β-DENSITY FUNCTION ON THE CLASS GROUP OF PROJECTIVE TORIC VARIETIES

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ABSTRACT. We prove the existence of a compactly supported, continuous (except at finitely many points) function $g_{I,\mathbf{m}} : [0,\infty) \longrightarrow \mathbb{R}$ for all monomial prime ideals I of R of height one where $(R,\mathbf{m})$ is the homogeneous coordinate ring associated to a projectively normal toric pair (X,D), such that

$$\int_0^\infty g_{I,\mathbf{m}}(\lambda)d\lambda = \beta(I,\mathbf{m}),$$

where $\beta(I,\mathbf{m})$ is the second coefficient of the Hilbert-Kunz function of I with respect to the maximal ideal $\mathbf{m}$, as proved by Huneke-McDermott-Monsky [HMM04]. Using the above result, for standard graded normal affine monoid rings we give a complete description of the class map $\tau_{\mathbf{m}} : \mathrm{Cl}(R) \longrightarrow \mathbb{R}$ introduced in [HMM04] to prove the existence of the second coefficient of the Hilbert-Kunz function. Moreover, we show the function $g_{I,\mathbf{m}}$ is multiplicative on Segre products with the expression involving the first two coefficients of the Hilbert plolynomial of the rings and the ideals.

1. Introduction

Let R be a Noetherian ring of prime characteristic $p>0$ and of dimension d and let $\eta \subseteq R$ be an ideal of finite colength. Let M be a finitely generated R-module. The Hilbert-Kunz function of M with respect to the ideal η is defined as

$$\mathrm{HK}(M,\eta)(n) := \ell(M/\eta^{[q]}M)$$

where $q = p^n$, the ideal $\eta^{[q]} = n$-th Frobenius power of the ideal η and $\ell(M/\eta^{[q]}M)$ denotes the length of the R-module $M/\eta^{[q]}M$. The limit

$$\lim_{n\to\infty} \frac{1}{q^d}\ell(M/\eta^{[q]}M) =: e_{HK}(M,\eta)$$

exists [Mon83] and is called the Hilbert-Kunz multiplicity of M with respect to the ideal η. In addition to the above conditions, when R is an excellent normal domain, Huneke, McDermott and Monsky [HMM04, Theorem 1] have shown the existence of a real number $\beta(M,\eta)$ such that

$$\mathrm{HK}(M,\eta)(n) = e_{HK}(M,\eta)q^d + \beta(M,\eta)q^{d-1} + O(q^{d-2}).$$

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In the course of the proof of the above result, they have asserted the existence of a homomorphism $\tau_\eta : \mathrm{Cl}(R) \longrightarrow \mathbb{R}$ on the class group of R, $\mathrm{Cl}(R)$, the quotient of the free abelian group on the height one prime ideals of R by the subgroup of principal divisors. Let M be a finitely generated R-module. Then M admits a finite filtration $0 \to \cdots M_{i-1} \to M_i \cdots \to M$ such that M_i/M_{i-1} is isomorphic to R/P_i with P_i prime ideals in R. Consider the divisor $-\sum P_i$ where the sum is taken over all P_i appearing in the quotients M_i/M_{i-1} that are of height one. The image of this divisor in the class group of R is independent of the filtration chosen for M, and is defined as the class of M, denoted by $c(M)$. Let M be a finitely generated torsion-free R-module. By [HMM04, Corollary 1.10], the limit

$$\tau_\eta(M) := \lim_{q\to\infty} \frac{1}{q^{d-1}} \left[\ell(M/\eta^{[q]}M) - \mathrm{rank}(M)\ell(R/\eta^{[q]})\right]$$

is well defined and depends only on $c(M)$, the class of M in $\mathrm{Cl}(R)$. When R is F-finite,

$$\beta(M,\eta) = \tau_\eta(c(M)) - \frac{\mathrm{rank}(M)}{p^d - p^{d-1}} \tau_\eta(c({}^1R)),$$

where ${}^1R$ denotes the finitely generated module R over itself with the action given by the first Frobenius homomorphism.

The result of Huneke–McDermott–Monsky was generalised by Hochster-Yao in [HY09] from normal rings to the equidimensional reduced rings such that the singular locus is given by an ideal of height at least 2. Chan and Kurano have proved the result for reduced rings regular in codimension one [CK16]. For a normal affine monoid R, Bruns in [Bru05] have proved that HK function is a quasi polynomial and gave another proof of the existence of the constant second coefficient $\beta(R, \mathbf{m})$.

In order to study $e_{HK}(M,\eta)$, when R is a standard graded ring ($\dim(R) \geq 2$) with a homogeneous ideal η of finite colength and M is a finitely generated non-negatively graded R-module, Trivedi has defined the notion of Hilbert-Kunz density function, and obtained its relation with the HK multiplicity [Tri18, Theorem 1.1]: *The sequence of functions* $\{f_n(M,\eta) : [0,\infty) \longrightarrow \mathbb{R}_{\geq 0}\}_n$ *given by*

$$f_n(M,\eta)(\lambda) = \frac{1}{q^{d-1}} \ell(M/\eta^{[q]}M)_{\lfloor q\lambda \rfloor}$$

converges uniformly to a compactly supported continuous function $f_{M,\eta} : [0,\infty) \to \mathbb{R}_{\geq 0}$, *such that*

$$e_{HK}(M,\eta) = \int_0^\infty f_{M,\eta}(\lambda) d\lambda.$$

We call $f_{M,\eta}$ the Hilbert-Kunz density function or the HK density function of M with respect to the ideal η. The existence of a uniformly converging sequence makes the density function a more refined and useful invariant (compared to e_{HK}) in the graded situation ([Tri17], [Tri19], [TW20]). Applying the theory of HK density functions to projective toric varieties (denoted here as toric pairs (X, D)), one obtains [MT19, Theorem 6.3] an algebraic characterization of the tiling property of the associated polytopes P_D (in the ambient lattice) in terms of the asymptotic growth of e_{HK}, i.e.,

$e_{HK}(R, \mathbf{m}^k)$ relative to $e_0(R, \mathbf{m}^k)$ (the Hilbert Samuel multiplicity of R with respect to the ideal $\mathbf{m}^k$) as $k \to \infty$.

Let (X, D) be a toric pair, i.e., X is a projective toric variety over an algebraically closed field of characteristic $p > 0$, with a very ample T-Cartier divisor D and let R be the homogeneous coordinate ring of X, with respect to the embedding given by the very ample line bundle $\mathcal{O}_X(D)$, with homogeneous maximal ideal $\mathbf{m}$. There is a convex lattice polytope P_D as in (2.1), a convex polyhedral cone C_D and a bounded body $\mathcal{P}_D$ as in (2.3), associated to a toric pair (X, D). Such a bounded body was introduced by K. Eto (see [Eto02]), in order to study the HK multiplicity for a toric ring, and he proved that e_{HK} is the relative volume of such a body (we use the notation rVol_n to denote the n-dimensional relative volume function). In [MT19], it was shown that the HK density function at λ is the relative volume of the $\{z = \lambda\}$ slice of $\mathcal{P}_D$.

Similar to the HK density function, for a 'projectively normal' toric pair (X, D) (i.e., (X, D) is a toric pair such that the coordinate ring R is an integrally closed domain), it was shown in [MT20] that there exists a *β-density function* $g_{R,\mathbf{m}} : [0, \infty) \to \mathbb{R}$ which similarly refines the β-invariant of [HMM04]. More precisely, it was shown that *the sequence of functions $\{g_n(R, \mathbf{m}) : [0, \infty) \longrightarrow \mathbb{R}\}_{n \in \mathbb{N}}$, given by*

$$g_n(R, \mathbf{m})(\lambda) = \frac{1}{q^{d-2}} \left(\ell(R/\boldsymbol{m}^{[q]})_{\lfloor q\lambda \rfloor} - f_{R,\mathbf{m}}(\lfloor q\lambda \rfloor / q) q^{d-1} \right), \tag{1.1}$$

converges uniformly to a compactly supported continuous (except possibly on a finite set) function $g_{R,\mathbf{m}}$ such that $\int_0^\infty g_{R,\mathbf{m}}(x)dx = \beta(R, \mathbf{m})$. It was shown that the β-density function $g_{R,\mathbf{m}}$ at λ is expressible in terms of the relative volume of the $\{z = \lambda\}$ slice of the boundary, $\partial(\mathcal{P}_D)$, of $\mathcal{P}_D$ (stated in this paper as Theorem 2.1).

