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Complementarity in atomic (finite-level quantum) systems: an information-theoretic approach

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Abstract. We develop an information theoretic interpretation of the number-phase complementarity in atomic systems, where phase is treated as a continuous positive operator valued measure (POVM). The relevant uncertainty principle is obtained as an upper bound on a sum of knowledge of these two observables for the case of two-level systems. A tighter bound characterizing the uncertainty relation is obtained numerically in terms of a weighted knowledge sum involving these variables. We point out that complementarity in these systems departs from mutual unbiasededness in two signalificant ways: first, the maximum knowledge of a POVM variable is less than log(dimension) bits; second, surprisingly, for higher dimensional systems, the unbiasedness may not be mutual but unidirectional in that phase remains unbiased with respect to number states, but not vice versa. Finally, we study the effect of non-dissipative and dissipative noise on these complementary variables for a single-qubit system.

1 Introduction

Two observables A and B of a d-level system are called complementary if knowledge of the measured value of A implies maximal uncertainty of the measured value of B, and vice versa [1,2]. Complementarity is an aspect of the Heisenberg uncertainty principle, which says that for any state ψ, the probability distributions obtained by measuring A and B cannot both be arbitrarily peaked if A and B are sufficiently non-commuting. Heisenberg uncertainty is traditionally expressed by the relation

$$\triangle_\psi A \triangle_\psi B \geq \frac{1}{2}|\langle [A,B]\rangle_\psi|, \tag{1}$$

where $(\triangle_\psi A)^2 = \langle A^2\rangle_\psi - (\langle A\rangle_\psi)^2$. However, this representation of the Heisenberg uncertainty relation has the disadvantage that the right hand side of Eq. (1) is not a fixed lower bound but is state dependent. For example, if ψ is an eigenstate of A, then both $\triangle_\psi A$ and the right hand side of Eq. (1) vanish, so that no restriction is imposed on the uncertainty in B. To improve this situation, an information theoretic (or "entropic") version of the Heisenberg uncertainty relationship has been proposed [1,2,3], which relies on Shannon entropy of measurement outcomes as a measure of uncertainty [4,5]. An application of this idea to obtain an entropic uncertainty relation for oscillator systems in the Pegg-Barnett scheme [6] has been made in Ref. [7], and for entropic uncertainty relations among more than two complementary variables, in Ref. [8].

Given two observables $A \equiv \sum_a a|a\rangle\langle a|$ and $B \equiv \sum_b b|b\rangle\langle b|$, let the entropy generated by measuring A or B on a state $|\psi\rangle$ be given by, respectively, $H(A)$ and $H(B)$. The information theoretic representation of the Heisenberg uncertainty principle states that $H(A)+H(B) \geq 2\log\left(\frac{1}{f(A,B)}\right)$, where $f(A,B) = \max_{a,b}|\langle a|b\rangle|$, and $H(\cdot)$ is the Shannon binary entropy. We note that $f(A,B) \geq d^{-1/2}$, where d is the (finite) dimension of the system. A pair of observables, A and B, for which $f(A,B) = d^{-1/2}$ are said to form mutually unbiased bases (MUB) [9,10]. Thus, any $|a\rangle$ is an equal amplitude superposition in the basis $\{|b\rangle\}$ and vice versa. Conventionally, two Hermitian observables are called complementary only if they are mutually unbiased. Given a mutually unbiased pair of Hermitian observables, A and B, the Heisenberg uncertainty relation takes the form

$$H(A) + H(B) \geq \log d. \tag{2}$$

A further advantage of the entropic version of the uncertainty principle over (1) is that unlike the latter, it is insensitive to eigenvalue relabeling, and depends only on the probability distribution obtained by measuring A or B on a given state [3].

Even the information theoretic representation (2) may not in general be suitable if A or B is not discrete, because the continuous analog of $H(A)$, which is $H_c(p) \equiv -\int_x dx\, p(x)\log[p(x)]$, is not positive definite, as can be seen from the case where the probability distribution is given by $p(x) = 2$ for $x \in [0, \frac{1}{2}]$ and $p(x) = 0$ for $x \in (\frac{1}{2}, 1]$, where we find $H_c(p) = -\log 2$. It is well possible that this pathological behaviour does not afflict classes of physical states of interest. In particular, we verified this in the case of the *phase distribution* of two- and four-level atomic systems. However, we are not aware that this is generically true. In any case, this potential problem can be generally overcome if the uncertainty principle is expressed in terms of relative entropy (also called Kullbäck-Leibler divergence, which is always positive) [11], instead of Shannon entropy. An example of where this finds application would be when one of the observables, say A, is bounded, and its conjugate B is described not as a Hermitian operator but as a continuous-valued POVM. An instance of this kind, considered below in detail, is the number and phase of an atomic system. Here we show that the relative entropic definition can be used to express complementarity of number and phase, where the notion of complementarity is extended to accomodate POVMs. We thus makes this intuitive notion more concrete. Here the 'number' variable is analogous to energy in oscillator systems (in the sense of having discrete eigenvalues with fixed difference between consecutive values) and amplitude of light field (eg., a laser, in the sense of being conjugate to a phase variable). We note that recourse to relative entropy is not necessary for a POVM of discrete variables [12], since Shannon entropy is well defined in this case.

The quantum description of phases [13] has a long history [6,14,15,16,17,18]. Pegg and Barnett [6], following Dirac [14], carried out a polar decomposition of the annihilation operator and defined a Hermitian phase operator in a finite-dimensional Hilbert space. In their scheme, the expectation value of a function of the phase operator is first carried out in a finite-dimensional Hilbert space, and then the dimension is taken to the limit of infinity. However, it is not possible to interpret this expectation value as that of a function of a Hermitian phase operator in an infinite-dimensional Hilbert space [19,20]. To circumvent this problem, the concept of phase distribution for the quantum phase has been introduced [19,21]. In this scheme, one associates a phase distribution to a given state such that the average of a function of the phase operator in the state, computed with the phase distribution, reproduces the results of Pegg and Barnett.

An interesting question to ask is how mutually unbiased observables behave in the presence of noise. Intuitively, one would expect that the uncertainty or entropy of each observable should be non-decreasing under the effect of noise. However, this is not generally true, as seen for example in the case of a quantum deleter [22,23], where uncertainty in the computational basis vanishes asymptotically during a qubit's dissipative interaction with a vacuum bath. Here we study number and phase of atomic systems subjected to both non-dissipative and dissipative noise. Noise can be thought of as a manifestation of an open system effect [24]. The total Hamiltonian is $H = H_S + H_R + H_{SR}$, where S stands for the system, R for the reservoir and SR for the system-reservoir interaction. The evolution of the system of interest S (in this case the atomic system) is studied taking into account the effect of its environment R, through the SR interaction term, making the resulting dynamics non-unitary. The open system effects can be broadly classified into non-dissipative, corresponding to the case where $[H_S, H_{SR}] = 0$ resulting in decoherence without dissipation or dissipative, corresponding to the case where $[H_S, H_{SR}] \neq 0$ resulting in decoherence along with dissipation [25].

