

# Complementarity in atomic and oscillator systems

R. Srikanth<sup>1,2,\*</sup> and Subhashish Banerjee<sup>3,†</sup>

<sup>1</sup>Poornaprajna Institute of Scientific Research, Sadashiva Nagar, Bangalore- 560 080, India.

<sup>2</sup>Raman Research Institute, Sadashiva Nagar, Bangalore - 560 080, India

Phone: +91 9844 593440

<sup>3</sup>Chennai Mathematical Institute, Padur PO, Siruseri 603103, India

We develop a unified, information theoretic interpretation of the number-phase complementarity that is applicable both to finite-dimensional (atomic) and infinite-dimensional (oscillator) systems. The relevant uncertainty principle is obtained as a lower bound on *entropy excess*, the difference between number entropy and phase knowledge, the latter defined as the relative entropy with respect to the uniform distribution.

PACS numbers: 03.65.Ta,03.65.Yz,03.67.-a

Two observables  $A$  and  $B$  of a  $d$ -level system are called complementary if knowledge of the measured value of  $A$  implies maximal uncertainty of the measured value of  $B$ , and vice versa [1]. Complementarity is an aspect of the Heisenberg uncertainty principle, which says that for any state  $\psi$ , the probability distributions obtained by measuring  $A$  and  $B$  cannot both be arbitrarily peaked if  $A$  and  $B$  are sufficiently non-commuting. Expressed in terms of measurement entropy the Heisenberg uncertainty principle takes the form:

$$H(A) + H(B) \geq \log d. \quad (1)$$

where  $H(A)$  and  $H(B)$  are the Shannon entropy of the measurement outcomes of a  $d$ -level quantum system [2, 3]. Eq. (1) has several advantages over the traditional uncertainty multiplicative form [1, 4–6].

More generally, given two observables  $A \equiv \sum_a a|a\rangle\langle a|$  and  $B \equiv \sum_b b|b\rangle\langle b|$ , let the entropy generated by measuring  $A$  or  $B$  on a state  $|\psi\rangle$  be given by, respectively,  $H(A)$  and  $H(B)$ . The information theoretic representation of the Heisenberg uncertainty principle states that  $H(A) + H(B) \geq 2 \log \left( \frac{1}{f(A,B)} \right)$ , where  $f(A, B) = \max_{a,b} |\langle a|b\rangle|$ , and  $H(\cdot)$  is the Shannon binary entropy. A pair of observables,  $A$  and  $B$ , for which  $f(A, B) = d^{-1/2}$  are said to form mutually unbiased bases (MUB) [7, 8]. Conventionally, two Hermitian observables are called complementary only if they are mutually unbiased.

An application of this idea to obtain an entropic uncertainty relation for oscillator systems in the Pegg-Barnett scheme [9] has been made in Ref. [10], and for higher entropic uncertainty relations in Ref. [11]. An algebraic treatment of the uncertainty relations, in terms of complementary subalgebras, is studied in Ref. [12].

An extension of Eq. (1) to the case where  $A$  or  $B$  is not discrete is considered in Ref. [13], where the problem that the Shannon entropy of a continuous random variable may be negative is circumvented by instead using relative entropy (also called Kullbäck-Leibler divergence, which is always positive) [14, 15] with respect to a uniform distribution. This quantity is a measure of knowledge [13]. Note that recourse to entropic knowledge may not be always necessary, and other ways might exist to circumvent the problem. In the case of position  $X$  and momentum  $P$ , the sum of functionals,  $H[X] + H[P]$  is always positive even though the summands may be negative [16]. Further, one might consider exponentiating the entropy to ensure positivity. But neither of these options is, in our opinion, intuitive from the perspective of interpretation. Finally, it may well turn out that for all physical states of a system, the continuous variables in question may never yield negative entropies. In Ref. [13], we numerically found this to be the case for the continued-valued phase observable in a two-level system. However, we know of no (published) proof that this is true in general, which was the motivation behind adopting the concept of entropic knowledge in the uncertainty relation.