In regard to Theorem 2.1, one would like to ask whether there exists the notion of β-density function (with respect to the homogeneous maximal ideal $\mathbf{m}$) for all finitely generated non-negatively graded R-modules M which refines the invariant $\beta(M, \mathbf{m})$. In this paper we answer this question affirmatively for monomial prime ideals of R of height one. Using this result, we define a 'τ-density function' $\alpha_{I,\mathbf{m}} : [0, \infty) \longrightarrow \mathbb{R}$ for these ideals which describe the value of the function $\tau_{\mathbf{m}} : [0, \infty) \longrightarrow \mathbb{R}$ for these ideals via a simple integral formula, i.e., $\int_0^\infty \alpha_{I,\mathbf{m}}(x)dx = \tau_{\mathbf{m}}(I)$. This gives a complete description of the homomorphism $\tau = \tau_{\mathbf{m}}$ since the class group of R is generated by its monomial prime ideals of height one.

Let $I = p_F$ be a monomial prime ideal of height one, associated to a facet F of P_D. To prove the existence of the β-density function for I with respect to the homogeneous maximal ideal $\mathbf{m}$, consider the sequence of functions $\{g_n(I, \mathbf{m}) : [0, \infty) \longrightarrow \mathbb{R}\}_n$, given by

$$g_n(I, \mathbf{m})(\lambda) = \frac{1}{q^{d-2}} \left(\ell(I/\mathbf{m}^{[q]}I)_{\lfloor \lambda q \rfloor} - f_{I,\mathbf{m}}(\lfloor \lambda q \rfloor / q) q^{d-1} \right).$$

Let $\sigma_F : \mathbb{R}^d \longrightarrow \mathbb{R}$ be the support function for the facet of C_D corresponding to the facet F of P_D and let $H_{F,\mu} = \{x \in \mathbb{R}^d \mid \sigma_F(x) = \mu\}$ for all $\mu \in \mathbb{Q}_{\geq 0}$. Also, let $\mu_{D,F} = \{\mu \in \mathbb{Q}_{>0} \mid u \in H_{F,\mu} \text{ for some } u \in P_D \cap \mathcal{M}\}$, where $\mathcal{M}$ is the ambient lattice associated to the torus $T \subset X$ (see Section 2). We prove the following main result.

Theorem 1.1. *Let (X, D) be a projectively normal toric pair of dimension ≥ 2 and let $(R, \mathbf{m})$ be the associated homogeneous coordinate ring. Let $I = p_F$ be a monomial prime ideal of height one, associated to a facet F of the polytope P_D. There exists a finite set $\upsilon_{\mathcal{P}_D,F} \subset [0, \infty)$ such that for any compact set $V \subset [0, \infty) \setminus \upsilon_{\mathcal{P}_D,F}$, the sequence of functions $\{g_n(I, \mathbf{m})|_V\}$ converges uniformly to a function $g_{I,\mathbf{m}}|_V$ where $g_{I,\mathbf{m}} : [0, \infty) \setminus \upsilon_{\mathcal{P}_D,F} \longrightarrow \mathbb{R}$ is a compactly supported continuous function given by*

$$g_{I,\mathbf{m}}(\lambda) = g_{R,\mathbf{m}}(\lambda) - f_{R/I,\mathbf{m}/I}(\lambda) + \sum_{\mu\in\mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D) \cap H_{F,\mu} \cap \{z = \lambda\}).$$

Here $f_{R/I,\mathbf{m}/I} : [0, \infty) \longrightarrow \mathbb{R}_{\geq 0}$ is the HK *density function of the graded ring R/I with respect to the homogeneous maximal ideal $\mathbf{m}/I$.*

Moreover,

$$\beta(I, \mathbf{m}) = \int_0^\infty g_{I,\mathbf{m}}(\lambda)d\lambda.$$

Now we give a brief sketch of the proof of Theorem 1.1. Since $f_{R,\mathbf{m}}(\lambda) = f_{I,\mathbf{m}}(\lambda)$ for all $\lambda \in [0, \infty)$ ([MT19, Proposition 2.14]), we note that

$$g_n(I, \mathbf{m})(\lambda) = g_n(R, \mathbf{m})(\lambda) + f_n(R/I, \mathbf{m}/I)(\lambda) + \psi_n(\lambda), \tag{1.2}$$

where the function $\psi_n : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$\psi_n(\lambda) = \frac{1}{q^{d-2}}\ell\left(\frac{m^{[q]} \cap I}{m^{[q]}I}\right)_{\lfloor q\lambda\rfloor}.$$

Thus we need to show the sequence of functions $\{\psi_n\}$ converges uniformly. We note that the proof of 'existence' of an invariant or a property in Hilbert-Kunz theory, often boils down to bounding the 'correction' term in a converging sequence. For example, for the proof of [HMM04, Theorem 1], for any torsion-free R-module M with $c(M) = 0$, they show that $\ell(M/\eta^{[q]}M) - \mathrm{rank}(M)\ell(R/\eta^{[q]}) = O(q^{d-2})$. The [HMM04, Lemma 1.2] is crucial for this proof which uses a similar order bound on the length $\ell(T/\eta^{[q]}T) = O(q^{\dim(T)})$ for any finitely generated R-module T, due to Monsky [Mon83]. To prove the existence of the Hilbert-Kunz density function, in [Tri18, Proposition 2.12] Trivedi shows $|f_n(M, \eta)(\lambda) - f_{n'}(M, \eta)(\lambda)| = O(1/q)$ for all $n' > n \gg 0$. In [MT20], for $\lambda \in [0, \infty)$ and $\lambda_n := \lfloor q\lambda\rfloor/q \notin v(\mathcal{P}_D)$ (Notations 3.6(2)), it is shown that $g_n(R, \mathbf{m})(\lambda) = g(R, \mathbf{m})(\lambda_n) + c(\lambda_n)/q$ with $|c(\lambda_n)| < \tilde{C}$, a constant independent of λ and n.

In this paper, we use a similar approach to bound the error term in the converging sequence of functions $\{\psi_n\}$. In particular, we show that (Lemma 4.5) there exists a finite set $\upsilon_{\mathcal{P}_D,F} \subset [0, \infty)$ such that for all $\lambda \in [0, \infty)$ and for all $n \in \mathbb{N}$ with $\lambda_n \notin \upsilon_{\mathcal{P}_D,F}$,

$$\psi_n(\lambda) = \sum_{\mu\in\mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D) \cap H_{F,\mu} \cap \{z = \lambda_n\}) + \frac{c_\lambda(n)}{q}$$

where $|c_\lambda(n)| \leq C$ for some constant C, independent of λ and n. Hence for any compact set $V \subset [0, \infty) \setminus \upsilon_{\mathcal{P}_D,F}$, the sequence of functions $\{\psi_n|_V\}$ converges uniformly to the function $\Psi_F|_V$, given by $\lambda \mapsto \sum_{\mu\in\mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D) \cap H_{F,\mu} \cap \{z = \lambda\})$. This observation along with Equation (1.2), Theorem 2.1 and the property of HK density function give us the proof of the first part of the main Theorem.

Now, since $f_{R,\mathbf{m}}(\lambda) = f_{I,\mathbf{m}}(\lambda)$ for all $\lambda \in [0, \infty)$, by [MT20, Lemma 40] we have

$$\int_0^\infty f_{I,\mathbf{m}}(\lfloor \lambda q \rfloor / q) d\lambda = e_{HK}(I, \mathbf{m}) + O(1/q^{d-2}).$$

Now, a similar approximation of the integral of the function $g_{I,\mathbf{m}}$ by the integral of the functions $g_n(I, \mathbf{m})$, as was approximated the integral of the function $g_{R,\mathbf{m}}$ by the integral of the functions $g_n(R, \mathbf{m})$ in [MT20], gives us that $\int_0^\infty g_{I,\mathbf{m}} = \beta(I, \mathbf{m})$.

