_T})$ and $\{\Psi_{0k}\} \subseteq C^\infty(\overline{M})$ such that f_k converges to f in $L_p(M_T)$ and Ψ_{0k} converges to Ψ_0 in $W_p^{2-\frac{2}{p}}(M)$. Define a sequence $\{\Psi_k\}$ such that,

$$\begin{aligned} \Psi_{k_t} &= \tilde{d}\Delta_M \Psi_k + f_k & \xi \in M, \quad 0 < t < T \\ \Psi_k &= \Psi_{0k} & \xi \in M, \quad t = 0 \end{aligned} \tag{4.10}$$

Now, transform system (4.10) over a bounded region in $\mathbb{R}^{n-1}$. Similar to the proof of Lemma 4.4, for each point $\xi \in M$ there exists an open set V_ξ of M containing ξ and a C^2 diffeomorphism $\phi_\xi : B(0,1) \xrightarrow{\text{onto}} V_\xi$. Corresponding to each k, let $\tilde{f}_k = f_k \circ \phi_\xi$, $\Phi_{0k} = \Psi_{0k} \circ \phi_\xi$ and using the Laplace Beltrami operator, (4.10) on $B(0,1) \subset U$ takes the form

$$\begin{aligned} \Phi_{kt} &= \frac{\tilde{d}}{\sqrt{det\ g}} \partial_j(g^{ij}\sqrt{det\ g}\ \partial_i \Phi_k) + \tilde{f}_k & x \in B(0,1), \quad 0 < t < T \\ \Phi_k &= \Phi_{0k} & x \in B(0,1), \quad t = 0 \end{aligned} \tag{4.11}$$

Consequently, in a bounded region $B(0,1) \times (0,T)$ of the Euclidean space, we consider (4.11) in the nondivergence form defined in (4.3) for each Φ_k, with $\tilde{f}$ replaced by $\tilde{f}_k$ and Φ_0 by Φ_{0k}.

Taking $0 < 2r < 1$, using a cut off function $\psi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$ defined in (4.5), and defining $Q = B(0, 2r)$, $Q_T = B(0, 2r) \times (0, T)$, $S_T = \partial B(0, r) \times (0, T)$, and $w_k = \psi\Phi_k$, we see that

$$\begin{aligned} \frac{\partial w_k}{\partial t} - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 w_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} a_i \frac{\partial w_k}{\partial x_i} &= \theta_k && (x,t) \in Q_T \\ w_k &= 0 && (x,t) \in S_T \\ w_k\big|_{t=0} &= \psi\Phi_{0k} && t = 0, x \in Q \end{aligned}$$

where,

$$\theta_k = \tilde{f}_k \psi - 2\sum_{i=1}^{n-1} a_{ij} \frac{\partial \Phi_k}{\partial x_i}\frac{\partial \psi}{\partial x_j} - \Phi_k \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \Phi_k \sum_{i=1}^{n-1} a_i \frac{\partial \psi}{\partial x_i}$$

Note that f_k and Ψ_{0k} are smooth functions. Therefore Lemma 4.4 guarantees $\Phi_k \in W_q^{2,1}(Q_T)$ for all $q \geq 2$. Thus $\theta_k \in L_q(Q_T)$ for all $q \geq 2$. Recall $\psi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$. Using Lemma 2.7 for $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\begin{aligned} \|\theta_k\|_{p,Q_T} &\leq \|\tilde{f}_k \psi\|_{p,Q_T} + M_1 \|\Phi_k\|_{p,Q_T} + M_2 \|\Phi_{kx}\|_{p,Q_T} \\ &\leq \|\tilde{f}_k\|_{p,Q_T} + M_1 \|\Phi_k\|_{p,Q_T} \\ &\quad + M_2(\epsilon \|\Phi_{kxx}\|_{p,Q_T} + c_\epsilon \|\Phi_k\|_{p,Q_T}) \end{aligned} \tag{4.12}$$

Here $M_1, M_2 > 0$ are independent of f and Ψ_0. At this point we need an estimate for $\|\Phi_k\|_{p,Q_T}$. From Lemma 2.4 for $1 < p \leq n < q$ there exists $C_\epsilon > 0$ such that

$$\|\Phi_k\|^p_{L_{\frac{pq}{q-p}}(Q_T)} \leq \epsilon(\|\Phi_{kx}\|^p_{p,Q_T} + \|\Phi_{kt}\|^p_{p,Q_T}) + C_\epsilon \|\Phi_k\|^p_{1,Q_T}$$

Since $p < \frac{pq}{q-p}$, from Hölder's inequality, ϵ and C_ϵ get scaled to $\tilde{\epsilon} > 0$ and $C_{\tilde{\epsilon}} > 0$ (with $\tilde{\epsilon} \to 0^+$ as $\epsilon \to 0^+$), and

$$\begin{aligned} \|\Phi_k\|_{p,Q_T} &\leq \tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) \\ &\quad + C_{\tilde{\epsilon}} \|\Phi_k\|_{1,Q_T} \end{aligned} \tag{4.13}$$

From (4.12) and (4.13),

$$\begin{aligned} \|\theta_k\|_{p,Q_T} &\leq (M_1 + M_2 c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}} \|\Phi_k\|_{1,Q_T}) \\ &\quad + \|\tilde{f}_k\|_{p,Q_T} + M_2 \epsilon \|\Phi_{kxx}\|_{p,Q_T} \end{aligned}$$

Recall $g_{i,j}$ are C^1 functions on the compact manifold M. Therefore $a_{i,j}$ and a_i satisfy the hypothesis (bounded continuous function in $\overline{Q_T}$) of Lemma 2.1. Using Lemma 2.1 for $p \neq \frac{3}{2}$,

$$\|w_k\|^{(2)}_{p,Q_T} \leq C_{p,T}(\|\theta_k\|_{p,Q_T} + \|\psi\Phi_{0k}\|^{(2-\frac{2}{p})}_{p,Q}) \tag{4.14}$$

where $C_{p,T}$ is independent of θ and $\psi\Phi_0$. Combining (4.12) and (4.14), we get

$$\begin{aligned} \|w_k\|^{(2)}_{p,Q_T} &\leq C_{p,T}(\|\theta_k\|_{p,Q_T} + \|\psi\Phi_{0k}\|^{(2-\frac{2}{p})}_{p,Q}) \\ &\leq C_{p,T}\{\|\tilde{f}_k\|_{p,Q_T} + M_2 \epsilon \|\Phi_{kxx}\|_{p,Q_T} \\ &\quad + (M_1 + M_2 c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}} \|\Phi_k\|_{1,Q_T}) \\ &\quad + \|\psi\Phi_{0k}\|^{(2-\frac{2}{p})}_{p,Q}\} \end{aligned}$$

Note that $w_k = \Phi_k$ on $W_T = B(0,r) \times (0,T)$. Thus

$$\begin{aligned} \|\Phi_k\|^{(2)}_{p,W_T} \le C_{p,T}\{&\|\tilde{f}_k\|_{p,Q_T} + M_2\epsilon\|\Phi_{kxx}\|_{p,Q_T} \\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Phi_{kt}\|_{p,Q_T} + \|\Phi_{kx}\|_{p,Q_T}) + C_{\tilde{\epsilon}}\|\Phi_k\|_{1,Q_T}) \\ &+ \|\psi\Phi_{0k}\|^{(2-\frac{2}{p})}_{p,Q}\} \end{aligned} \tag{4.15}$$

Observe (4.15) is over $B(0,r) \times (0,T) \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. To get an estimate on the manifold, apply the change of variable, $\|\Phi_k\|^{(2)}_{p,W_T} = \|\Psi_k|\det((\phi^{-1})')|\|^{(2)}_{p,\phi(W_T)}$ and using first mean value theorem of integration there exist $\hat{\xi} \in \phi(W_T)$, and $\tilde{C}_{p,T,\hat{\xi}}$ such that

$$\begin{aligned} \|\Psi_k\|^{(2)}_{p,\phi(W_T)} \le \tilde{C}_{p,\xi,T}\{&\|f_k\|_{p,\phi(Q_T)} + M_2\epsilon\|\Psi_{kxx}\|_{p,\phi(Q_T)} \\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Psi_{kt}\|_{p,\phi(Q_T)} + \|\Psi_{kx}\|_{p,\phi(Q_T)}) + C_{\tilde{\epsilon}}\|\Psi_k\|_{1,(\phi(Q_T))}) \\ &+ \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,\phi(Q)}\} \end{aligned} \tag{4.16}$$

So, an estimate in an open neighborhood of a point $\xi \in M$ can be obtained. As one varies the point ξ on M, there exist corresponding open neighborhoods V_ξ and a smooth diffemorphisms $\phi_\xi : B(0,r) \longrightarrow V_\xi$, which result in different $\tilde{C}_{p,\hat{\xi},T}$ for every V_ξ. Consider an open cover of M such that $M = \bigcup_{\xi \in M} V_\xi$. Since M is compact, there exists $\{\xi_1, \xi_2, ..., \xi_N\}$ such that $M \subset \bigcup_{\substack{\xi_j \in M \\ 1 \le j \le N}} V_{\xi_j}$ and $\tilde{C}_{p,\hat{\xi}_j,T}$ corrresponding to each V_{ξ_j}. Let $\hat{C}_{p,T} = \sum_{1 \le j \le N} \tilde{C}_{p,\hat{\xi}_j,T}$. Inequality (4.16) implies

$$\begin{aligned} \|\Psi_k\|^{(2)}_{p,M_T} \le \hat{C}_{p,T}\{&\|f_k\|_{p,M_T} + M_2\epsilon\|\Psi_{kxx}\|_{p,M_T} \\ &+ (M_1 + M_2c_\epsilon)(\tilde{\epsilon}(\|\Psi_{kt}\|_{p,M_T} + \|\Psi_{kx}\|_{p,M_T}) + C_{\tilde{\epsilon}}\|\Psi_k\|_{1,M_T}) \\ &+ \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,M}\} \end{aligned} \tag{4.17}$$

Also, a simple calculation gives

$$\|\Psi_k\|_{1,M_T} \le \|f_k\|_{1,M_T} + \|\Psi_{0k}\|_{1,M}$$

Now, choose $\epsilon > 0$ such that,

$$\max\{\hat{C}_{p,T}M_2\epsilon, \quad \hat{C}_{p,T}\tilde{\epsilon}(M_1 + M_2c_\epsilon)\} < \frac{1}{2}$$

For this choice of ϵ, (4.17) gives the $W^{2,1}_p$ estimates

$$\|\Psi_k\|^{(2)}_{p,M_T} \le \hat{C}_{p,T}(\|f_k\|_{p,M_T} + C_\epsilon(\|f_k\|_{1,M_T} + \|\Psi_{0k}\|_{1,M}) + \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,M})$$

$$\|\Psi_k\|^{(2)}_{p,M_T} \le \hat{K}_{p,T}(\|f_k\|_{p,M_T} + \|\Psi_{0k}\|^{(2-\frac{2}{p})}_{p,M}) \tag{4.18}$$

where $\hat{K}_{p,T} > 0$ is independent of f_k and Ψ_{0k}. It remains to show that the sequence $\{\Psi_k\}$ converges to a function Ψ in $W^{2,1}_p(M_T)$, and Ψ solves (3.2). From linearity and (4.18), if $m, l \in \mathbb{N}$ then $\Psi_m - \Psi_l$ satisfies

$$\begin{aligned} (\Psi_m - \Psi_l)_t &= \tilde{d}\Delta_M(\Psi_m - \Psi_l) + f_m - f_l & \xi \in M, \quad 0 < t < T \\ \Psi_m - \Psi_l &= \Psi_{0m} - \Psi_{0l} & \xi \in M, \quad t = 0 \end{aligned}$$

and

$$\|\Psi_m - \Psi_l\|^{(2)}_{p,M_T} \le \hat{K}_{p,T}(\|f_m - f_l\|_{p,M_T} + \|\Psi_{0m} - \Psi_{0l}\|^{(2-\frac{2}{q})}_{p,M})$$

This implies $\{\Psi_k\}$ is a Cauchy sequence in $W^{2,1}_p(M_T)$, so there is a function $\psi \in W^{2,1}_p(M_T)$ such that $\Psi_k \to \Psi$. Then f_k converges to f in $L_p(M_T)$, Ψ_{0k} converges to Ψ_0 in $W_p^{2-\frac{2}{p}}(M)$, and Ψ_k converges to $\Psi \in W^{2,1}_p(M_T)$. Therefore Ψ solves (3.2), and (4.18) implies

$$\|\Psi\|^{(2)}_{p,M_T} \le \hat{K}_{p,T}(\|f\|_{p,M_T} + \|\Psi_0\|^{(2-\frac{2}{p})}_{p,M})$$

Hence $\mathcal{F} = (1,\infty)$, and the proof of Theorem 3.5 is complete.

5. Hölder Estimates for the Neumann problem. The following result is a version of Theorem 9.1 with Neumann boundary conditions, referred to in chapter 4 of [19] on page 351.

LEMMA 5.1. *Let $p > 1$. Suppose $\theta \in L_p(\Omega \times (0,T))$, $\varphi_0 \in W_p^{(2-\frac{2}{p})}(\Omega)$ and $\gamma \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(M \times (0,T))$ with $p \neq 3$. In addition, when $p > 3$ assume*

$$d\frac{\partial \varphi_0}{\partial \eta} = \gamma \quad on\ M \times \{0\}$$

Then (3.3) has a unique solution $\varphi \in W^{2,1}_p(\Omega \times (0,T))$ and there exists C dependent upon Ω, p, T, and independent of θ, φ_0 and γ such that

$$\|\varphi\|^{(2)}_{p,(\Omega\times(0,T))} \le C(\|\theta\|_{p,(\Omega\times(0,T))} + \|\varphi_0\|^{(2-\frac{2}{p})}_{p,\Omega} + \|\gamma\|^{(1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p})}_{p,(\partial\Omega\times(0,T))})$$

DEFINITION 5.2. *φ is said to be a weak solution of system (3.3) from $V_2^{1,\frac{1}{2}}(\Omega_T)$ if and only if*

$$-\int_0^T\int_\Omega \varphi\nu_t - \int_0^T\int_{\partial\Omega} d\ \nu\frac{\partial\varphi}{\partial\eta} + \int_0^T\int_\Omega d\ \nabla\nu.\nabla\varphi - \int_0^T\int_\Omega \theta\nu = \int_\Omega \nu(x,0)\varphi(x,0)$$

for any $\nu \in W^{1,1}_2(\Omega_T)$ that is equal to zero for $t = T$.

