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A New Scalarization and Numerical Method for Constructing Weak Pareto Front of Multi-objective Optimization Problems*

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Abstract

A numerical technique is presented for constructing an approximation of the weak Pareto front of nonconvex multi-objective optimization problems, based on a new Tchebychev-type scalarization and its equivalent representations. First, existing results on the standard Tchebychev scalarization, the weak Pareto and Pareto minima, as well as the uniqueness of the optimal value in the Pareto front, are recalled and discussed for the case when the set of weak Pareto minima is the same as the set of Pareto minima, namely, when weak Pareto minima are also Pareto minima. Of the two algorithms we present, Algorithm 1 is based on this discussion. Algorithm 2, on the other hand, is based on the new scalarizations incorporating rays associated with the weights of the scalarization in the value (or objective) space, as constraints. We prove two relevant results for the new scalarization. The new scalarizations and the resulting Algorithm 2 are particularly effective in constructing an approximation of the weak Pareto sections of the front. We illustrate the working and capability of both algorithms by means of smooth and nonsmooth test problems with connected and disconnected Pareto fronts.

Key words: Multi-objective optimization, Pareto front, efficient set, Tchebychev scalarization, numerical methods, nonconvex optimization, nonsmooth optimization.

1 Introduction and Scalarization Techniques

We consider the following multi-objective optimization problem.

$$(P) \quad \min_{x \in X} (f_1(x), \dots, f_p(x)),$$

where $X \subset \mathbb{R}^n$, and the objective functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, p$, are continuous. Note that f_i can in general be nonsmooth and nonconvex. There are two main solution concepts associated with Problem (P), namely the Pareto minimum and the weak Pareto minimum. A

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point $x^* \in X$ is said to be a *Pareto minimum* if there exists no $x \in X$ such that $f(x) \neq f(x^*)$ and

$$f_i(x) \leq f_i(x^*), \quad \text{for all } i = 1, \dots, p.$$

On the other hand the vector $x^* \in X$ is said to be a *weak Pareto minimum* if there exists no $x \in X$ such that

$$f_i(x) < f_i(x^*), \quad \text{for all } i = 1, \dots, p.$$

The set of all objective function values at the Pareto and weak Pareto minima is said to be the *Pareto front* (or *efficient set*) of Problem (P) in the objective value space.

For computing a solution of the nonconvex multi-objective optimization problem (P), the following single-objective problem (i.e. scalarization) is often considered [23].

$$(P_w) \quad \min_{x \in X} \max\{w_1(f_1(x) - u_1^*), \dots, w_p(f_p(x) - u_p^*)\},$$

where w_i , $i = 1, \dots, p$, are referred to as weights, and u_i^* , $i = 1, \dots, p$, are the respective *utopian objective values*. A *utopian objective vector* u^* associated with Problem (P) consists of components u_i^* given as $u_i^* = f_i^* - \varepsilon_i$ where $\varepsilon_i > 0$ for all $i = 1, \dots, p$ and f_i^* is the optimal value of the optimization problem,

$$(P_i) \quad \min_{x \in X} f_i(x).$$

Let \bar{x}_i be a minimizer of Problem (P_i). Then $f_i^* = f_i(\bar{x}_i)$, $i = 1, \dots, p$.

Problem (P_w) is referred to as the *weighted Tchebychev problem* (or *Tchebychev scalarization*) because of the weighted Tchebychev norm $\max_i |w_i(f_i(x) - u_i^*)| = \max_i w_i(f_i(x) - u_i^*)$ appearing in the cost.

The following result is well-known.

Theorem 1 (Miettinen [23, Theorems 3.4.2 and 3.4.5]) The point x^* is a weak Pareto minimum of (P) if, and only if, x^* is a solution of (P_w) for some $w_1, \dots, w_p > 0$.

Theorem 1 lays the ground for an approximate construction of the Pareto front: One solves Problem (P_w) with a range of values for the weights, w_1, \dots, w_p , and “hopes” to generate points giving a good approximation of the Pareto front.

There has been a great deal of effort by the researchers in the area (especially in recent years) for developing methods to generate an approximation of the Pareto front, see e.g. [1, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 22, 24, 25, 27, 29, 30]. In most of these works, various types of scalarization are considered (notably except in [8, 14, 18, 15, 24, 27]), and one of the main concerns in these efforts is to get a more-or-less uniform distribution in the value space of the points found by solving the single-objective problem (or scalarization).

The following result from [26], which we re-word for our purposes and setting, leads to an interesting geometric interpretation.

Theorem 2 (Ogryczak [26, Theorem 1]) Suppose that the point x^* is a Pareto minimum of Problem (P) such that

$$w_1(f_1(x^*) - u_1^*) = \dots = w_p(f_p(x^*) - u_p^*), \quad (1)$$

for some $w_1, \dots, w_p > 0$. Define the optimal value vector $(\bar{f}_1, \dots, \bar{f}_p) := (f_1(x^*), \dots, f_p(x^*))$. Then $(\bar{f}_1, \dots, \bar{f}_p) = (f_1(\bar{x}), \dots, f_p(\bar{x}))$, where \bar{x} is a solution of Problem (P_w) for the same w_1, \dots, w_p , is the *unique* optimal value vector.