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2. Density functions on projective toric varieties

In this paper we work over an algebraically closed field K with char $p > 0$. Let $\mathcal{N}$ be a lattice (which is isomorphic to $\mathbb{Z}^{d-1}$) and let $\mathcal{M} = \mathrm{Hom}(\mathcal{N}, \mathbb{Z})$ denote the dual lattice with a dual pairing $\langle \ , \rangle$. Let $T = \mathrm{Spec}(K[\mathcal{M}])$ be the torus with character lattice $\mathcal{M}$ and let X be a complete toric variety over K with fan $\Delta \subset \mathcal{N} \otimes \mathbb{R} := \mathcal{N}_\mathbb{R}$. The irreducible subvarieties of codimension 1 of X which are stable under the action of the torus T correspond to the edges (one dimensional cones) of Δ. If $\tau_1, \ldots, \tau_n$ denote the edges of the fan Δ, then these divisors are the orbit closures $D_i = V(\tau_i)$. Let v_i be the first lattice point along the edge τ_i. A very ample T-Cartier divisor $D = \sum_i a_i D_i$ ($a_i \in \mathbb{Z}$) determines a convex lattice polytope in $\mathcal{M}_\mathbb{R} := \mathcal{M} \otimes \mathbb{R}$ defined by

$$P_D = \{u \in \mathcal{M}_\mathbb{R} \mid \langle u, v_i \rangle \geq -a_i \ \text{ for all } i \ \} \tag{2.1}$$

and the induced embedding of X in $\mathbb{P}^{l-1}$ is given by

$$\phi = \phi_D : X \to \mathbb{P}^{l-1}, \quad x \mapsto (\chi^{u_1}(x) : \cdots : \chi^{u_l}(x)),$$

where $P_D \cap \mathcal{M} = \{u_1, u_2, \ldots, u_l\}$ (for more detailed discussion, see [Ful93]).

The ring $K[\chi^{(u_1,1)}, \ldots, \chi^{(u_l,1)}]$ is the homogeneous coordinate ring of X with respect to this embedding. We have an isomorphism of graded rings [CLS11, Proposition 1.1.9]

$$\frac{K[Y_1, \ldots, Y_l]}{I} \simeq K[\chi^{(u_1,1)}, \ldots, \chi^{(u_l,1)}] =: R, \tag{2.2}$$

where, the kernel I is generated by the binomials of the form

$$Y_1^{a_1} Y_2^{a_2} \cdots Y_l^{a_l} - Y_1^{b_1} Y_2^{b_2} \cdots Y_l^{b_l}$$

where $a_1, \ldots, a_l, b_1, \ldots, b_l$ are nonnegative integers satisfying the equations

$$a_1 u_1 + \cdots + a_l u_l = b_1 u_1 + \cdots + b_l u_l \ \text{ and } \ a_1 + \cdots + a_l = b_1 + \cdots + b_l.$$

Due to this isomorphism, we can consider $R = K[S]$ as a standard graded ring with $\deg(\chi^{(u_i,1)}) = 1$, where S is the semigroup generated by $\langle (P_D \cap \mathcal{M}) \times \{z = 1\} \rangle$ in $\mathbb{R}^d$.

Let C_D be the cone generated by $\langle (P_D \cap \mathcal{M}) \times \{z = 1\} \rangle$ in $\mathbb{R}^d$. The prime ideals of the polytopal ring R is in one-to-one correspondence with faces of C_D, given by

$$C_F \leftrightarrow p_F := \text{ ideal of } R \text{ generated by the set of monomials } \{\chi^\nu \mid \nu \in S \setminus C_F\} \subset R$$

where C_F is the face of C_D corresponding to a face F of P_D [BG09, Proposition 2.36, Proposition 4.32]. The height one prime ideals correspond to the facets of P_D under this correspondence [BG09, Proposition 4.35]. In this case, the valuation v_{p_F} is the unique extension of the support form σ_F of C_D associated with the facet C_F. When (X, D) is a projectively normal toric pair, i.e., the associated homogeneous coordinate ring R is an integrally closed domain, the semigroup $S = C_D \cap \mathbb{Z}^d$ and the divisorial monomial ideals of R are exactly the R-submodules of $R = K[S]$ whose monomial basis is determined by a system

$$\{x \in \mathbb{R}^d \mid \sigma_F(x) \geq n_F, F \text{ is a facet of } P_D\}$$

for $n_F \in \mathbb{Z}$ [BG09, Theorem 4.53]. Let $\mathrm{Div}(S)$ denote the subgroup of $\mathrm{Div}(R)$ generated by monomial divisorial prime ideals and let $\mathrm{Princ}(S)$ be its subgroup generated by principal monomial ideals. The class group of the semigroup S, denoted $\mathrm{Cl}(S) = \mathrm{Div}(S)/\mathrm{Princ}(S)$ is generated by the classes of the ideals p_F where F runs over the set of facets of P_D [BG09, Corollary 4.55] and is isomorphic to the group $\mathrm{Cl}(R)$, the class group of R [BG09, Theorem 4.59].

For a toric pair (X, D), let

$$\mathcal{P}_{\mathcal{D}} = \{p \in C_D \mid p \notin (u, 1) + C_D, \text{for every } u \in P_D \cap \mathcal{M}\}. \tag{2.3}$$

By result of Eto we have $e_{HK}(R, \mathbf{m}) = \mathrm{Vol}_d(\mathcal{P}_D) = \mathrm{Vol}_d(\overline{\mathcal{P}}_D)$ [Eto02, Theorem 2]. Here Vol_n denotes the n-dimensional volume. Moreover,

$$\mathrm{HKd}(R, \mathbf{m})(\lambda) = \mathrm{Vol}_{d-1}(\mathcal{P}_D \cap \{z = \lambda\}) = \mathrm{Vol}_{d-1}(\overline{\mathcal{P}}_D \cap \{z = \lambda\})$$

for all $\lambda \in [0, \infty)$ [MT19, Theorem 1.1]. In particular, it is a piecewise polynomial function.

We recall the following result from [MT20]:

Theorem 2.1. [MT20, Theorem 2, Corollary 3] *Let $(R, \mathbf{m})$ be the homogeneous coordinate ring of dimension $d \geq 3$, associated to the projectively normal toric pair (X, D). Then there exists a finite set $v(\mathcal{P}_D) \subseteq \mathbb{R}_{\geq 0}$ such that, for any compact set $V \subseteq \mathbb{R}_{\geq 0} \setminus v(\mathcal{P}_D)$, the sequence $\{g_n|_V\}_n$ (as described in (1.1)) converges uniformly to $g_{R,\mathbf{m}}|_V$, where $g_{R,\mathbf{m}} : \mathbb{R}_{\geq 0} \setminus v(\mathcal{P}_D) \longrightarrow \mathbb{R}$ is a continuous function given by*

$$g_{R,\mathbf{m}}(\lambda) = \mathrm{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \partial(C_D) \cap \{z = \lambda\}\right) - \frac{\mathrm{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \{z = \lambda\}\right)}{2}.$$

Moreover, we have

$$\beta(R, \mathbf{m}) = \int_0^\infty g_{R,\mathbf{m}}(\lambda) d\lambda = \mathrm{rVol}_{d-1}\left(\partial(\mathcal{P}_D) \cap \partial(C_D)\right) - \frac{\mathrm{rVol}_{d-1}\left(\partial(\mathcal{P}_D)\right)}{2}.$$

Throughout the paper, we use the following notations.

Notations 2.2. (1) For a facet F of P_D, let C_F be the corresponding facet of C_D with supporting hyperplane H_F and support form $\sigma_F : \mathbb{R}^d \longrightarrow \mathbb{R}$. Hence $H_F = \{x \in \mathbb{R}^d \mid \sigma_F(x) = 0\}$ and $C_F = C_D \cap H_F$. Note that

$$C_D = \cap_{\{F \mid F \text{ is a facet of } P_D\}} \{x \in \mathbb{R}^d \mid \sigma_F(x) \geq 0\}.$$

(2) $C_u = (u, 1) + C_D$ for $u \in P_D \cap \mathcal{M}$.

(3) For the ideal $I = p_F$, we set
$$C_I = \{x \in C_D \mid \sigma_F(x) > 0\}.$$
(4) For a set $A \subset \mathcal{M}_{\mathbb{R}} \times \mathbb{R} \simeq \mathbb{R}^d$, we denote
$$A \cap \{z = \lambda\} := A \cap \{(\mathbf{x}, \lambda) \mid \mathbf{x} \in \mathbb{R}^{d-1}\}.$$
(5) For a bounded set $A \subset \mathbb{R}^d$, we set $L(A) = A \cap (\mathcal{M} \times \mathbb{Z})$ = the (finite) set of lattice points of A.
(6) For $m \in \mathbb{Z}$, let us denote the set of lattice points in the hyperplane $\{z = m\}$ by Λ_m, i.e., $\Lambda_m = \mathbb{Z}^{d-1} \times \{z = m\} \subseteq \mathbb{R}^d$.