We also need a notion of solution of (1.3) which was first introduced in the study of Dirichlet and Neumann problems for the Laplace operator in a bounded C^1 domain by Fabes, Jodeit and Rivier [9]. They used Calderon's result in [5] on L^p continuity of Cauchy integral operators for C^1 curves. Further in [10], Fabes and Riviere constructed solutions to the initial Neumann problem for the heat equation satisfying the zero initial condition in the form of a single layer heat potential, when densities belong to $L_p(M \times (0,T))$, $1 < p < \infty$. We will consider the solution to (1.3) in the sense of one which is constructed in [10].

The following result plays a crucial role for that construction of solution to make sense, and is proved in [10].

PROPOSITION 5.3. *Assume Ω is a C^1 domain and for $Q \in M$, η_Q being the unit outward normal to M at Q. For $0 < \epsilon < t$ set*

$$J_\epsilon(f)(Q,t) = \int_0^{t-\epsilon}\int_M \frac{\langle y - Q, \eta_Q\rangle}{(t-s)^{\frac{n}{2}+1}} \exp\left(-\frac{|Q-y|^2}{4(t-s)}\right) f(s,y)\ d\sigma\ ds$$

Then

1. *For every* $1 < p < \infty$ *there exists* $C_p > 0$ *such that* $\sup_{0<\epsilon<t} |J_\epsilon(f)(Q,t)| = J(f)(Q,t)$ *satisfies*

$$\|J(f)\|_{L_p(M\times(0,T))} \le C_p \|f\|_{L_p(M\times(0,T))} \text{ for all } f \in L_p(M\times(0,T))$$

2. $\lim_{\epsilon\to 0^+} J_\epsilon(f) = J(f)$ *exists in* $L_p(M\times(0,T))$ *and pointwise for almost every* $(Q,t) \in (M\times(0,T))$ *provided* $f \in L_p(M\times(0,T)), 1 < p < \infty$.
3. $c_n I + J$ *is invertible on* $L_p(M\times(0,T))$ *for each* $1 < p < \infty$ *and* $c_n \neq 0$.

We consider the case $d = 1$ below. The extension to arbitrary $d > 0$ is straightforward. For $Q \in M$, $(x,t) \in \Omega_T$ and $t > s$, consider

$$W(t-s,x,Q) = \frac{\exp\left(\frac{-|x-Q|^2}{4(t-s)}\right)}{(t-s)^{\frac{n}{2}}} \text{ and } g(Q,t) = -2[-c_n I + J]^{-1}\gamma(Q,t)$$

where c_n is given in [10].

Definition 5.4. *φ is said to be a classical solution of system (1.3) with $d = 1$ and, $\gamma \in L_p(M\times(0,T))$ for $p > 1$ if and only if*

$$\varphi(x,t) = \int_0^t \int_M W(t-s,x,Q)g(Q,s)\, d\sigma\, ds \text{ for all } (x,t) \in \Omega_T$$

Remark 4. *When $\theta = 0$ and $\varphi(x,0) = 0$, the weak solution of (3.3) is the same as the classical solution of (1.3).*

In order to prove the classical solution φ to (1.3) is Hölder continuous, let (x,T), $(y,\tau) \in \Omega_T$ such that

$$\varphi(x,T) = \int_0^T \int_M W(T-s,x,Q)g(Q,s)\, d\sigma\, ds$$

and

$$\varphi(y,\tau) = \int_0^\tau \int_M W(\tau-s,y,Q)g(Q,s)\, d\sigma\, ds$$

Without loss of generality we assume $0 < \tau < T$. Consider the difference

$$\varphi(x,T) - \varphi(y,\tau) = \int_0^\tau \int_M (W(T-s,x,Q) - W(\tau-s,y,Q))g(Q,s)\, d\sigma\, ds + \int_\tau^T \int_M W(T-s,x,Q)g(Q,s)\, d\sigma\, ds$$

Lemmas 5.5, 5.6 and 5.7 provide estimates needed to prove φ is Hölder continuous. Throughout the proofs we assume $p' = \frac{p}{p-1}$.

Lemma 5.5. *Let $p > n+1$. Suppose (x,T), $(y,\tau) \in \Omega_T$ with $0 < \tau < T$ and $\mathcal{R}^c = \{(Q,s) \in M\times(0,\tau) : |x-Q| + |T-s|^{\frac{1}{2}} < 2(|x-y| + |T-\tau|^{\frac{1}{2}})\}$. Then for $0 < a < 1 - \frac{n+1}{p}$ there exists $K_1 > 0$ depending on $p, n, \overline{\Omega}$, T and independent of $g \in L_p(M\times(0,T))$ such that*

$$\int_{\mathcal{R}^c} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(Q,s)|\, d\sigma\, ds \le K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^a \| g \|_{p,M\times[0,\tau]}$$

Proof.

$$\int_{\mathcal{R}^c} |(W(T-s,x,Q)-W(\tau-s,y,Q))g(Q,s)|\;d\sigma\;ds$$

$$= \int_{\mathcal{R}^c} \left| \frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}} - \frac{\exp\left(\frac{-|y-Q|^2}{4(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} \right| |g(Q,s)|\;d\sigma\;ds$$

$$\leq \left[\left(\int_{\mathcal{R}^c} \left(\frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}} \right)^{p'} \right)^{\frac{1}{p'}} + \left(\int_{\mathcal{R}^c} \left(\frac{\exp\left(\frac{-|y-Q|^2}{4(\tau-s)}\right)}{(\tau-s)^{\frac{n}{2}}} \right)^{p'} \right)^{\frac{1}{p'}} \right] \|g\|_{p,\mathcal{R}^c}$$

By hypothesis $p > n+1$. Pick $0 < \epsilon < \frac{p-(n+1)}{p-1}$, set $N = \frac{n-1-\epsilon}{2}$. Then there exists $c > 0$ such that $w^N \cdot \exp(-w) \leq c \cdot N$ for all $w \geq 0$. Consequently,

$$\left[\left(\int_{\mathcal{R}^c} \frac{\exp\left(\frac{-p'|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{np'}{2}}} \right)^{\frac{1}{p'}} + \left(\int_{\mathcal{R}^c} \frac{\exp\left(\frac{-p'|y-Q|^2}{4(\tau-s)}\right)}{(\tau-s)^{\frac{np'}{2}}} \right)^{\frac{1}{p'}} \right] \|g\|_{p,\mathcal{R}^c}$$

$$\leq \left[\left(\int_{\mathcal{R}^c} \frac{c \cdot N}{(T-s)^{\frac{np'}{2}} \left(\frac{p'|x-Q|^2}{4(T-s)}\right)^N} \right)^{\frac{1}{p'}} + \left(\int_{\mathcal{R}^c} \frac{c \cdot N}{(\tau-s)^{\frac{np'}{2}} \left(\frac{p'|y-Q|^2}{4(\tau-s)}\right)^N} \right)^{\frac{1}{p'}} \right] \|g\|_{p,\mathcal{R}^c}$$

$$\leq \Bigg[C_1 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|x-Q|^{n-1-\epsilon}}\;d\sigma \right)^{\frac{1}{p'}}$$

$$+C_2 \left(\int_0^\tau (\tau-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|y-Q|^{n-1-\epsilon}}\;d\sigma \right)^{\frac{1}{p'}} \Bigg] \parallel g \parallel_{p,\mathcal{R}^c}$$

where $A = \{Q \in M : |x-Q| < 2|x-y| + |T-\tau|^{\frac{1}{2}}\}$. Since $|T-\tau| < |T-s|, \mathcal{R}^c \subset A \times (0,\tau)$. Let $\rho_y = |y-Q|$, $\rho_x = |x-Q|$. Notice that in A, $0 < \rho_x < 2|x-y| + |T-\tau|^{\frac{1}{2}}$ and $0 < \rho_y < |x-y| + \rho_x < 3|x-y| + |T-\tau|^{\frac{1}{2}}$. Therefore,

$$\Bigg[C_1 \left(\int_0^\tau (\tau-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|y-Q|^{n-1-\epsilon}} d\sigma \right)^{\frac{1}{p'}}$$

$$+C_2 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_A \frac{1}{|x-Q|^{n-1-\epsilon}} d\sigma \right)^{\frac{1}{p'}} \Bigg] \parallel g \parallel_{p,\mathcal{R}^c}$$

$$\leq \Bigg[\tilde{C}_1 \left(\int_0^\tau (\tau-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_0^{3|x-y|+|T-\tau|^{\frac{1}{2}}} r^{\epsilon-1} dr \right)^{\frac{1}{p'}}$$

$$+\tilde{C}_2 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \int_0^{2|x-y|+|T-\tau|^{\frac{1}{2}}} r^{\epsilon-1} dr \right)^{\frac{1}{p'}} \Bigg] \|g\|_{p,\mathcal{R}^c}$$

$$\leq \left[\frac{\tilde{C}_1}{\epsilon^{\frac{1}{p'}}} (\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \left(3|x-y| + |T-\tau|^{\frac{1}{2}}\right)^{\frac{\epsilon}{p'}} \right.$$
$$\left. + \frac{\tilde{C}_2}{\epsilon^{\frac{1}{p'}}} \left(T^{\frac{n+1-\epsilon-np'}{2}} - (T-\tau)^{\frac{n+1-\epsilon-np'}{2}}\right)^{\frac{1}{p'}} \left(2|x-y| + |T-\tau|^{\frac{1}{2}}\right)^{\frac{\epsilon}{p'}} \right] \|g\|_{p,\mathcal{R}^c}$$

By hypothesis, $p' < \frac{n+1-\epsilon}{n}$. Therefore, there exists $K_1 > 0$ depends on p, n and T such that

$$\int_{\mathcal{R}^c} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(Q,s)| \; d\sigma \; ds$$
$$\leq K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^{\frac{\epsilon(p-1)}{p}} \| \; g \; \|_{p,M\times[0,\tau]}.$$

The result follows since $0 < \epsilon < \frac{ap}{p-1}$ is arbitrary.□

The proof of the following Lemma makes use of Brown's corollary to Theorem 3.1 in [4]. This also provides a proof for the remark made in [4] after Lemma 3.4.

LEMMA 5.6. *Let $p > n+1$. Suppose (x,T),$(y,\tau) \in \Omega_T$ and $\mathcal{R} = \{(Q,s) \in M \times (0,\tau) : 2(|x-y| + |T-\tau|^{\frac{1}{2}}) < |x-Q| + |T-s|^{\frac{1}{2}}\}$. Then for $0 < a < 1 - \frac{n+1}{p}$ there exists $K_2 > 0$ depending on $p, n, \overline{\Omega}$, T and independent of $g \in L_p(M \times (0,T))$ such that,*

$$\int_{\mathcal{R}} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(Q,s)| \; d\sigma \; ds$$
$$\leq K_2 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^a \| \; g \; \|_{p,M\times[0,\tau]} \; .$$

Proof. Using the Theorem 3.1 in [4], we have

$$\int_{\mathcal{R}} |(W(T-s,x,Q) - W(\tau-s,y,Q))g(Q,s)| \; d\sigma \; ds$$
$$\leq \int_{\mathcal{R}} C \left(\frac{|T-\tau|^{\frac{1}{2}} + |x-y|}{|T-s|^{\frac{1}{2}} + |x-Q|}\right) (1 + (T-s)^{\frac{-n}{2}}) \exp\left(\frac{-|x-Q|^2}{4(T-s)}\right) |g(Q,s)| \; d\sigma \; ds$$
$$\leq D_1 \left(\frac{1}{2}\right)^{1-a} \int_{\mathcal{R}} \left(\frac{|T-\tau|^{\frac{1}{2}} + |x-y|}{|T-s|^{\frac{1}{2}} + |x-Q|}\right)^a \frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}} |g(Q,s)| \; d\sigma \; ds$$
$$\leq \tilde{D}_1 \int_{\mathcal{R}} \frac{1}{|x-Q|^a} \frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}} |g(Q,s)| \; d\sigma \; ds$$

where $D_1 = C(T^{\frac{n}{2}} + 1)$ and $\tilde{D}_1 = D_1 \left(\frac{1}{2}\right)^{1-a} \left(|T-\tau|^{\frac{1}{2}} + |x-y|\right)^a$. By hypothesis, $n + 1 - (n+a)p' > 0$. Pick $0 < \epsilon < (n+1) - (n+a)p'$ and set $N = \frac{n-1-\epsilon-ap'}{2}$. Then there exists $c > 0$

such that $w^N \cdot \exp(-w) \leq c \cdot N$ for all $w \geq 0$. Consequently,

$$
\begin{aligned}
&\tilde{D}_1 \left(\int_{\mathcal{R}} \frac{1}{|x-Q|^{ap'}} \frac{\exp\left(\frac{-p'|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{np'}{2}}} \, d\sigma \, ds \right)^{\frac{1}{p'}} \| g \|_{p,\mathcal{R}} \\
&\leq \tilde{D}_1 \left(\int_{\mathcal{R}} \frac{1}{|x-Q|^{ap'}} \frac{c \cdot N}{(T-s)^{\frac{np'}{2}} \left(\frac{p'|x-Q|^2}{4(T-s)}\right)^N} \right)^{\frac{1}{p'}} \| g \|_{p,\mathcal{R}} \\
&\leq \tilde{c}\tilde{D}_1 \left(\int_0^\tau \int_M \frac{(T-s)^{\frac{n-1-\epsilon-ap'}{2}}}{(T-s)^{\frac{np'}{2}}} \frac{1}{|x-Q|^{n-1-\epsilon}} \, d\sigma \, ds \right)^{\frac{1}{p'}} \| g(s,Q) \|_{p,M\times[0,\tau]} \\
&\leq \tilde{c}\tilde{D}_1 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-ap'-np'}{2}} \, ds \cdot \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} \, d\sigma \right)^{\frac{1}{p'}} \| g \|_{p,M\times[0,\tau]}
\end{aligned}
$$