An example where our re-expressed entropic uncertainty finds application would be when one of the observables, say  $A$ , is bounded, and its conjugate  $B$  is described not as a Hermitian operator but as a *continuous-valued* POVM. (There are no continuous observables corresponding to *projective measurements* in finite-dimensional Hilbert spaces.) A particular case of discrete-continuous conjugacy, considered in detail in Ref. [13], is the number and phase of an atomic system. This generalization of the entropic uncertainty principle to cover discrete-continuous systems still suffers from the restriction that the system must be finite dimensional, since in the case of an infinite-dimensional system, such as an oscillator, entropic knowledge of the number distribution can diverge, making it unsuitable for infinite-dimensional systems. Therefore to set up an entropic version of the uncertainty principle, that unifies and is

---

\*Electronic address: srik@ppisr.res.in

†Electronic address: subhashish@cmi.ac.in

applicable to all systems, including infinite dimensional and/or continuous-variable systems, it may be advantageous to use a combination of entropy and knowledge, in particular, the difference between entropy of the discrete, infinite observable and between phase knowledge. This is discussed in detail below.

The quantum description of phases [18] has a long history of [9, 19–23]; see also Refs. [24, 25]. In the quantum theory of detection, the issue of quant phases appears quite fundamentally as a lack of phase-shift self-adjoint operator that is covariant under shifts generated by the number observable [26, 27]. In a recent approach, which we adopt, the concept of phase distribution for the quantum phase has been introduced [24, 28–30]. This distribution is the expectation value of the *canonical phase*, the POVM obtained by setting all entries in the phase matrix (defining the generalized phase POVM) equal to 1 [31]. This reference also discusses different forms of complementarity of number and phase, depending on the definition of phase chosen. In particular, number and canonical phase are not complementary, but satisfy the weaker condition of *value complementarity*, as defined there. Our present work can be considered as an entropic interpretation of the value complementarity of number and canonical phase. In this section, we briefly recapitulate, for convenience, some useful formulas of quantum phase distributions for oscillator systems. For the case of atomic systems, the basic formulas were presented in [13].

We define a phase distribution  $\mathcal{P}(\theta)$  for a given density operator  $\rho$ , which in our case would be the reduced density matrix, as

$$\mathcal{P}(\theta) = \frac{1}{2\pi} \langle \theta | \rho | \theta \rangle = \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \rho_{m,n} e^{i(n-m)\theta}, \quad 0 \leq \theta < 2\pi \quad (2)$$

where the states  $|\theta\rangle$  are the eigenstates of the Susskind-Glogower [20] phase operator corresponding to eigenvalues of unit magnitude and are defined in terms of the number states  $|n\rangle$  as  $|\theta\rangle = \sum_{n=0}^{\infty} e^{in\theta} |n\rangle$ , and  $|\theta\rangle\langle\theta|$  is the canonical phase POVM. The sum in Eq. (2) is assumed to converge. The phase distribution is positive definite and normalized to unity with  $\int_{\theta} |\theta\rangle\langle\theta| d\theta = 1$ .

The complementary number distribution is

$$p(m) = \langle m | \rho | m \rangle, \quad (3)$$

where  $|m\rangle$  is the number (Fock) state. Analogous results exist for atomic states, with the Susskind-Glogower states replaced by atomic coherent states [33, 34], and number states by Wigner-Dicke states [35, 36].