Note that

$$w_1 (f_1 - u_1^*) = \cdots = w_p (f_p - u_p^*), \quad (2)$$

where f_1, \dots, f_p are the coordinates of the value space, represents a *ray*, i.e. a line segment, with direction $(1/w_1, \dots, 1/w_p)$, emanating from the utopia point (u_1^*, \dots, u_p^*) . Let $f = (f_1, \dots, f_p)$ and $v = (1/w_1, \dots, 1/w_p)$. Then a parametric equation for the ray can simply be written as

$$f = tv + u^*, \quad t \geq 0. \quad (3)$$

Theorem 2, combined with Theorem 1, tells us that if the ray given by (1) intersects the Pareto front for some given weights w_1, \dots, w_p , and if the point at the intersection is a Pareto minimum (but not only a weak Pareto minimum), then a solution of Problem (P_w) yields the intersection point on the Pareto front. This result is useful in generating a good approximation of the Pareto front under a special case: In the case when the weak Pareto minima in the front are also Pareto minima, one can use a grid of values for the weights, w_1, \dots, w_p , for a “more-or-less evenly spaced” rays and an appropriate choice of the utopia point as a reference point. This idea forms a basis of the methods presented by [9, 10, 11, 25] recently. Our approach differs from these earlier works in that we propose a new scalarization (PR_w) (see below) with which it is possible to construct an approximation of weak Pareto fronts. We explain a few other advantages of the new scalarization when we introduce it, in sequel.

Note that Problem (P_w) can equivalently be written, by using a standard trick from mathematical programming, as

$$(P1_w) \begin{cases} \min_{x \in X} & \gamma \\ \text{subject to} & w_i (f_i(x) - u_i^*) \leq \gamma, \quad i = 1, \dots, p. \end{cases}$$

Problem $(P1_w)$ is referred to as *goal-attainment method* [23, 25], as well as *Pascoletti-Serafini (P-S) scalarization* [9, 10, 11]. With Problem $(P1_w)$, one has the advantage over Problem (P_w) in that, in the case when Problem (P) is smooth, Problem $(P1_w)$ is also smooth, for which powerful numerical techniques can be employed to find a solution. However this comes at the expense of p additional constraints and one extra (slack) decision variable, γ , when compared with Problem (P_w) . Therefore, in the case when Problem (P) is nonsmooth, Problem (P_w) should be preferred.

Never the less, it is interesting to note that, in view of Equation (1) and the discussion following Theorems 1 and 2 above, in the case when the set of weak Pareto points is the same as the set of Pareto points in the front, one can re-write Problem $(P1_w)$ simply as

$$(P2_w) \begin{cases} \min_{x \in X} & \gamma \\ \text{subject to} & w_i (f_i(x) - u_i^*) = \gamma, \quad i = 1, \dots, p. \end{cases}$$

Problem $(P2_w)$ is referred to as the *modified Pascoletti-Serafini scalarization* in [9].

In the case when the set of weak Pareto points is not the same as the set of Pareto points in the front, i.e., when there are weak Pareto minima in the front which are not Pareto minima, there is no guarantee anymore that the weak Pareto minimum found by solving Problem (P_w) , or by solving Problems $(P1_w)$ and $(P2_w)$, is at the intersection of the ray associated with the chosen weights. This is detrimental to finding a good spread of points in the Pareto front, which is later illustrated by means of a test problem in Figure 2(a). Therefore it is necessary to add the expression for the ray as a constraint to the scalarization.

In this paper we propose a new Tchebychev-type scalarization, which is the scalarization (P_w) subject to the ray associated with the choice of weights of the scalarization and a utopia point as the reference point. Namely, we propose

$$(PR_w) \begin{cases} \min_{x \in X} \max_{1 \leq i \leq p} w_i (f_i(x) - u_i^*), \\ \text{subject to} & w_i (f_i(x) - u_i^*) - w_{i+1} (f_{i+1}(x) - u_{i+1}^*) = 0, \quad i = 1, \dots, p-1. \end{cases}$$

Note that the $(p-1)$ equality constraints here represent a ray as in (2). We call this new scalarization *Tchebychev scalarization along rays*.

For completeness and comparison purposes, we present two algorithms in the paper: Algorithm 1 implements the results in Theorems 1 and 2 using the scalarization (P_w) ; Algorithm 2 implements the result in Theorem 4 using the scalarization (PR_w) . We apply both algorithms to smooth and nonsmooth problems with connected and disconnected Pareto fronts.

It is important to note that the modified P-S scalarization was introduced in [9] more from a theoretical view point rather than a practical view point. Our approach is largely motivated from a practical point of view. Furthermore, the version of the modified P-S problem that we state in our paper as $(P2_w)$ is a special instance of the family of parametric problem introduced in [9] which is denoted as $(\overline{SP}(a, r))$. In our case $a = u^*$ the vector consisting of utopian values and $r = (1/w_1, \dots, 1/w_p)$. Of course we have chosen $w_i > 0$ for all i .

In [9] no relation between weak minimal points and the solutions of the modified P-S scalarization problem is given. It is however not apparent that there is any relationship between the weak Pareto points and the solutions of the modified P-S scalarization scheme. The reason for this might be as follows. In the original P-S scalarization, the ordering cone of the multi-objective problem plays a major role. In fact, the assumption that the ordering cone has a non-empty interior is significant in showing that the solution of the P-S scalarization problem corresponds to a weak Pareto point. See [9, Chapter 2] for details. However in the modified P-S scheme we replace the ordering cone with the cone $\{0\}$. This cone has no interior and this seems, at this moment, to be the factor stopping us from relating the weak Pareto points with the solution of the modified P-S problem.

