Controlled bidirectional remote state preparation in noisy environment: A generalized view

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March 14, 2018

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Abstract

It is shown that a realistic, controlled bidirectional remote state preparation is possible using a large class of entangled quantum states having a particular structure. Existing protocols of probabilistic, deterministic and joint remote state preparation are generalized to obtain the corresponding protocols of controlled bidirectional remote state preparation (CBRSP). A general way of incorporating the effects of two well known noise processes, the amplitude-damping and phase-damping noise, on the probabilistic CBRSP process is studied in detail by considering that noise only affects the travel qubits of the quantum channel used for the probabilistic CBRSP process. Also indicated is how to account for the effect of these noise channels on deterministic and joint remote state CBRSP protocols.

Keywords: Remote state preparation, controlled bidirectional communication, quantum communication, amplitude-damping noise, phase-damping noise.

1 Introduction

The concept of quantum teleportation was introduced by Bennett et al. in 1993 [1]. In teleportation an unknown quantum state is transmitted from a sender (Alice) to a receiver (Bob) by using a shared entanglement and two bits of classical communication. In this process, the unknown quantum state does not travel through the quantum channel and thus this process does not have any classical analogue. This extremely interesting nonclassical nature of the quantum teleportation phenomenon drew considerable attention from the quantum communication community and consequently a large number of modified teleportation schemes have been proposed (for a review see Chapter 7 of Ref. [2]). For example, a large number of proposals have been made for quantum information splitting (QIS) or quantum secret sharing (QSS) [3], controlled teleportation (CT) [4, 5], hierarchical quantum information splitting (HQIS) [6, 7], remote state preparation (RSP) [8, 9, 10]. All these schemes may be viewed as modified teleportation protocols. Considerable attention has been devoted to bidirectional controlled quantum state teleportation (BCST) [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], controlled remote state preparation [21] and joint remote state preparation (JRSP) [22, 23, 24], among others.

The original scheme of quantum teleportation was a one-way scheme in which Alice teleports an unknown qubit to Bob by using two bits of classical communication and an entangled state already shared by them. Subsequently, it was modified by Huelga *et al.* [25, 26] and others to obtain schemes for bidirectional state teleportation (BST), where both Alice and Bob can simultaneously transmit unknown qubits to each other. It was also established that BST is useful for the implementation of nonlocal quantum gates or quantum remote controls. Recently, BST schemes have been generalized to obtain a set of schemes for bidirectional controlled state teleportation (BCST) [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. These BCST schemes are three party schemes where BST is possible provided the supervisor/controller (Charlie) discloses his information (measurement outcome). To be more precise, in a usual BST scheme, Alice and Bob can simultaneously transmit unknown quantum states to each other and reconstruct the state received by them without the help of a third party (Charlie), whereas in a BCST scheme also Alice and Bob can simultaneously transmit unknown quantum states to each other as in BST, but could not reconstruct the state received by them until the supervisor Charlie (third party) allows them to do so.

In another line of research, conventional teleportation schemes were modified to address a specific scenario, where the sender (Alice) knows the state to be teleported, whereas the receiver (Bob) is completely unaware of it. Such modified teleportation schemes are referred to as schemes for the RSP. Thus, an RSP scheme may be viewed as a scheme for teleportation of a known quantum state. RSP is possible with different kinds of quantum channels and it's not limited to the remote preparation of known single qubit states. In fact, several schemes for remote preparation of known multipartite quantum states are also reported in the recent past. Initially, the RSP scheme was designed for a single qubit state [8]. The scheme

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was probabilistic, but it was shown that the task of remote preparation of a known single qubit can be performed using a bit of classical communication and a shared entanglement. Subsequently, this idea was generalized to a deterministic RSP [27], wherein the remote state could be prepared with unit success probability. However, the reduction in classical communication with respect to a quantum teleportation scheme that was achieved in the original scheme was lost here. The idea of RSP has been generalized, recently, in many other ways. For example, RSP schemes are proposed for remote preparation of a 4-qubit GHZ state [28], multi-qubit GHZ state [29], 4-qubit cluster-type state [30, 31], arbitrary two qubit state [32] and W state [33, 34]. Further, schemes for JRSP have been constructed. A JRSP scheme involves at least three parties. The simplest JRSP designed by An [35] can be described as follows. Sender1, Sender2 and receiver share a GHZ state. Sender1 and Sender2 want to jointly transmit a qubit $|\psi\rangle = a|0\rangle + b\exp(i\phi)|1\rangle$ to the receiver, but Sender1 knows $(a,\, b)$ and Sender2 knows ϕ. In such a case Sender1 measures his/her qubit using $\{|u_0\rangle = a|0\rangle + b|1\rangle,\, |u_1\rangle = b|0\rangle - a|1\rangle\}$ basis set and announces the result. Subsequently, Sender2 measures his/her qubit using $\{|v_0\rangle = \frac{|0\rangle+\exp(i\phi)|1\rangle}{\sqrt{2}},\, |v_1\rangle = \frac{\exp(-i\phi)|0\rangle-|1\rangle}{\sqrt{2}}\}$ basis set (depending upon Sender1's measurement outcome Sender2 may have to apply a unitary operation before measuring his/her qubit in $\{|v_0\rangle, |v_1\rangle\}$ basis) and announces the result. Using measurement outcomes of the senders, the receiver can apply appropriate unitary operations and reconstruct the unknown quantum state, which is jointly transmitted by the senders. Extending this idea JRSP schemes were proposed for, among others, W and W-type states [33, 34], arbitrary two qubit state [32] and 4-qubit cluster like state [22]. Further generalizing these ideas, schemes were proposed for controlled RSP [36], controlled JRSP [37], multi-party controlled JRSP [24], deterministic JRSP [38] and JRSP with a passive receiver [23]. Thus, a large number of variants of the RSP scheme have been investigated in a one-directional scenario. However, except a recent work by Cao and An [39], no serious effort has yet been made to realize these features of RSP in bidirectional scenarios. Such a scenario is practically relevant and can be visualized easily in analogy with BCST as follows. Consider that both Alice and Bob wish to remotely prepare quantum states that are known to the senders, but unknown to the receivers and receivers can reconstruct the state only when the supervisor Charlie allows them to do so. A scheme for realizing this task would be referred to as the controlled bidirectional remote state preparation (CBRSP). Such a scheme of CBRSP has been proposed in [39] by using a 5-qubit state

$$|Q\rangle_{A_1A_2B_1B_2C_1} = \frac{1}{2}\left(|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle\right)_{A_1A_2B_1B_2C_1}, \tag{1}$$

where the subscripts A, B and C indicate the qubits of Alice (Sender), Bob (Receiver) and Charlie (Controller), respectively. For our convenience, we can replace A, B and C by S, R, and C, respectively, and after particle swapping rewrite the above state as

$$|Q\rangle_{S_1R_1S_2R_2C_1} = \frac{1}{\sqrt{2}}\left(|\psi^+\rangle_{S_1R_1}|\psi^+\rangle_{S_2R_2}|+\rangle_{C_1} + |\psi^-\rangle_{S_1R_1}|\psi^-\rangle_{S_2R_2}|-\rangle_{C_1}\right), \tag{2}$$

where $|\psi^\pm\rangle = \frac{|00\rangle\pm|11\rangle}{\sqrt{2}}$. Clearly, this is one of the states shown to be useful for BCST in [13]. To be precise, in Ref. [13], it was argued that the general form of the states that are useful for BCST may be described as

$$|\psi\rangle_{S_1R_1S_2R_2C_1} = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle_{S_1R_1}|\psi_2\rangle_{S_2R_2}|a\rangle_{C_1} \pm |\psi_3\rangle_{S_1R_1}|\psi_4\rangle_{S_2R_2}|b\rangle_{C_1}\right), \tag{3}$$