Lemma 2.3. *Let $C_F \subset C_D$ be the cone generated by the facet F of P_D. Then*
(1) *for $q \in \mathbb{N}$, we have $C_F \setminus \cup_{u \in L(P_D)} q(u,1) + C_D = C_F \setminus \cup_{u \in L(F)} q(u,1) + C_F$.*
(2) $[C_D \setminus \cup_{u \in L(P_D)} C_u] \cap C_F = C_F \setminus \cup_{u \in L(P_D)} C_u = C_F \setminus \cup_{u \in L(F)} (u,1) + C_F$.

Proof. Proof of Part (1): Since $\cup_{u \in L(P_D)} q(u,1) + C_D \supset \cup_{u \in L(F)} q(u,1) + C_F$, it is enough to show
$$C_F \cap [\cup_{u \in L(P_D)} q(u,1) + C_D] = \cup_{u \in L(F)} q(u,1) + C_F.$$
Let $x \in C_F \cap [\cup_{u \in L(P_D)} q(u,1) + C_D]$. Choose $u_0 \in L(P_D)$ such that $x = q(u_0,1) + y$ for some $y \in C_D$. Since $x \in C_F$, we have $0 = \sigma_F(x) = q\sigma_F(u_0,1) + \sigma_F(y)$. This implies $\sigma_F(u_0,1) = \sigma_F(y) = 0$, i.e., $(u_0,1), y \in C_F$. Hence $x \in \cup_{u \in L(F)} q(u,1) + C_F$. The reverse inclusion follows since $\sigma_F((u,1) + y) = 0$ for all $u \in L(F)$ and $y \in C_F$.

Proof of Part (2): The first equality is obvious. The second equality follows from Part (1). □

Remark 2.4. *Let $(R, \mathbf{m})$ be the homogeneous coordinate ring of dimension $d \geq 3$, associated to the projectively normal toric pair (X, D). Let I be a monomial prime ideal of height one, associated to a facet F of the polytope P_D and let $f_{\overline{R},\overline{\mathbf{m}}}$ be* HK *density function of the standard graded ring $\overline{R} := R/I$ with respect to its homogeneous maximal ideal $\overline{\mathbf{m}} = \mathbf{m}/I$. For $\lambda \in [0, \infty)$ and $q = p^n, n \in \mathbb{N}$, we have $f_{\overline{R},\overline{\mathbf{m}}}(\lambda) =$ $\lim_n f_n(\overline{R}, \overline{\mathbf{m}})(\lambda)$*
$$= \lim_n \frac{1}{q^{d-2}} \ell\left(\frac{R}{m^{[q]} + I}\right)_{\lfloor q\lambda \rfloor} = \lim_n \frac{1}{q^{d-2}} \# \left[(C_F \setminus \cup_{u \in L(P_D)} q(u,1) + C_D) \cap \Lambda_{\lfloor q\lambda \rfloor}\right]$$
$$= \lim_n \frac{1}{q^{d-2}} \# \left[(C_F \setminus \cup_{u \in L(F)} q(u,1) + C_F) \cap \Lambda_{\lfloor q\lambda \rfloor}\right].$$

3. The boundary of $\mathcal{P}_D$ parallel to the facet C_F of the cone C_D

In this section, we study the set $\partial(\mathcal{P}_D)$, the set $\partial(\mathcal{P}_D) \cap \{x \in C_D \mid \sigma_F(x) = \mu\}$, where $\mu = \sigma_F(u,1)$ for some $u \in L(P_D \setminus F)$. We also study the coefficient of the Ehrhart quasi-polynomial of certain polytopes lying inside $\partial(\mathcal{P}_D)$. We set the following notations first:

Notations 3.1. (1) For a convex polytope Q, let $v(Q)$ = {vertices of Q} and $\mathcal{F}(Q)$ = {facets of Q}.
(2) For a convex polytope $Q \subset \mathbb{R}^d$, and for $\lambda \in [0, \infty]$ we set $Q_\lambda = Q \cap \{z = \lambda\}$.

(3) For a set $F \subseteq \mathbb{R}^d$, $\partial(F)$ = boundary of F in $\mathbb{R}^d$ and $F^\circ = F \setminus \partial(F)$ = interior of F in $\mathbb{R}^d$.
(4) For a set $F \subseteq \mathbb{R}^d$, $\partial_C(F)$ = boundary of F in C_D in the subspace topology of C_D, thinking of $C_D \subseteq \mathbb{R}^d$.
(5) For a set $F \subseteq \mathbb{R}^d$, we denote $A(F)$ = affine hull of F in $\mathbb{R}^d$, the smallest affine set containing F, i.e., $A(F) = \{\sum_i^m a_i f_i \mid m \in \mathbb{N}, a_i \in \mathbb{R}, f_i \in F, \sum_{i=1}^m a_i = 1\}$.
(6) For a set $F \subseteq \mathbb{R}^d$, we say $y \in \mathrm{relint}(F)$, the relative interior of F, if there exists $\epsilon > 0$ such that $B_d(y,\epsilon) \cap A(F) \subseteq F$. Here $B_d(y,\epsilon)$ denotes the d-dimensional ball of radius ϵ around y.
(7) For a facet F of P_D, and for $\mu \in \mathbb{Q}_{>0}$, we set $H_{F,\mu} = \{x \in \mathbb{R}^d \mid \sigma_F(x) = \mu\}$.
(8) Let $\mu_{D,F} := \{\mu \in \mathbb{Q}_{>0} \mid \sigma_F(u,1) = \mu \text{ for some } u \in L(P_D \setminus F)\}$.
(9) $\partial_{D,F} = \cup_{\mu\in\mu_{D,F}} \partial(\mathcal{P}_D) \cap H_{F,\mu}$.

For a toric pair (X, D), a decomposition of $C_D = \cup_{j=1}^s F_j$ was given in [MT19, Lemma 4.5], (for $d \geq 3$, as $d = 2$ corresponds to $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$, for $n \geq 1$, which is easy to handle directly), where F_j's are d-dimensional cones such that, each $P_j := F_j \cap \overline{\mathcal{P}}_D$ is a convex rational polytope and is a closure of $P_j' := F_j \cap \mathcal{P}_D$. In [MT20], the boundary of $\mathcal{P}_D$ was studied and described in terms of the facets of P_j's. We recall the decomposition of C_D and few properties of $\partial(\mathcal{P}_D)$ from [MT19] and [MT20] which are relevant for this work.

The cone $F_j \in \{d\text{-dimensional cones}\}$, which is the closure of a connected component of $C_D \setminus \cup_{iu} H_{iu}$, where the hyperplanes H_{iu} are given by

$$H_{iu} = \text{the affine hull of } \ \{(v_{ik}, 1), (u, 1), (\mathbf{0}) \mid v_{ik} \in v(C_{0i}), \quad u \in L(P_D)\},$$

where $C_{0i} \in \{(d-3) \text{ dimensional faces of } P_D\}$ and $\mathbf{0}$ is the origin of $\mathbb{R}^d$. For $u \in L(P_D)$, let

$$P_j' = F_j \cap \cap_{u\in L(P_D)} (C_u)^c = F_j \cap \cap_{u\in L(P_D)} [C_D \setminus C_u],$$

which is a convex set [MT19, Lemma 4.5] and $P_j = \overline{F_j \cap \cap_{u\in L(P_D)} (C_D \setminus C_u)}$ is the d-dimensional convex rational polytope which is the closure of P_j' in C_D (which equals the closure in $\mathbb{R}^d$).