In [9] it has been shown that a Pareto minimum point corresponds to the solution of the modified P-S scalarization problem under certain conditions which might not be so easy to verify in practice. To implement the approach in [9] using the modified P-S scheme, one needs to consider a hyperplane in the image space and then have rays originating from each point chosen in the hyperplane. These rays are required to intersect the Pareto front. In our approach we do not require constructing a hyperplane, and we can locate the weak Pareto minimum points quite easily in most cases, for example when the Pareto front is connected.

The rest of our paper is organized as follows. In Section 2, we prove two results about the new Tchebychev scalarization along rays, namely, Theorems 3 and 4. In Section 3, we present Algorithms 1 and 2. In Section 4, we illustrate both algorithms on three test problems and provide discussion. In Section 5, we make conclusions and point to future directions which might be interesting to pursue.

2 Results on Tchebychev Scalarization Along Rays

In this section we prove two results in Theorems 3 and 4 concerning the new Tchebychev scalarization along rays. In particular, Theorem 4 lays the ground for Algorithm 2.

The following theorem is analogous to the first part of Theorem 1 (i.e. Theorem 3.4.5 in [23]).

Theorem 3 If x^* is a weak Pareto minimum of (P) then x^* is a solution of (PR_w) for some $w_1, \dots, w_p > 0$.

Proof. Let $x^* \in X$ be a weak Pareto minimum. Let

$$w_i = \frac{\beta}{f_i(x^*) - u_i^*}, \quad i = 1, \dots, p, \quad (4)$$

where β is some fixed positive number. With this choice of w_i , $x^* \in X$ satisfies the equality constraints in (PR_w) . Suppose that $x^* \in X$ is not a solution of (PR_w) . Then there exists $\hat{x} \in X$ satisfying the equality constraints such that

$$\max_{1 \leq i \leq p} w_i (f_i(\hat{x}) - u_i^*) < \max_{1 \leq i \leq p} w_i (f_i(x^*) - u_i^*) = \beta$$

i.e.

$$w_i (f_i(\hat{x}) - u_i^*) < \beta, \quad \text{for all } i = 1, \dots, p.$$

Substituting the weights in (4), one gets

$$\frac{\beta}{f_i(x^*) - u_i^*} (f_i(\hat{x}) - u_i^*) < \beta,$$

and, after re-arranging,

$$f_i(\hat{x}) < f_i(x^*) \quad \text{for all } i = 1, \dots, p, \quad (5)$$

because $(f_i(x^*) - u_i^*)$ and β are positive. Inequality (5) is a contradiction with the fact that x^* is a weak Pareto minimum. \square

Remark 1 Note that the converse of Theorem 3 is not true, unless the Pareto front is connected. This can be illustrated as follows. Consider a bi-objective problem. Suppose that x_1^* and x_2^* are solutions of (P_1) and (P_2) , respectively, and let $f_1^* := f_1(x_1^*)$, $\bar{f}_2 := f_2(x_1^*)$, $f_2^* := f_2(x_2^*)$, and $\bar{f}_1 := f_1(x_2^*)$. The Pareto front is *connected* if and only if the line segment given by $f_2 = u_2^* + \alpha(f_1 - u_1^*)$ in the value space intersects the Pareto front for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, where

$$\alpha_{\min} = \arctan\left(\frac{f_2^* - u_2^*}{f_1^* - u_1^*}\right) \quad \text{and} \quad \alpha_{\max} = \arctan\left(\frac{\bar{f}_2 - u_2^*}{\bar{f}_1 - u_1^*}\right), \quad (6)$$

see Figure 1(a). Suppose that the Pareto front is not connected. Then there exists u_1^* , u_2^* , w_1 and w_2 such that the line segment defined by

$$w_1 (f_1(x) - u_1^*) - w_2 (f_2(x) - u_2^*) = 0, \quad \text{for all } x \in X,$$

in the objective space does not intersect the Pareto front - see Figure 1(b). Therefore a solution of (PR_w) with this equality constraint is not a Pareto minimum.

We can prove the converse of Theorem 3, if we require the Pareto front to be connected.

Theorem 4 Suppose that the Pareto front associated with Problem (P) is connected. If x^* is a solution of (PR_w) for some $w_1, \dots, w_p > 0$, then x^* is a weak Pareto minimum of (P).

Proof. Since x^* is a solution of (PR_w) for some $w_1, \dots, w_p > 0$,

$$w_i (f_i(x^*) - u_i^*) - w_{i+1} (f_{i+1}(x^*) - u_{i+1}^*) = 0, \quad i = 1, \dots, p-1,$$