Then by change of variable, there exists $C, \alpha > 0$ such that

$$
\begin{aligned}
&\tilde{D}_1 \left(\int_0^\tau (T-s)^{\frac{n-1-\epsilon-ap'-np'}{2}} \, ds \cdot \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} \, d\sigma \right)^{\frac{1}{p'}} \| g \|_{p,M\times[0,\tau]} \\
&\leq C\tilde{D}_1 \left((T)^{\frac{n-1-\epsilon-ap'-np'}{2}+1} \cdot \int_0^\alpha \frac{1}{r^{1-\epsilon}} dr \right)^{\frac{1}{p'}} \| g \|_{p,M\times[0,\tau]}
\end{aligned}
$$

The result follows. ∎

LEMMA 5.7. *Let $p > n+1$, and suppose $(x,T),(y,\tau) \in \Omega_T$. Then for $0 < a < \frac{1}{2} - \frac{n+1}{2p}$ there exists $K_3 > 0$, depending on $p, n, \overline{\Omega}$ and T, and independent of $g \in L_p(M \times (0,T))$ such that,*

$$
\int_\tau^T \int_M |W(T-s,x,Q)g(Q,s)| \, d\sigma \, ds \leq K_3 (T-\tau)^a \, \| g \|_{p,M\times[\tau,T]}
$$

Proof. By hypothesis $p > n+1$. Pick $0 < \epsilon < n+1-np'$ and set $N = \frac{n-1-\epsilon}{2}$. Then there exists $c > 0$ such that $w^N \cdot \exp(-w) \leq c \cdot N$ for all $w \geq 0$. Consequently,

$$
\begin{aligned}
&\int_\tau^T \int_M |W(T-s,x,Q)g(Q,s)| \, d\sigma \, ds \\
&\leq \int_\tau^T \int_M \frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}} |g(Q,s)| \, d\sigma \, ds \\
&\leq C_3 \int_\tau^T \int_M \frac{\tilde{C}(T-s)^{\frac{n-1-\epsilon}{2}}}{(T-s)^{\frac{n}{2}}} \cdot \frac{1}{|x-Q|^{n-1-\epsilon}} |g(Q,s)| \, d\sigma \, ds \\
&\leq C_3 \left(\int_\tau^T (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \cdot \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} d\sigma \right)^{\frac{1}{p'}} \| g \|_{p,M\times[\tau,T]}
\end{aligned}
$$

Similarly, by change of variable there exist $\tilde{C}_3, \alpha > 0$ such that

$$\begin{aligned} & C_3 \left(\int_\tau^T (T-s)^{\frac{n-1-\epsilon-np'}{2}} ds \cdot \int_M \frac{1}{|x-Q|^{n-1-\epsilon}} d\sigma \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[\tau,T]} \\ & \leq \tilde{C}_3 \left(|(T-\tau)^{\frac{n-1-\epsilon-np'}{2}+1}| \cdot \int_0^\alpha \frac{1}{r^{1-\epsilon}} dr \right)^{\frac{1}{p'}} \parallel g \parallel_{p,M\times[\tau,T]} \\ & \leq K_3 (T-\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \parallel g \parallel_{p,M\times[\tau,T]} \end{aligned}$$

where $K_3 > 0$, depends on $p, n, \overline{\Omega}$ and T, and independent of $g \in L_p(M \times (0,T))$. The result follows since $0 < \epsilon < n+1-np'$ is arbitrary, and $\frac{n+1-np'}{2p'} = \frac{1}{2} - \frac{n+1}{2p}$. □

Proposition 5.8. *Suppose $\gamma \in L_p(M \times (0,T))$ for $p > n+1$. Then the classical solution of (1.3) is Hölder continuous on $\overline{\Omega} \times (0,\hat{T})$ with Hölder exponent $0 < a < 1 - \frac{n+1}{p}$, and there exists $\tilde{K}_p > 0$, depending on $p, n, \overline{\Omega}$ and T, and independent of γ such that*

$$|\varphi(x,T) - \varphi(y,\tau)| \leq \tilde{K}_p \Big(|T-\tau|^{\frac{1}{2}} + |x-y| \Big)^a \parallel \gamma \parallel_{p,M\times(0,T)}$$

for all $(x,T), (y,\tau) \in \Omega_T$.

Proof. We prove this proposition for $d = 1$. The extension to arbitrary $d > 0$ follows from a simple change of variables. Let $\tilde{\Omega}$ be an open subset of Ω with smooth boundary such that the closure of $\tilde{\Omega}$ is contained in Ω. It is straightforward matter to apply cut-off functions and Theorem 9.1 in [19] to obtain an estimate for φ in $W_p^{2,1}(\tilde{\Omega} \times (0,T))$. Moreover, there exists $L_{p,\tilde{\Omega},T}$ independent of γ such that

$$\|\varphi\|_{p,\tilde{\Omega}_T}^{(2)} \leq L_{p,\tilde{\Omega},T} \|\gamma\|_{p,M_T}$$

Since $p > n+1$, $W_p^{2,1}(\tilde{\Omega} \times (0,T))$ embeds continuously into the space of Hölder continuous functions (see [19]). As a result we have Hölder continuity of the solution to (1.3) away from M_T. We want to extend this behavior to points near M_T.

Pick points (x,T), $(y,\tau) \in \Omega_T$. We know from Fabes and Riviere [10] that the solution of (1.3) is given by

$$\varphi(x,T) = \int_0^T \int_M W(T-s,x,Q) g(Q,s) \; d\sigma \; ds$$

where $W(T-s,x,Q) = \frac{\exp\left(\frac{-|x-Q|^2}{4(T-s)}\right)}{(T-s)^{\frac{n}{2}}}$, $g(Q,t) = [I+J]^{-1}\gamma(Q,t)$ and

$$J(g)(Q,t) = \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \int_M \frac{\langle y-Q, \eta_Q \rangle}{(t-s)^{\frac{n}{2}+1}} \exp\left(-\frac{|Q-y|^2}{4(t-s)} \right) g(s,y) \; d\sigma \; ds$$

for almost every $Q \in M$ (for smooth manifold it is true for all Q), η_Q being the unit outward normal to M at Q.

$$
\begin{aligned}
|\varphi(x,T)-\varphi(y,\tau)| &= |\int_0^\tau \int_M (W(T-s,x,Q)-W(\tau-s,y,Q))g(Q,s)\, d\sigma\, ds \\
&\quad + \int_\tau^T \int_M W(T-s,x,Q)g(Q,s)\, d\sigma\, ds| \\
&\le |\int_{\mathcal{R}^c} (W(T-s,x,Q)-W(\tau-s,y,Q))g(Q,s)\, d\sigma\, ds| \\
&\quad + |\int_{\mathcal{R}} (W(T-s,x,Q)-W(\tau-s,y,Q))g(Q,s)\, d\sigma\, ds| \\
&\quad + \int_\tau^T \int_M C(1+(T-s)^{\frac{-n}{2}}) \exp\left(\frac{-|x-Q|^2}{4(T-s)}\right) |g(Q,s)|\, d\sigma\, ds
\end{aligned}
$$

Where $\mathcal{R}$ and $\mathcal{R}^c$ are given in Lemmas 5.6 and 5.7. Now using Lemma 5.5, Lemmas 5.6 and 5.7 for $0 < a < 1 - \frac{n+1}{p}$, there exists $K_1, K_2, K_3 > 0$ depending on $p, n, \overline{\Omega}$, T and independent of $g \in L_p(M \times (0,T))$, such that

$$
\begin{aligned}
|\varphi(x,T)-\varphi(y,\tau)| &\le K_1 \left(|x-y| + |T-\tau|^{\frac{1}{2}}\right)^a \parallel g \parallel_{p,M\times(0,\tau)} \\
&\quad + K_2 \Big(|T-\tau|^{\frac{1}{2}} + |x-y|\Big)^a \parallel g \parallel_{p,M\times(0,\tau)} \\
&\quad + K_3 (T-\tau)^{\frac{n+1-\epsilon-np'}{2p'}} \parallel g \parallel_{p,M\times(\tau,T)}
\end{aligned}
$$

So,

$$
|\varphi(x,T)-\varphi(y,\tau)| \le \tilde{K}_p \Big(|T-\tau|^{\frac{1}{2}} + |x-y|\Big)^a \parallel g \parallel_{p,M\times(0,T)}
$$

□

Now we combine Hölder estimates and Theorem 9.1 in chapter 4 of [19] to get the existence of a Hölder continuous solution to system (3.3) for any finite time $T > 0$.

Proof of Theorem 3.6: Chapter 4, Theorem 5.1 in [19] implies (3.3) has the unique weak solution. In order to get Hölder estimates, we break (3.3) into two sub systems. To this end, consider

$$
\begin{aligned}
& {\varphi_2}_t = d\Delta\varphi_2 + \theta && x \in \Omega, \quad 0 < t < T \\
& d\frac{\partial \varphi_2}{\partial \eta} = d\frac{\partial \varphi_0}{\partial \eta} && x \in M, \quad 0 < t < T \\
& \varphi_2 = \varphi_0 && x \in \Omega, \quad t = 0
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
& {\varphi_1}_t = d\Delta\varphi_1 && x \in \Omega, \quad 0 < t < T \\
& d\frac{\partial \varphi_1}{\partial \eta} = \gamma - d\frac{\partial \varphi_0}{\partial \eta} && x \in M, \quad 0 < t < T \\
& \varphi_1 = 0 && x \in \Omega, \quad t = 0
\end{aligned}
\tag{5.2}
$$

From Lemma 5.1 there exists a unique solution of (5.1) in $W_p^{2,1}(\Omega \times (0,T))$, and a constant $C_1(T,p) > 0$ independent of θ and φ_0 such that

$$\|\varphi_2\|^{(2)}_{p,\Omega\times(0,T)} \le C_1(T,p)(\|\theta\|_{p,\Omega\times(0,T)} + \|\frac{\partial \varphi_0}{\partial \eta}\|^{(1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p})}_{p,(\partial\Omega\times(0,T))}) + \|\varphi_0\|^{(2)}_{p,\Omega}$$

Using proposition 5.8, there exists $C_2(T,0) > 0$ independent of γ and φ_0 so that the unique weak solution to (5.2) satisfies,

$$|\varphi_1|^{(\beta)}_{\Omega\times(0,T)} \le C_2(T,p)\left[\|\gamma\|_{p,M\times(0,T)} + \|\frac{\partial\varphi_0}{\partial\eta}\|_{p,M\times(0,T)}\right]$$

where $0 < \beta < 1 - \frac{n+1}{p}$. By linearity, $\varphi = \varphi_1 + \varphi_2$ solves (3.3). Moreover, for $p > n+1$, $W_p^{2,1}(\Omega\times(0,T))$ embeds continuously into $C^{\beta,\frac{\beta}{2}}(\overline{\Omega}_T)$. So, there exists $C(T,p) > 0$ independent of θ, γ and φ_0 such that

$$|\varphi|^{(\beta)}_{\Omega\times(0,T)} \le C(T,p)(\|\theta\|_{p,\Omega\times(0,T)} + \|\gamma\|_{p,M\times(0,T)} + \|\varphi_0\|^{(2)}_{p,\Omega}) \tag{5.3}$$

REMARK 5. *We will use these Hölder estimates to obtain sup norm estimates, and local existence results for* (3.1).