Defining entropic knowledge  $R[f]$  of random variable  $f$  as its relative entropy with respect to the uniform distribution  $\frac{1}{d}$ , i.e.,  $R[f] \equiv S(f(j) || \frac{1}{d}) = \sum_j f(j) \log(df(j))$ , we can recast Heisenberg uncertainty principle in terms of entropy  $H$  and knowledge  $R$ , as shown by the theorem:

**Theorem 1** *Given two Hermitian observables  $A$  and  $B$  that form a pair of MUB in a finite dimensional Hilbert space, the uncertainty relation (1) can be expressed as*

$$X(A, B) \equiv H(A) - R(B) \geq 0. \quad (4)$$

**Proof.** Let the distribution obtained by measuring  $A$  and  $B$  on a given state be, respectively,  $\{p_j\}$  and  $\{q_k\}$ . Denoting  $H(A) \equiv -\sum_j p_j \log_2 p_j$ , the l.h.s of Eq. (4) is given by  $H(A) - S(B || \frac{1}{d}) = H(A) - \sum_k q_k \log(dq_k) = H(A) + H(B) - \log d \geq 2 \log\left(\frac{1}{f(A,B)}\right) - \log d$ , where the last equation follows from Ref. [1]. For a pair of MUB [4, 5],  $f(A, B) = d^{-1/2}$ , from which the theorem follows. ■

Physically Eq. (4) expresses that ignorance of one of two MUB variables is at least as large as the knowledge of the other. Some properties of  $X$  are evident. Clearly,  $X(A, B) = X(B, A)$ . It is not difficult to see that  $X(A, B)$  attains its largest value of  $\log(d)$  when  $A$  and  $B$  are MUBs (and the state is an eigenstate of a third MUB), and its minimum value of  $-\log(d)$  when  $A$  and  $B$  are commuting (and the state is an eigenstate of either observable). We may quantify the ‘degree of complementarity’ in the following way. Since the above bound is tight, we define  $X_{\min}(A, B) \equiv 2 \log\left(\frac{1}{f(A,B)}\right) - \log d$  as the smallest value of  $X(A, B)$  over all possible states for a given pair of Hermitian observables  $A$  and  $B$ . This is a monotonically decreasing function in the interval  $[d^{-1/2}, 1]$ , going from 0 to  $-\log(d)$ . Two Hermitian observables  $A$  and  $B$  are maximally complementary (i.e., form an MUB) if  $X_{\min}(A, B) = 0$ . are minimally complementary (i.e., are compatible) if  $X_{\min}(A, B) = -\log(d)$ .

A point worth noting about Eq. (4) is that it contains no explicit mention of dimension  $d$ . What is remarkable is that we find this situation persists even when one of  $A$  or  $B$  is not discrete, but a continuous-valued POVM

(for discrete-valued POVMs, cf. Ref. [37]), and furthermore, the system is no longer finite dimensional but instead infinite dimensional. The only additional requirement is that the continuous-valued variable should be set as  $B$  (the knowledge- rather than the ignorance-variable), since  $H(B)$  can potentially be negative for such variables. This makes  $X(A, B) \geq 0$  as a very succinct and general statement of the uncertainty principle. By contrast, because there is no prior guarantee that measurement entropy  $H(\cdot)$  will be non-negative for a continuous-valued observable, it is not obvious that the version of the Heisenberg uncertainty principle given by (1) is generally applicable, and furthermore, because there is no prior guarantee that measurement entropic knowledge  $R(\cdot)$  will be well-defined for infinite-dimensional variables, the version  $R(A) + R(B) \leq \log(d)$  of Ref. [13] is also not obviously generally applicable.

One catch is that on account of the POVM-nature [26] of  $B$ ,  $R(B)$  may have a maximum value less than  $\log(d)$  in the finite dimensional case. A generalization of the concept of ‘maximal complementarity’ or ‘MUBness’ would be to apply those terms to  $A$  and  $B$ , when one of them is a POVM, where the maximal knowledge of the measured value of  $A$  implies minimal knowledge of the measured value of  $B$ , and vice versa, but with maximum knowledge no longer being required to be as high as  $\log d$  bits.