where quantum states $|a\rangle$ and $|b\rangle$ satisfy $\langle a|b\rangle = \delta_{a,b}$, $|\psi_i\rangle \in \{|\psi^+\rangle, |\psi^-\rangle, |\phi^+\rangle, |\phi^-\rangle : |\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle\}$, $|\psi^\pm\rangle = \frac{|00\rangle\pm|11\rangle}{\sqrt{2}}$, $|\phi^\pm\rangle = \frac{|01\rangle\pm|10\rangle}{\sqrt{2}}$ and as before S, R, and C represent sender, receiver and controller[1]. Here $|\psi_i\rangle$ is a Bell state and the condition

$$|\psi_1\rangle \neq |\psi_3\rangle, |\psi_2\rangle \neq |\psi_4\rangle \tag{4}$$

ensures that Charlie's qubit is appropriately entangled with the remaining 4 qubits [13]. To be precise, the receiver and the sender are unaware of the entangled (Bell) states they share unless the controller measures his qubit in $\{|a\rangle, |b\rangle\}$ basis and discloses the result. We know that a shared Bell state and two bits of classical communication is sufficient for teleportation, but without the disclosure of the controller, the receiver will not be able to choose appropriate unitary operations that will be required to reconstruct the state. In what follows, we will extend our earlier work on BCST [13] and Cao and An's recent work on CBRSP [39] to show that if the state (3) satisfies the condition (4) then on the disclosure of the outcome of Charlie's measurement in $\{|a\rangle, |b\rangle\}$ basis, Alice and Bob will know with certainty which two Bell states they share and consequently they will be able to implement a scheme for probabilistic RSP or deterministic RSP in a controlled bidirectional manner.

In the discussion so far we have seen that probabilistic and deterministic RSP, JRSP, controlled RSP and controlled JRSP are studied in detail for one directional cases. We have also noted that RSP and teleportation are closely linked phenomena as RSP can be viewed as teleportation of a known qubit. Interestingly, a set of schemes for BCST have been proposed, but only a single deterministic scheme for CBRSP using 5-qubit cluster states is proposed till date. Neither any scheme for probabilistic CBRSP, nor a scheme for controlled joint BRSP (CJBRSP) is proposed until now. Keeping, these facts in mind, the present paper aims to provide schemes for (i) probabilistic CBRSP, (ii) deterministic CBRSP and (iii) deterministic CJBRSP using quantum states of the generic form (3). Further, we aim to show that the schemes for probabilistic CBRSP, deterministic CBRSP and deterministic CJBRSP can be realized using infinitely many different quantum channels as the state described

[1] It is sufficient to consider $|a\rangle$ and $|b\rangle$ as single qubit states, but it is not essential. One may consider them as 2-qubit in particular or n-qubit (n>1) states in general, but that would only introduce additional complexity.

by (3) can be constructed in 144 different ways for each choice of $\{|a\rangle, |b\rangle\}$ basis and $\{|a\rangle, |b\rangle\}$ can be chosen in infinitely many ways [13]. Thus, the recently proposed Cao and An scheme of deterministic CBRSP is just a special case of infinitely many possibilities and their proposal, which is limited to deterministic CBRSP can be extended to probabilistic CBRSP and deterministic CJBRSP, too.

A practical implementation of any RSP protocol would imply taking into account the effect of the ambient environment on the basic processes that constitute the protocol. For example, all control protocols, as discussed here, depend upon the information sent to the sender, receiver by the controller. This information is encoded in the travel qubits, i.e., the qubits prepared by the controller and sent to different receivers and senders. These travel qubits traverse through physical space and would be susceptible to the influence of the ambient environment, resulting in noise. Here we model such a situation by studying the effect of two well known noise models, viz., the amplitude-damping and phase-damping channels [40, 41, 42] on a probabilistic CBRSP protocol, and indicate how these effects can be taken into account for a deterministic CBRSP as well as for deterministic CJBRSP.

The paper is organized as follows. In Section 2, we propose a protocol of probabilistic CBRSP using quantum states of the form (3). In Section 3, we show that the same states can be used to realize deterministic CBRSP. We propose a protocol of deterministic CJBRSP, in Section 4. In Section 5, we discuss the effect of the amplitude-damping and the phase-damping noise channels on the probabilistic CBRSP scheme described in this paper. Finally, we conclude in Section 6.

2 Probabilistic controlled bidirectional remote state preparation

Let us assume that the sender (S) and receiver (R) share an entangled state $|\phi^-\rangle_{SR} = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)_{SR}$. Now the sender wishes to transmit a known qubit $|\psi\rangle = a|0\rangle + b\exp(i\phi)|1\rangle$ to the receiver. Here, the sender knows the values of a, b and ϕ. However, the receiver is completely unaware of these values.

Now we introduce a new basis set $\{|q_1\rangle, |q_2\rangle\}$, where

$$\begin{array}{rcl} |q_1\rangle & = & a|0\rangle + b\exp(i\phi)|1\rangle = |\psi\rangle, \\ |q_2\rangle & = & b\exp(-i\phi)|0\rangle - a|1\rangle = U_{RSP}|\psi\rangle. \end{array} \tag{5}$$

Using $\{|q_1\rangle, |q_2\rangle\}$ basis set we can write

$$\begin{array}{rcl} |0\rangle & = & a|q_1\rangle + b\exp(i\phi)|q_2\rangle, \\ |1\rangle & = & b\exp(-i\phi)|q_1\rangle - a|q_2\rangle, \end{array} \tag{6}$$

and using (6) we can express the entangled state $|\phi^-\rangle_{SR} = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)_{SR}$ shared by the receiver and sender as follows:

$$\begin{array}{rcl} |\phi^-\rangle_{SR} & = & \frac{1}{\sqrt{2}}\left(|0\rangle_S|1\rangle_R - |1\rangle_S|0\rangle_R\right) \\ & = & \frac{1}{\sqrt{2}}\left(\left(a|q_1\rangle + b\exp(i\phi)|q_2\rangle\right)_S|1\rangle_R - \left(b\exp(-i\phi)|q_1\rangle - a|q_2\rangle\right)_S|0\rangle_R\right)\right) \\ & = & \frac{1}{\sqrt{2}}|q_1\rangle_S\left(a|1\rangle - b\exp(-i\phi)|0\rangle\right)_R + \frac{1}{\sqrt{2}}|q_2\rangle_S\left(a|0\rangle + b\exp(i\phi)|1\rangle\right)_R \\ & = & \frac{1}{\sqrt{2}}\left(|q_2q_1\rangle_{SR} - |q_1q_2\rangle_{SR}\right). \end{array} \tag{7}$$

Now the sender measures his/her qubit in $\{|q_1\rangle, |q_2\rangle\}$ basis set and communicates the result to the receiver. From (7) it is clear that if the sender's measurement outcome is $|q_2\rangle$ then the receiver does not need to do anything to reconstruct the qubit unknown to him, but if the outcome of the sender's measurement is $|q_1\rangle$ then the protocol fails as without the knowledge of ϕ, the receiver will not be able to transform $|q_2\rangle$ into $|q_1\rangle = |\psi\rangle$. Thus, if the sender and receiver share a prior entanglement, then the teleportation of a known qubit requires only one projective measurement and one bit of classical communication. Therefore, probabilistic remote state preparation requires fewer resources than conventional teleportation. Now we may note that RSP is also possible when other Bell states are used as initially shared entanglement. To be precise, in $\{|q_1\rangle, |q_2\rangle\}$ basis, we can write

$$\begin{array}{rcl} |\phi^+\rangle_{SR} & = & \frac{1}{\sqrt{2}}\left(|0\rangle_S|1\rangle_R + |1\rangle_S|0\rangle_R\right) = \frac{1}{\sqrt{2}}Z_R\left(-|q_2q_1\rangle_{SR} + |q_1q_2\rangle_{SR}\right), \\ |\psi^+\rangle_{SR} & = & \frac{1}{\sqrt{2}}\left(|0\rangle_S|0\rangle_R + |1\rangle_S|1\rangle_R\right) = \frac{1}{\sqrt{2}}iY_R\left(-|q_2q_1\rangle_{SR} + |q_1q_2\rangle_{SR}\right), \\ |\psi^-\rangle_{SR} & = & \frac{1}{\sqrt{2}}\left(|0\rangle_S|0\rangle_R - |1\rangle_S|1\rangle_R\right) = \frac{1}{\sqrt{2}}X_R\left(|q_2q_1\rangle_{SR} - |q_1q_2\rangle_{SR}\right). \end{array} \tag{8}$$