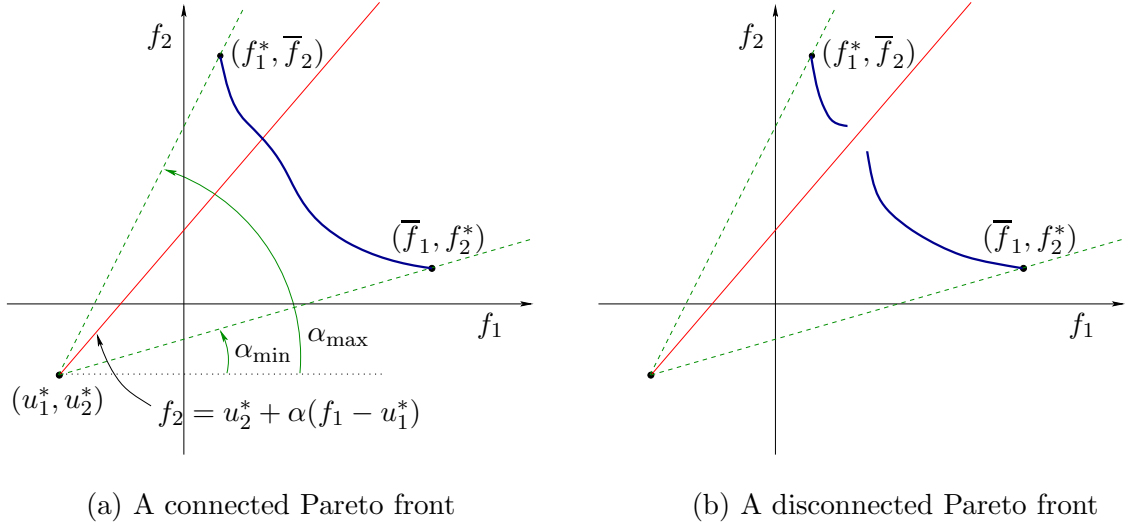


Figure 1: Connected and Disconnected Pareto fronts for the biobjective case, and the ray emanating from a utopia point – see Remark 1.

holds. Then one can write down the equalities in (2) which in turn define in the value space the ray $f = u^* + tv$, $t \geq 0$, in (3). So $f(x^*) = (f_1(x^*), \dots, f_p(x^*))$ is a point in the ray, for some $t \geq 0$.

We claim that x^* solves (P), i.e. x^* is a weak Pareto minimum. In other words, we claim that $f(x^*)$ is an intersection point of the ray with the Pareto front.

Assume on the contrary that x^* is not a weak Pareto minimum. Then, because the Pareto front is connected, there exists $\hat{x} \in X$ such that $f(\hat{x})$ is an intersection point of the ray with the Pareto front and that

$$f_i(\hat{x}) - u_i^* < f_i(x^*) - u_i^* \quad \text{for all } i = 1, \dots, p.$$

Since $w_1, \dots, w_p > 0$, we have

$$w_i (f_i(\hat{x}) - u_i^*) < w_i (f_i(x^*) - u_i^*) \quad \text{for all } i = 1, \dots, p.$$

Hence

$$\max_{1 \leq i \leq p} w_i (f_i(\hat{x}) - u_i^*) < \max_{1 \leq i \leq p} w_i (f_i(x^*) - u_i^*).$$

This contradicts the fact that x^* solves (PR_w) . □

3 Two Algorithms

Here we provide two algorithms for constructing an approximation of the Pareto front of biobjective problems. In each algorithm, we generate a grid of weights in a similar way they are generated in [7, 9, 25]. The procedures we describe here can be generalized to three or more objective cases by taking the methods of weight grid generation described in [7, 9, 25], as a basis, which should be considered as future work.

Algorithm 1 uses the results given in Theorems 1 and 2, in that a solution of Problem (P_w) gives a unique point in the Pareto front in the value (i.e. objective) space. Although Algorithm 1 does not utilize rays, if the solution of Problem (P_w) is Pareto (or weak Pareto which is also Pareto), then the point found is along the ray associated with the weights chosen, in the value space. If the solution of Problem (P_w) is only weak Pareto, but not Pareto,

Theorem 1 still guarantees that the solution point will be in the Pareto front, albeit Theorem 2 will no longer ensure that the solution point will be in the ray associated by the chosen weights. The latter situation can result in a non-uniform spread of the points found, as later depicted in Figure 2(a).

To address the difficulties of Algorithm 1 mentioned above, we propose Algorithm 2, which uses the result given in Theorem 4: If a ray associated with the chosen weights intersects the Pareto front, Problem (PR_w) yields a point in the Pareto front in the value space, even if the solution point is only a weak Pareto minimum.

Algorithm 2 uses rays formed by a range of values of the weights, just like Algorithm 1 does. In fact, Steps 0.0– $k.1$ of both algorithms, for varying the values of the weights corresponding to a range of rays, are identical. In principle, there is no way to know in advance if a ray intersects the Pareto front, especially in the case when the Pareto front is disconnected. So, a solution of Problem (PR_w) (if it exists) may not be a Pareto minimum. We eliminate such solutions by carrying out a “weeding out” procedure in the final step of Algorithm 2.

Algorithm 1 (Tchebychev)

Step 0.0 (Initialization) Choose the utopia parameters, $\varepsilon_1, \varepsilon_2 > 0$. Set the number of discrete points, $(N + 1)$, in the Pareto front. Set $k := 1$.

Step 0.1 (Boundary of the front)

- (a) Find \bar{x} that solves Problem (P_1) . Let $f_1^* := f_1(\bar{x})$ and $\bar{f}_2 := f_2(\bar{x})$.
Mark a boundary point in the Pareto front: $\bar{f}^0 := (f_1(\bar{x}), f_2(\bar{x}))$
- (b) Find \bar{x} that solves Problem (P_2) . Let $\bar{f}_1 := f_1(\bar{x})$ and $f_2^* := f_2(\bar{x})$.
Mark a boundary point in the Pareto front: $\bar{f}^N := (f_1(\bar{x}), f_2(\bar{x}))$.

Step 0.2 (Utopia point) Set $u^* := (u_1^*, u_2^*)$ with $u_i^* := f_i^* - \varepsilon_i$, $i = 1, 2$.