For the phase variable given by the POVM  $\phi$  and probability distribution  $\mathcal{P}(\phi)$ , entropic knowledge is given by the functional [28, 32]:

$$R[\mathcal{P}(\phi)] = \int_0^{2\pi} d\phi \mathcal{P}(\phi) \log[2\pi\mathcal{P}(\phi)], \quad (5)$$

where the  $\log(\cdot)$  refers to the binary base.

It is at first not obvious that Eq. (4) holds for infinite dimensional systems. Based on a result due to Białynicki-Birula and Mycielski [38], which in turn uses the concept of the  $(p, q)$ -norm of the Fourier transformation found by Beckner [16] for all values of  $p$ , for an oscillator system, we can show that it is indeed the case. In particular,

$$-\int_{-\pi}^{\pi} d\phi P(\phi) \log(P(\phi)) - \sum_{m=0}^{\infty} p_m \log(p_m) \geq \log(2\pi) \quad (6)$$

Here it is worth noting that, along the lines of Ref. [38], one may obtain analogous entropic uncertainty relations between phase and number of quanta, as well as between energy and time [17].

Setting the ‘number variable’  $m$  in Eq. (6) as  $A$ , and the phase variable  $\phi$  as  $B$ , and noting that the first term in the l.h.s of Eq. (6), using Eq. (5), is just  $\log(2\pi) - R[P(\phi)]$ , we obtain

$$X[m, \phi] \equiv H[m] - R[\phi] \geq 0, \quad (7)$$

which is Eq. (4) applied to an infinite-dimensional system that includes a non-Hermitian POVM (phase  $\phi$ ). Eq. (7) expresses the fact ignorance of variable  $m$  is at least as great as knowledge of its complementary partner,  $\phi$ . Restricting our attention only to number-phase complementarity, we find on comparing Eqs. (4) and (7) that the statement  $X \geq 0$  as a description of the Heisenberg uncertainty relation holds good both for finite and infinite dimensional systems. This version of the Heisenberg uncertainty principle may be called *the principle of entropy excess*. This thus renders physically intuitive the result of Białynicki-Birula and Mycielski [38], derived using elements of advanced functional analysis. As related work, we cite an information theoretic interpretation of uncertainty in the context of phase resolution in harmonic oscillator systems, in [39]. Also, the number-phase complementarity, for a harmonic oscillator system, using information exclusion relations has been studied in [40].

As pointed out earlier, the knowledge-sum approach cannot be applied to infinite dimensional systems, whereas the principle of entropy excess can be applied to finite as well as infinite dimensional systems, making it a more flexible tool for describing number-phase complementarity in a host of systems. Here we apply the principle of entropy excess (7) to number-phase complementarity in (finite-level) atomic systems, briefly revisiting results obtained earlier [13] from the perspective of an upper bound on the knowledge-sum of complementary variables, as well as to an infinite dimensional harmonic oscillator, thereby highlighting its greater scope.

When applied to a finite level (atomic) system, the relation (7) still leaves some room for improvement [13]. For example, in the case of qubits (two-level systems), number states saturate the bound because they satisfy  $H(m) = R(\phi) = 0$ . The corresponding states of the phase variable (which maximize phase knowledge and minimize number knowledge) are the equatorial states, for which  $H(m) = 1$ , but  $R[\phi] \approx 0.245 < 1$  [13]. Following this reference, one way to address this problem is to modify (7) to the inequality

$$X^\mu[m, \phi] \equiv H[m] - \mu R[\phi] \geq 0 \quad (8)$$

for all pure states in  $\mathbf{C}^2$ , where parameter  $\mu (> 0)$  is chosen to be the largest value such that inequality (8) is satisfied over all state space.

From the concavity of  $H[m]$  and the convexity of  $R[\phi]$ , it follows that Eq. (8) holds for any mixed state. Figure (1), illustrates the tighter bound imposed by  $X_\mu[m, \phi]$  than  $X[m, \phi]$ . In Figure (2), the number entropy  $H[m]$ , phase knowledge  $R[\phi]$  and entropy excess  $X[m, \phi]$  (Eq. (7)) are depicted for a harmonic oscillator starting out in the usual coherent state  $|\alpha\rangle$ .