Thus, if we consider that the sender and receiver share a Bell state; to prepare a remote state at the receiver's end, sender measures his/her qubit in $\{|q_1\rangle, |q_2\rangle\}$ basis and announces the outcome, then the receiver can reconstruct the state (in those cases where the output of the sender's measurement is $|q_2\rangle$) by applying a unitary operator on his/her (receiver's) qubit, provided the receiver knows which Bell state he/she shared with the sender. The specific relation between the initially shared entangled state and receiver's operation can be obtained from Eq. (7)-(8) and the same is summarized in Table 1. Now, if we consider that Charlie prepares the 5-qubit state (3) and sends qubits S_1, R_2 to Alice and R_1, S_2 to Bob. In this situation Alice and Bob share two Bell states that can be used for bidirectional remote state preparation (say the first Bell state is shared for Alice to Bob transmission and the second one for Bob to Alice transmission). Both of the senders can make the necessary measurements in the rotated bases. To be precise, if Alice wishes to remotely prepare $|\psi\rangle = a|0\rangle + b\exp(i\phi)|1\rangle$ at Bob's end, then she should measure S_1 qubit in $\{|q_1\rangle, |q_2\rangle\}$ basis as given in (5). Similarly, if Bob wishes to remotely prepare

	Initial state shared by the sender and receiver			
	$\|\psi^+\rangle$	$\|\psi^-\rangle$	$\|\phi^+\rangle$	$\|\phi^-\rangle$
Sender's measurement outcome in $\{\|q_1\rangle, \|q_2\rangle\}$ basis	Receiver's operation	Receiver's operation	Receiver's operation	Receiver's operation
$\|q_2\rangle$	iY	X	Z	I
$\|q_1\rangle$	Protocol fails			

Table 1: Relation between the measurement outcomes of the sender and the unitary operations applied by the receiver to implement probabilistic remote state preparation using different initial states.

$|\psi\rangle = a'|0\rangle + b'\exp(i\phi)|1\rangle$ at Alice's end, he should measure his qubit S_2 in $\{|q_1'\rangle, |q_2'\rangle\}$ basis where $|q_1'\rangle = a'|0\rangle + b'\exp(i\phi')|1\rangle$ and $|q_2'\rangle = b'\exp(-i\phi')|0\rangle - a'|1\rangle$. However, the receivers will not be able to apply the required unitary operator unless they know which Bell states were initially shared by them. For this information, they have to wait for Charlie's announcement of a measurement outcome which he obtains by measuring his qubit in $\{|a\rangle, |b\rangle\}$ basis. Specifically, from (3) we can see that if Charlie's measurement outcome is $|a\rangle$ then Alice to Bob (Bob to Alice) RSP channel is $|\psi_1\rangle$ ($|\psi_2\rangle$). Similarly, if Charlie's measurement outcome is $|b\rangle$ then Alice to Bob (Bob to Alice) RSP channel is $|\psi_3\rangle$ ($|\psi_4\rangle$). Thus, Charlie can control the bidirectional remote state preparation protocol described here. Further, since the protocol succeeds only when the sender's measurement outcome is $|q_2\rangle$ ($|q_2'\rangle$), thus the protocol is of probabilistic nature. In brief, we have obtained a generalized Pati-type scheme for probabilistic CBRSP.

3 Deterministic controlled bidirectional remote state preparation

The CBRSP scheme described in the previous section and its parent scheme (one-directional Pati scheme) was probabilistic. In [27] the Pati scheme was generalized to obtain a one-directional deterministic scheme. In their original scheme [27], the sender (S) and receiver (R) start with a shared Bell state $|\psi^+\rangle = \frac{(|00\rangle+|11\rangle)_{SR}}{\sqrt{2}}$, where the sender (S) has the first qubit and the receiver (R) has the second qubit. The sender also prepares another ancillary qubit in state $|0\rangle$ which is indexed with the subscript S'. Subsequently, the sender applies a CONT operation using the S qubit as the control qubit and the S' qubit as the target qubit to obtain a combined state

$$\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|00\rangle+|11\rangle)_{SR}}{\sqrt{2}}|0\rangle_{S'} = \frac{(|000\rangle+|111\rangle)_{SS'R}}{\sqrt{2}} = \text{GHZ}^{0+}, \tag{9}$$

which is nothing but a GHZ state. For our convenience, we have indexed the GHZ states produced in this way with a superscript 0+, where 0 is the decimal value of the first component of the superposition that forms the GHZ state (i.e., decimal value of 000) and the + sign denotes the relative phase between the two components of the superposition. Before, we describe the original protocol of [27] in detail, we would like to note a few things for our convenience. Firstly, a Bell state can be expressed in general as $|\psi\rangle_{\text{Bell}} = \frac{|ij\rangle\pm|\bar{i}\bar{j}\rangle}{\sqrt{2}}$ with $i, j \in \{0, 1\}\}$. Thus,

$$\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|ij\rangle\pm|\bar{i}\bar{j}\rangle)_{\text{SR}}}{\sqrt{2}}|0\rangle_{S'} = \frac{(|iij\rangle\pm|\bar{i}\bar{i}\bar{j}\rangle)_{SS'R}}{\sqrt{2}} = \text{GHZ}^{\text{x}\pm}, \tag{10}$$

where x is the decimal value of binary number iij and $\pm$ denotes the relative phase between the two components of the superposition. This would lead to 4 GHZ states depending upon the choice of initial Bell state. Similarly, we can obtain 4 more GHZ states if the sender prepares the ancillary qubit in the state $|1\rangle$. Specifically, the relation between the GHZ states produced and the Bell state used are as follows:

$$\begin{array}{ccccccc}
\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}|\psi^\pm\rangle_{SR}|0\rangle_{S'} & = & \begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|00\rangle\pm|11\rangle)_{SR}}{\sqrt{2}}|0\rangle_{S'} & = & \text{GHZ}^{0\pm} & = & \frac{(|000\rangle\pm|111\rangle)_{SS'R}}{\sqrt{2}},\\
\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}|\phi^\pm\rangle_{SR}|0\rangle_{S'} & = & \begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|01\rangle\pm|10\rangle)_{SR}}{\sqrt{2}}|0\rangle_{S'} & = & \text{GHZ}^{1\pm} & = & \frac{(|001\rangle\pm|110\rangle)_{SS'R}}{\sqrt{2}},\\
\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}|\psi^\pm\rangle_{SR}|1\rangle_{S'} & = & \begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|00\rangle\pm|11\rangle)_{SR}}{\sqrt{2}}|1\rangle_{S'} & = & \text{GHZ}^{2\pm} & = & \frac{(|010\rangle\pm|101\rangle)_{SS'R}}{\sqrt{2}},\\
\begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}|\phi^\pm\rangle_{SR}|1\rangle_{S'} & = & \begin{array}{c}\text{CNOT}\\ S\rightarrow S'\end{array}\frac{(|01\rangle\pm|10\rangle)_{SR}}{\sqrt{2}}|1\rangle_{S'} & = & \text{GHZ}^{3\pm} & = & \frac{(|011\rangle\pm|100\rangle)_{SS'R}}{\sqrt{2}}.
\end{array} \tag{11}$$