A class of observables that may be measured repeatedly with arbitrary precision, with the influence of the measurement apparatus on the system being confined strictly to the conjugate observables, is called quantum non-demolition (QND) or back-action evasive observables [26,27,28,29]. Such a measurement scheme was originally introduced in the context of the detection of gravitational waves [30,31]. The non-dissipative open system effect described above would be a QND effect. Since they describe dephasing without dissipation, a study of phase diffusion in such a situation would be important from the context of a number of experimental situations. A study of the quantum phase diffusion in a number of QND systems was carried out in Ref. [32] using the phase distribution approach. In Ref. [23], the above study was extended to include the effect of dissipation on phase diffusion. This would be under the rubric of a dissipative open system effect, described above.

In this paper we study three broad, related problems: first, we formulate a novel characterization of the Heisenberg uncertainty relationship in terms of Kullbäck-Leibler divergence (or relative entropy). Second, we motivate it by applying it to a study of complementarity in an angular momentum system, which involves a continuous variable POVM; lastly, we study the behavior of complementary variables when subjected to dissipative and non-dissipative (purely dephasing) noise.

The plan of the paper is as follows. In Section 2, we introduce the concept of phase distribution in an atomic system which would be used subsequently. In Section 3, we motivate and develop an information theoretic representation of complementarity as applied to a two-level atomic system, with a brief discussion of a four-level atomic system. Since any system of interest would, inevitably, be surrounded by an environment which would effect its dynamics, it is of relevance to discuss the above ideas of complementarity in the context of open quantum systems. We do this in Section 4 by recapitulating relevant work [23,25,32,33,34] on open quantum systems, of relevance here. Section 4.1 deals with the non-dissipative open system effect, described by the phase damping channel [4,25,32,33], and Section 4.2 discusses the dissipative open system effect, described by the squeezed generalized amplitude damping channel [33,34]. The reason for the above terminologies is the connection of the dynamics generated by these processes with the noise effects pertinent to quantum information [33]. For completeness, we relegate some technical details pertaining to these noisy channels to Appendix A and B, where the physical processes underlying these channels are also briefly discussed. In Section 5 we make our conclusions and discuss some open questions coming out of our work.

2 Phase distribution

It is not possible to interpret the expectation value of a function of the phase operator, in the Pegg and Barnett scheme [6], as the expectation value of a function of a Hermitian phase operator in an infinite-dimensional Hilbert space [19,20]. This motivates the introduction of the phase distribution for oscillator systems [19,21]. Interestingly, the concept of phase distribution can also be extended to atomic systems [35], which we study here . The phase distribution $\mathcal{P}(\phi)$, ϕ being related to the phase of the dipole moment of the system, is given by

$$\mathcal{P}(\phi) = \frac{2j+1}{4\pi} \int_0^{\pi} d\theta \sin(\theta) Q(\theta, \phi), \tag{3}$$

where $\mathcal{P}(\phi) > 0$ and is normalized to unity, i.e., $\int_0^{2\pi} d\phi \mathcal{P}(\phi) = 1$. In the above, j is the angular momentum of the atomic system. The quantity ϕ is important in the context of atomic coherences and the interferometry based on such coherences [35]. Here $Q(\theta, \phi)$ is defined as

$$Q(\theta, \phi) = \langle \theta, \phi | \rho^s | \theta, \phi \rangle, \tag{4}$$

where $|\theta, \phi\rangle$ are the atomic coherent states [36,37] given by an expansion over the Wigner-Dicke states [38], which are the simultaneous eigenstates of the angular momentum operators J^2 and J_Z, as

$$|\theta, \phi\rangle = \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} (\sin(\theta/2))^{j+m} (\cos(\theta/2))^{j-m} |j, m\rangle e^{-i(j+m)\phi}. \tag{5}$$

It can be shown that the angular momentum operators J_ξ, J_η and J_ζ (obtained by rotating the operators J_x, J_y and J_z through an angle θ about an axis $\hat{n} = (\sin\phi, -\cos\phi, 0)$), being mutually non-commuting, obey an uncertainty relationship of the type $\langle J_\xi^2\rangle\langle J_\eta^2\rangle \geq \frac{1}{4}\langle J_\zeta^2\rangle$. Atomic coherent states (obtained by rotating the Wigner-Dicke states via θ and ϕ as above) are precisely those states that saturate this bound, from which the name is derived [35]. For two level systems, they exhaust all pure states, whereas for larger dimensions, this is no longer true. Using Eq. (4) in Eq. (3), with insertions of partitions of unity in terms of the Wigner-Dicke states, we can write the phase distribution function as [32]

$$\mathcal{P}(\phi) = \frac{2j+1}{4\pi}\int_0^\pi d\theta \sin\theta \sum_{n,m=-j}^{j} \langle\theta,\phi|j,n\rangle\langle j,n|\rho^s(t)|j,m\rangle\langle j,m|\theta,\phi\rangle. \quad (6)$$

The phase distribution $\mathcal{P}(\phi)$, taking into account the environmental effects, have been studied in detail for QND as well as dissipative systems in [32,23] for physically interesting initial conditions of the system S, i.e., (a). Wigner-Dicke state, (b). atomic coherent state and (c). atomic squeezed state.

3 Information theoretic representation of complementarity

The relative entropy associated with a discrete distribution $f(j)$ with respect to a distribution $g(j)$ defined over the same index set, is given by

$$S(f||g) = \sum_j f(j)\log\left(\frac{f(j)}{g(j)}\right). \quad (7)$$

It can be thought of as a measure of 'distance' of distribution f from distribution g in that $S(f||g) \geq 0$, where the equality holds if and only if $f(j) = g(j)$ [4]. Consider random variable F with probability distribution f. We will define $R(F)$ as the relative entropy of f with respect to the uniform distribution $\frac{1}{d}$, i.e.,

$$R(F) \equiv R[f(j)] = \sum_j f(j)\log(df(j)). \quad (8)$$

As a measure of distance from a uniform distribution, which has maximal entropy, $R(F)$ can be interpreted as a measure of *knowledge*, as against uncertainty, of the random variable described by distribution f. The following theorem re-casts Heisenberg uncertainty principle in terms of relative entropy.

Theorem 1 *Given two mutually unbiased Hermitian observables A and B, the uncertainty relation (2) is equivalent to*

$$R(A) + R(B) \leq \log d, \quad (9)$$

where d is the (finite) dimension of the system.

Proof. Let the distribution obtained by measuring A and B on a given state be, respectively, $\{p_j\}$ and $\{q_k\}$. The l.h.s is given by

$$\begin{aligned} S\left(A||\frac{1}{d}\right) + S\left(B||\frac{1}{d}\right) &= \sum_j p_j\log(dp_j) + \sum_k q_k\log(dq_k) \\ &= -[H(A)+H(B)] + 2\log d \quad (10) \\ &\leq -2\log\left(\frac{1}{f(A,B)}\right) + 2\log d. \quad (11) \end{aligned}$$

This is the general result for any two non-commuting Hermitian observables. If A and B are mutually unbiased, then $f(A,B) = d^{-\frac{1}{2}}$, and the theorem follows. It follows from the concavity of H, and thus from the convexity of R, that the inequality Eq. (9) derived for pure states holds also for mixed states. ■

Physically, Eq. (9) expresses the fact that simultaneous knowledge of A and B is bounded above by $\log d$. This is in contrast to inequality (2), which is bounded below, being a statement on the sum of ignorances or uncertainties. Both are equivalent ways of expressing the fact that the probability distributions obtained by measuring A and B on several identical copies of a given state cannot both peak simultaneously.