6. Proof of Theorems 3.2 and 3.3.

6.1. Local Existence.

THEOREM 6.1. *Suppose F, G and H are Lipschitz. Then* (3.1) *has a unique global solution.*

Proof. Let $T > 0$, Fix $(u_0, v_0) \in W_p^2(\Omega) \times W_p^2(\Omega)$ such that they satisfy the compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M. \tag{6.1}$$

Set

$$X = \{(u,v) \in C(\overline{\Omega}\times[0,T]) \times C(M\times[0,T]) : u(x,0) = 0, \forall\, x\in\overline{\Omega}, v(x,0) = 0, \forall\, x \in M\}$$

Note $(X, \|\cdot\|_\infty)$ is a Banach space. Let $(u,v) \in X$. Now consider

$$\begin{aligned}
U_t &= D\Delta U + H(u+u_0) && x\in\Omega, \quad 0<t<T\\
V_t &= \tilde{D}\Delta_M V + F(u+u_0, v+v_0) && x\in M, \quad 0<t<T\\
D\frac{\partial U}{\partial\eta} &= G(u+u_0, v+v_0) && x\in M, \quad 0<t<T\\
U &= u_0 && x\in\Omega, \quad t=0\\
V &= v_0 && x\in M, \quad t=0
\end{aligned} \tag{6.2}$$

From Theorems 3.5 and 3.6, (6.2) possesses a unique weak solution $(U,V) \in V_2^{1,\frac{1}{2}}(\Omega_T) \times W_p^{2,1}(M_T)$. Furthermore, from embeddings, $(U,V) \in C(\overline{\Omega}\times[0,T]) \times C(M\times[0,T])$. Define

$$S : X \to X \text{ via } S(u,v) = (U - u_0, V - v_0),$$

where (U, V) solves (6.2). We will see that S is continuous and compact. Let (u, v), $(\tilde{u}, \tilde{v}) \in X$. Then

$$S(u, v) - S(\tilde{u}, \tilde{v}) = (U - \tilde{U}, V - \tilde{V})$$

Using linearity, $(U - \tilde{U}, V - \tilde{V})$ solves

$$\begin{aligned}
U_t - \tilde{U}_t &= D\Delta(U - \tilde{U}) + H(u + u_0) - H(\tilde{u} + u_0) && x \in \Omega, \quad 0 < t < T \\
V_t - \tilde{V}_t &= \tilde{D}\Delta_M(V - \tilde{V}) + F(u + u_0, v + v_0) - F(\tilde{u} + u_0, \tilde{v} + v_0) && x \in M, \quad 0 < t < T \\
D\frac{\partial(U - \tilde{U})}{\partial\eta} &= G(u + u_0, v + v_0) - G(\tilde{u} + u_0, \tilde{v} + v_0) && x \in M, \quad 0 < t < T \\
U - \tilde{U} &= 0 && x \in \Omega, \quad t = 0 \\
V - \tilde{V} &= 0 && x \in M, \quad t = 0
\end{aligned}$$

From Theorem 3.6, if $p > n + 1$ there exists K independent of $H, G, F, u, v, \tilde{u}, \tilde{v}$ such that

$$\begin{aligned}
\|U - \tilde{U}\|_{\infty,\Omega_T} + \|V - \tilde{V}\|_{\infty,M_T} \le K\,(&\|F(u + u_0, v + v_0) - F(\tilde{u} + u_0, \tilde{v} + v_0)\|_{p,M_T} \\
&+ \|G(u + u_0, v + v_0) - G(\tilde{u} + u_0, \tilde{v} + v_0)\|_{p,M_T} \\
&+ \|H(u + u_0) - H(\tilde{u} + u_0)\|_{p,\Omega_T})
\end{aligned}$$

Using the boundedness of Ω and M, there exists $\tilde{K} > 0$ such that

$$\begin{aligned}
\|U - \tilde{U}\|_{\infty,\Omega_T} + \|V - \tilde{V}\|_{\infty,M_T} \le \tilde{K}\,(&\|F(u + u_0, v + v_0) - F(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,M_T} \\
&+ \|G(u + u_0, v + v_0) - G(\tilde{u} + u_0, \tilde{v} + v_0)\|_{\infty,M_T} \\
&+ \|H(u + u_0) - H(\tilde{u} + u_0)\|_{\infty,\Omega_T})
\end{aligned}$$

Since, F, G, H are Lipschitz functions there exists $\tilde{M} > 0$ such that

$$\|U - \tilde{U}\|_{\infty,\Omega_T} + \|V - \tilde{V}\|_{\infty,M_T} \le \tilde{M}(\|u - \tilde{u}\|_{\infty,\overline{\Omega}_T} + \|v - \tilde{v})\|_{\infty,M_T})$$

Therefore S is continuous with respect to the sup norm. Moreover, for $p > n+1$, from Theorem 3.5, 3.6, and Lemma 2.6, there exists $\hat{C}(T, p) > 0$, independent of $F(u+u_0, v+v_0), G(u+u_0, v+v_0), H(u + u_0), u_0$ and v_0 such that for all $0 < \alpha < 1 - \frac{n}{p}$, $0 < \beta < 1 - \frac{n+1}{p}$,

$$\begin{aligned}
|U|^{(\beta)}_{\Omega_T} + |V|^{(\alpha)}_{M_T} \le \hat{C}(T, p)(&\|H(u + u_0)\|_{p,\Omega_T} + \|G(u + u_0, v + v_0)\|_{p,M_T} \\
&+ \|F(u + u_0, v + v_0)\|_{p,M_T} + \|v_0\|^{(2)}_{p,M} + \|u_0\|^{(2)}_{p,\Omega})
\end{aligned} \tag{6.3}$$

Using (6.3), S maps bounded sets in X to precompact sets, and hence S is compact with respect to the sup norm. Now we show S has a fixed point. To this end, we show that the set A=$\{(u, v) \in X : (u, v) = \lambda S(u, v)$ for some $0 < \lambda \le 1\}$ is bounded in X with respect to the sup norm. Let $(u, v) \in$ A. Then there exists $0 < \lambda \le 1$ such that $(\frac{u}{\lambda}, \frac{v}{\lambda}) = S(u, v)$. Therefore if $(\hat{u}, \hat{v}) = (u + \lambda u_0, v + \lambda v_0)$ then

$$\begin{aligned}
\hat{u}_t &= D\Delta\hat{u} + \lambda H(u + u_0) && x \in \Omega, \quad 0 < t < T \\
\hat{v}_t &= \tilde{D}\Delta_M\hat{v} + \lambda F(u + u_0, v + v_0) && x \in M, \quad 0 < t < T \\
D\frac{\partial\hat{u}}{\partial\eta} &= \lambda G(u + u_0, v + v_0) && x \in M, \quad 0 < t < T \\
\hat{u} &= \lambda u_0 && x \in \Omega, \quad t = 0 \\
\hat{v} &= \lambda v_0 && x \in M, \quad t = 0
\end{aligned}$$

From Theorem 3.6 and H, F and G being Lipschitz, there exists $N > 0$ such that $\|(\hat{u},\hat{v})\|_\infty \le N$, with N independent of λ, u and v. Since $\|(u,v)\|_\infty \le \|(\hat{u},\hat{v})\|_\infty \le N$, hence boundedness of the set is accomplished. Thus, applying Schaefer's theorem (see [8]), we conclude S has a fixed point (U,V). Further, $(U+u_0, V+v_0)$ is a solution of (3.1). Moreover, bootstrapping the regularity of this solution using well known estimates, we obtained a *solution* to (3.1) according to Definition 3.1.

Finally, we show the solution of (3.1) is unique. Suppose $(u,v), (\hat{u},\hat{v})$ solve (3.1). Then, $(u-\hat{u}, v-\hat{v})$ satisfies

$$\begin{aligned}
u_t - \hat{u}_t &= D\Delta(u-\hat{u}) + H(u) - H(\hat{u}) && x\in\Omega, \quad t>0\\
v_t - \hat{v}_t &= \tilde{D}\Delta_M(v-\hat{v}) + F(u,v) - F(\hat{u},\hat{v}) && x\in M, \quad t>0\\
D\frac{\partial(u-\hat{u})}{\partial\eta} &= G(u,v) - G(\hat{u},\hat{v}) && x\in M, \quad t>0\\
u-\hat{u} &= 0 && x\in\Omega, \quad t=0\\
v-\hat{v} &= 0 && x\in M, \quad t=0
\end{aligned}$$

Taking the dot product of the $v_t - \hat{v}_t$ equation with $(v-\hat{v})$, and the $u_t - \hat{u}_t$ equation with $(u-\hat{u})$, and integrating over M and Ω respectively, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2) + D\|\nabla(u-\hat{u})\|_{2,\Omega}^2\\
&\qquad\le \|v-\hat{v}\|_{2,M}\|F(u,v)-F(\hat{u},\hat{v})\|_{2,M} + \|u-\hat{u}\|_{2,\Omega}\|H(u)-H(\hat{u})\|_{2,\Omega}\\
&\qquad\quad + \|u-\hat{u}\|_{2,M}\|G(u,v)-G(\hat{u},\hat{v})\|_{2,M}\\
&\qquad\le K\|v-\hat{v}\|_{2,M}\left(\|u-\hat{u}\|_{2,M} + \|v-\hat{v}\|_{2,M}\right)\\
&\qquad\quad + K\|u-\hat{u}\|_{2,M}\left(\|u-\hat{u}\|_{2,M} + \|v-\hat{v}\|_{2,M}\right) + +K\|u-\hat{u}\|_{2,\Omega}^2\\
&\qquad\le K(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,M}^2)\\
&\qquad\quad + 2K\|u-\hat{u}\|_{2,M}\|v-\hat{v}\|_{2,M} + K\|u-\hat{u}\|_{2,\Omega}^2\\
&\qquad\le 2K(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,M}^2) + K\|u-\hat{u}\|_{2,\Omega}^2
\end{aligned}$$

From Lemma 2.4, for $p=2$ and $\epsilon = \frac{d_{min}}{2K} = \frac{\min\{d_j : 1\le j\le k\}}{2K}$, we have

$$\|u-\hat{u}\|_{2,M}^2 \le \frac{d_{min}}{2K}\|\nabla(u-\hat{u})\|_{2,\Omega}^2 + \tilde{C}_\epsilon\|u-\hat{u}\|_{2,\Omega}^2 \tag{6.4}$$

Using (6.4)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2\right) &\le 2K\|v-\hat{v}\|_{2,M}^2 + K(1+2\tilde{C}_\epsilon)\|u-\hat{u}\|_{2,\Omega}^2\\
&\le C_{\epsilon,k}\left(\|v-\hat{v}\|_{2,M}^2 + \|u-\hat{u}\|_{2,\Omega}^2\right)
\end{aligned}$$

Observe, $(u-\hat{u}) = (v-\hat{v}) = 0$ at $t=0$ and $\left(\|u-\hat{u}\|_{2,\Omega}^2 + \|v-\hat{v}\|_{2,M}^2\right) \ge 0$. Therefore, applying Gronwall's inequality, $v=\hat{v}$ and $u=\hat{u}$. Hence system (3.1) has the unique global solution. □

Proof of Theorem 3.2: Recall that $u_0 \in W_p^2(\Omega)$ and $v_0 \in W_p^2(M)$ with $p>n$, and u_0, v_0 satisfies the compatibility condition for $p>3$. From Sobolev imbedding (see [11], [19]), u_0, v_0 are bounded functions. Therefore there exists $\tilde{r}>0$ such that $\|u_0(\cdot)\|_{\infty,\Omega} \le \tilde{r}$, $\|v_0(\cdot)\|_{\infty,M} \le \tilde{r}$.

For each $r > \tilde{r}$, we define cut off functions $\phi_r \in C_0^\infty(\mathbb{R}^k, [0,1])$ and $\psi_r \in C_0^\infty((\mathbb{R}^k \times \mathbb{R}^m), [0,1])$ such that $\phi_r(z) = 1$ for all $|z| \le r$, and $\phi_r(z) = 0$ for all $|z| > 2r$. Similarly

$\psi_r(z,w) = 1$ when $|z| \le r$ and $|w| \le r$, and $\psi_r(z,w) = 0$ when $|z| > 2r$, or $|w| > 2r$. In addition, we define $H_r = H\phi_r, F_r = F\psi_r$ and $G_r = G\psi_r$. From construction, $H_r(z) = H(z), F_r(z,w) = F(z,w)$ and $G_r(z,w) = G(z,w)$ when $|z| \le r$ and $|w| \le r$. Also, there exists $M_r > 0$ such that H_r, G_r and F_r are Lipschitz functions with Lipschitz coefficient M_r. Consider the "restricted" system

$$\begin{aligned} u_t &= D\Delta u + H_r(u) && x \in \Omega, \quad t > 0 \\ v_t &= \tilde{D}\Delta_M v + F_r(u,v) && x \in M, \quad t > 0 \\ D\frac{\partial u}{\partial \eta} &= G_r(u,v) && x \in M, \quad t > 0 \\ u &= u_0 && x \in \Omega, \quad t = 0 \\ v &= v_0 && x \in M, \quad t = 0 \end{aligned} \tag{6.5}$$

From Theorem 6.1, (6.5) has a unique global solution (u_r, v_r). If $\|u(\cdot,t)\|_{\infty,\Omega}, \|v(\cdot,t)\|_{\infty,M} \le r$ for all $t \ge 0$, then (u_r, v_r) is a global solution to (3.1). If not, there exists $T_r > 0$ such that

$$\|u_r(\cdot,t)\|_{\infty,\Omega} + \|v_r(\cdot,t)\|_{\infty,M} \le r \quad \forall t \in [0, T_r]$$

and for all $\tau > T_r$ there exists t such that $T_r < t < \tau$, and $x \in \overline{\Omega}$ and $z \in M$, such that

$$|u_r(x,t)| + |v_r(z,t)| > r$$

Note that T_r is increasing with respect to r. Let $T_{\max} = \lim_{r\to\infty} T_r$. Now we define (u,v) as follows. Given $0 < t < T_{\max}$, there exists $r > 0$ such that $t < T_r \le T_{\max}$. For all $x \in \overline{\Omega}$, $u(x,t) = u_r(x,t)$, and for all $x \in M$, $v(x,t) = v_r(x,t)$. Furthermore (u,v) solves (3.1) with $T = T_{\max}$. Also, uniqueness of (u_r, v_r) implies uniqueness of (u,v). It remains to show that the solution of (3.1) is maximal and if $T_{\max} < \infty$ then

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t\to T_{\max}^-} \|v(\cdot,t)\|_{\infty,M} = \infty.$$

Suppose $T_{\max} < \infty$ and set,

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{\infty,\Omega} + \limsup_{t\to T_{\max}^-} \|v(\cdot,t)\|_{\infty,M} = R.$$

If $R = \infty$ then (u,v) is a maximal solution. If $R < \infty$ there exists $L > 0$ such that

$$\|u\|_{\infty,\Omega\times(0,T_{\max})} + \|v\|_{\infty,M\times(0,T_{\max})} \le L.$$

As a result, $T_{2L} > T_{\max}$, contradicting the construction of T_{2L}. $\square$

Now we prove that under some extra assumptions that the solution to (3.1) is componentwise nonnegative. Consider the system

$$\begin{aligned} u_t &= D\Delta u + H(u^+) && x \in \Omega, \quad 0 < t < T \\ v_t &= \tilde{D}\Delta_M v + F(u^+, v^+) && x \in M, \quad 0 < t < T \\ D\frac{\partial u}{\partial \eta} &= G(u^+, v^+) && x \in M, \quad 0 < t < T \\ u &= u_0 && x \in \Omega, \quad t = 0 \\ v &= v_0 && x \in M, \quad t = 0 \end{aligned} \tag{6.6}$$

where $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$.