Therefore

$$\mathcal{P}_D = \cup_{j=1}^s P_j' \quad \text{and} \quad \overline{\mathcal{P}}_D = \cup_{j=1}^s P_j,$$

where $P_1, \ldots, P_s$ are distinct polytopes, whose interiors are disjoint. Moreover, facets of each P_j are transversal to the z-hyperplane, i.e., $\dim(\partial(P_j) \cap \{z = \lambda\}) < d - 1$ for all $\lambda \in \mathbb{R}$ and for all j. Note that

$$P_j = \overline{F_j \setminus \cup_{u\in L(P_D)} C_u} = \overline{\cap_{u\in L(P_D)} F_j \setminus C_u} = P_j' \sqcup \left(\cup_{u\in L(P_D)} \partial_C(C_u) \cap P_j\right)$$

and $\partial_C(C_u) \cap P_j = \cup_{\{E \mid E\in\mathcal{F}(C_u),\ E\not\subseteq\partial(C_D)\}} E \cap P_j$ [MT20, Lemma 8]. Moreover, for any facet $E \in \mathcal{F}(P_j)$, either $E \subset E_{j_i}$, for some facet $E_{j_i} \in \mathcal{F}(F_j)$; or $F \subset F_{u_\nu}$, for some facet $F_{u_\nu} \in \mathcal{F}(C_u)$ and $u \in L(P_D)$. In the later case $F = P_j \cap F_{u_\nu} = P_j \cap A(F_{u_\nu})$, where $F_{u_\nu} \not\subseteq \partial(C_D)$ [MT20, Lemma 9]. Finally we record [MT20, Lemma 10] which gives the explicit description of $\partial(\mathcal{P}_D)$ as follows:

Lemma 3.2. (1) $\partial(\mathcal{P}_D) = \cup_{\{E\in\mathcal{F}(P_j) \mid E\neq P_i\cap P_j\}} E$. *In particular*
(2) $\partial(\mathcal{P}_D) = \bigcup_{\{E\in\mathcal{F}(C_D)\}} E \cap \overline{\mathcal{P}}_D \cup \bigcup_{\{E\in\mathcal{F}(C_u), u\in L(P_D)\}} E \cap \overline{\mathcal{P}}_D$.

Lemma 3.3. *Suppose $v \in L(P_D) \setminus L(F')$ for some $F' \in \mathcal{F}(P_D)$.*

(1) *Then there exists a d-dimensional cone F_j occurring in the decomposition of C_D such that $(v,1) \in F_j$ and $[(v,1) + C_{F'}] \cap F_j^\circ \neq \emptyset$.*

(2) *Moreover, $\dim(\partial(P_j) \cap [(v,1) + C_{F'}]) = d-1$, i.e., there exists $\tilde{E} \in \mathcal{F}(P_j)$ such that $\tilde{E} \subset (v,1) + C_{F'}$.*

Proof. Proof of Part (1): We choose small $\epsilon > 0$ such that $B_d((v,1),\epsilon) \cap F_j = \emptyset$ for all cones F_j in the decomposition of C_D with $(v,1) \notin F_j$. Therefore $[(v,1) + C_{F'}] \cap B_d((v,1),\epsilon) \subseteq \cup_{(v,1)\in F_j} F_j$. If

$$[(v,1) + C_{F'}] \cap B_d((v,1),\epsilon) \subseteq \cup_{(v,1)\in F_j} \partial F_j,$$

then $d-1 = \dim([(v,1)+C_{F'}] \cap B_d((v,1),\epsilon)) = \dim([(v,1)+C_{F'}] \cap B_d((v,1),\epsilon) \cap \partial F_{j_0})$ for some F_{j_0} containing $(v,1)$. Hence

$$A((v,1) + C_{F'}) = A([(v,1) + C_{F'}] \cap B_d((v,1),\epsilon)) = A(F'')$$

for some $F'' \in \mathcal{F}(F_{j_0})$. This is a contradiction since F'' passes through origin in $\mathbb{R}^d$, whereas $\mathbf{0} \notin A((v,1) + C_{F'})$ since $(v,1) \notin C_{F'}$. Hence $[(v,1) + C_{F'}] \cap F_j^\circ \neq \emptyset$ for some F_j containing $(v,1)$.

Proof of Part (2): We take a cone F_j such that $(v,1) \in F_j$ and $[(v,1)+C_{F'}] \cap F_j^\circ \neq \emptyset$. Note that $\dim([(v,1)+C_{F'}] \cap F_j^\circ) = d-1$. By the claim in the proof of [MT20, Lemma 9(2)], we have $[(v,1) + C_{F'}] \cap F_j = A((v,1) + C_{F'}) \cap F_j$. Hence

$$[(v,1) + C_{F'}] \cap F_j^\circ \subseteq \mathrm{relint}((v,1) + C_{F'}),$$

and $\dim(\mathrm{relint}((v,1) + C_{F'}) \cap F_j^\circ) = d-1$. Since $(v,1) \in F_j$, this implies for any ball $B_d((v,1),\epsilon)$ around $(v,1)$ of radius $\epsilon > 0$, we have

$$\dim(\mathrm{relint}((v,1) + C_{F'}) \cap F_j^\circ \cap B_d((v,1),\epsilon)) = d-1. \tag{3.1}$$

Since $(v,1) \notin (u,1) + C_D$ for all $u \in L(P_D) \setminus \{v\}$, we can take small $\tilde{\epsilon} > 0$ such that $B_d((v,1),\tilde{\epsilon}) \cap [(u,1) + C_D] = \emptyset$ for all $u \in L(P_D) \setminus \{v\}$. For any

$$y \in \mathrm{relint}((v,1) + C_{F'}) \cap F_j^\circ \cap B_d((v,1),\tilde{\epsilon}),$$

we may choose $\epsilon_y > 0$ small enough such that $B_d(y,\epsilon_y) \subseteq F_j^\circ \cap B_d((v,1),\tilde{\epsilon})$ and $B_d(y,\epsilon_y) \cap A((v,1) + C_{F'}) \subseteq \mathrm{relint}((v,1) + C_{F'})$. Therefore,

$$B_d(y,\epsilon_y) \cap P'_j = B_d(y,\epsilon_y) \setminus \cup_{u\in L(P_D)} (u,1) + C_D = B_d(y,\epsilon_y) \setminus (v,1) + C_D \neq \emptyset.$$

Hence $\mathrm{relint}((v,1) + C_{F'}) \cap F_j^\circ \cap B_d((v,1),\tilde{\epsilon}) \subseteq \partial(P'_j) = \partial(P_j)$. From (3.1), we have $\dim([(v,1) + C_{F'}] \cap \partial(P_j)) = d-1$. □

Lemma 3.4. (1) *For $\mu \in \mathbb{Q}_{>0}$, let*

$$A_{F,\mu} = (\cup_{\substack{u\in L(P_D),\\ \sigma_F(u,1)=\mu}} (u,1) + C_F) \setminus (\cup_{\substack{v\in L(P_D),\\ \sigma_F(v,1)<\mu}} (v,1) + C_D).$$

Then

$$A_{F,\mu} \subseteq \partial(\mathcal{P}_D) \cap H_{F,\mu} \textit{ for all } \mu \in \mathbb{Q}_{>0}.$$

(2) $\partial(\mathcal{P}_D) \cap H_{F,\mu} \subseteq A_{F,\mu} \cup B_{F,\mu} \cup [\partial(\mathcal{P}_D) \cap \partial(C_D) \cap H_{F,\mu}]$ *where*

$$B_{F,\mu} = \Big[\cup_{\substack{v\in L(P_D),\sigma_F(v,1)<\mu \\ F\neq F'\in\mathcal{F}(P_D)}} (v,1) + C_{F'} \Big] \cap H_{F,\mu} \cap \partial(\mathcal{P}_D).$$

Proof. Proof of Part (1): Let $x \in A_{F,\mu}$, i.e., $x \in ((u,1)+C_F) \setminus (\cup_{\substack{v\in L(P_D),\\ \sigma_F(v,1)<\mu}} C_v)$ for some $u \in L(P_D \setminus F)$ with $\sigma_F(u,1)=\mu$. Choose a small neighbourhood around x of radius ϵ, $B_d(x,\epsilon) \subset \mathbb{R}^d \setminus (\cup_{\substack{v\in L(P_D),\\ \sigma_F(v,1)<\mu}} C_v)$. Note that if $B_d(x,\epsilon) \cap \{y \in C_D \mid \sigma_F(y) < \mu\} = \emptyset$ then $B_d(x,\epsilon) \cap C_D \subset \{\sigma_F \geq \mu\}$ which is a contradiction since $\sigma_F(x) = \mu > 0$ and we have $\tilde{x} \in C_D$ with $\sigma_F(\tilde{x}) < \mu$. Hence,

$$\emptyset \neq B_d(x,\epsilon) \cap \{y \in C_D \mid \sigma_F(y) < \mu\} \subset C_D \setminus (\cup_{v\in L(P_D)} C_v) = \mathcal{P}_D,$$

which gives $x \in \overline{\mathcal{P}}_D$. Since $A_{F,\mu} \cap \mathcal{P}_D = \emptyset$, this implies $A_{F,\mu} \subseteq \partial(\mathcal{P}_D) \cap H_{F,\mu}$.