Now we continue with the original idea of [27]. After creating GHZ state (9), the sender (who wishes to remotely prepare a quantum state $|\psi\rangle = a|0\rangle + b\exp(i\phi)|1\rangle$) measures his/her first qubit in a new basis $\{|u_0\rangle = a|0\rangle + b|1\rangle, |u_1\rangle = b|0\rangle - a|1\rangle\}$. Using this, we can write $|0\rangle = a|u_0\rangle + b|u_1\rangle$ and $|1\rangle = b|u_0\rangle - a|u_1\rangle$ and consequently the GHZ state (9) can be expressed as

$$\begin{array}{lcl}\text{GHZ}^{0+} & = & \frac{(|000\rangle+|111\rangle)_{SS'R}}{\sqrt{2}}\\ & = & \frac{1}{\sqrt{2}}\left[|u_0\rangle\left(a|00\rangle+b|11\rangle\right)+|u_1\rangle\left(b|00\rangle-a|11\rangle\right)\right]_{SS'R}\end{array}. \tag{12}$$

		Initial state shared by the sender and receiver			
		$\|\psi^{+}\rangle$	$\|\psi^{-}\rangle$	$\|\phi^{+}\rangle$	$\|\phi^{-}\rangle$
Sender's measurement outcome in $\{\|u_0\rangle,\|u_1\rangle\}$ basis	Sender's measurement outcome in $\{\|v_0\rangle,\|v_1\rangle\}$ basis	Receiver's operation	Receiver's operation	Receiver's operation	Receiver's operation
$\|u_0\rangle$	$\|v_0\rangle$	I	Z	X	iY
$\|u_0\rangle$	$\|v_1\rangle$	Z	I	iY	X
$\|u_1\rangle$	$\|v_0\rangle$	iY	X	Z	I
$\|u_1\rangle$	$\|v_1\rangle$	X	iY	I	Z

Table 2: Relation between the measurement outcomes of the sender and the unitary operations applied by the receiver to implement deterministic remote state preparation using different initial states.

From Eq. (12) we can easily see that if sender's measurement of the first qubit in $\{|u_0\rangle,|u_1\rangle\}$ basis yields $|u_0\rangle$ then the state of the remaining qubits collapses to $|\Psi_0\rangle_{S'R}=(a|00\rangle+b|11\rangle)_{S'R}$ and if the measurement yields $|u_1\rangle$ then the state of the remaining qubits collapses to $|\Psi_1\rangle_{S'R}=(b|00\rangle-a|11\rangle)_{S'R}$. In [27], the sender follows two different routes depending on the outcome of the previous measurement. To be precise, when he/she obtains $|u_0\rangle$ he/she applies a phase gate $\Pi=\left(\begin{array}{cc}1 & 0\\ 0 & \exp\left(2i\phi\right)\end{array}\right)$ on the qubit S' to transform $|\Psi_0\rangle_{S'R}$ to

$$|\Psi_0'\rangle_{S'R}=\Pi_{S'}|\Psi_0\rangle_{S'R}=\left(a|00\rangle+b\exp\left(2i\phi\right)|11\rangle\right)_{S'R}. \tag{13}$$

However, if his/her measurement yields $|u_1\rangle$, then he/she does nothing (i.e., keeps the state $|\Psi_1\rangle_{S'R}=(b|00\rangle-a|11\rangle)_{S'R}$ unchanged). Subsequently he/she measures the S' qubit in $\left\{|v_0\rangle=\frac{|0\rangle+\exp(i\phi)|1\rangle}{\sqrt{2}},|v_1\rangle=\frac{\exp(-i\phi)|0\rangle-|1\rangle}{\sqrt{2}}\right\}$ basis. By expressing S' qubit in $\{|v_0\rangle,|v_1\rangle\}$ basis, the states $|\Psi_0'\rangle_{S'R}$ and $|\Psi_1\rangle_{S'R}$ can be rewritten as follows

$$|\Psi_0'\rangle_{S'R}=\frac{1}{\sqrt{2}}\left[|v_0\rangle_{S'}\left(a|0\rangle+b\exp(i\phi)|1\rangle\right)_R+\exp(i\phi)|v_1\rangle_{S'}\left(a|0\rangle-b\exp(i\phi)|1\rangle\right)_R\right], \tag{14}$$

and

$$|\Psi_1\rangle_{S'R}=\frac{1}{\sqrt{2}}\left[\exp(-i\phi)|v_0\rangle_{S'}\left(b\exp(i\phi)|0\rangle-a|1\rangle\right)_R+|v_1\rangle_{S'}\left(b\exp(i\phi)|0\rangle+a|1\rangle\right)_R\right]. \tag{15}$$

From Eq. (14)-(15) we can clearly conclude that if the sender's measurements yield $|u_0\rangle|v_0\rangle$, $|u_0\rangle|v_1\rangle$, $|u_1\rangle|v_0\rangle$, and $|u_1\rangle|v_1\rangle$ respectively, then the receiver can reconstruct the transmitted state by applying I, Z, iY, and X operators, respectively. Thus we have a protocol for deterministic RSP using an initially shared Bell state which was prepared in $|\psi^{+}\rangle=\frac{(|00\rangle+|11\rangle)_{SR}}{\sqrt{2}}$. It is easy to check that if the receiver and sender start from other Bell states, then, also the above protocol succeeds in deterministic RSP. However, the unitary operations to be performed by the receiver are different for different initial shared states as shown in Table 2. Now, it is easy to observe that if Alice and Bob start with a quantum state of the form (3) and in a manner analogous to the probabilistic CBRSP described in the previous section use one Bell state for Alice to Bob transmission and the other one for Bob to Alice transmission, then both of them will be able to remotely prepare their quantum states provided they know which Bell states they share. Thus, only after Charlie measures his qubit in $\{|a\rangle,|b\rangle\}$ basis and announces the result, Alice and Bob will know which operations from the Table 2 are to be used. However, once Alice and Bob know Charlie's measurement outcome, Alice (Bob) can deterministically prepare her (his) quantum state at Bob's (Alice's) side. Thus all states of the form (3) would lead to deterministic CBRSP. Now it appears obvious why a special case of (3) used by Cao and An in Ref. [39] lead to the first ever protocol of CBRSP. Here we observe that there exists infinitely many 5-qubit states that can be used for CBRSP and have provided a general structure of those states as (3).

4 Deterministic controlled joint bidirectional remote state preparation

In an usual JRSP scheme a quantum state $a|0\rangle+b\exp(i\phi)|1\rangle$ is jointly prepared at the receiver's end by two senders. One of the senders knows the value of a, b and the other one knows the value of ϕ. Taking a careful look into the scheme presented in [27] of deterministic RSP, described above, one can quickly recognize that the projective measurement performed on qubit S (in $\{|u_0\rangle,|u_1\rangle\}$ basis) only requires the knowledge of a, b, while application of unitary operator Π and measurement in $\{|v_0\rangle,|v_1\rangle\}$ basis performed on the qubit S' only requires the knowledge of ϕ. Thus, two different parties can perform this operation. Specifically, we may provide access of qubit S to a person having knowledge of a,b and access of qubit S' to another person having knowledge of ϕ. To see that this is sufficient for JRSP, consider that Sender1 prepares a GHZ^{0+} state in such a way that he/she keeps the first qubit (S), and sends the second (S') and third (R) qubits to Sender2 and the

		Initial state shared by Sender1, Sender2 and receiver			
		GHZ^{0+} or GHZ^{2+}	GHZ^{0-} or GHZ^{2-}	GHZ^{1+} or GHZ^{3+}	GHZ^{1-} or GHZ^{3-}
Sender1's measurement outcome in $\{\|u_0\rangle, \|u_1\rangle\}$ basis	Sender2's measurement outcome in $\{\|v_0\rangle, \|v_1\rangle\}$ basis	Receiver's operation	Receiver's operation	Receiver's operation	Receiver's operation
$\|u_0\rangle$	$\|v_0\rangle$	I	Z	X	iY
$\|u_0\rangle$	$\|v_1\rangle$	Z	I	iY	X
$\|u_1\rangle$	$\|v_0\rangle$	iY	X	Z	I
$\|u_1\rangle$	$\|v_1\rangle$	X	iY	I	Z