PROPOSITION 6.2. *Suppose F, G and H are locally Lipschitz, quasi positive functions, and u_0, v_0 are componentwise nonnegative functions. Then (6.6) has a unique componentwise nonnegative solution.*

Proof. Note that $F(u^+, v^+)$, $G(u^+, v^+)$ and $H(u^+)$ are locally Lipschitz functions of u and v. Therefore from Theorem 3.2 there exists a unique maximal solution to (6.6) on $(0, T_{\max})$. Consider (6.6) componentwise. Multiply the v_{it} equation by v_i^- and the u_{jt} equation by u_j^-,

$$v_i^- \frac{\partial v_i}{\partial t} = \tilde{d}_i v_i^- \Delta_M v_i + v_i^- F_i(u^+, v^+) \tag{6.7}$$

$$u_j^- \frac{\partial u_j}{\partial t} = d_j u_j^- \Delta u_j + u_j^- H_j(u^+) \tag{6.8}$$

Since $w^- \frac{dw}{dt} = \frac{-1}{2} \frac{d}{dt}(w^-)^2$,

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(v_i^-)^2 + \frac{1}{2}\frac{\partial}{\partial t}(u_j^-)^2 = &-\tilde{d}_i v_i^- \Delta_M v_i - v_i^- F_i(u^+, v^+) \\ &- d_j u_j^- \Delta u_j - u_j^- H_j(u^+) \end{aligned}$$

Integrating (6.7) and (6.8) over M and Ω respectively, gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_i^-(\cdot,t)\|_{2,M}^2 + \frac{1}{2}\frac{d}{dt}\|u_j^-(\cdot,t)\|_{2,\Omega}^2 + \tilde{d}_i \int_M |\nabla v_i^-|^2 \, d\sigma + d_j \int_\Omega |\nabla u_j^-|^2 \, dx \\ = -\int_\Omega u_j^- H_j(u^+) \, dx - \int_M u_j^- G_j(u^+, v^+) \, d\sigma - \int_M v_i^- F_i(u^+, v^+) \, d\sigma \end{aligned}$$

Since F, G and H are quasi-positive and $\tilde{d}_i, d_j > 0$,

$$\frac{1}{2}\frac{d}{dt}\|v_i^-(\cdot,t)\|_{2,M}^2 + \frac{1}{2}\frac{d}{dt} \parallel u_j^-(\cdot,t) \parallel_{2,\Omega}^2 \leq 0$$

Therefore, the solution (u, v) is componentwise nonnegative. ∎

COROLLARY 6.3. *Suppose F, G and H are locally Lipschitz, quasi positive functions, and u_0, v_0 are componentwise nonnegative functions. Then the unique solution (u, v) of (3.1) is componentwise nonnegative.*

Proof. From Theorem 3.2 and Proposition 6.2, there exists a unique, componentwise nonnegative and maximal solution (u, v) to (6.6). In fact (u, v) also solves (3.1). The result follows. ∎

6.2. Bootstrapping Strategy. The following system will play a central role in duality arguments.

$$\begin{aligned} &\Psi_t = -\tilde{d}\Delta_M \Psi - \tilde{\vartheta} && (x,t) \in M \times (\tau, T) \\ &\Psi = 0 && x \in M, t = T \end{aligned} \tag{6.9a}$$

$$\begin{aligned} &\varphi_t = -d\Delta\varphi - \vartheta && (x,t) \in \Omega \times (\tau, T) \\ &\kappa_1 d\frac{\partial \varphi}{\partial \eta} + \kappa_2 \varphi = \Psi && (x,t) \in M \times (\tau, T) \\ &\varphi = 0 && x \in \Omega, \quad t = T \end{aligned} \tag{6.9b}$$

Here, $p > 1$, $0 < \tau < T$, $\tilde{\vartheta} \in L_p(M \times (\tau, T))$ and $\tilde{\vartheta} \geq 0$, and $\vartheta \in L_p(\Omega \times (\tau, T))$ and $\vartheta \geq 0$. Also $d > 0$, $\tilde{d} > 0$, and $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $\kappa_1 \geq 0$ and $\kappa_1\kappa_2 \neq 0$. Lemmas 6.4 to 6.8 provide helpful estimates.

Lemma 6.4. *(6.9a) has a unique nonnegative solution $\Psi \in {W_p}^{2,1}(M \times (\tau, T))$ and there exists $C_{p,T} > 0$ independent of $\tilde{\vartheta}$ such that*

$$\|\Psi\|^{(2)}_{p,M\times(\tau,T)} \leq C_{p,T}\|\tilde{\theta}\|_{p,M\times(\tau,T)}$$

Proof. The result follows from Theorem 3.5 and the comparison principle. □

Lemma 6.5. *Let $p > 1$, $\kappa_1 \geq 0$ and if $\kappa_1 = 0$ then $\kappa_2 > 0$. Suppose Ψ is the unique nonnegative solution of (6.9a). Then (6.9b) has a unique nonnegative solution $\varphi \in W_p^{2,1}(\Omega \times (\tau, T))$. Moreover, there exists $C_{p,T} > 0$ independent of ϑ and $\tilde{\vartheta}$ and dependent on $d, \tilde{d}, \kappa_1$ and κ_2 such that*

$$\|\varphi\|^{(2)}_{p,(\Omega\times(\tau,T))} \leq C_{p,T}(\|\tilde{\theta}\|_{p,M\times(\tau,T)} + \|\theta\|_{p,\Omega\times(\tau,T)})$$

Proof. The result follows from Lemma 6.4, Sobolev embedding and similar arguments of proof on page 342, section 9 of chapter 4 in [19], and the comparison principle. □

Remark 6. *If $p > n + 2$ and $\kappa_1 > 0$, then $\nabla\varphi$ is Hölder continuous in x and t. See the Corollary after Theorem 9.1, (page 342) chapter 4 of [19].*

Lemma 6.6. *Suppose $l > 0$ is a non integral number, $\kappa_1 > 0$, $d > 0$, $\vartheta \in C^{l,\frac{l}{2}}(\overline{\Omega} \times [\tau, T])$, $\tilde{\vartheta} \in C^{l,\frac{l}{2}}(M \times [\tau, T])$, $\varphi(x, T) \in C^{2+l}(\overline{\Omega})$ and $\Psi \in C^{l+1,\frac{(l+1)}{2}}(M \times [\tau, T])$. Then (6.9b) has a unique solution in $C^{l+2,\frac{l}{2}+1}(\overline{\Omega} \times [\tau, T])$. Moreover there exists $c > 0$ independent of Ψ and ϑ such that*

$$|\varphi|^{(l+2)}_{\Omega\times[\tau,T]} \leq c\left(|\vartheta|^{(l)}_{\Omega\times[\tau,T]} + |\Psi|^{(l+1)}_{M\times(\tau,T)}\right)$$

Proof. See Theorem 5.3 in chapter 4 of [19]. □

Lemma 6.7. *Suppose $1 < p < \infty$, $\kappa_1 > 0$, and r, s are positive integers. If $q \geq p$ and $2 - 2r - s - \left(\frac{1}{p} - \frac{1}{q}\right)(n + 2) \geq 0$ then there exists $\tilde{K} > 0$ depending on Ω, r, s, n, p such that*

$$\|D_t^r D_x^s \varphi\|_{q,\Omega\times(\tau,T)} \leq \tilde{K}\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)}$$

for all $\varphi \in W_p^{2,1}(\Omega \times (\tau, T))$.

Proof. See Lemma 3.3 in chapter 2 of [19]. □

Lemma 6.8. *Suppose $1 < p < \infty$, $\kappa_1 > 0$, and r, s, m are positive integers satisfying $2r + s < 2m - \frac{2}{p}$. There exists $c > 0$ independent of $\varphi \in {W_p}^{2m,m}(\Omega \times (\tau, T))$ such that*

$$D_t^r D_x^s \varphi|_{t=\tau} \in {W_p}^{2m-2r-s-\frac{2}{p}}(\Omega) \text{ and } \|\varphi\|^{(2m-2r-s-\frac{2}{p})}_{p,\Omega} \leq c\|\varphi\|^{(2m)}_{p,\Omega\times(\tau,T)}$$

In addition, when $2r+s<2m-\frac{1}{p}$,

$$D_t^r D_x^s \varphi|_{M\times(\tau,T)} \in W_p^{2m-2r-s-\frac{1}{p},\ m-r-\frac{s}{2}-\frac{1}{2p}}(M\times(\tau,T))$$
$$\text{and } \|\varphi\|_{p,M\times(\tau,T)}^{(2m-2r-s-\frac{1}{p})} \le c\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2m)}$$

Proof. See Lemma 3.4 in chapter 2 of [19]. ∎

Lemma 6.9. *Let* $p>1$, $\kappa_1=0$ *and suppose* $0\le\vartheta\in L_p(\Omega\times(\tau,T))$, *and* Ψ *is a unique solution of* (6.9a). *Then* $\Psi\in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(M\times(\tau,T))$, *and* (6.9b) *has a unique solution* $\varphi\in W_p^{2,1}(\Omega\times(\tau,T))$. *Moreover, there exists* $C_{p,T}>0$ *independent of* ϑ *and dependent on d, and* κ_2 *such that*

$$\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)} \le C_{p,T}(\|\vartheta\|_{p,\Omega\times(\tau,T)}+\|\tilde{\vartheta}\|_{p,M\times(\tau,T)})$$

Proof. The result follows from Theorem 9.1 in chapter 4 of [19], Lemma 6.4, and Sobolev embedding. ∎

Remark 7. *If* $p>\frac{n+2}{2}$, $\kappa_1=0$ *and* φ *satisfies system* (6.9b), *then* φ *is a Hölder continuous function in* x *and* t. *See the Corollary after Theorem 9.1, chapter 4 of [19].*

Remark 8. *By Lemma 6.4, Lemma 6.5, Lemma 6.8, and Sobolev embedding, we have* $\varphi(\cdot,\tau)\in W_p^{2-\frac{2}{p}}(\Omega)$, $\Psi(\cdot,\tau)\in W_p^{2-\frac{2}{p}}(M)$, *and there exists* $c>0$ *independent of* φ, Ψ *such that*

$$\|\varphi(\cdot,\tau)\|_{p,\Omega}^{(2-\frac{2}{p})} \le c(\|\vartheta\|_{p,\Omega\times(\tau,T)}+\|\tilde{\vartheta}\|_{p,M\times(\tau,T)})$$

$$\|\Psi(\cdot,\tau)\|_{p,M}^{(2-\frac{2}{p})} \le c\|\vartheta\|_{p,\Omega\times(\tau,T)}$$

respectively. Moreover, if $p>n$ *there exists* $c>0$ *independent of* φ, Ψ *such that*

$$\|\varphi\|_{\infty,\Omega\times(\tau,T)} \le c\|\varphi(\cdot,\tau)\|_{p,\Omega}^{(2-\frac{2}{p})}$$

$$\|\Psi\|_{\infty,M\times(\tau,T)} \le c\|\Psi(\cdot,\tau)\|_{p,M}^{(2-\frac{2}{p})}$$

respectively.

Lemma 6.10. *Let* $1<p<n+2$ *and* $1<q\le\frac{(n+1)p}{n+2-p}$. *There exists a constant* $\hat{C}>0$ *depending on* $p, T-\tau, M$ *and* n *such that if* $\varphi\in W_p^{2,1}(\Omega\times(\tau,T))$, *then*

$$\left\|\frac{\partial\varphi}{\partial\eta}\right\|_{q,M\times(\tau,T)} \le \hat{C}\|\varphi\|_{p,\Omega\times(\tau,T)}^{(2)}$$

Proof. It suffices to consider the case when φ is smooth in $\overline{\Omega}\times[\tau,T]$, as such functions are dense in $W_p^{2,1}(\Omega\times(\tau,T))$. M is a $C^{2+\mu}$, $n-1$ dimensional manifold ($\mu>0$). Therefore, for every $\hat{\xi}\in M$ there exists $\epsilon_{\hat{\xi}}>0$, an open set $V\subset\mathbb{R}^n$ containing 0, and a $C^{2+\mu}$ diffeomorphism $\psi:V\to B(\hat{\xi},\epsilon_{\hat{\xi}})$ such that $\psi(\mathbf{0})=\hat{\xi}$, $\psi(\{x\in V: x_n>0\})=B(\hat{\xi},\epsilon_{\hat{\xi}})\cap\Omega$ and $\psi(\{x\in V: x_n=0\})=B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M$. Since ψ is a C^2 diffeomorphism, $(\psi^{-1})_n$, the nth component of ψ^{-1}, is differentiable in $B(\hat{\xi},\epsilon_{\hat{\xi}})$, and by definition of ψ, $(\psi^{-1})_n(\xi)=0$ if and only if $\xi\in B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M$. Further, $\nabla(\psi^{-1})_n(\xi)$ is nonzero and orthogonal to $B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M$ at each $\xi\in B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M$. Without loss of generality, we assume the outward unit normal is given by

$$\eta(\xi)=\frac{\nabla(\psi^{-1})_n(\xi)}{|(\nabla\psi^{-1})_n(\xi)|}\quad\forall\,\xi\in B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M$$

We know,

$$\frac{\partial\varphi}{\partial\eta}(\xi,t)=\nabla_\xi\varphi(\xi,t)\cdot\eta(\xi)\quad\forall\,(\xi,t)\in B(\hat{\xi},\epsilon_{\hat{\xi}})\cap M\times(\tau,T).$$