In terms of R, two Hermitian observables A and B of a d-level system are called mutually unbiased if the maximal knowledge of the measured value of A, given by $\log d$ bits, implies minimal knowledge of the measured value of B, given by 0 bits, and vice versa. In anticipation of the introduction of POVMs instead of Hermitian observables, we will find it convenient to weaken the definition of mutual unbiasedness and call two variables A and B (of which one or both of them may be a POVM) as *quasi-mutually unbiased* if the maximal knowledge of the measured value of A implies minimal knowledge of the measured value of B, and vice versa. The maximum knowledge no longer being $\log d$ bits, but lesser, the pair A and B may be called quasi-mutually unbiased bases (quasi-MUB's), an extension of the concept of MUB from the case of orthonormal bases to that of non-orthonormal bases.

If two observables are not mutually unbiased, then $\log d$ does not bound from above the knowledge sum $R_T \equiv R(A) + R(B)$, and there exist states such that the corresponding sum satisfies $R_T > \log d$. Intuitively, this is because in the case of two observables that are not mutually unbiased, knowledge of the two observables pertaining to a given state may simultaneously peak. For example, consider the qubit observables σ_z and $\mathbf{n}\cdot\sigma$ in the Hilbert space $\mathbb{C}^2$, where $\mathbf{n} = (\sin\theta, 0, \cos\theta)$ and σ is the vector of Pauli matrices. It can be seen using Eq. (10) that any eigenstate of $\mathbf{n}\cdot\sigma$ corresponds to the knowledge sum $R_T = 2 - H(\cos^2(\theta/2))$. This sum is greater than one, except for $\theta = \pi/2$, which corresponds to the mutually unbiased observable σ_x.

Eq. (7) has a natural extension to the continuous case, given by

$$S(f||g) = \int dp\ f(p) \log\left(\frac{f(p)}{g(p)}\right). \tag{12}$$

As in the discrete case, we define $R(f)$ as relative entropy setting $g(p)$ to a continuous constant function. In particular, the relative entropy of $\mathcal{P}(\phi)$ with respect to a uniform distribution $\frac{1}{2\pi}$ [32,23] over ϕ is given by the functional

$$R[\mathcal{P}(\phi)] = \int_0^{2\pi} d\phi\ \mathcal{P}(\phi) \log[2\pi\mathcal{P}(\phi)], \tag{13}$$

where the $\log(\cdot)$ refers to the binary base.

We define minimum entropy states with respect to an observable as states that yield the minimum entropy when the observable is measured on them. In the context of relative entropy, these states can be generalized to what may be called maximum knowledge (MXK) states, which are applicable even when the measured variable is continuous. For projector valued measurements (PVMs), clearly any eigenstate is a MXK state, with a corresponding entropy of zero and knowledge $R = \log d$. PVMs, projectors to the eigenstates of a Hermitian operator representing an observable, satisfy three axiomatic requirements: they are positive operators that form a partition of unity; further they satisfy the orthonormalcy condition $\hat{P}_j\hat{P}_k = \delta_{jk}\hat{P}_j$, where $\hat{P}_j$ is a measurement operator. The last property implies the idempotency of projectors, which captures the idea that projective measurements are repeatable. From a quantum information perspective, it is useful to consider generalized measurements in which the operator elements M_m may not be orthonormal, but satisfy the completeness condition $\sum_m M_m^\dagger M_m = I$ and $M_m^\dagger M_m \geq 0$ [4]. In the context of a qubit, for a generalized measurement, the knowledge corresponding to a MXK state can be less than 1, i.e., $R(|\mathrm{MXK}\rangle) \leq 1$. For a PVM, we have $R(|\mathrm{MXK}\rangle) = 1$, whereas a POVM considered here is a measurement strategy such that $R(|\mathrm{MXK}\rangle) < 1$. The reason is that whereas PVM is an orthonormal resolution of unity, a POVM forms a non-orthonormal resolution of unity [39]. POVMs are useful elsewhere, in quantum information, as general measurement strategies for optimally distinguishing states [4].

A plot of $R_\phi \equiv R[\mathcal{P}(\phi)]$ for a two-level atomic system in an atomic coherent state $|\alpha', \beta'\rangle$ with $\mathcal{P}(\phi) = \frac{1}{2\pi}\left[1 + \frac{\pi}{4}\sin(\alpha')\cos(\beta' - \phi)\right]$ [32,23], is given by the dashed curve in Figure (1). We note that R_ϕ has no dependence on β' because β' occurs in $\mathcal{P}(\phi)$ only as the translation $\phi - \beta'$, and R_ϕ is translation invariant, i.e., unchanged under the transformation $\phi \longrightarrow \phi + \Delta$. The maximum knowledge R_ϕ of about 0.245 occurs at $\alpha' = \pi/2$. The corresponding continuous family of states $|\pi/2, \beta'\rangle$ forms the MXK states or *quasi-eigenstates* of the phase observable. These are equatorial states on the Bloch sphere, having the form $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi_0}|1\rangle)$. That R_ϕ is less that 1 for these states reflects the fact that here phase ϕ is a POVM.

In analogy with the oscillator case, the Wigner-Dicke or excitation states may be thought of as 'number states', thereby making J_z the 'number observable', whose distribution is $p(m)$, given in Eq. (14). The 'number' distribution given by

$$p(m) = \langle j, m|\rho^s(t)|j, m\rangle, \tag{14}$$

is considered as complementary to $\mathcal{P}(\phi)$ [35]. It is of interest to ask whether they are complementary in the sense of MUBs.

In the manner of Eq. (8), we can define $R_m \equiv R[p(m)]$ as knowledge of the number variable. We note that J_z and phase ϕ have a reciprocal behavior reminiscent of MUBs: the eigenstates of J_z, i.e., Wigner-Dicke states, correspond to minimal knowledge $R_\phi(= 0)$, as seen from the dashed curve in Figure (1). This can be seen by noting that for the Wigner-Dicke states $|j, \tilde{m}\rangle$, the phase distribution is [32]

$$\mathcal{P}(\phi) = \frac{2j+1}{2\pi}\binom{2j}{j+\tilde{m}}\mathcal{B}\left[j+\tilde{m}+1, j-\tilde{m}+1\right] = \frac{1}{2\pi}, \tag{15}$$

where $\mathcal{B}$ stands for the Beta function. Thus, it follows via Eq. (13) that the knowledge R_ϕ vanishes. Conversely, we note that the states which minimize R_ϕ are the Wigner-Dicke states. To see this, we observe that if $\mathcal{P}(\phi)$ is constant, then in Eq. (6), each term in the summation, which is proportional to $e^{i(m-n)\phi}$, must individually be independent of ϕ. Since ϕ is arbitrary, this is possible only if $m = n$, i.e., the state ρ^s is diagonal in the Wigner-Dicke basis. Thus, MXK states of m correspond precisely to minimum knowledge (MNK) states of ϕ.

The plot of relative entropy R_m for all atomic coherent states is given by the bold curve in Figure (1). The equatorial states on the Bloch sphere, the MXK of ϕ, are precisely equivalent to the MNK states of m (characterized by $R_m = 0$), as can be seen from comparing the dashed and bold curves in Figure (1). Thus number and phase share with MUBs the reciprocal property that maximum knowledge of one of them is simultaneous with minimal knowledge of the other, but differs from MUBs in that the maximum possible knowledge of ϕ is less than $\log(d) = 1$ bit, essentially on account of its POVM nature.

Two variables form a quasi-MUB if any MXK state of either variable is an MNK state of the other, where the knowledge of the MXK state may be less than $\log d$ bits. Thus, J_z and ϕ are quasi-MUB's (but not MUB's), and are complementary in the extended sense.