Step 0.3 (Range of angles for the rays) Compute α_{\min} and α_{\max} using (6). Set the increment, $\Delta\alpha := (\alpha_{\max} - \alpha_{\min})/N$.

Step $k.1$ (Angle and weights for a ray) Set $\alpha := \alpha_{\min} + k \Delta\alpha$. Set $w_1 := \sin \alpha$ and $w_2 := \cos \alpha$.

Step $k.2$ (A Pareto minimum) Find \bar{x} that solves Problem (P_w) .
Assign a point in the Pareto front: $\bar{f}^k := (f_1(\bar{x}), f_2(\bar{x}))$.

Step $k.3$ (Stopping criterion) If $k = N$ then STOP. Otherwise, set $k := k + 1$, and go to Step $k.1$.

Algorithm 2 (Tchebychev Along Rays)

Step 0.0–Step $k.1$ Do the same as in Steps 0.0– $k.1$ of Algorithm 1.

Step $k.2$ (A candidate for a Pareto minimum) Find \bar{x} that solves Problem (PR_w) .
Assign a candidate point for the Pareto front: $\bar{f}^k := (f_1(\bar{x}), f_2(\bar{x}))$.

Step $k.3$ (Completion of cycle) If $k = N$ then go to Step $(N + 1)$. Otherwise, set $k := k + 1$, and go to Step $k.1$.

Step $(N + 1)$ (Weeding out non-Pareto points) For each $i = 0, \dots, N$, if there exists $j = 0, \dots, N$ such that $f_1^j < f_1^i$ and $f_2^j < f_2^i$, then eliminate \bar{f}^i .

In either algorithm, in order to get a good spread of points in the Pareto front, it would be useful to choose the weights $w_1, w_2 > 0$ uniformly (e.g. over a grid) and make the utopia point (u_1^*, u_2^*) far enough from the ideal point, and thus, far from the front (by choosing the utopia parameters, $\varepsilon_1, \varepsilon_2 > 0$, large enough). Numerical illustrations in the next section will clarify this comment further.

In Step $k.2$ of both Algorithms 1 and 2, it is necessary to employ a nonsmooth numerical technique. In the case the (original) multi-objective problem is nonsmooth, a nonsmooth numerical method would also be necessary for Step 0.1. In the application of Algorithms 1 and 2 to the test problems that we study in the next section, we have used the deflected subgradient method [3, 4, 5, 6]. In solving the subproblems of the deflected subgradient method, we have used MATLAB's `fminsearch` which implements the Nelder-Mead method [21], as well as the freeware `SolvOpt` [19] which implements Shor's r-Algorithm [28].

It should be pointed out that if the approximation of the front is not done over a sufficiently fine grid, i.e., if the resulting approximation points are not close enough to one another, then the weeding out procedure may not eliminate some of the non-Pareto points. Never the less, in the numerical experiments presented in the next section, the weeding out procedure seems to be serving the purpose reasonably well.

Another approach which has similar geometric features to those of ours is the *Normal Boundary Intersection* (NBI) method due to Das and Dennis [7]. Just like our approach, the NBI method finds a point in the lower-boundary of the feasible objective set and may generate a point which is not a Pareto or weak Pareto minimum. Thus the weeding process has to be common to both. There are however significant differences. The multiobjective optimization problem that the NBI approach studies has a compact feasible set, although theoretically there is no such requirement in our case. We only require the feasible set to be closed. Our method and the NBI method fundamentally differ in the way they find the lower boundary of the feasible objective space. In the NBI approach one first constructs a set called CHIM (convex hull of individual minima), and then moves from a point in CHIM, along a direction normal to the set CHIM, towards the lower part of the boundary of the convex objective space. In our case, however, geometrically speaking, we find a point of the lower boundary of the objective feasible set by using rays emanating from a utopia point. The subproblems, or the scalarization used, in the two methods, also differ: while we employ a new scalarization, called Tchebychev scalarization along rays, the NBI method uses a scalarization akin to $(P2_w)$.

4 Numerical Illustration

In this section we illustrate numerical implementations of Algorithms 1 and 2 we presented in the previous section on three test problems.

4.1 Problem 1

A test problem introduced in [29] has also been used in many other studies – see e.g. [8, 9, 10]. We modify this problem by adding a nonsmooth (third) constraint as follows.

$$\begin{aligned} \min \quad & (x_1, x_2) \\ \text{subject to} \quad & -x_1^2 - x_2^2 + 1 + 0.1 \cos(16 \arctan(x_1/x_2)) \leq 0, \\ & (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \leq 0, \\ & 0.2 - \max\{|x_1 - 0.6|, |x_2 - 0.7|\} \leq 0, \\ & 0 \leq x_1, x_2 \leq \pi. \end{aligned}$$

As a result of the addition of the nonsmooth constraint, the Pareto front has a portion where the weak Pareto points are not Pareto. By devising this nonsmooth version of the problem, which has not been studied elsewhere, we would like to illustrate the effectiveness of Algorithm 2.