Proof of Part (2): It is enough to show that $[\partial(\mathcal{P}_D) \setminus \partial(C_D)] \cap H_{F,\mu} \subseteq A_{F,\mu} \cup B_{F,\mu}$. Let $x \in [\partial(\mathcal{P}_D) \setminus \partial(C_D)] \cap H_{F,\mu}$. Then by Lemma 3.2(2), $x \in E$ where $E \in \mathcal{F}(C_u)$ for some $u \in L(P_D)$. We split the proof in two cases.
Case (1): Suppose $E = (u,1) + C_F$ for some $u \in L(P_D)$. This implies $\sigma_F(u,1) = \mu$. Suppose $x \notin A_{F,\mu}$. This implies $x \in (v,1) + C_D$ for some $v \in L(P_D)$ with $\sigma_F(v,1) < \mu$. Since $x \in \partial(\mathcal{P}_D)$, we have $x \notin (v,1) + C_D^\circ, i.e., x \in \cup_{F'\in\mathcal{F}(P_D)}(v,1) + C_{F'}$. But $x \notin (v,1) + C_F$ since $\sigma_F(v,1) < \mu = \sigma_F(x)$. Hence $x \in \left[\cup_{F\neq F'\in\mathcal{F}(P_D)} (v,1) + C_{F'}\right] \cap H_{F,\mu} \cap \partial(\mathcal{P}_D) \subseteq B_{F,\mu}$.
Case (2): Suppose $x \notin (u,1) + C_F$ for all $u \in L(P_D)$. Then $E = (v,1) + C_{F'}$ for some $v \in L(P_D)$ and $F' \in \mathcal{F}(P_D)$ with $F \neq F'$. Since $x \notin (v,1) + C_F$ we must have $\sigma_F(v,1) < \mu$. Hence $x \in B_{F,\mu}$. This proves Part (2). □

Definition 3.5. *We recall the definition of Ehrhart quasi-polynomial of a convex polytope $P \subset \mathbb{R}^d$. The function $i(P,-) : \mathbb{N} \longrightarrow \mathbb{N}$ given by*

$$i(P,n) := \#(nP \cap \mathbb{Z}^d) = \sum_{j=0}^{\dim(P)} C_j(P,n)n^j,$$

is a quasi-polynomial of degree $\dim(P)$*, i.e., the coefficient $C_j(P,n)$ of n^j is periodic in n for all $j = 0,\ldots,n$, and $C_{\dim(P)}$ is not identically zero. Moreover $C_{\dim(P)} =$* $\mathrm{rVol}_{\dim(P)}(P)$ *if $A(P) \cap \mathbb{Z}^d \neq \emptyset$.*

Notations 3.6. (1) In the rest of the paper, for a bounded set $Q \subset \mathbb{R}^d$ and for $n, m \in \mathbb{N}$, we define

$$i(Q,n,m) := \#(nQ \cap \{z = m\} \cap \mathbb{Z}^d), \tag{3.2}$$

where z is the d^{th} coordinate function on $\mathbb{R}^d$.

(2) Let $v(\mathcal{P}_D) := \cup_{j=1}^s \pi(v(P_j))$, where $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is the projection given by projecting to the last coordinate z and the set $\pi(v(P_j)) = \{\rho_{j_1},\ldots,\rho_{j_{m_j}}\}$, with $\rho_{j_1} < \rho_{j_2} < \cdots < \rho_{j_{m_j}}$.
(3) Let $S = \{m/q \mid q = p^n,\ m,n \in \mathbb{Z}_{\geq 0}\} \setminus v(\mathcal{P}_D)$.

Lemma 3.7. *Let $\mu \in \mathbb{Q}_{>0}$. There exists a finite set $S_F \subset [0,\infty)$ such that for $\lambda \in S \setminus S_F$ and $q = p^n, n \in \mathbb{N}$, such that $q\lambda \in \mathbb{Z}_{\geq 0}$,*

(1) *there exists a constant $C_\mu > 0$ (independent of $\lambda \in S$ and $n \in \mathbb{N}$) such that*

$$i([\partial(\mathcal{P}_D) \cap H_{F,\mu}] \setminus A_{F,\mu}, q, q\lambda) = c_\mu(\lambda,n)q^{d-3}$$

for some constant $c_\mu(\lambda,n)$ *with* $|c_\mu(\lambda,n)| < C_\mu$.

(2) *there exists a constant* $C_1 > 0$ *(independent of* $\lambda \in S \setminus S_F$ *and* $n \in \mathbb{N}$*) such that*

$$i(\partial_{D,F}, q, q\lambda) = i(A_F, q, q\lambda) + c_\lambda^{(1)}(n) q^{d-3}$$

for some constant $c_\lambda^{(1)}(n)$ *with* $|c_\lambda^{(1)}(n)| < C_1$*. Here* $A_F = \cup_{\mu \in \mu_{D,F}} A_{F,\mu}$ *and* $\partial_{D,F}$ *is as in Notations* 3.1(9).

Proof. Proof of Part (1): By Lemma 3.4(2),

$$[\partial(\mathcal{P}_D) \cap H_{F,\mu}] \setminus A_{F,\mu} \subseteq B_{F,\mu} \cup [\partial(\mathcal{P}_D) \cap \partial(C_D) \cap H_{F,\mu}].$$

Note that $B_{F,\mu} = \Big[\cup_{\substack{v \in L(P_D), \sigma_F(v,1)<\mu \\ F \neq F' \in \mathcal{F}(P_D)}} (v,1) + C_{F'} \Big] \cap H_{F,\mu} \cap \partial(\mathcal{P}_D)$

$$= \bigcup_{\substack{\{E \in \mathcal{F}(P_j) \mid E \neq P_i \cap P_j\}_j \\ \{F' \in \mathcal{F}(P_D) \mid F' \neq F\} \\ \{v \in L(P_D) \mid \sigma_F(v,1) < \mu\}}} E \cap ((v,1) + C_{F'}) \cap H_{F,\mu}. \tag{3.3}$$

The second equality follows from the description of $\partial(\mathcal{P}_D)$ in Lemma 3.2(1). Note that

$$\partial(\mathcal{P}_D) \cap \partial(C_D) \cap H_{F,\mu} = \cup_{\{F' \in \mathcal{F}(P_D) \mid F' \neq F\}} C_{F'} \cap \partial(\mathcal{P}_D) \cap H_{F,\mu}.$$

Again by Lemma 3.2(1),

$$\partial(\mathcal{P}_D) \cap \partial(C_D) \cap H_{F,\mu} = \bigcup_{\substack{\{E \in \mathcal{F}(P_j) \mid E \neq P_i \cap P_j\} \\ \{F' \in \mathcal{F}(P_D) \mid F' \neq F\}}} E \cap C_{F'} \cap H_{F,\mu}. \tag{3.4}$$

For each convex rational polytope Q appearing in the union (in the right hand side) of Equation (3.3) and Equation (3.4), we have $\dim(Q) \leq d-2$, since the facet $C_{F'}$ is transversal to $H_{F,\mu}$ for all $F \neq F' \in \mathcal{F}(P_D)$. Write $B_{F,\mu} \cup [\partial(\mathcal{P}_D) \cap \partial(C_D) \cap H_{F,\mu}] = \cup_{\gamma \in \Gamma} Q_\gamma$ where Γ is a finite index set indexing the finitely many rational polytopes appearing in Equation (3.3) and Equation (3.4). Since $\dim(Q_\gamma) \leq d-2$, if $\dim(Q_\gamma \cap \{z = \lambda_\gamma\}) = d-2$ for some $\lambda_\gamma \in [0,\infty)$, then $Q_\gamma \subset \{z = \lambda_\gamma\}$ [MT20, Lemma 14(1)]. Hence for atmost one $\lambda \in [0,\infty)$, we have $\dim(Q_\gamma \cap \{z = \lambda\}) = d-2$. Let $S_{F,\mu}$ denote the (finite) set of all such λ 's, i.e., $S_{F,\mu} := \{\lambda \in [0,\infty) \mid \dim(Q_\gamma \cap \{z = \lambda\}) = d-2$ for some $\gamma \in \Gamma\}$.