We note that as number increases, with increase in  $\alpha$ , so does  $H[m]$  (since the variance of a Poisson distribution equals its mean), whereas phase  $\phi$  becomes increasingly certain, leading to increase in  $R[\phi]$ . Through a numerical search, we found that  $\mu \approx 4.085$  for dimension  $d = 2$  and  $\mu \approx 1.973$  for  $d = 4$ . And when  $d = \infty$ , we find analytically  $\mu = 1$ , as can be seen from the discussion leading up to Eq. (7). From the above numerical-analytical pattern, we conjecture that as the system dimension increases from two to infinity,  $\mu$  falls monotonically from about 4 to 1.

Thus the principle of entropy excess, incorporating knowledge and entropy, emerges as a flexible measure by which number-phase complementarity of finite as well as infinite dimensional systems can be studied in a unified manner.

- 
- [1] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103–1106 (1988).  
 [2] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge 2000).  
 [3] A. Galindo, M.A. Martin-Delgado, Rev. Mod. Phys. **74**, 347–423 (2000); M. Ohya and D. Petz, *Quantum Entropy and Its Use* (Springer-Verlag, New York, 1993).  
 [4] K. Kraus, Phys. Rev. D **35**, 3070–3075 (1987).  
 [5] D. Deutsch, Phys. Rev. Lett. **50**, 631–633 (1983).  
 [6] H. Partovi, Phys. Rev. Lett. **50**, 1883 (1983).  
 [7] I. D. Ivanovic, J. Phys. A **14**, 3241–3246 (1981).  
 [8] T. Durt, J. Phys. A: Math. Gen. **38**, 5267–5284 (2005).  
 [9] D. T. Pegg and S. M. Barnett, J. Mod. Opt. **36**, 7–19 (1989); Phys. Rev. A **39**, 1665–1675 (1989).  
 [10] S. Abe, Phys. Lett. A **166**, 163–167 (1992).  
 [11] S. Wehner and A. Winter, eprint arXiv:0710.1185.  
 [12] D. Petz, Reports on Mathematical Physics: **59**, 209–224 (2007).  
 [13] R. Srikanth and S. Banerjee, Euro Phys. J. D. **53**, 217–227 (2009); eprint arXiv:0711.0875.  
 [14] S. Kullback and R. A. Leibler, Ann. of Math. Stat. **22**, 79–86 (1951).  
 [15] P. Hayden, R. Jozsa, D. Petz and A. Winter, Commun. Math. Phys. **246**, 359–374 (2004); eprint arXiv:quant-ph/0304007.  
 [16] W. Beckner, Ann. Math. **102**, 159–182 (1975).  
 [17] M. Grabowski, Phys. Lett. A **124**, 19–21 (1987).  
 [18] V. Perinova, A. Luks and J. Perina, *Phase in Optics* (World Scientific, Singapore, 1998).  
 [19] P. A. M. Dirac, Proc. R. Soc. Lond. A **114**, 243–265 (1927).  
 [20] L. Susskind and J. Glogower, Physics **1**, 49–61 (1964).  
 [21] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411–440 (1968).  
 [22] J. H. Shapiro, S. R. Shepard and N. C. Wong, Phys. Rev. Lett. **62**, 2377–2380 (1989).  
 [23] W. P. Schleich and S. M. Barnett (eds.), *Quantum Phase and Phase Dependent Measurements*, Physica Scripta Special issue **T48** (1993).  
 [24] J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795–3818 (1991).  
 [25] M. J. W. Hall, Quantum Opt. **3**, 7–12 (1991).  
 [26] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North Holland 1982).  
 [27] J.-P. Pellonpää, PhD thesis, 2002. "Covariant Phase Observables in Quantum Mechanics" <https://oa.doria.fi/handle/10024/5808?show=full>  
 [28] S. Banerjee, J. Ghosh and R. Ghosh, Phys. Rev. A **75**, 062106 (2007).  
 [29] G. S. Agarwal, S. Chaturvedi, K. Tara and V. Srinivasan, Phys. Rev. A **45**, 4904–4910 (1992).  
 [30] G. S. Agarwal and R. P. Singh, Phys. Lett. A **217**, 215–218 (1996).  
 [31] P. Busch, P. Lahti, J.-P. Pellonpää and K. Ylänen, J. of Phys. A: Math. Gen **34**, 5923–5935 (2001).  
 [32] S. Banerjee and R. Srikanth, Phys. Rev. A **76**, 062109 (2007).  
 [33] G. S. Agarwal and R. R. Puri, Phys. Rev. A **41**, 3782–3791 (1990).  
 [34] M. A. Rashid, J. Math. Phys. **19**, 1391–1396 (1978).  
 [35] F. T. Arecchi, E. Courtens, R. Gilmore and H. Thomas, Phys. Rev. A **6**, 2211–2237 (1972).  
 [36] R. H. Dicke, Phys. Rev. **93**, 99–110 (1954).  
 [37] S. Massar, Phys. Rev. A **76**, 042114 (2007); eprint quant-ph/0703036.  
 [38] I. Białynicki-Birula and J. Mycielski, Comm. Math. Phys. **44**, 129–132 (1975).  
 [39] M. J. W. Hall, Journal of Modern Optics **40**, 809–824 (1993).  
 [40] M. J. W. Hall, Phys. Rev. Lett. **74**, 3307–3311 (1995); Phys. Rev. A **55**, 100–113 (1997).