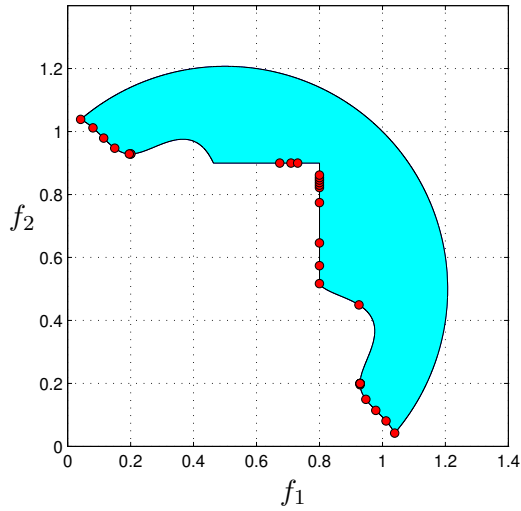
In both Algorithms 1 and 2, the utopia parameters are taken to be $\varepsilon_1 = 5$ and $\varepsilon_2 = 5$. So the utopia point, $u^* \approx (-4.9583, -4.9583)$. We set $N = 30$. At portions of the Pareto front where the points are either Pareto or weak Pareto which are also Pareto, that portion of the front can be obtained by Algorithm 1 with a nice spread of points, as shown in Figure 2(a). It is worth noting that these points coincide with those obtained along rays by Algorithm 2, which are depicted in Figure 2(b). However, the portion of the front where weak Pareto points are not Pareto cannot be generated well by Algorithm 1, as can be seen in Figure 2(a). Algorithm 2 can generate the complete portion with weak Pareto points, as illustrated in Figure 2(b) before weeding out. Figure 2(c) depicts an approximation of the front generated by Algorithm 2, after weeding out. To emphasize the capability of Algorithm 2 further, we also construct an approximation of the Pareto front with twice as many rays, i.e. with $N = 60$, as shown in Figure 2(d).

It should be noted that because the utopia (or reference) point is far enough from the front, the rays, although emanating from a single point, appear to be almost parallel. This, together with equal increments in the weights, helps a more-or-less uniform spread of the points computed.

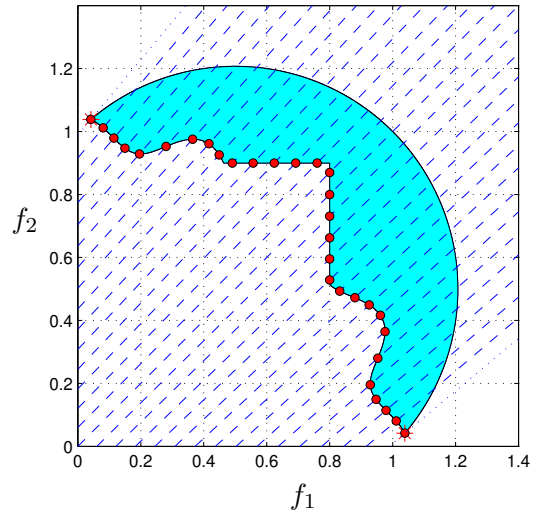
4.2 Problem 2

We consider another problem with a disconnected Pareto front, which is taken from [27]. This problem was also used in [15] to illustrate a technique for finding Pareto points without using a scalarization. We provide the problem as an example where Algorithm 2 can help even in the case when all weak Pareto points in the front are also Pareto points (see Figure 3(c) for the resulting “full” front) where Algorithm 1 was supposed to do the job, but could not.

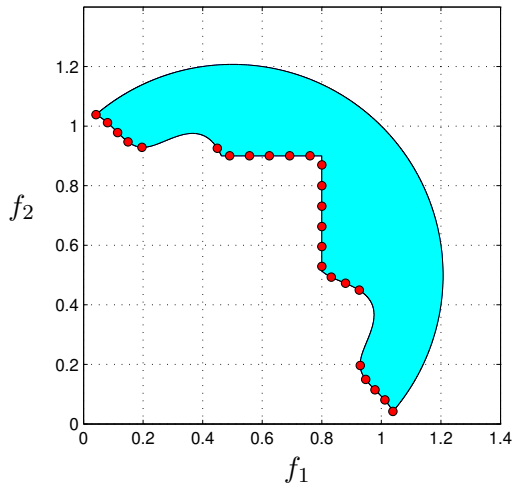
In both algorithms, the utopia parameters are taken as $\varepsilon_1 = 10$ and $\varepsilon_2 = 10$. So the utopia point, $u^* \approx (-3.1666, -10.000)$. Also $N = 30$. According to the underlying theory Algorithm 1 is expected to do the job; however because of the large jump from one portion of the Pareto front to the other, the initial guess for the nonsmooth solver within the deflected subgradient method is not good enough to find a global minimum of Problem (P_w) in Step *k*.2. As a result the whole of the second (upper) portion of the front is missed - see Figure 3(a). On the other hand, in Step *k*.2 of Algorithm 2, at least local Pareto points are found consecutively, leading to a “discovery” of the second (upper) portion of the front - see Figure 3(b). Consequently, the weeding out procedure yields the desired Pareto front as shown in Figure 3(c). The results also illustrate that the spread of the points found in the front is more-or-less uniform. In both [27] and [15], where no scalarization is used for finding



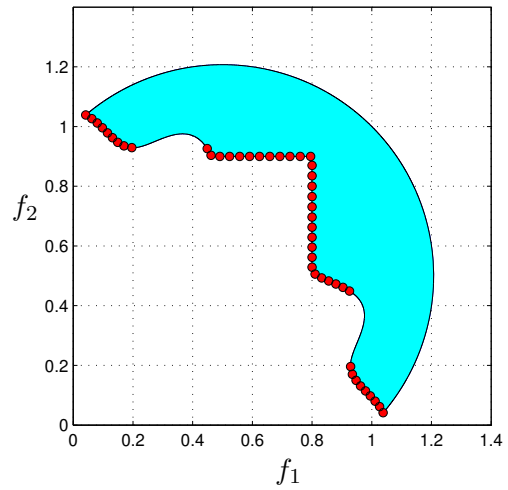
(a) Weak Pareto minima found by Alg. 1 (using no rays)