From the dot-dashed curve in this Figure, we numerically find an expression of the uncertainty principle to be

$$R_T \equiv R_\phi + R_m \leq 1 \tag{16}$$

for all states (pure or in general mixed) in $\mathbb{C}^2$, in analogy with Eq. (9). The inequality is saturated for the Wigner-Dicke states.

As an expression of the uncertainty principle, the relation (16) still leaves some room for improvement. First, it is not a tight bound. In particular, for equatorial states it permits R_ϕ to be as high as 1, whereas as seen from the dashed curve in Figure 1, the maximum value of $R_\phi \approx 0.245$. We note that the bound cannot be tightened simply by decreasing the r.h.s, since it is saturated for Wigner-Dicke states. Further, the variable ϕ takes values in the interval

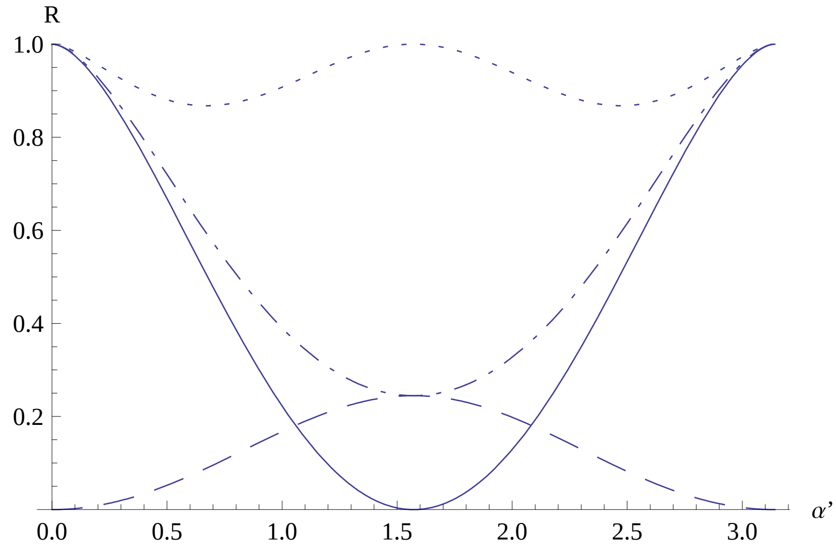

Fig. 1. Plot of knowledge R with respect to α' of a qubit starting in an atomic coherent state $|\alpha', \beta'\rangle$. Knowledge R is symmetric with respect to β', which is therefore not depicted in the figure. The individual curves are: $R_\phi \equiv R[\mathcal{P}(\phi)]$ (dashed curve), $R_m \equiv R[\mathcal{P}(m)]$ (bold curve), the knowledge sum $R_\phi + R_m$ (dot-dashed) and the weighted knowledge sum $R_S(\mu) = \mu R_\phi + R_m$ with $\mu = 4.085$ (dotted curve). The maximum of R_ϕ is not 1 but 0.245 bits, which occurs at $\alpha' = \pi/2$, i.e., the equatorial states on the Bloch sphere, which are thus the MXK states or quasi-eigenstates of ϕ. The minimum value of $R_\phi = 0$ occurs at $\alpha' = 0$ and π, corresponding to the Wigner-Dicke states.

$[0, 2\pi]$ irrespective of the dimensionality of the Hilbert space, unlike m, which takes d values. Consequently, R_ϕ, unlike R_m, is not seen to be bounded by the dimension of the Hilbert space in a straightforward way. To see that in general $R[p(x)]$ increases without bound, consider the probability distribution $p(x) = x_0 > 1$ in $x \in [0, \frac{1}{x_0}]$ and $p(x) = 0$ in $x \in (\frac{1}{x_0}, 1]$, for which we find $R(p(x)||1) = \log x_0$.

One way to address these problems is to generalize (16) to a family of inequalities, parametrized by $\mu > 0$, of the form

$$R_S(\mu) \equiv \mu R_\phi + R_m \leq 1 \tag{17}$$

for all states in $\mathbb{C}^2$. We find that the largest value of μ such that inequality (17) is satified over all state space is $(r_\phi)^{-1} \approx 4.085$, where r_ϕ is the value of R_ϕ for the equatorial states, the MXK states of ϕ. A plot of $R_S(1/r_\phi)$ over pure states is shown as the dotted curve in Figure 1. Comparing this with the dot-dashed curve in Figure 1, we find that $R_S(\mu)$ is bounded more tightly than $R_T \equiv R_S(1)$.

From Figure 1, we find that the two Wigner-Dicke states and all equatorial states may be regarded as *coherent* with respect to the number-phase pair, in that they maximize the knowledge sum and are thus closest to classical states. We note of course that this definition of state coherence differs from the conventional one for atomic states, defined with respect to angular momentum operators. Unless we use μR_ϕ in place of R_ϕ, only the Wigner-Dicke states could be called coherent in the new sense.

We now briefly extend the entropic version of complementarity to a higher spin system, which is seen to present a new feature. We consider a spin-3/2 (four-level) system, whose general state is given by the ansatz

$$|\psi\rangle = r_\alpha e^{i\theta_\alpha}|\frac{3}{2}, -\frac{3}{2}\rangle + r_\beta e^{i\theta_\beta}|\frac{3}{2}, -\frac{1}{2}\rangle + r_\gamma e^{i\theta_\gamma}|\frac{3}{2}, +\frac{1}{2}\rangle + r_\delta|\frac{3}{2}, +\frac{3}{2}\rangle) \tag{18}$$

where $r_\alpha^2 + r_\beta^2 + r_\gamma^2 + r_\delta^2 = 1$, and a global phase is omitted. Using Eq. (18) in Eq. (6), we find

$$
\begin{aligned}
P(\phi) = \frac{1}{\pi}\Bigg[\frac{1}{2} &+ \left(\frac{15\pi r_\alpha r_\beta}{32\sqrt{3}}\right)\cos(\phi-\theta_\alpha\theta_\beta) + \left(\frac{r_\alpha r_\gamma}{\sqrt{3}}\right)\cos(2\phi-\theta_\alpha\theta_\gamma) \\
&+ \left(\frac{3\pi r_\alpha r_\delta}{32}\right)\cos(3\phi-\theta_\alpha) + \left(\frac{9\pi r_\beta r_\gamma}{32}\right)\cos(\phi-\theta_\beta+\theta_\gamma) + \left(\frac{r_\beta r_\delta}{\sqrt{3}}\right)\cos(2\phi-\theta_\beta) \\
&+ \left(\frac{15\pi r_\gamma r_\delta}{32\sqrt{3}}\right)\cos(\phi-\theta_\gamma)\Bigg]. \qquad (19)
\end{aligned}
$$

As before, we compute 'number' knowledge $R_m(r_\alpha, r_\beta, r_\gamma)$ using Eq. (8), and phase knowledge $R_\phi(r_\alpha, r_\beta, r_\gamma, \theta_\alpha, \theta_\beta, \theta_\gamma)$ using Eq. (13). It may be verified that for 'number' states (for which r_α or r_β or r_γ or r_δ is 1), $R_\phi = 0$. In fact, it may be seen from Eqs. (5), (6) and (15), that a general property of atomic systems is that a Wigner-Dicke state is equivalent to a MNK phase state in any finite dimension. On the other hand, numerically searching over all possible states of the form (18), we find that the maximum value 0.86 bits of R_ϕ occurs at $\Psi(r_\alpha = 0.36, r_\beta = 0.61, r_\gamma = 0.61, \theta_\alpha = \pi, \theta_\beta = 0, \theta_\gamma = \pi)$, which is not an equal amplitude superposition of 'number' states. Thus, remarkably, for the spin-3/2 case, MXK phase states do not correspond to MNK 'number' states, even though the converse is true. We expect that this unidirectionally (as against mutually) unbiased behavior will persist even for higher spin systems. Phase and 'number' therefore do not here form a quasi-MUB as defined for the single qubit case, and may be considered complementary only in an even more weak sense. This is in contrast to the case where the observables are Hermitian, where five MUBs are known to exist in four dimensions [10].