Now

$$i([\partial(\mathcal{P}_D) \cap H_{F,\mu}] \setminus A_{F,\mu}, q, q\lambda) \leq i(\cup_{\gamma \in \Gamma} Q_\gamma, q, q\lambda)$$

$$= \sum_{\gamma \in \Gamma} i(Q_\gamma, q, q\lambda) + \sum_{\alpha \in \Gamma'} \epsilon_\alpha i(Q'_\alpha, q, q\lambda) \tag{3.5}$$

where Γ' is a index set indexing the rational polytopes which are (finite) intersection of rational polytopes from the set $\{Q_\gamma \mid \gamma \in \Gamma\}$ and $\epsilon_\alpha \in \{-1, 1\}$ depending on $\alpha \in \Gamma'$. By [MT20, Lemma 49], for all $\lambda \in S \setminus S_{F,\mu}$ and for all $Q = Q_{\gamma_1} \cap \cdots \cap Q_{\gamma_k}$ $(k \geq 1)$, where $\gamma_i \in \Gamma$, there exists positive constant C_Q (independent of λ and n) such that

$$i(Q, q, q\lambda) = i(Q_\lambda, q) = c_{Q_\lambda}(n) q^{d-3}$$

for some constant $c_{Q_\lambda}(n)$ with $|c_{Q_\lambda}(n)| < C_Q$. Hence the assertion in Part (1) follows from Equation (3.5).

Proof of Part (2): We set $S_F := \cup_{\mu\in\mu_{D,F}} S_{F,\mu}$. The proof follows immediately from Part (1) since the set $\mu_{D,F}$ is finite. □

Lemma 3.8. *Let Q be the convex polytope $E \cap H_{F,\mu}$ for $\mu \in \mu_{D,F}$ and $E \in \mathcal{F}(P_j)$ for some $j \in \{1, \dots, s\}$. Let $\lambda \in S$. Suppose, $q = p^n$ for some $n \in \mathbb{N}$ such that $q\lambda \in \mathbb{Z}$. Then*

1. $C_{d-2}(Q_\lambda, q) = \mathrm{rVol}_{d-2}(Q_\lambda)$.
2. *If $\dim(Q) = d-1$, then for all $j = 1, \dots, d-3$, we have $C_j(Q_\lambda, q) < \tilde{C}_Q$ for some constant $\tilde{C}_Q$ independent of λ and n.*
3. (a) *If $\dim(Q) = d-2$ and Q is transversal to the $\{z = 0\}$ hyperplane, or*
   (b) *if $\dim(Q) < d-2$,*
   then $i(Q_\lambda, q) \le C'_Q q^{d-3}$ for some constant C'_Q independent of λ and n.

Proof. We set $m = q\lambda$.

Proof of Part (1): We know $\dim(E) = d-1$ and E is transversal to the $\{z = 0\}$ hyperplane. Hence $\dim(Q_\lambda) \le \dim(E_\lambda) \le d-2$. If $\dim(Q_\lambda) = d-2$, then $\dim(Q_\lambda) = \dim(E_\lambda)$ and

$$A(qQ_\lambda) \cap \mathbb{Z}^d = A(qE_\lambda) \cap \mathbb{Z}^d = A(qE_\lambda \cap \{z = m\}) \cap \mathbb{Z}^d \neq \emptyset,$$

by [MT20, Lemma 14(3)]. Therefore, by the proof of Case (a), [MT20, Lemma 33(1)], we have $C_{d-2}(Q_\lambda, q) = \mathrm{rVol}_{d-2}(Q_\lambda)$. If $\dim(Q_\lambda) < d-2$, then by the proof of Case (b), [MT20, Lemma 33(1)], we have $C_{d-2}(Q_\lambda, q) = 0 = \mathrm{rVol}_{d-2}(Q_\lambda)$.

Proof of Part (2) follows from the proof of Part (a) of [MT20, Lemma 33(2)].

Proof of Part (3) follows from the proof of Part (b) of [MT20, Lemma 33(2)]. □

Definition 3.9. *We define the set*

$$T_F = \cup_{\substack{\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j \\ \mu\in\mu_{D,F}}} \{\lambda \in [0,\infty) \mid \dim(E \cap H_{F,\mu}) = \dim((E \cap H_{F,\mu})_\lambda) = d-2\}.$$

Note that the set T_F is finite.

Remark 3.10. Recall the set S_F defined in the proof of Lemma 3.7. We remark that $S_F \subseteq v(\mathcal{P}_D) \cup T_F$. To prove this we first note that $S_F = S_1 \cup S_2$, where

$$S_1 = \bigcup_{\substack{\mu\in\mu_{D,F} \\ \{E\in\mathcal{F}(P_j)\ \mid\ E\neq P_i\cap P_j\}_j \\ \{F'\in\mathcal{F}(P_D)\ \mid\ F'\neq F\}}} \{\lambda \in [0,\infty) \mid \dim((E \cap C_{F'} \cap H_{F,\mu})_\lambda) = d-2\}$$

and

$$S_2 = \bigcup_{\substack{\mu\in\mu_{D,F}, v\in L(P_D) \\ \{E\in\mathcal{F}(P_j)\ \mid\ E\neq P_i\cap P_j\}_j \\ \{F'\in\mathcal{F}(P_D)\ \mid\ F'\neq F\}}} \{\lambda \in [0,\infty) \mid \dim((E \cap ((v,1) + C_{F'}) \cap H_{F,\mu})_\lambda) = d-2\}.$$

Hence, $S_F \subseteq S_{F,1} \cup S_{F,2} \cup T_F$ where

$$S_{F,1} = \bigcup_{\substack{\mu\in\mu_{D,F} \\ \{E\in\mathcal{F}(P_j)\ \mid\ E\neq P_i\cap P_j\}_j \\ \{F'\in\mathcal{F}(P_D)\ \mid\ F'\neq F\}}} \{\lambda \in [0,\infty) \mid E \subseteq H_{F,\mu}, \dim((E \cap C_{F'})_\lambda) = d-2\}$$

and

$$S_{F,2} = \bigcup_{\substack{\mu\in\mu_{D,F}, v\in L(P_D) \\ \{E\in\mathcal{F}(P_j)\ \mid\ E\neq P_i\cap P_j\}_j \\ \{F'\in\mathcal{F}(P_D)\ \mid\ F'\neq F\}}} \{\lambda\in[0,\infty) \mid E\subseteq H_{F,\mu}, \dim((E\cap[(v,1)+C_{F'}])_\lambda) = d-2\}.$$

It is enough to show $S_{F,1}\cup S_{F,2}\subseteq v(\mathcal{P}_D)$. Suppose $\lambda\in S_{F,1}$, i.e., $\dim((E\cap C_{F'})_\lambda) = d-2$ for some $F'\in\mathcal{F}(P_D)$ with $F'\neq F$ and $E\in\mathcal{F}(P_j)$ such that $E\subseteq H_{F,\mu}$ for some $\mu\in\mu_{D,F}$ and $E\neq P_i\cap P_j$ for all $i\in\{1,\dots,s\}$. Since $\dim(C_{F'}\cap\mathcal{P}_D) = d-1$, there exists $\tilde{E}\in\mathcal{F}(P_k)$ for some $k\in\{1,\dots,s\}$ such that $\tilde{E}\subseteq C_{F'}$. This implies $E\cap\tilde{E}\subseteq\{z=\lambda\}$, hence $\lambda\in v(\mathcal{P}_D)$.

Now suppose $\lambda\in S_{F,2}$ and $\dim((E\cap[(v,1)+C_{F'}])_\lambda) = d-2$ where $F'\in\mathcal{F}(P_D)$ with $F'\neq F$, $v\in L(P_D)$ and $E\in\mathcal{F}(P_j)$ for some $j\in\{1,\dots,s\}$ such that $E\subseteq H_{F,\mu}$ and $E\neq P_i\cap P_j$ for all $i\in\{1,\dots,s\}$. If $v\in L(F')$, then $\lambda\in S_{F,1}\subseteq v(\mathcal{P}_D)$. Therefore, we assume $v\in L(P_D)\setminus L(F')$. By Lemma 3.3, there exists $E_1\in\mathcal{F}(P_k)$ for some $k\in\{1,\dots,s\}$ such that $E_1\subset(v,1)+C_{F'}$. Hence $E\cap E_1\subseteq\{z=\lambda\}$, i.e., $\lambda\in v(\mathcal{P}_D)$.

4. β-density function for $I = p_F$

In the rest of the paper, we assume (X, D) is a projectively normal toric pair.