(b) Weak Pareto and non-Pareto points found in Alg. 2 (rays shown by dashed lines)

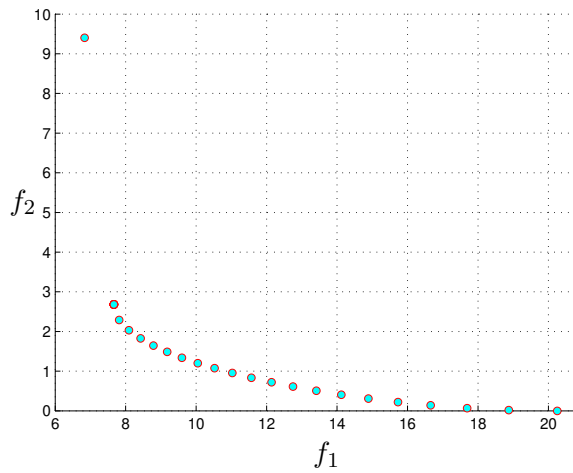


(c) Weak Pareto points after *weeding out* non-Pareto points along rays

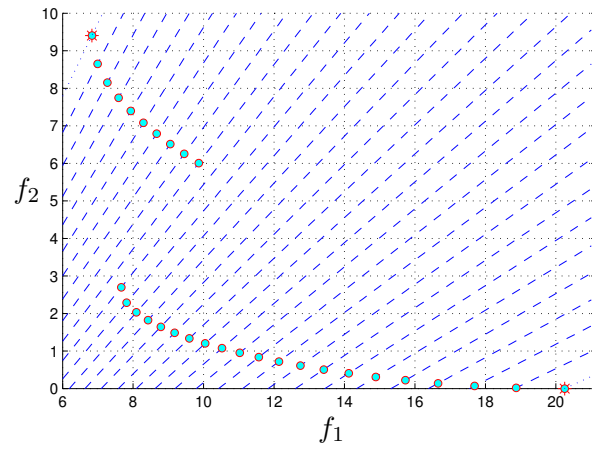


(d) A refined approximation of the front using Alg. 2 with $N = 60$

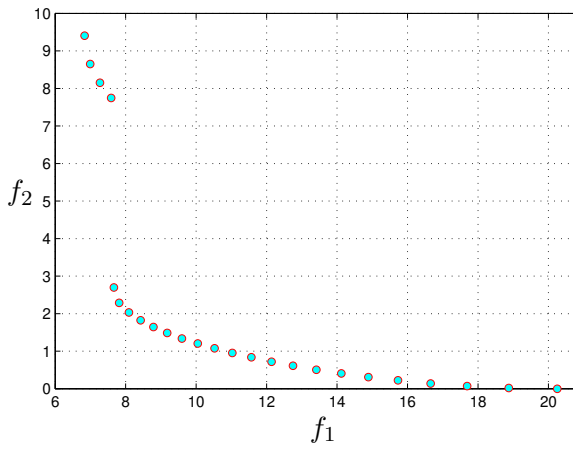
Figure 2: An illustration of improvement of an approximation of the Pareto front for Problem 1, from the set in (a) to the set in (c), with $N = 30$, and a refinement in (d) with $N = 60$.



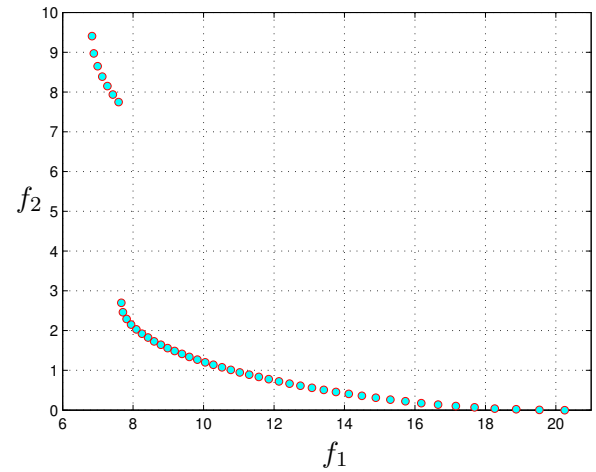
(a) Weak Pareto minima found by Alg. 1 (using no rays)



(b) Weak Pareto and non-Pareto points found in Alg. 2 (rays shown by dashed lines)



(c) Weak Pareto points after *weeding out* non-Pareto points along rays



(d) A refined approximation of the front using Alg. 2 with $N = 60$

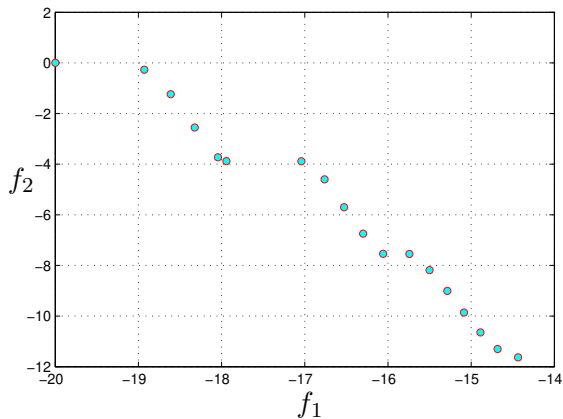
Figure 3: An illustration of improvement of an approximation of the Pareto front for Problem 2, from the set in (a) to the set in (c), with $N = 30$, and a refinement in (d) with $N = 60$.