As in the two-level case, one way to address this problem is to generalize (16) to a family of inequalities, parametrized by $\mu_2 > 0$, of the form

$$R_S(\mu_2) \equiv \mu_2 R_\phi + R_m \leq 2 \qquad (20)$$

over all states in $\mathbb{C}^4$. Our strategy is to numerically search over all states of the form (18)– other than the Wiger-Dicke states, where $R_m = 2$ and $R_\phi = 0$– in order to determine the largest value of μ_2 such that inequality (20) is *just* satified, i.e., the inequality must be satisfied at all points, with the equality being valid for at least one point. By this method, we find $\mu_2 = 1.973$ with the maximum $R_S(\mu_2)$ of 2 occuring at $\psi_p \equiv \psi(r_\alpha = 0.24, r_\beta = 0.64, r_\gamma = 0.68, \theta_\alpha = \pi, \theta_\beta = 0, \theta_\gamma = \pi)$. As states that maximize the knowledge sum $R_S(\mu_2)$, we may regard ψ_p and the Wigner-Dicke states as coherent states from the viewpoint of number-phase entropy.

4 Application to open systems

Here we study the effect of noise coming from open quantum system effects, on the atomic number-phase complementarity developed in the previous section. The noise effects we consider come from non-dissipative as well as dissipative interactions of the atomic system S with its environment which is modelled as a bath of harmonic oscillators starting in a squeezed thermal state [25,33,34]. This enables us to consider the effect of bath squeezing on the complementarity. We briefly recapitulate previous work [23,25,32,33,34] related to the effect of various noisy channels on the 'number' and phase distributions. In Section 4.1 we consider the effect of the phase damping channel which is the information theoretic analogue of the non-dissipative open system effect [25,33] while in Section 4.2 we consider the effect of the squeezed generalized amplitude damping channel which is the information theoretic analogue of the dissipative open system effect [33,34]. Intuitively, one would expect that open system effects, like measurements, cannot increase the knowledge sum. Interestingly, we find that this is not true for certain regimes of the squeezed generalized amplitude damping channel.

4.1 Phase damping channel

The 'number' and phase distributions for a qubit starting from an atomic coherent state $|\alpha', \beta'\rangle$, and subjected to a phase damping channel due to its interaction with a squeezed thermal bath, are [23,25,32]

$$
\begin{aligned}
p(m) &= \binom{2j}{j+m} (\sin(\alpha'/2))^{2(j+m)} (\cos(\alpha'/2))^{2(j-m)} \\
\mathcal{P}(\phi) &= \frac{1}{2\pi}\left[1 + \frac{\pi}{4}\sin(\alpha')\cos(\beta' + \omega t - \phi) e^{-(\hbar\omega)^2 \gamma(t)}\right].
\end{aligned} \tag{21}
$$

R_ϕ (Eq. (13)) is invariant under the translation of $\phi \longrightarrow \phi + a$. Setting $a = -\beta' - \omega t$, we find that R_ϕ is independent of β'. A derivation of Eq. (21) can be found in Refs. [32,23]. For completeness, the expression for $\gamma(t)$ in Eq. (21) is given in Appendix A and the physical process underlying the phase damping channel discussed.

Figure 2 depicts the effect of phase damping noise on the knowledge sum R_S. Comparing it with the noiseless case (dotted curve in Figure 1), we find a reduction in the total knowledge R_S, as expected. It follows from Eq. (21) that R_m remains unaffected under the action of this channel. Thus, the effect of noise on R_S is due entirely to its effect on R_ϕ, which decreases in the presence of noise *for all pure states* (because β' does not play any role and because the plot represents all possible values of α').

Figure 2 shows that squeezing has the detrimental effect of impairing phase knowledge for all regimes of the parameter space. This is in marked contrast to the case of the squeezed generalized amplitude damping noise, discussed in Section (4.2). Thus, squeezing, like temperature, has the overall detrimental effect of impairing R_S. This is consistent for the case of a QND interaction (which generates a phase damping channel [25,33]) of the system with its environment, i.e., $[H_S, H_{SR}] = 0$, as also corrorborated by the observation that squeezing and temperature concurrently impair geometric phase [33] and phase diffusion [32,23] and suggests that squeezing, like temperature, should adversely affect channel capacity for phase damping noise.

4.2 Squeezed generalized amplitude damping channel

The 'number' and phase distributions for a qubit starting from an atomic coherent state $|\alpha', \beta'\rangle$, and subjected to a squeezed generalized amplitude damping channel [34] due to its interaction with a squeezed thermal bath, are [23],

$$
p(m = 1/2, t) = \frac{1}{2}\left[\left(1 - \frac{\gamma_0}{\gamma^\beta}\right) + \left(1 + \frac{\gamma_0}{\gamma^\beta}\right) e^{-\gamma^\beta t}\right] \sin^2(\alpha'/2) + \frac{\gamma_-}{\gamma^\beta}\left(1 - e^{-\gamma^\beta t}\right)\cos^2(\alpha'/2), \tag{22}
$$

and

$$
\begin{aligned}
\mathcal{P}(\phi) = \frac{1}{2\pi}\Big[1 + \frac{\pi}{4\alpha}\sin(\alpha')\Big\{&\alpha\cosh(\alpha t)\cos(\phi - \beta') + \omega\sinh(\alpha t)\sin(\phi - \beta') \\
&-\gamma_0\chi\sinh(\alpha t)\cos(\varPhi + \beta' + \phi)\Big\} e^{-\frac{\gamma^\beta t}{2}}\Big].
\end{aligned} \tag{23}
$$

A derivation of Eqs. (22) and (23) can be found in Refs. [23]. For completeness, the parameters appearing in these equations are given in Appendix B where a brief discussion of the physical process behind the squeezed generalized amplitude damping channel is also made.