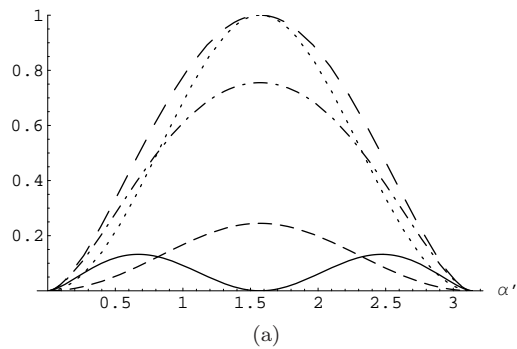


FIG. 1: Entropy excess of a two level atomic system starting in an atomic coherent state  $|\alpha', \beta'\rangle$ , as a function of  $\alpha'$ , with  $\beta' = 0.0$ , two parameters covering the Bloch sphere of a two-level system in the notation of Ref. [13]. The large-dashed (resp., small-dashed) line represents  $H[m]$  (resp.,  $R[\phi]$ ). The dotted-curve represents  $\mu R[\phi]$  (where  $\mu = 4.085$ ). The solid (resp., dot-dashed) curve represents the entropy excess  $X_\mu$  (resp.  $X$ ).

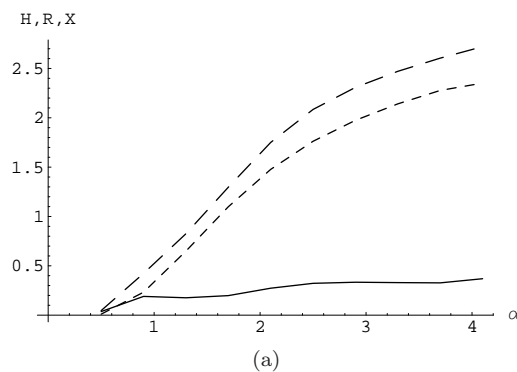


FIG. 2: Number entropy  $H[m]$  (large-dashed line), phase knowledge  $R[\phi]$  (small-dashed line) and entropy excess  $X[m, \phi]$  (Eq. (7), bold line) plotted as a function of the parameter  $\alpha$  for a harmonic oscillator system initially in a coherent state  $|\alpha\rangle$ .