Table 3: Table for reconstruction of the quantum state for JRSP. Protocol to be followed for shared states of the form $GHZ_i \in \left\{GHZ^{0\pm}, GHZ^{1\pm}\right\}$ is slightly different from that for the shared state of the form $GHZ_i \in \left\{GHZ^{2\pm}, GHZ^{3\pm}\right\}$.

receiver, respectively. Subsequently, Sender1 measures his/her qubit in $\{|u_0\rangle, |u_1\rangle\}$ basis and communicates the result to the Sender2 and the receiver. Sender2 applies Π gate on his/her qubit if required (i.e., iff the outcome of Sender1's measurement is $|u_0\rangle$) and measures it in $\{|v_0\rangle, |v_1\rangle\}$ basis and communicates the result to the receiver. The receiver can use Table 2 or equivalently Table 3 to apply an appropriate operation to reconstruct $a|0\rangle + b\exp(i\phi)|1\rangle$ at his/her end. Now it is not difficult to observe that by combining Eqs. (3) and (11) we can obtain

$$\begin{array}{ll} & \begin{array}{c}\text{CNOT}\\ S_1\rightarrow S_1'\end{array}\begin{array}{c}\text{CNOT}\\ S_2\rightarrow S_2'\end{array}|\psi\rangle_{S_1R_1S_2R_2C_1}|00\rangle_{S_1'S_2'}\\ = & \frac{1}{\sqrt{2}}\begin{array}{c}\text{CNOT}\\ S_1\rightarrow S_1'\end{array}\begin{array}{c}\text{CNOT}\\ S_2\rightarrow S_2'\end{array}\left(|\psi_1\rangle_{S_1R_1}|\psi_2\rangle_{S_2R_2}|a\rangle_{C_1} \pm |\psi_3\rangle_{S_1R_1}|\psi_4\rangle_{S_2R_2}|b\rangle_{C_1}\right)|00\rangle_{S_1'S_2'},\\ = & \frac{1}{\sqrt{2}}\left(|GHZ_1\rangle_{S_1S_1'R_1}|GHZ_2\rangle_{S_2S_2'R_2}|a\rangle_{C_1} \pm |GHZ_3\rangle_{S_1S_1'R_1}|GHZ_4\rangle_{S_2S_2'R_2}|b\rangle_{C_1}\right),\end{array} \tag{16}$$

with $GHZ_1 \neq GHZ_3$ and $GHZ_2 \neq GHZ_4$ and $GHZ_i \in \left\{GHZ^{0\pm}, GHZ^{1\pm}\right\}$ with $i \in \{1,2,3,4\}$. The 7-qubit state (16) that originated from our 5-qubit channel (3) is clearly sufficient for the deterministic CJBRSP. As from Table 3 we can see that without the knowledge of the GHZ_i shared by Sender1, Sender2 and the receiver, it will be impossible for the receiver to reconstruct the quantum state transmitted. On disclosure of controller's measurement, shared states reduce to the product of two GHZ states. One of them can be used for JRSP in one direction and the other for the JRSP in the other direction. Following the same logic as followed in [13], we can show that for each choice of basis set $\{|a\rangle, |b\rangle\}$ there are 144 alternative ways to satisfy the condition $GHZ_1 \neq GHZ_3$ and $GHZ_2 \neq GHZ_4$ and $GHZ_i \in \left\{GHZ^{0\pm}, GHZ^{1\pm}\right\}$ with $i \in \{1,2,3,4\}$ (without $\pm$ sign) and thus to construct alternative quantum channels of the form (16) (cf. [13]). Interestingly, (16) does not exhaust all the possibilities. For example, we can think of quantum states of the form

$$\begin{array}{c}\text{CNOT}\\ S_1\rightarrow S_1'\end{array}\begin{array}{c}\text{CNOT}\\ S_2\rightarrow S_2'\end{array}|\psi\rangle_{S_1R_1S_2R_2C_1}|11\rangle_{S_1'S_2'} = \frac{1}{\sqrt{2}}\left(|GHZ_1\rangle_{S_1S_1'R_1}|GHZ_2\rangle_{S_2S_2'R_2}|a\rangle_{C_1} \pm |GHZ_3\rangle_{S_1S_1'R_1}|GHZ_4\rangle_{S_2S_2'R_2}|b\rangle_{C_1}\right) \tag{17}$$

with $GHZ_1 \neq GHZ_3$ and $GHZ_2 \neq GHZ_4$ and $GHZ_i \in \left\{GHZ^{2\pm}, GHZ^{3\pm}\right\}$ with $i \in \{1,2,3,4\}$. Once again we will obtain 144 alternative quantum channels for each choice of basis set $\{|a\rangle, |b\rangle\}$. All these states are also useful for CJBRSP. However, in this case we have to slightly modify the intrinsic protocol of JRSP. To be precise, if we assume that Sender1, Sender2 and receiver share a state $|\psi\rangle_{SS'R} = GHZ \in \left\{GHZ^{2\pm}, GHZ^{3\pm}\right\}$, and measurement of Sender1 on S using $\{|u_0\rangle, |u_1\rangle\}$ basis yields $|u_0\rangle$ ($|u_1\rangle$) then Sender2 does nothing (applies unitary operator Π) on the qubit S' before measuring the qubit using $\{|v_0\rangle, |v_1\rangle\}$ basis. Subsequently, using these measurement outcomes, the receiver will be able to reconstruct the unknown state by applying appropriate unitary operators described in Table 3. Combining the above two quantum channels (i.e., combining Eqns. (16) and (17)) now we can also think of a very general quantum channel for CJBRSP of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|GHZ_1\rangle_{S_1S_1'R_1}|GHZ_2\rangle_{S_2S_2'R_2}|a\rangle_{C_1} \pm |GHZ_3\rangle_{S_1S_1'R_1}|GHZ_4\rangle_{S_2S_2'R_2}|b\rangle_{C_1}\right) \tag{18}$$

with $GHZ_1 \neq GHZ_3$ and $GHZ_2 \neq GHZ_4$ and $GHZ_i \in \left\{GHZ^{0\pm}, GHZ^{1\pm}, GHZ^{2\pm}, GHZ^{3\pm}\right\}$ with $i \in \{1,2,3,4\}$. Such a state will obviously work as a quantum channel for CJBRSP. To be precise, in a particular implementation of the scheme all parties know $GHZ_1, GHZ_2, GHZ_3, GHZ_4$, and Sender2 can always apply his/her operation; however in the most general case (for example, consider that $GHZ_1 = GHZ^{1+}$ and $GHZ_3 = GHZ^{3+}$) Sender2 may have to wait till the disclosure of the controller to decide in which case he/she will apply the Π operation.

5 Effect of the amplitude-damping noise and the phase-damping noise on the CBRSP process

In this section, we consider the effect of noise on remotely prepared quantum states when the travel qubits pass through either the amplitude-damping noisy environment or the phase-damping noisy environment. The amplitude-damping noise model is characterized by the following Kraus operators [43]

$$E_0^A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta_A} \end{bmatrix}, \qquad E_1^A = \begin{bmatrix} 0 & \sqrt{\eta_A} \\ 0 & 0 \end{bmatrix}, \tag{19}$$

where η_A $(0 \leq \eta_A \leq 1)$ describes the probability of error due to amplitude-damping noisy environment when a travel qubit pass through it. η_A is also referred to as decoherence rate. Similarly, phase-damping noise model is characterized by the following Kraus operators

$$E_0^P = \sqrt{1-\eta_P}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1^P = \sqrt{\eta_P}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_2^P = \sqrt{\eta_P}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{20}$$

where η_P $(0 \leq \eta_P \leq 1)$ is the decoherence rate for the phase-damping noise.