Now in order to transform $\frac{\partial\varphi(\xi,t)}{\partial\eta}$ back to $\mathbb{R}^n$, pick $L>0$, such that $E=\underbrace{[-L,L]\times[-L,L]\times...\times[-L,L]}_{(n-1)\text{ times}}\times[0,L]\subset V$, and define $\tilde{\varphi}$ such that

$$\tilde{\varphi}(x,t)=-\int_0^{x_n}\nabla_x\varphi(\psi(x',z),t)^TD(\psi(x',z))\eta(\psi(x',z))\;dz\quad\forall\;x=(x',z)\in E$$

where $x'\in\underbrace{[-L,L]\times[-L,L]\times...\times[-L,L]}_{(n-1)\text{ times}}$. We know $\varphi\in W_p^{2,1}(\Omega\times(\tau,T))$. Therefore from Lemma 6.7, there exists $0<\alpha<L$ and $K_{\hat{\xi}}>0$, depending on Ω,n,p such that

$$\int_{S_\alpha}\left|\frac{\partial\tilde{\varphi}((x',\alpha),t)}{\partial x_n}\right|^r\,d\sigma dt<K_{\hat{\xi}}\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)}\quad\forall\;1<r\le\frac{(n+2)p}{n+2-p}\tag{6.9}$$

where $S_\alpha=E|_{x_n=\alpha}\times(\tau,T)$ and $S_{x_n}=E|_{0\le x_n\le\alpha}\times(\tau,T)$. Using the fundamental theorem of calculus,

$$\int_{E\times(\tau,T)}\left|\frac{\partial\tilde{\varphi}((x',0),t)}{\partial x_n}\right|^q\,d\sigma\;dt\le\int_{S_\alpha}\left|\frac{\partial\tilde{\varphi}((x',\alpha),t)}{\partial x_n}\right|^q\,d\sigma\;dt$$
$$+\,q\int_{S_{x_n}}\left|\frac{\partial\tilde{\varphi}((x',s),t)}{\partial x_n}\right|^{q-1}\cdot\left|\frac{\partial^2\tilde{\varphi}((x',s),t)}{\partial x_n^2}\right|\,d\sigma\;dt$$

Using (6.9),

$$\int_{E\times(\tau,T)}\left|\frac{\partial\tilde{\varphi}((x',0),t)}{\partial x_n}\right|^q\,d\sigma\;dt\le K_{\hat{\xi}}(\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)})^q$$
$$+\,q\int_{S_{x_n}}\left|\frac{\partial\tilde{\varphi}((x',s),t)}{\partial x_n}\right|^{q-1}\cdot\left|\frac{\partial^2\tilde{\varphi}((x',s),t)}{\partial x_n^2}\right|\,d\sigma\;dt$$

Applying Hölder inequality,

$$\int_{E\times(\tau,T)}\left|\frac{\partial\tilde{\varphi}((x',0),t)}{\partial x_n}\right|^q\,d\sigma\;dt\le K_{\hat{\xi}}(\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)})^q$$
$$+\,q\left(\int_{S_{x_n}}\left|\frac{\partial\tilde{\varphi}((x',s),t)}{\partial x_n}\right|^{\frac{(q-1)p}{p-1}}\,d\sigma\;dt\right)^{\frac{p-1}{p}}\left(\int_{S_{x_n}}\left|\frac{\partial^2\tilde{\varphi}((x',s),t)}{\partial x_n^2}\right|^p\,d\sigma\;dt\right)^{\frac{1}{p}}$$

Recall $\frac{\partial^2\tilde{\varphi}}{\partial x_n^2} \in L_p(S_{x_n})$. So using Lemma 6.7 we have

$$\int_{E\times(\tau,T)} \left|\frac{\partial\tilde{\varphi}((x',0),t)}{\partial x_n}\right|^q \, d\sigma \, dt \le \hat{K}(\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)})^q \tag{6.10}$$

Now, M is a compact manifold. Therefore there exists set $A = \{P_1, ..., P_N\} \subset M$ such that $M \subset \cup_{1\le i\le N} B(P_i, \epsilon_{P_i})$. Let V_i, $\hat{K}_i$ and α_i be the open sets and constants respectively obtained above when $\hat{\xi} = P_i$. Then,

$$\begin{aligned}
\left(\int_\tau^T \int_M \left|\frac{\partial\varphi}{\partial\eta}\right|^q \, d\sigma \, dt\right)^{\frac{1}{q}} &\le \left(\sum_{P_i\in A} \int_\tau^T \int_{B(P_i,\epsilon)} \left|\frac{\partial\varphi}{\partial\eta}\right|^q \, d\sigma \, dt\right)^{\frac{1}{q}} \\
&\le C\left(\sum_{P_i\in A} \int_\tau^T \int_{V_i|_{x_n=0}} \left|\frac{\partial\tilde{\varphi}}{\partial x_n}\right|^q \, d\sigma \, dt\right)^{\frac{1}{q}} \\
&\le C \sum_{P_i\in A} \tilde{K}_i \|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)}
\end{aligned}$$

Therefore, for some $\hat{C} > 0$, depending only upon p, τ, T, M and n, we get

$$\left\|\frac{\partial\varphi}{\partial\eta}\right\|_{q,M\times(\tau,T)} \le \hat{C}\|\varphi\|^{(2)}_{p,\Omega\times(\tau,T)} \quad \text{for all } 1 < q \le \frac{(n+1)p}{n+2-p}$$

□

The following Lemma plays a key role in bootstrapping L_p estimates of solutions to (3.1).

LEMMA 6.11. *Assume the hypothesis of Corollary 6.3, and suppose (u, v) is the unique, maximal nonnegative solution to (3.1) and $T_{\max} < \infty$. If $1 \le j \le k$ and $1 \le i \le m$, such that $(V_{i,j}1)$ holds, then there exists $K_{T_{\max}} > 0$ such that*

$$\|u_j(\cdot,t)\|_{1,\Omega} + \|v_i(\cdot,t)\|_{1,M} + \|u_j\|_{1,M\times(0,T_{\max})} \le K_{T_{\max}} \quad \textit{for all } \ 0 \le t < T_{max}.$$

Proof. For simplicity, take $\sigma = 1$ in $(V_{i,j}1)$. Let $0 < T < T_{\max}$, and consider the system

$$\begin{aligned}
\varphi_t &= -d\Delta\varphi && (x,t) \in \Omega \times (0,T) \\
d\frac{\partial\varphi}{\partial\eta} &= \alpha\varphi + 1 && (x,t) \in M \times (0,T) \\
\varphi &= \varphi_T && x \in \Omega, \quad t = T
\end{aligned} \tag{6.11}$$

where α is given in $(V_{i,j}1)$, $d > 0$, and $\varphi_T \in C^{2+\Upsilon}(\overline{\Omega})$ for some $\Upsilon > 0$, is nonnegative and satisfies the compatibility condition

$$d\frac{\partial\varphi_T}{\partial\eta} = \alpha\varphi_T + 1 \quad \text{on } M \times \{T\}$$

From Lemma 6.6, $\varphi \in C^{2+\Upsilon,1+\frac{\Upsilon}{2}}(\overline{\Omega} \times [0,T])$ and therefore by standard sequential argument $\varphi \in C^{2+\Upsilon,1+\frac{\Upsilon}{2}}(M \times [0,T])$. Also, note that $g(s) = \alpha s + 1$ satisfies $g(0) \ge 0$. Therefore, Proposition 6.2 implies $\varphi \ge 0$. Now having enough regularity for φ on $M \times [0,T]$, consider

$$\Delta_M\varphi = -\frac{1}{\sqrt{det\ g}}\partial_j(g^{ij}\sqrt{det\ g}\ \partial_i\varphi)$$

where g is the metric on M and $g^{i,j}$ is ith row and jth column entry of the inverse of matrix associated to metric g. Further let $\tilde{\vartheta} = -\varphi_t - \tilde{d}\Delta_M \varphi$. Then,

$$\begin{aligned}\int_0^T \int_M v_i \tilde{\vartheta} &= \int_0^T \int_\Omega u_j(-\varphi_t - d\Delta\varphi) + \int_0^T \int_M v_i(-\varphi_t - \tilde{d}\Delta_M \varphi) \\ &= \int_0^T \int_\Omega \varphi(u_{j_t} - d\Delta u_j) + \int_0^T \int_M \varphi(v_{it} - \tilde{d}\Delta_M v_i) - d\int_0^T \int_M u_j \frac{\partial \varphi}{\partial \eta} + d\int_0^T \int_M \frac{\partial u_j}{\partial \eta}\varphi \\ &\quad + \int_\Omega u_j(x,0)\varphi(x,0) + \int_M v_i(\zeta,0)\varphi(x,0) - \int_\Omega u_j(x,T)\varphi_T - \int_M v_i(\zeta,T)\varphi_T\end{aligned}$$

Using $d\frac{\partial\varphi}{\partial\eta} = \alpha\varphi + 1$

$$\begin{aligned}\int_0^T \int_M u_j \le \int_0^T \int_\Omega \varphi H_j(u) + \int_0^T \int_M (F_i(u,v) + G_j(u,v))\varphi \\ + \int_\Omega u_j(x,0)\varphi(x,0) + \int_M v_i(\zeta,0)\varphi(x,0) - \int_0^T \int_M v_i\tilde{\vartheta}\end{aligned}$$

Using $(V_{i,j}1)$,

$$\begin{aligned}\int_0^T \int_M u_j \le \int_0^T \int_\Omega \beta\varphi(u_j+1) + \int_0^T \int_M \alpha(v_i+1)\varphi \\ + \int_\Omega u_j(x,0)\varphi(x,0) + \int_M v_i(\zeta,0)\varphi(x,0)) - \int_0^T \int_M v_i\tilde{\vartheta}\end{aligned} \tag{6.12}$$

Now, integrating the u_j equation over Ω and the v_i equation over M,

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u_j + \int_M v_i\right) &= d\int_\Omega \Delta u_j + \int_\Omega H_j(u) + \tilde{d}\int_M \Delta v_i + \int_M F_j(u,v) \\ &\le \beta \int_\Omega (u_j+1) + \int_M (G_j(u,v) + F_i(u,v)) \\ &\le \beta \int_\Omega (u_j+1) + \alpha \int_M (u_j + v_i + 1) \qquad (6.13)\end{aligned}$$

Integrating (6.13) over $(0,t)$ with $0 < t \le T < T_{\max}$, and using (6.12), gives

$$\int_\Omega u_j(x,t) + \int_M v_i(\zeta,t) \le \tilde{\beta}\int_0^t \int_\Omega u_j + \tilde{\alpha}\int_0^t \int_M v_i + \tilde{L}(t) \tag{6.14}$$

where

$$\begin{aligned}\tilde{L}(t) = \alpha|M|t + \beta|\Omega|t + \alpha\beta\|\varphi\|_{1,\Omega\times(0,t)} + \alpha^2\|\varphi\|_{1,M\times(0,t)} + \alpha\|u_j(x,0)\|_{1,\Omega}\cdot\|\varphi(x,0)\|_{\infty,\Omega} \\ + \|v_i(\zeta,0)\|_{1,M} + \alpha\|v_i(\zeta,0)\|_{1,M}\cdot\|\varphi(x,0)\|_{\infty,M} + \|u_j(x,0)\|_{1,\Omega}\end{aligned}$$

$$\tilde{\alpha}(t) = \alpha^2\|\varphi\|_{\infty,M\times(0,t)} + \alpha + \alpha\|\tilde{\theta}\|_{\infty,M\times(0,t)} \quad \text{and} \quad \tilde{\beta}(t) = \beta + \alpha\beta\|\varphi\|_{\infty,\Omega\times(0,t)}$$

Applying Generalized Gronwall's inequality to (6.14) gives the bound for the first two integrals on the RHS of (6.14), and then substituting this bound gives

$$\begin{aligned}\int_\Omega u_j(x,t) + \int_M v_i(\zeta,t) &\le \tilde{L}(t) + \int_0^t (\tilde{\alpha}(s) + \tilde{\beta}(s))\tilde{L}(s)\exp\left(\int_s^t \tilde{\alpha}(r) + \tilde{\beta}(r)dr\right)\, ds \\ &\le C_{T_{\max}}\end{aligned}$$

for all $0 \le t < T < T_{max}$. Substituting this estimate of u_j on Ω and v_i on M in (6.12) yields

$$\int_0^T \int_M u_j \le \beta \left(\|\varphi\|_{\infty,\Omega\times(0,T)} \|u_j\|_{1,\Omega\times(0,T)} + |\Omega| T \|\varphi\|_{\infty,\Omega\times(0,T)} \right) \\ + \alpha \left(\|\varphi\|_{\infty,M\times(0,T)} \|v_i\|_{1,M\times(0,T)} + |M| T \|\varphi\|_{\infty,M\times(0,T)} \right) \\ + \|u_j(\cdot,0)\|_{1,\Omega} \|\varphi(\cdot,0)\|_{\infty,\Omega} + \|v_i(\cdot,0)\|_{1,M} \|\varphi(\cdot,0)\|_{\infty,M} + \|v_i\|_{1,M} \|\tilde{\theta}\|_{\infty,M}$$

Since $T < T_{\max}$ is arbitrary, the conclusion of the theorem holds. ☐

Lemma 6.12. *Assume the hypothesis of Corollary 6.3 holds. Suppose (u, v) is the unique, maximal nonnegative solution to (3.1) and $T_{\max} < \infty$. If $1 \le j \le k$ and $1 \le i \le m$, such that $(V_{i,j}1)$ and $(V_{i,j}2)$ holds, and for $q > 1$, $v_i \in L_q(M \times (0, T_{\max}))$, then $u_j \in L_q(M \times (0, T_{\max}))$ and $u_j \in L_q(\Omega \times (0, T_{\max}))$.*