Figures 3(a) and (b) depict the effect of squeezed generalized amplitude damping noise on μR_ϕ (Eq. (17)), without and with bath squeezing, respectively. Comparing them with the noiseless case of Figure 1 (which, it may be noted, is unscaled by μ), we find as expected that noise impairs phase knowledge. However, comparing Figure 3(b) with (a), we find that squeezing has the beneficial effect of relatively improving phase knowledge for certain regimes

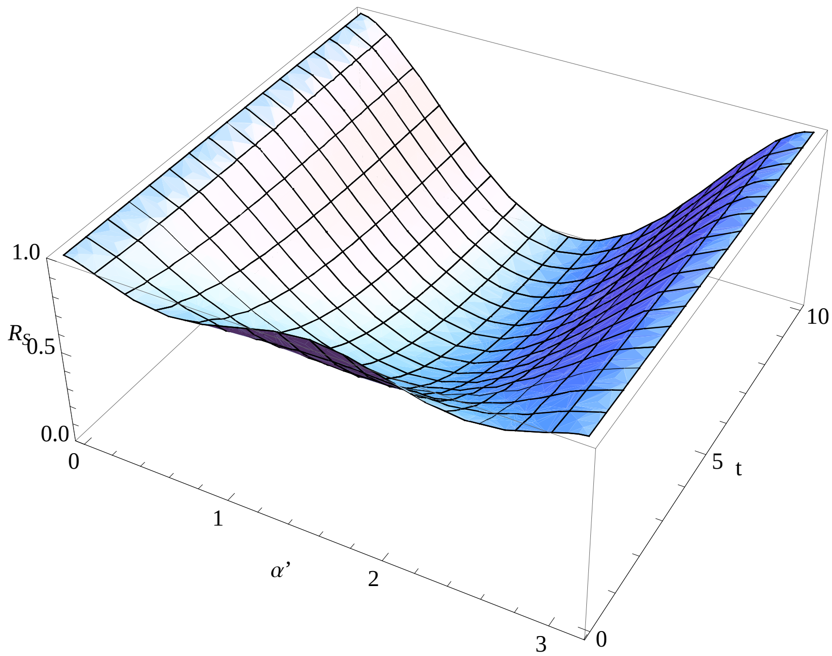

Fig. 2. Plot of $R_S = \mu R_\phi + R_m$ for a qubit starting from an atomic coherent state $|\alpha', \beta' = \pi/4\rangle$, subjected to a phase damping channel, with temperature (in units where $\hbar \equiv k_B \equiv 1$) $T = 2$, $\gamma_0 = 0.025$, $\omega_c = 100$ and $\omega = 1.0$, with respect to bath exposure time and bath squeezing parameters (Eq. (36)) $r = 1$ and $a = 0.0$. For the phase damping channel, R_m remains invariant. Both R_m and R_ϕ, and hence R_S, are independent of β'.

of the parameter space, and the detrimental effect of relatively impairing them in others. This property can be shown to improve the classical channel capacity [34]. Further, bath squeezing is seen to render R_ϕ dependent on β', because, as evident from Eq. (23), β' no longer appears as a translation in ϕ when the squeezing parameter χ (Eq. (39)) is non-vanishing. On the other hand, it follows from Eq. (22) that R_m is independent of β', so that R_S is dependent on β'. This stands in contrast to that of the phase damping channel, where inspite of squeezing, R_S remains independent of β' and, furthermore, squeezing impairs knowledge of ϕ in all regimes of the parameter space.

A point worth noting is that, in contrast to the phase damping channel, in a squeezed generalized amplitude damping channel, R_m and R_S are not necessarily non-increasing functions of time. Figure 4(a) depicts the effect of squeezed generalized amplitude damping channel on R_S, by bringing out the behavior of R_S as a function of bath exposure time. The dashed curve shows that squeezing has a detrimental effect on the knowledge sum, as one would usually expect. A surprising departure from this behavior may be noted for the case of the bold curve, which corresponds to the action of a dissipative interaction with an unsqueezed vacuum bath, where the knowledge sum R_S increases to 1. This counterintuitive behavior is due to the quantum deleting action, a contractive map whereby any initial state, including a mixed state, is asymptotically prepared in the pure state $|\frac{1}{2}, -\frac{1}{2}\rangle$ for vanishing temperature, and a mixture of $|0\rangle$ and $|1\rangle$ states for finite temperature, where the asymptotic mixture is determined purely by the environmental parameters of T and r, and not by the system's initial state [22]. A similar effect was noted in [40], where in a study of quantum state diffusion of an open system it was shown that for a specific noise, due to a particular system-reservoir interaction, there can be a reduction in the quantum dispersion entropy leading to localization.

It follows from the complementarity relation Eq. (17), that in the asymptotic limit of the deleting action, R_ϕ goes to 0 for both $|0\rangle$ and $|1\rangle$, and hence also, by the convexity of R, for any mixture that is diagonal in this basis. More generally, it is seen from Figure 4(b) that for

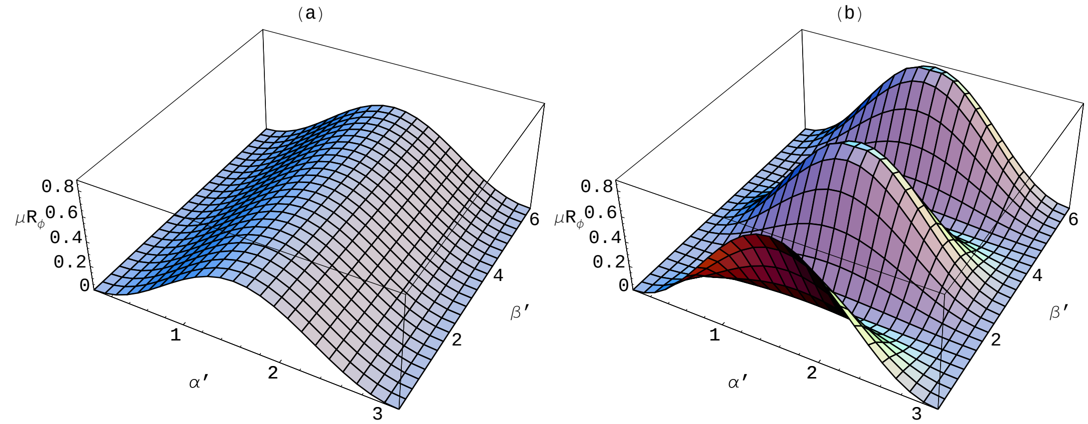

Fig. 3. Plot of μR_ϕ (scaled R_ϕ [Eq. (17)]) for a qubit starting from an atomic coherent state $|\alpha', \beta'\rangle$, and subjected to a squeezed generalized amplitude damping channel with temperature (in units where $\hbar \equiv k_B \equiv 1$) $T = 300$, $\gamma_0 = 0.01$, bath exposure time $t = 0.1$, and $\omega = 1.0$. Figure (a) is for the case of zero bath squeezing, and (b) for the case of bath squeezing parameters (Eq. (39)) $r = 1$ and $\Phi = \pi/8$. Squeezing has the effect of breaking translation symmetry in β' and improving phase knowledge (i.e., reducing phase uncertainty) for certain regimes of the parameter space.