In Section 2, we have proposed a scheme for probabilistic CBRSP using a 5-qubit quantum channel $|\psi\rangle_{S_1R_1S_2R_2C_1}$ having a general form described by (3). As the state is pure it is straightforward to obtain the corresponding density matrix

$$\rho = |\psi\rangle_{S_1R_1S_2R_2C_1\,S_1R_1S_2R_2C_1}\langle\psi|.$$

Now the effect of the noisy environment described by (19) or (20) on the density operator ρ is

$$\rho_k = \sum_{i,j} E_{i,S_1}^k \otimes E_{j,R_1}^k \otimes E_{j,S_2}^k \otimes E_{i,R_2}^k \otimes I_{2,C_1}\rho\left(E_{i,S_1}^k \otimes E_{j,R_1}^k \otimes E_{j,S_2}^k \otimes E_{i,R_2}^k \otimes I_{2,C_1}\right)^\dagger, \tag{21}$$

where I_2 is a 2×2 identity matrix, $k \in \{A, P\}$. For for $k = A$, i.e., for amplitude-damping noise $i, j \in \{1, 2\}$, while for $k = P$, i.e., for phase-damping noise $i, j \in \{1, 2, 3\}$, and the second subscripts on the Kraus operators are included to specify the specific qubit on which it is to be operated. In the construction of (21) we have considered that the qubit of the controller (C_1) is not affected by the noise as it is not transmitted through the noisy environment. Further, it is assumed that both the qubits sent to Alice (i.e., S_1 and R_2 qubits) are affected by the same Kraus operator and similarly, the qubits R_1 and S_2 sent to Bob are also affected by the same Kraus operator. As a consequence of the noisy environment, the initial quantum channel which was pure gets transformed into a mixed state. Senders and receivers faithfully apply the probabilistic CBRSP scheme on ρ_k. To be precise, we assume that Alice (Sender1) and Bob (Sender2) wish to remotely prepare qubits $a_1|0\rangle + b_1\exp(i\phi_1)|1\rangle$ and $a_2|0\rangle + b_2\exp(i\phi_2)|1\rangle$ at the side of Bob (Receiver1) and Alice (Receiver2), respectively. To do so in accordance with the probabilistic CBRSP scheme described in the present work, S_1 qubit is measured by Alice using $\{|q_1\rangle_{S_1} = a_1|0\rangle + b_1\exp(i\phi_1)|1\rangle,\ |q_2\rangle_{S_1} = b_1\exp(-i\phi_1)|0\rangle - a_1|1\rangle\}$ basis, S_2 qubit is measured by Bob using $\{|q_1\rangle_{S_2} = a_2|0\rangle + b_2\exp(i\phi_2)|1\rangle,\ |q_2\rangle_{S_2} = b_2\exp(-i\phi_2)|0\rangle - a_2|1\rangle\}$, and C_1 qubit is measured by Charlie\controller using $\{|a\rangle, |b\rangle\}$ basis. As in a probabilistic CBRSP scheme, the RSP in a specific direction succeeds only when corresponding senders measurement yields $|q_2\rangle$. For a successful probabilistic CBRSP, measurements on S_1 and S_2 should yield $|q_2\rangle_{S_1}$ and $|q_2\rangle_{S_2}$, respectively. For our convenience, we assume that the measurement of controller yields $|b\rangle$. Thus, to selectively choose these outcomes we have to apply the operator

$$U = (|q_2\rangle_{S_1S_1}\langle q_2|) \otimes I_{2,R_1}\,(|q_2\rangle_{S_2S_2}\langle q_2|) \otimes I_{2,R_2} \otimes |b\rangle_{C_1C_1}\langle b|$$

on ρ_k yielding an unnormalized quantum state

$$\rho_{k_1} = U\rho_k U^\dagger,$$

which can be normalized to yield a quantum state

$$\rho_{k_2} = \frac{\rho_{k_1}}{\mathrm{Tr}\left(\rho_{k_1}\right)}.$$

Now the combined states of the qubits R_1 and R_2 or ρ_{k_3} can be obtained form ρ_{k_2}, by tracing out the qubits that are already measured. Specifically,

$$\rho_{k_3} = \mathrm{Tr}_{S_1S_2C_1}\left(\rho_{k_2}\right).$$

Depending upon the specific choice of the initial quantum channel we may have to apply appropriate Pauli operators on ρ_{k_3} to obtain the final quantum state $\rho_{k,\mathrm{out}}$ which is the product of the quantum states produced on the side of the Receivers 1 and 2 in a noisy environment. Specific noise model is characterized through the index k. We have already assumed that Alice (Sender1) and Bob (Sender2) wish to remotely prepare qubits $a_1|0\rangle + b_1\exp(i\phi_1)|1\rangle$ and $a_2|0\rangle + b_2\exp(i\phi_2)|1\rangle$ at the side of Bob (Receiver1) and Alice (Receiver2), respectively. Thus, the expected final state in the absence of noise is a