Pareto points, no uniformity in the points found in the front can be ensured. Finally, we present a refined approximation of the front in Figure 3(d) found by Algorithm 2, where we have taken twice as many rays, namely we have set $N = 60$.

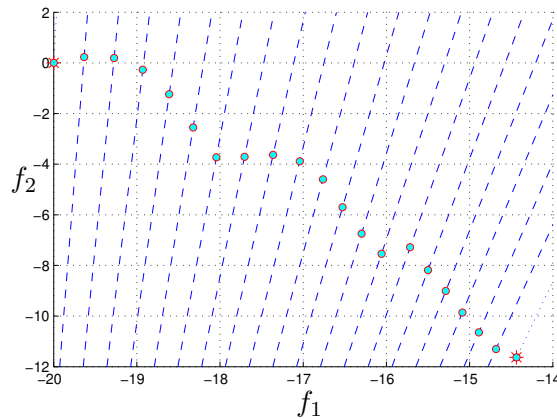
4.3 Problem 3

We consider a problem from [20] as a case when Algorithms 1 and 2 work equally well. The same problem was also studied in [8].

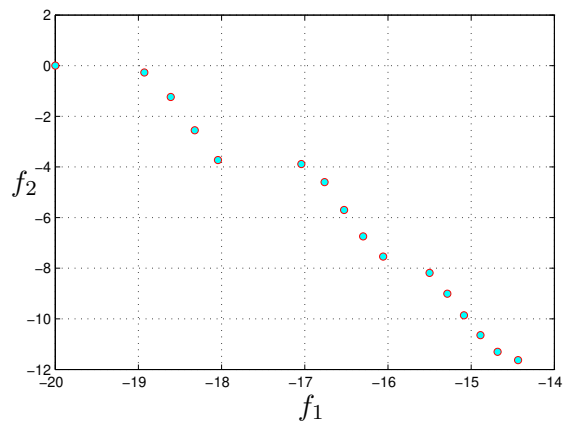
In both algorithms, the utopia parameters are taken as $\varepsilon_1 = 1$ and $\varepsilon_2 = 45$. So the utopia point, $u^* \approx (-21.000, -56.627)$. Also $N = 30$. Note that ε_1 and ε_2 have different orders of magnitudes; however, the chosen values help in getting an even spread of the points in the Pareto front - see Figure 4.



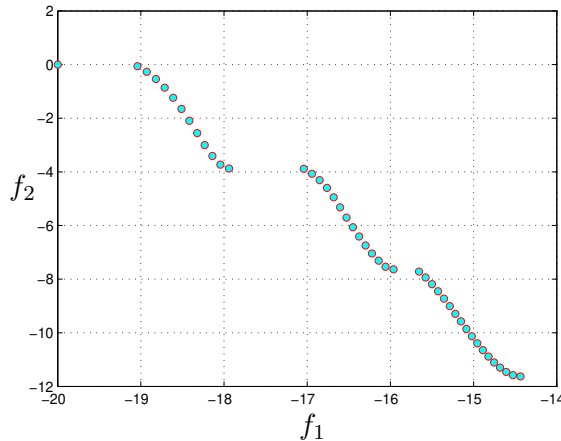
(a) Weak Pareto minima found by Alg. 1 (using no rays)



(b) Weak Pareto and non-Pareto points found in Alg. 2 (rays shown by dashed lines)



(c) Weak Pareto points after *weeding out* non-Pareto points along rays



(d) A refined approximation of the front using Alg. 2 with $N = 60$

Figure 4: An illustration of the similar fronts obtained for Problem 3 by Algorithms 1 and 2. A refinement in (d) with $N = 60$ is also shown.

All weak Pareto points are also Pareto points. So both Algorithms 1 and 2 give more-or-less the same front - see Figures 4(a) and (c). In fact, Algorithm 1 in this case accentuates the weak Pareto points more, because, for any values of weights corresponding to rays which do not intersect the front, Algorithm 1 finds a weak Pareto point at the boundary of a disconnected front. Algorithm 1 do not seem to have the problem it had for Problem 2, because the gaps between the disconnected portions of the front do not appear to be so large.

Finally, we present a refined approximation of the Pareto front with twice as many rays, i.e. with $n = 60$, in Figure 4(d). Accuracy and the uniformity of the points depicted in the figure looks to be better than those presented in [8].

5 Conclusion

We have presented a new scalarization technique, called Tchebychev scalarization along rays, and the associated Algorithm 2. We have also provided Algorithm 1, which is based on the

classical Tchebychev scalarization. Algorithm 1 is effective for problems with a Pareto front where the set of weak Pareto points is the same as the set of Pareto points. Algorithm 2 is based on the new Tchebychev scalarization along rays (in the value space) associated with the weights of the scalarization. Although we have presented Algorithms 1 and 2 for biobjective problems, they can be generalized to problems with more than two objectives, using the techniques employed in this paper, as well as those given in [7, 9, 25], as a basis. It would be of particular interest to apply the techniques developed in this paper to nonconvex optimal control problems along lines similar to those given in [2] for convex optimal control problems.

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