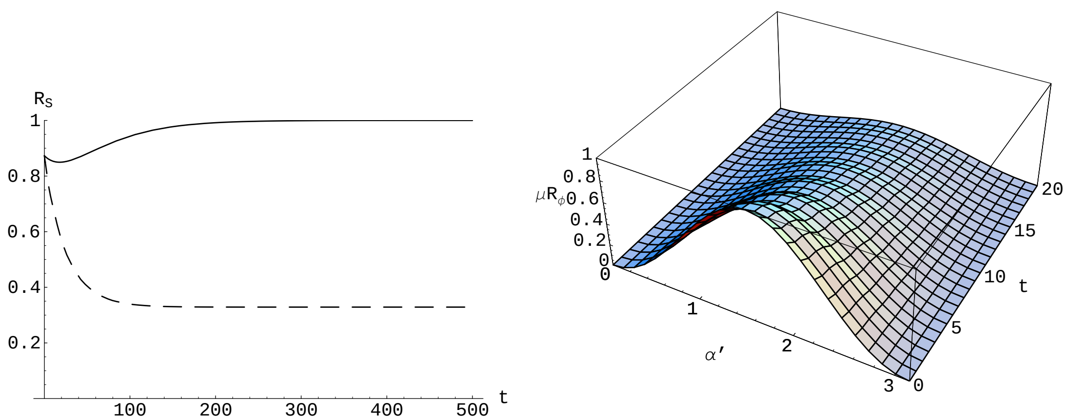

Fig. 4. (a) Plot of $R_S \equiv \mu R_\phi + R_m$ with respect to time for a qubit starting from an atomic coherent state $|\alpha' = \pi/4, \beta' = \pi/4\rangle$ and subjected to a squeezed generalized amplitude damping channel. The temperature (in units where $\hbar \equiv k_B \equiv 1$) $T = 0$, $\gamma_0 = 0.025$, and $\omega = 1.0$. The bath squeezing parameters (Eq. (39)) are $\Phi = 0$, and $r = 0$ ($r = 0.5$) for the case of bold (dashed) curves. (b) Plot of μR_ϕ for all α' as a function of time for a qubit starting from a coherent state $|\alpha', \pi/2\rangle$ and subjected a squeezed amplitude damping channel with the same parameters as above, with $T = \Phi = 0$, $r = 0.5$ and $\gamma = 0.05$. The R_ϕ-decreasing effect of increasing T or r is qualitatively the same.

all initial pure states, R_ϕ falls monotonically. This is to be expected since this noise prepares an asymptotic state that lies on the z-axis of the Bloch sphere, which implies by the convexity property of R_ϕ and the fact that $R_\phi = 0$ for the north and south pole states, that asymptotically $R_\phi = 0$ for *all initial pure states*.

5 Conclusions

In this work, we have investigated the number-phase complementarity in atomic systems from an entropic perspective through the number and phase distributions. Here number distribution refers to the probability distribution of measurement in the Wigner-Dicke basis (Eq. (14)), while phase distribution is defined by Eq. (6). We derive an uncertainty principle in terms of the Kullbäck-Leibler or relative entropy R of number and phase with respect to a uniform distribution. Since R can be regarded as a measure of knowledge of a random variable, the entropic uncertainty principle takes the form of an upper bound on the sum of number knowledge (R_m) and phase knowledge (R_ϕ). The choice of relative entropy over Shannon entropy was motivated by the fact that the latter is not strictly positive when applied to continuous probability distributions.

In the single-qubit case, number and phase are regarded as quasi-MUBs in the sense that any state maximizing knowledge of one variable simultaneously minimizes knowledge of the other, but maximum phase knowledge is strictly less than 1 bit (and less than $\log d$ bits in d dimensions).

Since R_ϕ is strictly less than one, the relative entropic formulation of the uncertainty principle does not tightly constrain R_m. We define a family of inequalities, parametrized by μ (Eq. (17)), that improves the upper bound on R_m. When $\mu = 1$, we obtain Eq. (16), and get the tightest bound for equatorial states (with the right hand side saturated) when $\mu \approx 4.085$. We briefly study the extension of the above concepts to a four-level system, where we find that the sense in which number and phase are said to be complementary must be further weakened to include unidirectional (but not mutual) unbiasedness. In particular, whereas phase is unbiased with respect to number, the converse is not true.

Finally, we study the complementary behavior of number and phase of a qubit subjected to the influence of its environment. For a qubit starting from an atomic coherent state $|\alpha', \beta'\rangle$, the translation symmetry of R_ϕ in β' is broken by the introduction of squeezing in the bath, for the case of a dissipative system-bath interaction (Figure 3(b)), but not in the case of a non-dissipative interaction. In the case of a purely decohering interaction, characterized by a phase damping channel, we find that noise invariably impairs the knowledge sum for these complementary variables (Eq. (17)), as expected. However, in the case of a squeezed generalized amplitude channel, the knowledge sum can increase in certain regimes. As a particularly dramatic illustration, when an initially maximally mixed state $\frac{1}{2}(|\frac{1}{2},\frac{1}{2}\rangle\langle\frac{1}{2},\frac{1}{2}| + |\frac{1}{2},-\frac{1}{2}\rangle\langle\frac{1}{2},-\frac{1}{2}|)$ is subjected to an unsqueezed vacuum bath, R_S rises from 0 to 1 asymptotically.

These results could be potentially useful for applications in quantum communication and quantum cryptography [41] involving atomic systems. The present work brings forth a number of open questions concerning an information theoretic study of complementarity in atomic systems involving continuous-valued POVMs, of which we list some here. Of immediate interest is the question whether the Shannon entropy of $\mathcal{P}(\phi)$ remains positive for all possible pure and mixed states. If yes, then one may revert back from the use of the knowledge variable R to that of entropy S. Also of interest is to analytically derive the bounds on the weighted knowledge sum, which we have obtained here numerically. Finally, it is of interest to explore the full scope and implications of one-way biasedness, and its connection to complementarity.

A Phase damping channel

Consider the Hamiltonian

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= H_S + \sum_k \hbar\omega_k b_k^\dagger b_k + H_S \sum_k g_k (b_k + b_k^\dagger) + H_S^2 \sum_k \frac{g_k^2}{\hbar\omega_k}. \end{aligned} \tag{24}$$

Here H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system-reservoir interaction, respectively. H_S is a generic system Hamiltonian which can be specified depending on the physical situation. $b_k^\dagger$, b_k denote the creation and annihilation operators for the reservoir oscillator of frequency ω_k, g_k stands for the coupling constant (assumed real) for the interaction of the oscillator field with the system. The last term on the right-hand side of Eq. (1) is a renormalization inducing 'counter term'. Since $[H_S, H_{SR}] = 0$, the Hamiltonian (1) is of QND type. The system-plus-reservoir composite is closed and hence obeys a unitary evolution given by

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}, \tag{25}$$

where

$$\rho(0) = \rho^s(0)\rho_R(0), \tag{26}$$

i.e., we assume separable initial conditions. Here $\rho_R(0)$ is the initial density matrix of the reservoir which we take to be a squeezed thermal bath given by

$$\rho_R(0) = S(r,\Phi)\rho_{th}S^\dagger(r,\Phi), \tag{27}$$

where

$$\rho_{th} = \prod_k \left[1 - e^{-\beta\hbar\omega_k}\right] e^{-\beta\hbar\omega_k b_k^\dagger b_k} \tag{28}$$

is the density matrix of the thermal bath at temperature T, with $\beta \equiv 1/(k_B T)$, k_B being the Boltzmann constant, and

$$S(r_k,\Phi_k) = \exp\left[r_k\left(\frac{b_k^2}{2}e^{-2i\Phi_k} - \frac{b_k^{\dagger 2}}{2}e^{2i\Phi_k}\right)\right] \tag{29}$$

is the squeezing operator with r_k, Φ_k being the squeezing parameters [42]. Here we take the system to be a two-level atomic system, with the Hamiltonian

$$H_S = \frac{\hbar\omega}{2}\sigma_z, \tag{30}$$

σ_z being the usual Pauli matrix. The reduced density matrix of the system, in the basis of the Wigner-Dicke states $|j,m\rangle$, after time t is [33]

$$\rho^s_{m,n}(t) = \begin{pmatrix} \cos^2(\frac{\theta_0}{2}) & \frac{1}{2}\sin(\theta_0)e^{-i(\omega t+\phi_0)}e^{-(\hbar\omega)^2\gamma(t)} \\ \frac{1}{2}\sin(\theta_0)e^{i(\omega t+\phi_0)}e^{-(\hbar\omega)^2\gamma(t)} & \sin^2(\frac{\theta_0}{2}) \end{pmatrix}, \tag{31}$$

from which the Bloch vectors can be extracted to yield

$$\begin{aligned} \langle\sigma_x(t)\rangle &= \sin(\theta_0)\cos(\omega t+\phi_0)e^{-(\hbar\omega)^2\gamma(t)}, \\ \langle\sigma_y(t)\rangle &= \sin(\theta_0)\sin(\omega t+\phi_0)e^{-(\hbar\omega)^2\gamma(t)}, \\ \langle\sigma_z(t)\rangle &= \cos(\theta_0). \end{aligned} \tag{32}$$