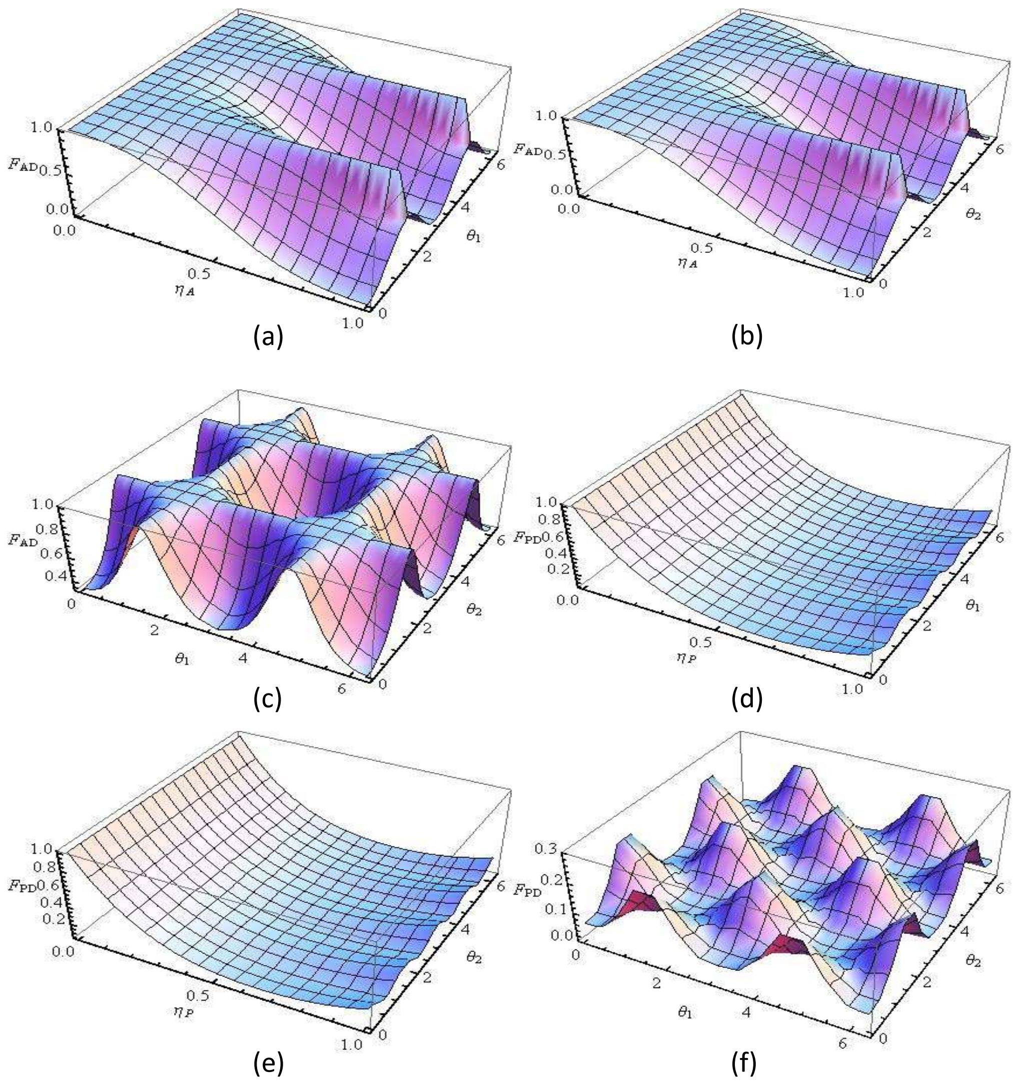

Figure 1: (Color online) Effect of noise on probabilistic CBRSP scheme is visualized through variation of fidelity $F_{\rm AD}$ (for amplitude-damping noise model) and $F_{\rm PD}$ (for phase-damping noise model) with respect to amplitude information of the states to be prepared remotely (i.e., θ_i) and decoherence rates (i.e., η_i) for various situations: (a) amplitude-damping noise with $\theta_2 = \frac{\pi}{4}$, (b) amplitude-damping noise with $\theta_1 = \frac{\pi}{4}$, (c) amplitude-damping noise with $\eta_A = 0.5$, (d) phase-damping noise with $\theta_2 = \frac{\pi}{4}$, (e) phase-damping noise with $\theta_1 = \frac{\pi}{4}$, (f) phase-damping noise with $\eta_P = 0.5$.

product state where Alice (Receiver2) will have qubit $a_2|0\rangle + b_2\exp(i\phi_2)|1\rangle$ in her possession and Bob (Receiver1) will have $a_1|0\rangle + b_1\exp(i\phi_1)|1\rangle$ in his possession. As a consequence, in the ideal situation (i.e., in the absence of noise) in all successful cases of CBRSP the final state would be

$$|T\rangle_{R_1R_2} = (a_1|0\rangle + b_1\exp(i\phi_1)|1\rangle) \otimes (a_2|0\rangle + b_2\exp(i\phi_2)|1\rangle).$$

For computational convenience, we assume that $a_i = \sin\theta_i$ and $b_i = \cos\theta_i$ with $i \in \{1, 2\}$. Thus,

$$|T\rangle_{R_1R_2} = \sin\theta_1\sin\theta_2|00\rangle + \cos\theta_2\sin\theta_1\exp(i\phi_2)|01\rangle + \cos\theta_1\sin\theta_2\exp(i\phi_1)|10\rangle + \cos\theta_1\cos\theta_2\exp(\phi_1+\phi_2)|11\rangle.$$

We can visualize the effect of noise by comparing the quantum state $\rho_{k,\rm out}$ prepared in the noisy environment with the state $|T\rangle_{R_1R_2}$ using fidelity

$$F = \langle T|\rho_{k,\rm out}|T\rangle, \tag{22}$$

which is square of the usual definition of fidelity of two quantum states ρ and σ defined as $F(\sigma,\rho) = Tr\sqrt{\sigma^{\frac{1}{2}}\rho\sigma^{\frac{1}{2}}}$. In the present paper, we have used (22) as the definition of fidelity.

To study the effect of amplitude-damping and phase-damping noise models we assume that the following specific quantum state of the general from (3) is used as a quantum channel for probabilistic CBRSP

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi^+\rangle_{S_1R_1}|\psi^+\rangle_{S_2R_2}|0\rangle_{C_1} + |\phi^-\rangle_{S_1R_1}|\phi^-\rangle_{S_2R_2}|1\rangle_{C_1}\right). \tag{23}$$

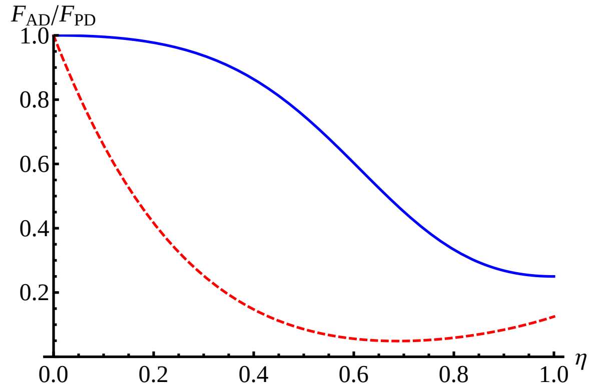

Figure 2: (Color online) Comparison of the effect of amplitude-damping noise (solid line) with phase damping noise (dashed line) by assuming $\eta_A = \eta_P = \eta$ and $\theta_1 = \theta_2 = \frac{\pi}{4}$. In this situation, fidelity for amplitude-damping noise is always greater than that for the phase-damping noise for the same value of decoherence rate η. Fidelity for amplitude-damping noise monotonically decreases with decoherence rate η, but fidelity for phase-damping is found to increase with η, after initially decreasing.

For this particular choice of quantum channel, the above described method of obtaining $\rho_{k,\rm out}$ yields

$$\begin{array}{l}\rho_{A,\rm out}\\ = N_A\left(\begin{array}{cccc} \rho_{A,11} & 4\sin^2\theta_1\sin2\theta_2\exp(-i\phi_2) & 4\sin2\theta_1\sin^2\theta_2\exp(-i\phi_1) & 2\sin2\theta_1\sin2\theta_2\exp(-i\phi_{12})\\ 4\sin^2\theta_1\sin2\theta_2\exp(i\phi_2) & 8\cos^2\theta_2\sin^2\theta_1 & 2\sin2\theta_1\sin2\theta_2\exp(-i\Delta\phi) & 4\cos^2\theta_2\sin2\theta_1\exp(-i\phi_1)\\ 4\sin2\theta_1\sin^2\theta_2\exp(i\phi_1) & 2\sin2\theta_1\sin2\theta_2\exp(i\Delta\phi) & 8\cos^2\theta_1\sin^2\theta_2 & 4\cos^2\theta_1\sin2\theta_2\exp(-i\phi_2)\\ 2\sin2\theta_1\sin2\theta_2\exp(i\phi_{12}) & 4\cos^2\theta_2\sin2\theta_1\exp(i\phi_1) & 4\cos^2\theta_1\sin2\theta_2\exp(i\phi_2) & 8\cos^2\theta_1\cos^2\theta_2 \end{array}\right),\end{array} \tag{24}$$

and

$$\begin{array}{rcl}\rho_{P,\rm out} & = & \frac{(-1+\eta_P)^4}{4}\\ & \times & \left(\begin{array}{cccc} 4\sin^2\theta_1\sin^2\theta_2 & 2\sin^2\theta_1\sin2\theta_2\exp(-i\phi_2) & 2\sin2\theta_1\sin^2\theta_2\exp(-i\phi_1) & \sin2\theta_1\sin2\theta_2\exp(-i\phi_{12})\\ 2\sin^2\theta_1\sin2\theta_2\exp(i\phi_2) & \frac{4\left(1-2\eta_P+2\eta_P^2\right)^2}{(-1+\eta_P)^4}\cos^2\theta_2\sin^2\theta_1 & \sin2\theta_1\sin2\theta_2\exp(-i\Delta\phi) & 2\cos^2\theta_2\sin2\theta_1\exp(-i\phi_1)\\ 2\sin2\theta_1\sin^2\theta_2\exp(i\phi_1) & \sin2\theta_1\sin2\theta_2\exp(i\Delta\phi) & \frac{4\left(1-2\eta_P+2\eta_P^2\right)^2}{(-1+\eta_P)^4}\cos^2\theta_1\sin^2\theta_2 & 2\cos^2\theta_1\sin2\theta_2\exp(-i\phi_2)\\ \sin2\theta_1\sin2\theta_2\exp(i\phi_{12}) & 2\cos^2\theta_2\sin2\theta_1\exp(i\phi_1) & 2\cos^2\theta_1\sin2\theta_2\exp(i\phi_2) & 4\cos^2\theta_1\cos^2\theta_2 \end{array}\right)\end{array} \tag{25}$$