Proof. Let $0 < t < T \le T_{\max}$. Multiplying the u_{j_t} equation by u_j^{q-1}, we get

$$\int_0^t \int_\Omega u_j^{q-1} u_{j_t} = d \int_0^t \int_\Omega u_j^{q-1} \Delta u_j + \int_0^t \int_\Omega u_j^{q-1} H_j(u) \\ = d \int_0^t \int_M u_j^{q-1} \frac{\partial u_j}{\partial \eta} - d \int_0^t \int_\Omega (q-1) u_j^{q-2} |\nabla u_j|^2 + \int_0^t \int_\Omega u_j^{q-1} H_j(u)$$

Using $(V_{i,j}2)$

$$\int_\Omega \frac{u_j^q}{q} + d \int_0^t \int_\Omega \frac{4(q-1)}{q^2} |\nabla u_j^{\frac{q}{2}}|^2 \le K_g \int_0^t \int_M u_j^{q-1}(u_j + v_i + 1) + \beta \int_0^t \int_\Omega (u_j + 1) u_j^{q-1} \\ + \int_\Omega \frac{u_{j0}^q}{q} \\ \le K_g \left(\int_0^t \int_M u_j^q + v_i u_j^{q-1} + u_j^{q-1} \right) + \beta \left(\int_0^t \int_\Omega u_j^q + u_j^{q-1} \right) \\ + \int_\Omega \frac{u_{j0}^q}{q} \tag{6.15}$$

Applying Young's inequality in (6.15)

$$\int_\Omega \frac{u_j^q}{q} + d \int_0^t \int_\Omega \frac{4(q-1)}{q^2} |\nabla u_j^{\frac{q}{2}}|^2 \le K_g \left(\frac{3q-2}{q} \right) \int_0^t \int_M u_j^q + \left(\beta + t|\Omega| \frac{\beta}{q} \right) \int_0^t \int_\Omega u_j^q \\ + \int_\Omega \frac{u_{j0}^q}{q} + K_g \left(\frac{1}{q} \right) \int_0^t \int_M v_i^q + \frac{t|M|}{q} \tag{6.16}$$

Also, for $1 < q \le \infty$, for all $\epsilon > 0$ and $t \le T \le T_{max}$, from Lemma 2.4, for $v = u^{\frac{q}{2}}$ there exists $C_\epsilon > 0$ such that,

$$\int_0^t \int_M u_j^q \le C_\epsilon \int_0^t \int_\Omega u_j^q + \epsilon \int_0^t \int_\Omega |\nabla u_j^{\frac{q}{2}}|^2 \tag{6.17}$$

Using (6.17) and (6.16) for appropriate $\epsilon > 0$, gives

$$\frac{1}{q} \frac{d}{dt} \int_0^t \int_\Omega u_j^q \le \tilde{K}_1 \int_0^t \int_\Omega u_j^q + \tilde{K}_2(T) \tag{6.18}$$

for

$$\tilde{K}_2(T) = K_g \left(\frac{1}{q}\right) \int_0^T \int_M v_i^q + \frac{T|M|}{q}$$

and $\tilde{K}_1 > 0$ depending on t, where $t \leq T \leq T_{max}$. Therefore from Gronwall's Inequailty

$$\int_\Omega {u_j}^q(x,t) \leq \tilde{K}_2(T) + \int_0^T \tilde{K}_1(s)\tilde{K}_2(s) \exp\left(\int_s^T \tilde{K}_1(r)dr\right) ds \tag{6.19}$$

To obtain estimates on boundary, we use (6.16) to obtain

$$\begin{aligned}\epsilon \int_0^T \int_\Omega |\nabla u_j^{\frac{q}{2}}|^2 \leq \left(\frac{q^2}{4d(q-1)}\right) 3K_g\epsilon \int_0^T \int_M u_j^q + \epsilon\left(\frac{q^2}{4d(q-1)}\right)\left(\beta + T|\Omega|\frac{\beta}{q}\right)\int_0^T\int_\Omega u_j^q \\ + \left(\frac{q^2}{4d(q-1)}\right)\left(\epsilon \int_\Omega \frac{{u_j}_0^q}{q} + \epsilon K_g \left(\frac{1}{q}\right)\int_0^T \int_M v_i^q + \epsilon \frac{T|M|}{q}\right)\end{aligned} \tag{6.20}$$

Using (6.17), (6.20) and (6.19) we have,

$$\begin{aligned}\int_0^T \int_M u_j^q \leq C_\epsilon \int_0^T \int_\Omega u_j^q + 3K_g\left(\frac{q^2}{4d(q-1)}\right)\epsilon \int_0^T \int_M u_j^q + \epsilon\left(\frac{q^2}{4d(q-1)}\right)\left(\beta + T|\Omega|\frac{\beta}{q}\right)\int_0^T\int_\Omega u_j^q \\ + \left(\frac{q^2}{4d(q-1)}\right)\left(\epsilon \int_\Omega \frac{{u_j}_0^q}{q} + \epsilon K_g \left(\frac{1}{q}\right)\int_0^T \int_M v_i^q + \epsilon \frac{T|M|}{q}\right)\end{aligned}$$

Now choosing ϵ such that

$$1 - 3K_g\left(\frac{q^2}{4d(q-1)}\right)\epsilon > 0$$

and using the estimate above for u_j on $(\Omega \times (0,T))$, we have $u_j \in L_q(M \times (0,T))$. Since T is arbitrary, $u_j \in L_q(M \times (0, T_{max}))$ □

Lemma 6.13. *Assume the hypothesis of Corollary 6.3, and suppose (u, v) is the unique, maximal nonnegative solution to (3.1) and $T_{\max} < \infty$. If $1 \leq j \leq k$ and $1 \leq i \leq m$ so that $(V_{i,j}1)$ and $(V_{i,j}2)$ hold, then for all $p > 1$ and $0 < T < T_{\max}$, there exists $C_{p,T} > 0$, such that*

$$\begin{aligned}&\|u_j\|_{p,\Omega\times(0,T_{max})} + \|v_i\|_{p,M\times(0,T_{max})} \\ &\qquad \leq C_{p,T_{max}}\left(\|u_j\|_{1,M\times(0,T_{max})} + \|u_j\|_{1,\Omega\times(0,T_{max})} + \|v_i\|_{1,M\times(0,T_{max})}\right)\end{aligned}$$

Proof. First we show there exists $r > 1$ such that if $q \geq 1$ such that $u_j \in L_q(\Omega \times (0, T_{max}))$ and $v_i \in L_q(M \times (0, T_{max}))$ then $u_j \in L_{rq}(\Omega \times (0, T_{max}))$ and $v_i \in L_{rq}(M \times (0, T_{max}))$. Consider the system (6.9*a*) and (6.9*b*) with $\kappa_1 = 0$, $\kappa_2 = 1$, $\tilde{\vartheta} \geq 0$, $\tilde{\vartheta} \in L_p(M \times (0, T_{max}))$ with $\|\tilde{\vartheta}\|_{p,(M\times(0,T_{max}))} = 1$, $\vartheta \geq 0$, and $\vartheta \in L_p(\Omega \times (0, T_{max}))$ with $\|\vartheta\|_{p,(\Omega\times(0,T_{max}))} = 1$. Multiplying u_j with ϑ and v_i with $\tilde{\vartheta}$ and for $0 < T \leq T_{max}$, integrating over $\Omega \times (0, T)$ and

$M\times(0,T)$ respectively, gives

$$\begin{aligned}\int_0^T\int_\Omega u_j\vartheta+\int_0^T\int_M v_i\tilde{\vartheta} &= \int_0^T\int_\Omega u_j(-\varphi_t-d\Delta\varphi)+\int_0^T\int_M v_i(-\Psi_t-\tilde{d}\Delta_M\Psi)\\ &= \int_0^T\int_\Omega \varphi(u_{j_t}-d\Delta u_j)+\int_0^T\int_M \Psi(v_{it}-\tilde{d}\Delta_M v_i)\\ &\quad - d\int_0^T\int_M u_j\frac{\partial\varphi}{\partial\eta}+d\int_0^T\int_M\frac{\partial u_j}{\partial\eta}\varphi+\int_\Omega u_j(x,0)\varphi(x,0)\\ &\quad +\int_M v_i(x,0)\Psi(x,0)-\int_M v_i(x,T)\Psi(x,T)-\int_\Omega u_j(x,T)\varphi(x,T)\end{aligned}$$

Since $\Psi(x,T)=0$ and $\varphi(x,T)=0$,

$$\begin{aligned}\int_0^T\int_\Omega u_j\vartheta+\int_0^T\int_M v_i\tilde{\vartheta} &\le \int_0^T\int_\Omega \varphi H_j(u)+\int_0^T\int_M (F_j(u,v)+G_i(u,v))\Psi\\ &\quad - d\int_0^T\int_M u_j\frac{\partial\varphi}{\partial\eta}+\int_\Omega u_j(x,0)\varphi(x,0)\\ &\quad +\int_M v_i(x,0)\Psi(x,0)\end{aligned}$$

Using $(V_{i,j}1)$,

$$\begin{aligned}\int_0^T\int_\Omega u_j\vartheta+\int_0^T\int_M v_i\tilde{\vartheta} &\le \int_0^T\int_\Omega \beta\varphi(u_j+1)+\int_0^T\int_M \alpha(u_j+v_i+1)\Psi\\ &\quad - d\int_0^T\int_M u_j\frac{\partial\varphi}{\partial\eta}+\int_\Omega u_j(x,0)\varphi(x,0)\\ &\quad +\int_M v_i(x,0)\Psi(x,0)\end{aligned}\tag{6.21}$$

Now we break the argument in two cases.

Case 1: Suppose $q=1$. Then $u_j\in L_1(\Omega\times(0,T_{max}))$ and $u_j,v_i\in L_1(M\times(0,T_{max}))$. Let $\epsilon>0$ and set $p=n+2+\epsilon$. Set $p'=\frac{n+2+\epsilon}{n+1+\epsilon}$ (conjugate of p). Remarks 6 and 8, and Lemma 6.11 imply all of the integrals on the right hand side of (6.21) are finite. Application of Hölder's inequality in (6.21), yields $v_i\in L_{p'}(M\times(0,T))$, and there exists $C_{p,T}>0$ such that

$$\|u_j\|_{p',\Omega\times(0,T)}+\|v_i\|_{p',M\times(0,T)}\le C_{p,T}(\|u_j\|_{1,\Omega\times(0,T_{max})}+\|v_i\|_{1,M\times(0,T_{max})}+\|u_j\|_{1,M\times(0,T_{max})})$$

Since $T\le T_{max}$ is arbitrary, therefore, Lemma 6.12 implies $u_j\in L_{p'}(M\times(0,T_{max}))$. So for this case, $r=\frac{n+2+\epsilon}{n+1+\epsilon}$.

Case 2: Suppose $q>1$ such that $u_j\in L_q(\Omega\times(0,T_{max}))$ and $u_j,v_i\in L_q(M\times(0,T_{max}))$. Recall $p>1$, $0\le\tilde{\vartheta}\in L_p(M\times(0,T_{max}))$ with $\|\tilde{\vartheta}\|_{p,(M\times(0,T_{max}))}=1$ and $0\le\vartheta\in L_p(\Omega\times(0,T_{max}))$ with $\|\vartheta\|_{p,(\Omega\times(0,T_{max}))}=1$. Also $p'=\frac{p}{p-1}$, $q'=\frac{q}{q-1}$. Note $T\le T_{max}$ is arbitrary. Applying Hölder's inequality in (6.21) and using Lemma 6.10, yields

$$\begin{aligned}&\|u_j\|_{p',\Omega\times(0,T_{max})}+\|v_i\|_{p',M\times(0,T_{max})}\\ &\qquad\le C_{p,T_{max}}(\|u_j\|_{q,\Omega\times(0,T_{max})}+\|v_i\|_{q,M\times(0,T_{max})}+\|u_j\|_{q,M\times(0,T_{max})})\end{aligned}$$

provided $p'\le\frac{(n+2)q}{n+1}$. So, in this case, $r=\frac{(n+2)}{n+1}$.