Here $\gamma(t)$ comes due to the interaction with the environment and for the case of an Ohmic bath with spectral density

$$I(\omega) = \frac{\gamma_0}{\pi}\omega e^{-\omega/\omega_c}, \tag{33}$$

where γ_0 and ω_c are two bath parameters characterizing the quantum noise, it can shown that using Eq. (33) one can obtain [25] in the $T = 0$ limit,

$$\gamma(t) = \frac{\gamma_0}{2\pi}\cosh(2r)\ln(1+\omega_c^2 t^2) - \frac{\gamma_0}{4\pi}\sinh(2r)\ln\left[\frac{\left(1+4\omega_c^2(t-a)^2\right)}{\left(1+\omega_c^2(t-2a)^2\right)^2}\right] - \frac{\gamma_0}{4\pi}\sinh(2r)\ln(1+4a^2\omega_c^2), \tag{34}$$

where the resulting integrals are defined only for $t > 2a$. In the high T limit, $\gamma(t)$ can be shown to be [25]

$$\gamma(t) = \frac{\gamma_0 k_B T}{\pi\hbar\omega_c}\cosh(2r)\left[2\omega_c t\tan^{-1}(\omega_c t) + \ln\left(\frac{1}{1+\omega_c^2 t^2}\right)\right] - \frac{\gamma_0 k_B T}{2\pi\hbar\omega_c}\sinh(2r)$$
$$\times\left[4\omega_c(t-a)\tan^{-1}\left(2\omega_c(t-a)\right) - 4\omega_c(t-2a)\tan^{-1}\left(\omega_c(t-2a)\right) + 4a\omega_c\tan^{-1}\left(2a\omega_c\right)\right.$$
$$\left. + \ln\left(\frac{\left[1+\omega_c^2(t-2a)^2\right]^2}{\left[1+4\omega_c^2(t-a)^2\right]}\right) + \ln\left(\frac{1}{1+4a^2\omega_c^2}\right)\right], \tag{35}$$

where, again, the resulting integrals are defined for $t > 2a$. Here we have for simplicity taken the squeezed bath parameters as

$$\cosh\left(2r(\omega)\right) = \cosh(2r), \quad \sinh\left(2r(\omega)\right) = \sinh(2r),$$
$$\Phi(\omega) = a\omega, \tag{36}$$

where a is a constant depending upon the squeezed bath. The results pertaining to a thermal bath can be obtained from the above equations by setting the squeezing parameters r and Φ to zero. σ_x, σ_y, σ_z are the standard Pauli matrices. It can be easily seen from the above Bloch vector equations that the QND evolution causes a coplanar, fixed by the polar angle θ_0, in-spiral towards the z-axis of the Bloch sphere. This is the characteristic of a phase-damping channel [4].

B Squeezed generalized amplitude damping channel

Here the reduced dynamics of the two level atomic system (30) interacting with a squeezed thermal bath under a weak Born-Markov and rotating wave approximation is studied. This implies that here the system interacts with its environment via a non-QND interaction, i.e., $[H_S, H_{SR}] \neq 0$ such that along with a loss in phase information, energy dissipation also takes place. The evolution has a Lindblad form which in the interaction picture is given by [24]

$$\frac{d}{dt}\rho^s(t) = \gamma_0(N+1)\left(\sigma_-\rho^s(t)\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho^s(t) - \frac{1}{2}\rho^s(t)\sigma_+\sigma_-\right)$$
$$+\gamma_0 N\left(\sigma_+\rho^s(t)\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho^s(t) - \frac{1}{2}\rho^s(t)\sigma_-\sigma_+\right)$$
$$-\gamma_0 M\sigma_+\rho^s(t)\sigma_+ - \gamma_0 M^*\sigma_-\rho^s(t)\sigma_-. \tag{37}$$

Here

$$N = N_{\rm th}(\cosh^2 r + \sinh^2 r) + \sinh^2 r, \tag{38}$$

$$M = -\frac{1}{2}\sinh(2r)e^{i\Phi}(2N_{th}+1), \tag{39}$$

and

$$N_{\rm th} = \frac{1}{e^{\hbar\omega/(k_B T)} - 1}, \tag{40}$$

where $N_{\rm th}$ is the Planck distribution giving the number of thermal photons at the frequency ω, and r, Φ are squeezing parameters of the bath. The case of a thermal bath without squeezing can be obtained from the above expressions by setting these squeezing parameters to zero. γ_0

is a constant typically denoting the system-environment coupling strength. This equation can be expressed in a manifestly Lindblad form as

$$\frac{d}{dt}\rho^s(t) = \sum_{j=1}^{2} \left(2R_j \rho^s R_j^\dagger - R_j^\dagger R_j \rho^s - \rho^s R_j^\dagger R_j \right), \tag{41}$$

where $R_1 = (\gamma_0(N_{\rm th}+1)/2)^{1/2} R$, $R_2 = (\gamma_0 N_{\rm th}/2)^{1/2} R^\dagger$. Here $R = \sigma_- \cosh(r) + e^{i\Phi}\sigma_+ \sinh(r)$, and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. If $T = 0$, so that $N_{\rm th} = 0$, then R_2 vanishes, and a single Lindblad operator suffices. The fact that the above equation can be expressed in the form (41) guarantees a Kraus or operator- sum representation [4] for the evolution of the reduced density matrix. It can be seen that the reduced density matrix, obtained by solving Eq. (37) in the Bloch form, shrinks towards the asymptotic equilibrium state ρ_{asymp}, given by

$$\rho_{asymp} = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}, \tag{42}$$

where $p = \frac{1}{2}\left[1 + \frac{1}{(2N+1)}\right]$. For the case of zero squeezing and zero temperature, this action corresponds to an amplitude-damping channel [4,33] with the Bloch sphere shrinking to a point representing the state $|0\rangle$ (the south pole of the Bloch sphere) while for the case of finite T but zero squeezing, the above action corresponds to a generalized amplitude-damping channel [4,33] with the Bloch sphere shrinking to a point along the line joining the south pole to the center of the Bloch sphere. The center of the Bloch sphere is reached in the limit of infinite temperature. Thus, the interaction with the environment provides a contractive map, such that the asymptotic state is pure ($p = 1$), corresponding to the deletion action [22], or mixed ($p < 1$), depending on environmental conditions. For finite T and bath squeezing, the above corresponds to a squeezed generalized amplitude damping channel [34].

In Eq. (23), the parameter α is given by

$$\alpha = \sqrt{\gamma_0^2 |M|^2 - \omega^2}, \tag{43}$$

while

$$\gamma^\beta = \gamma_0(2N+1). \tag{44}$$

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