where $\phi_{12} = \phi_1 + \phi_2$, $\Delta\phi = (\phi_1 - \phi_2)$,

$$\begin{array}{rcl}\rho_{A,11} & = & \frac{1}{(-1+\eta)^2}\left(2 - 4\eta_A + 6\eta_A^2 + 2\left(-1 + 2\eta_A + \eta_A^2\right)\cos2\theta_1 + \left(1 - 2\eta_A + 3\eta_A^2\right)\cos\left(2(\theta_1-\theta_2)\right) - 2\cos2\theta_2 + 4\eta_A\cos2\theta_2\right.\\ & + & \left. 2\eta_A^2\cos2\theta_2 + \cos\left(2(\theta_1+\theta_2)\right) - 2\eta_A\cos\left(2(\theta_1+\theta_2)\right) + 3\eta_A^2\cos\left(2(\theta_1+\theta_2)\right)\right),\end{array}$$

and

$$N_A = \frac{(-1+\eta_A)^2}{2\times\left(4 - 8\eta_A + 6\eta_A^2 + 2\eta_A^2\cos2\theta_1 + \eta_A^2\cos2(\theta_1-\theta_2) + 2\eta_A^2\cos2\theta_2 + \eta_A^2\cos2(\theta_1+\theta_2)\right)}.$$

Using (22) and (24) we obtain the fidelity of the quantum state prepared using the proposed probabilistic CBRSP scheme under amplitude-damping noise as

$$F_{\rm AD} = \frac{64 - 128\eta_A + 66\eta_A^2 - 2\eta_A^2\cos4\theta_1 + \eta_A^2\cos\left(4(\theta_1-\theta_2)\right) - 2\eta_A^2\cos4\theta_2 + \eta_A^2\cos\left(4(\theta_1+\theta_2)\right)}{16\left(4 - 8\eta_A + 6\eta_A^2 + 2\eta_A^2\cos2\theta_1 + \eta_A^2\cos\left(2(\theta_1-\theta_2)\right) + 2\eta_A^2\cos2\theta_2 + \eta_A^2\cos\left(2(\theta_1+\theta_2)\right)\right)}. \tag{26}$$

Similarly, by using (22) and (25) we obtain the fidelity of the quantum state prepared using the proposed probabilistic CBRSP scheme under phase-damping noise as

$$\begin{array}{rcl}F_{\rm PD} & = & \frac{1}{64}\left(64 - 256\eta_P + 420\eta_P^2 - 328\eta_P^3 + 118\eta_P^4 + 6\eta_P^2\left(2 - 4\eta_P + 3\eta_P^2\right)\cos4\theta_1 - 16\eta_P^2\left(2 - 4\eta_P + 3\eta_P^2\right)\cos\left(2(\theta_1-\theta_2)\right)\right.\\ & + & 2\eta_P^2\cos\left(4(\theta_1-\theta_2)\right) - 4\eta_P^3\cos\left(4(\theta_1-\theta_2)\right) + 3\eta_P^4\cos\left(4(\theta_1-\theta_2)\right) + 12\eta_P^2\cos4\theta_2 - 24\eta_P^3\cos4\theta_2\\ & + & 18\eta_P^4\cos4\theta_2 - 32\eta^2\cos\left(2(\theta_1+\theta_2)\right) + 64\eta_P^3\cos\left(2(\theta_1+\theta_2)\right) - 48\eta_P^4\cos\left(2(\theta_1+\theta_2)\right)\\ & + & \left. 2\eta_P^2\cos\left(4(\theta_1+\theta_2)\right) - 4\eta_P^3\cos\left(4(\theta_1+\theta_2)\right) + 3\eta_P^4\cos\left(4(\theta_1+\theta_2)\right)\right).\end{array} \tag{27}$$

From (26) and (27) we can see that both the fidelities $F_{\rm AD}$ for amplitude-damping noise and $F_{\rm PD}$ for phase-damping noise depend only on the decoherence rate η_k and the amplitude information (i.e., a_i and b_i) of the states that were attempted to be prepared remotely and fidelities are independent of the corresponding phase information ϕ_i. A similar observation was recently reported in Ref. [43] in the context of JRSP in noisy environments. The method adopted to study the effect of noise in the present paper and in Ref. [43] is quite general and can be easily applied to other schemes of quantum communication in general and to the schemes of bidirectional quantum communication in particular. For example, if we wish to extend the present discussion to the case of the deterministic CBRSP scheme described in Section 3 then we have to use a 7-qubit state, but the Kraus operators would still operate on the same qubits as in the case of the probabilistic CBRSP as the additional qubits used for deterministic CBRSP are prepared locally by the senders and these qubits are not exposed to the noisy environment. However, in case we wish to implement the scheme of deterministic CJBRSP as described in Section 4, we have to apply Kraus operators on 6 qubits (except the qubit of controller) of a 7-qubit quantum channel of the general form (18). Fidelities for deterministic CBRSP, CJBRSP and/or controlled bidirectional teleportation can be obtained easily by following the procedure adopted in this work. However, here we restrict ourselves to the case of probabilistic CBRSP alone. In case the probabilistic CBRSP scheme, realized using the quantum channel (23), is exposed to different noisy environments, then the fidelity corresponding to various noise models would depend on the corresponding parameters, as shown in Fig. 1. Specifically, Fig. 1 a-c (d-f) clearly illustrates the effect of amplitude-damping (phase-damping) noise on the fidelity $F_{\rm AD}$ ($F_{\rm PD}$) and variation of the fidelity with θ_i (or equivalently a_i and b_i) and decoherence rate η_k. We can easily observe that fidelity $F_{\rm AD}$ always decreases with decoherence η_A, (c.f. Fig. 1 a-b), but similar character is not observed in phase-damping channel where we can observe that initially fidelity decreases with η_P and after a point it starts increasing (c.f. Fig. 1 d-e). These characteristics can be visualized more clearly in Fig. 2, where we have compared the effect of amplitude-damping noise with phase damping noise by assuming $\eta_A = \eta_P = \eta$ and $\theta_1 = \theta_2 = \frac{\pi}{4}$. In this situation, fidelity of amplitude-damping channel (solid line in Fig. 2) is always more than that of the phase-damping channel (dashed line in Fig. 2) for the same value of decoherence rate η. Thus we see that information loss is less when the travel qubits are transferred through the amplitude-damping channel as compared to the phase-damping channel, in consistence with the work in [43] where similar considerations were applied to a JRSP process.

6 Conclusion

We have provided protocols of probabilistic CBRSP, deterministic CBRSP and deterministic CJBRSP. Interestingly, the probabilistic CBRSP requires a lesser amount of classical communications compared to the BCST schemes [13] which are deterministic. This advantage of lesser classical communication is lost in the deterministic CBRSP. This observation is analogous to the one-directional case. However, the operations used in deterministic CBRSP are such that it can be modified quickly into the protocol of deterministic CJBRSP. Interestingly, it is shown that the above protocols can be realized using infinitely many alternative quantum channels. Further, the deterministic protocols described above can also be turned in to probabilistic protocols of bidirectional RSP by considering the GHZ states shared by the sender(s) and receiver to be non-maximally entangled. In such a situation we will obtain a controlled bidirectional version of the recently proposed RSP scheme of Wei et al. [44]. The protocols presented here are also interesting because of its potential applications in several practical situations discussed in earlier works on RSP. Further, the presented protocol is experimentally realizable using presently available technologies and for the first time, to the best of our knowledge, effect of noise on a bidirectional quantum communication protocol is described. The effect of amplitude-damping and phase-damping noise, on our protocols, makes the present study much more realistic compared to the existing works as in practice we cannot have a noise-free quantum channel. In addition, the method used here for the study of effect of noise is quite general and it is possible to apply this approach to study the effect of noise in other similar schemes of quantum communication.

Acknowledgment: AP thanks Department of Science and Technology (DST), India for support provided through the DST project No. SR/S2/LOP-0012/2010. He also thanks N. B. An for some technical discussion.

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