Now, by repeating the above argument for rq instead of q, we get $v_i \in L_{r^m q}(M\times(0,T_{max}))$, $u_j \in L_{r^m q}(\Omega \times (0,T_{max}))$, for all $m > 1$. As $r > 1$, $\lim_{m\to\infty} r^m q \to \infty$, and as a result, $v_i \in L_p(M \times (0,T_{max}))$ for all $p > 1$. Hence from Lemma 6.12, $u_j \in L_p(M \times (0,T_{max}))$ and $u_j \in L_p(\Omega \times (0,T_{max}))$ for all $p > 1$, and there exists $C_{p,T} > 0$ such that

$$\|u_j\|_{p,\Omega\times(0,T)} + \|v_i\|_{p,M\times(0,T)} \le C_{p,T}\left(\|u_j\|_{q,M\times(0,T_{max})} + \|u_j\|_{q,\Omega\times(0,T_{max})} + \|v_i\|_{q,M\times(0,T_{max})}\right)$$

Again as $T \le T_{max}$ is arbitrary, we get

$$\begin{aligned}&\|u_j\|_{p,\Omega\times(0,T_{max})} + \|v_i\|_{p,M\times(0,T_{max})}\\ &\qquad\le C_{p,T_{max}}\left(\|u_j\|_{1,M\times(0,T_{max})} + \|u_j\|_{1,\Omega\times(0,T_{max})} + \|v_i\|_{1,M\times(0,T_{max})}\right)\end{aligned}$$

□

6.3. Global Existence.

Proof of Theorem 3.3: From Theorem 3.2 and Corollary 6.3, we already have a componentwise nonnegative, unique, maximal solution of (3.1). If $T_{\max} = \infty$, then we are done. So, by way of contradiction assume $T_{\max} < \infty$. From Lemma 6.13, we have L_p estimates for our solution for all $p \ge 1$, on $\Omega \times (0,T_{\max})$ and $M \times (0,T_{\max})$. We know from $(V_{i,j}2)$ and $(V_{i,j}3)$ that F_j and G_i are polynomially bounded above for each i and j. Let U and V solve

$$\begin{aligned}
U_t &= d_j\Delta U + \beta(u_j+1) && (x,t)\in\Omega\times(0,T_{max})\\
V_t &= \tilde{d}_i\Delta_M V + K_f(u_j+v_i+1)^l && (x,t)\in M\times(0,T_{max})\\
d_j\frac{\partial U}{\partial\eta} &= K_g(u_j+v_i+1) && (x,t)\in M\times(0,T_{max})\\
U &= U_0 && x\in\Omega,\quad t=0\\
V &= V_0 && x\in M,\quad t=0
\end{aligned}\tag{6.22}$$

Here, d_j and $\tilde{d}_i$ are the jth and ith column entry of diagonal matrix D and $\tilde{D}$ respectively. Also, U_0 and V_0 satisfy the compatibility condition, are component-wise nonnegative functions, and $(u_0)_j \le U_0$ and $(v_0)_i \le V_0$. For all $q \ge 1$, $K_f(u_j+v_i+1)^l$ and $K_g(u_j+v_i+1)$ lie in $L_q(M\times(0,T_{max}))$. Using Theorem 3.6, the solution of (6.22) is sup norm bounded. Therefore, by the Maximum Principle [26], the solution of (3.1) is bounded for finite time. Therefore Theorem 3.2 implies $T_{max} = \infty$. □

7. Examples and an Open Question. In this section we give some examples to support our theory.

Example 1. As described in [35], during bacterial cytokinesis, a proteinaceous contractile, called the Z ring assembles in the cell middle. The Z ring moves to the membrane and contracts, when triggered, to form two identical daughter cells. Positiong the Z ring in the middle of the cell involves two independent processes, referred to as Min system inhibition and nucleoid occlusion ([32], [33] Sun and Margolin 2001). In this example, we only discuss the Min system inhibits process. The Min system involves proteins MinC, MinD and MinE ([28] Raskin and de Boer 1999). MinC inhibits Z ring assembly while the action of MinD and MinE serve to exclude MinC from the middle of cell region. This promotes the assembly of the Z ring at the middle of the cell. In [35] the authors considered the Min subsystem involving 6 chemical reactions and 5 components, under specific rates and parameters and performed a numerical investigation using a finite volume method on a one dimensional mathematical model. Table 7.1 shows the assumed chemical reactions. The model was developed in [35] within the context of a cylindrical cell consisting of 2 subsystems; one involving Min oscillations and the other involving FtsZ reactions. The Min subsystem consists of ATP-bound cytosolic MinD, ADP-bound cytosolic MinD, membrane-bound MinD, cytosolic MinE, and membrane bound MinD:MinE complex. Those are denoted D_{cyt}^{ATP}, D_{cyt}^{ADP}, D_{mem}^{ATP}, E_{cyt}, and $E : D_{mem}^{ATP}$, respectively. This essentially constitutes the one dimensional version of the problem. These Min proteins react with certain reaction rates that are illustrated in Table 7.1. These reactions lead to five component model

Table 1: Reactions and Reaction Rates

Chemicals	Reactions	Reaction Rates
Min D	$D_{cyt}^{ADP} \xrightarrow{k_1} D_{cyt}^{ATP}$	$R_{exc} = k_1[D_{cyt}^{ADP}]$
Min D	$D_{cyt}^{ATP} \xrightarrow{k_2} D_{mem}^{ATP}$	$R_{Dcyt} = k_2[D_{cyt}^{ATP}]$
	$D_{cyt}^{ATP} \xrightarrow{k_3[D_{mem}^{ATP}]} D_{mem}^{ATP}$	$R_{Dmem} = k_3[D_{mem}^{ATP}][D_{cyt}^{ATP}]$
Min E	$E_{cyt} + D_{mem}^{ATP} \xrightarrow{k_4} E : D_{mem}^{ATP}$	$R_{Ecyt} = k_4[E_{cyt}][D_{mem}^{ATP}]$
	$E_{cyt} + D_{mem}^{ATP} \xrightarrow{k_5[E:D_{mem}^{ATP}]^2} E : D_{mem}^{ATP}$	$R_{Emem} = k_5[D_{mem}^{ATP}][E_{cyt}][E : D_{mem}^{ATP}]^2$
Min E	$E : D_{mem}^{ATP} \xrightarrow{k_6} E + D_{cyt}^{ADP}$	$R_{exp} = k_6[E : D_{mem}^{ATP}]$

with $(u, v) = (u_1, u_2, u_3, v_1, v_2)$, where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} [D_{cyt}^{ATP}] \\ [D_{cyt}^{ADP}] \\ [E_{cyt}] \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} [D_{mem}^{ATP}] \\ [E : D_{mem}^{ATP}] \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} \sigma_{Dmem} & 0 \\ 0 & \sigma_{E:Dmem} \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_{Dcyt} & 0 & 0 \\ 0 & \sigma_{ADyct} & 0 \\ 0 & 0 & \sigma_{Ecyt} \end{pmatrix}$$

$$G(u, v) = \begin{pmatrix} G_1(u, v) \\ G_2(u, v) \\ G_3(u, v) \end{pmatrix} = \begin{pmatrix} -R_{Dcyt} - R_{Dmem} \\ R_{exp} \\ R_{exp} - R_{Ecyt} - R_{Emem} \end{pmatrix} = \begin{pmatrix} -k_2u_1 - k_3v_1u_1 \\ k_6v_2 \\ k_6v_2 - k_4u_3v_1 - k_5v_1u_3{v_2}^2 \end{pmatrix},$$

$$F(u, v) = \begin{pmatrix} F_1(u, v) \\ F_2(u, v) \end{pmatrix} = \begin{pmatrix} R_{Dcyt} + R_{Dmem} - R_{Ecyt} - R_{Emem} \\ -R_{exp} + R_{Ecyt} + R_{Emem} \end{pmatrix} = \begin{pmatrix} k_2u_1 + k_3v_1u_1 - k_4u_3v_1 - k_5v_1u_3{v_2}^2 \\ -k_6v_2 + k_4u_3v_1 + k_5v_1u_3{v_2}^2 \end{pmatrix},$$

$$H(u) = \begin{pmatrix} H_1(u) \\ H_2(u) \\ H_3(u) \end{pmatrix} = \begin{pmatrix} R_{exc} \\ -R_{exc} \\ 0 \end{pmatrix} = \begin{pmatrix} k_1u_2 \\ -k_1u_2 \\ 0 \end{pmatrix},$$

and $u_0 = (u_{0j}) \in W_p^2(\Omega)$, $v_0 = (v_{0i}) \in W_p^2(M)$ are componentwise nonnegative functions with $p > n$. Also, u_0 and v_0 satisfy the compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

Here expressions of the form k_α and σ_β are positive constants. Note F, G and H are quasi positive functions. In the multidimensional setting, the concentration densities satisfy the reaction-diffusion system given by

$$\begin{aligned} u_t &= D\Delta u + H(u) && x\in\Omega, \quad 0<t<T\\ v_t &= \tilde{D}\Delta_M v + F(u,v) && x\in M, \quad 0<t<T\\ D\frac{\partial u}{\partial \eta} &= G(u,v) && x\in M, \quad 0<t<T\\ u &= u_0 && x\in\Omega, \quad t=0\\ v &= v_0 && x\in M, \quad t=0 \end{aligned}$$

Our local existence result holds for any number of finite components. Therefore, from Theorem 3.2, this system has a unique maximal componentwise nonnegative solution. In this example, if we take two specific components at a time, we are able to obtain L_p estimates for each of the components. For that purpose we apply our results to (u_3, v_2), u_2 and (u_1, v_1). In order to prove global existence, we assume $T_{max} < \infty$. Otherwise, we are done.

Consider (u_3, v_2). It is easy to see that for $j = 3$ and $i = 2$, the hypothesis of Lemma 6.13 is satisfied, since $G_3 + F_2 \leq 0$, G_3 is linearly bounded, and $H_3 = 0$. As a result, $u_3 \in L_p(\Omega_{T_{max}})$ and $v_2 \in L_p(M_{T_{max}})$ for all $p > 1$. Using Theorem 3.6 and the comparison principle, u_2 is Hölder continuous on $\Omega_{T_{\max}}$ for $p > n + 1$. Finally, consider (u_1, v_1). Clearly for $j = 1$ and $i = 1$, the hypothesis of Lemma 6.13 is satisfied, since $G_1 + F_1 \leq 0$, G_1 is linearly bounded, and H_1 is bounded. Therefore, $u_1 \in L_p(\Omega_{T_{max}})$ and $v_1 \in L_p(M_{T_{max}})$ for all $p > 1$.

We already have for all $1 \leq i \leq 2$ and $1 \leq j \leq 3$, $(u_j, v_i) \in L_p(\Omega_{T_{max}}) \times L_p(M_{T_{max}})$ for all $p \geq 1$. Therefore there exists $\tilde{p} > 1$ such that $G_j \in L_{\tilde{p}}(\Omega_{T_{max}})$ for all $p \geq \tilde{p}$, and $F_i \in L_{\tilde{p}}(M_{T_{max}})$ for all $p \geq \tilde{p}$. Consequently from Theorem 3.6, the solution is bounded, which contradicts the conclusion of Theorem 3.2. As a result, the system has a global solution.

Example 2. Consider the model considered by Rätz and Röger[29] for signaling networks. They formulated a mathematical model that couples reaction-diffusion in the inner volume to a reaction-diffusion system on the membrane via a flux condition. More specifically, consider the system (3.1) with $k = 1, m = 2$, where

$$G(u,v) = -q = -b_6\frac{|B|}{|M|}u(c_{max} - v_1 - v_2)_+ + b_{-6}v_2, \quad H(u) = 0$$

$$F(u,v) = \begin{pmatrix} F_1(u,v)\\ F_2(u,v)\end{pmatrix} = \begin{pmatrix} k_1 v_2 g_0\left(1 - \frac{K_5 v_1 g_0}{1+K_5 v_1}\right) + k_2 v_2 \frac{K_5 v_1 g_0}{1+K_5 v_1} - k_3\frac{v_1}{v_1+k_4}\\ -k_1 v_2 g_0\left(1 - \frac{K_5 v_1 g_0}{1+K_5 v_1}\right) - k_2 v_2 \frac{K_5 v_1 g_0}{1+K_5 v_1} + k_3\frac{v_1}{v_1+k_4} + q\end{pmatrix}$$

and $u_0 = (u_{0j}) \in W_p^{(2)}(\Omega)$, $v_0 = (v_{0i}) \in W_p^{(2)}(M)$ with $p > n$ are componentwise nonnegative. Also, u_0 and v_0 satisfy the compatibility condition

$$D\frac{\partial u_0}{\partial \eta} = G(u_0, v_0) \quad \text{on } M.$$

Here $k_\alpha, K_\alpha, g_0, c_{max}, b_{-6}$ are same positive constants as described in [29]. We note F, G and H are quasi positive functions. From Theorem 3.2, this system has a unique componentwise nonnegative maximal solution. In order to get global existence, we assume $T_{max} < \infty$. In order to obtain L_p estimates for each of the components, consider (u, v_2). It is easy to see that $G + F_2 \leq k_3$, $H = 0$, and G is linearly bounded above. So the hypothesis of Lemma 6.13 is satisfied. As a result, $u \in L_p(\Omega_{T_{max}})$ and $v_2 \in L_p(M_{T_{max}})$ for all $p > 1$. Now v_1, satisfies the hypothesis of Theorem 3.5. Therefore $v_1 \in W_p^{2,1}(M_{T_{max}})$ for all $p > 1$.

We already have for all $1 \le i \le 2$, $(u, v_i) \in L_p(\Omega_{T_{max}}) \times L_p(M_{T_{max}})$ for all $p \ge 1$. Therefore $G \in L_p(\Omega_{T_{max}})$ for all $p \ge 1$, and $F_i \in L_p(M_{T_{max}})$ for all $p \ge 1$. Consequently, from Theorem 3.6, the solution is bounded, which contradicts the conclusion of Theorem 3.2. As a result the system has a global solution.

Example 3. We look at a simple model to illustrate an interesting open question. Consider the system

$$\begin{aligned} u_t &= \Delta u && x \in \Omega, \quad 0 < t < T \\ v_t &= \Delta_M v + u^2 v^2 && x \in M, \quad 0 < t < T \\ \frac{\partial u}{\partial \eta} &= -u^2 v^2 && x \in M, \quad 0 < t < T \\ u &= u_0 && x \in \Omega, \quad t = 0 \\ v &= v_0 && x \in M, \quad t = 0 \end{aligned} \tag{7.1}$$

where u_0 and v_0 are nonnegative and smooth, and satisfy the compatibility condition. Clearly $H(u) = 0$, $G(u, v) = u^2 v^2$ and $F(u, v) = -u^2 v^2$ satisfy the hypothesis of Theorem 3.3 with $F + G \le 0$ and $G(u, v) \le 0$. Therefore (7.1) has a unique global componentwise nonnegative global solution. However, suppose we make a small change, and consider the system

$$\begin{aligned} u_t &= \Delta u && x \in \Omega, \quad 0 < t < T \\ v_t &= \Delta_M v - u^2 v^2 && x \in M, \quad 0 < t < T \\ \frac{\partial u}{\partial \eta} &= u^2 v^2 && x \in M, \quad 0 < t < T \\ u &= u_0 && x \in \Omega, \quad t = 0 \\ v &= v_0 && x \in M, \quad t = 0 \end{aligned} \tag{7.2}$$

Then we can show there exists a unique maximal componentwise nonnegative solution. We can also obtain L_1 estimates for u and v. Furthermore, it is easy to see that v is uniformly bounded. But our theory cannot be used to determine whether (7.2) has a global solution, and this remains an open question. More generally, it is not known whether replacing G_j in condition $(V_{i,j}2)$ with F_i will result in a theorem similar to Theorem 3.3.

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