Lemma 4.1. *The ideal $I = p_F$ is generated by the set $\{\chi^{(u,1)} \mid u\in L(P_D\setminus F)\}$.*

Proof. For $\mathbf{x}\in\mathbb{Z}^{d-1}$ and any integer $m\geq 2$ with $(\mathbf{x},m)\in C_I\cap(\Lambda_m)$, it is enough to show there exists $u\in P_D\cap M$ such that $(\mathbf{x},m)-(u,1)\in C_I\cap(\Lambda_{m-1})$. Now $(\mathbf{x},m) = \sum_{u\in L(P_D)} a_u(u,1)$ for $a_u\in\mathbb{Z}_{\geq 0}$ (since P_D is a normal polytope) and

$$1 < m = \sum_{u\in L(P_D)} a_u = \sum_{u\in L(P_D\setminus F)} a_u + \sum_{u\in L(F)} a_u.$$

If $\sum_{u\in L(F)} a_u\geq 1$, then choose $u_0\in L(F)$ such that $a_{u_0}\geq 1$. Since $(\mathbf{x},m)\in C_I$, we have $\sum_{u\in L(P_D\setminus F)} a_u > 0$, hence $(\mathbf{x},m)-(u_0,1)\in C_I\cap(\Lambda_{m-1})$. If $\sum_{u\in F} a_u = 0$, then $\sum_{u\in L(P_D\setminus F)} a_u = m > 1$. We choose $u_0\in L(P_D\setminus F)$ such that $a_{u_0}\geq 1$. Then $(\mathbf{x},m)-(u_0,1) = \sum_{u_0\neq u\in L(P_D\setminus F)} a_u(u,1)+(a_{u_0}-1)(u_0,1)$ and $\sigma_F((\mathbf{x},m)-(u_0,1)) > 0$, i.e., $(\mathbf{x},m)-(u_0,1)\in C_I\cap(\Lambda_{m-1})$. □

Definition 4.2. *For the monomial prime ideal $I = p_F$ of R,*

(1) *we define a sequence of functions $\{\psi_n : [0,\infty)\longrightarrow\mathbb{R}_{\geq 0}\}_{n\in\mathbb{N}}$ given by*

$$\psi_n(\lambda) = \frac{1}{q^{d-2}}\ell\left(\frac{m^{[q]}\cap I}{m^{[q]}I}\right)_{\lfloor q\lambda\rfloor}.$$

(2) *We define the 'small density function' $\Psi_F : [0,\infty)\longrightarrow\mathbb{R}_{\geq 0}$, given by*

$$\Psi_F(\lambda) = \sum_{\mu\in\mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D)\cap H_{F,\mu}\cap\{z=\lambda\}) = \mathrm{rVol}_{d-2}(\partial_{D,F}\cap\{z=\lambda\}).$$

Here for $Q = \cup_i Q_i$, a finite union of convex rational polytopes $Q_i\subset\mathbb{R}^d$ with $\dim(Q_i)\leq d'$, such that $\dim(Q_i\cap Q_j) < d'$, for $Q_i\neq Q_j$, we define $\mathrm{rVol}_{d'}Q =$

$\sum_i \mathrm{rVol}_{d'} Q_i$ and $\mathrm{rVol}_{d''} Q = 0$, if $d'' > d'$. For a detailed discussion of the definition of relative volume, see [MT20, Appendix A, Definition 47].

Remark 4.3. *Recall the set T_F described in Definition 3.9. Note that for all $\lambda \in [0,\infty) \setminus T_F$,*

$$\Psi_F(\lambda) = \sum_{\mu \in \mu_{D,F}} \sum_{\{E \in \mathcal{F}(P_j) | E \neq P_i \cap P_j\}_j} \mathrm{rVol}_{d-2}(E \cap H_{F,\mu} \cap \{z = \lambda\}).$$

Remark 4.4. *Suppose $Q = E \cap H_{F,\mu}$ for some $E \in \mathcal{F}(P_j)$, $\mu \in \mu_{D,F}$ and suppose the function $\psi_Q : [0,\infty) \longrightarrow \mathbb{R}_{\geq 0}$, given by*

$$\psi_Q(\lambda) = \mathrm{rVol}_{d-2}(Q \cap \{z = \lambda\}).$$

If $\dim(Q) = d-1$, then $E \subseteq H_{F,\mu}$ and $Q = E$. Therefore $\psi_Q : [0,\infty) \setminus v(\mathcal{P}_D) \longrightarrow \mathbb{R}_{\geq 0}$, given by $\lambda \longrightarrow \mathrm{rVol}_{d-2}(Q \cap \{z = \lambda\})$ is continuous, by [MT20, Remark 36]. *If $\dim(Q) = d-2$ and Q is transversal to the $\{z = 0\}$ hyperplane or $\dim(Q) \leq d-3$, then $\dim(Q_\lambda) \leq d-3$, hence $\psi_Q = 0$ on $[0,\infty)$. If $\dim(Q) = d-2$ and $\dim(Q \cap \{z = \lambda_0\}) = d-2$ for some $\lambda_0 \in [0,\infty)$, then $Q \subseteq \{z = \lambda_0\}$. Hence $\psi_Q(\lambda_0) = \mathrm{rVol}_{d-2}(Q \cap \{z = \lambda_0\})$ and $\psi_Q(\lambda) = 0$ for all $\lambda \neq \lambda_0$. Hence, by Remark 4.3, the function $\Psi_F : [0,\infty) \setminus (v(\mathcal{P}_D) \cup T_F) \longrightarrow \mathbb{R}_{\geq 0}$, given by $\lambda \mapsto \mathrm{rVol}_{d-2}(\partial_{D,F} \cap \{z = \lambda\})$ is continuous. Moreover, Ψ_F is a compactly supported and piecewise polynomial function* [MT20, Remark 36].

Lemma 4.5. *For all $\lambda \in [0,\infty)$ and $q = p^n \in \mathbb{N}$ with $\lambda_n := \lfloor q\lambda \rfloor / q \in S \setminus (v(\mathcal{P}_D) \cup T_F)$, we have*

$$\psi_n(\lambda) = \Psi_F(\lambda_n) + c_\lambda(n)/q, \text{ for some constant } c_\lambda(n),$$

such that $|c_\lambda(n)| < C$, where C is a constant independent of λ and $n \in \mathbb{N}$.

Proof. For $\lambda \in \mathbb{R}_{\geq 0}$ and $q = p^n$, let $m = \lfloor q\lambda \rfloor$. Note that

$$\ell(m^{[q]} \cap I)_m = \# \left[\left((\cup_{u \in L(P_D)} q(u,1) + C_D) \setminus C_F \right) \cap (\Lambda_m) \right]$$

$$= \# \left[(\cup_{u \in L(P_D)} q(u,1) + C_D) \setminus (C_F \cap [\cup_{u \in L(P_D)} q(u,1) + C_D]) \cap (\Lambda_m) \right].$$

By proof of Lemma 2.3, we have

$$\begin{aligned} \ell(m^{[q]} \cap I)_m &= \# \left[(\cup_{u \in L(P_D)} q(u,1) + C_D) \setminus (\cup_{u \in L(F)} q(u,1) + C_F) \cap (\Lambda_m) \right] \\ &= \# \left[(\cup_{u \in L(P_D \setminus F)} q(u,1) + C_D) \cup (\cup_{u \in L(F)} q(u,1) + [C_D \setminus C_F]) \cap (\Lambda_m) \right]. \quad (4.1) \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned} \ell(m^{[q]} I)_m &= \# \left[(\cup_{u \in L(P_D), v \in L(P_D \setminus F)} q(u,1) + (v,1) + C_D) \cap (\Lambda_m) \right] \\ &= \# \left[(\cup_{u \in L(P_D)} q(u,1) + [C_D \setminus C_F]) \cap (\Lambda_m) \right]. \quad (4.2) \end{aligned}$$

The last equation follows since, (X,D) is projectively normal, i.e., P_D is a normal polytope.

From Equation (4.1) and Equation (4.2), we have

$$\psi_n(\lambda) = \frac{1}{q^{d-2}}\#\big[(\cup_{u\in L(P_D\setminus F)}q(u,1)+C_F)\setminus(\cup_{u\in L(P_D)}q(u,1)+[C_D\setminus C_F])\cap(\Lambda_m)\big]$$

$$= \frac{1}{q^{d-2}}\#\big[\bigcup_{\mu\in\mu_{D,F}}(\cup_{\substack{u\in L(P_D),\\ \sigma_F(u,1)=\mu}} q(u,1)+C_F)\setminus(\cup_{\substack{v\in L(P_D),\\ \sigma_F(v,1)<\mu}} q(v,1)+C_D)\cap(\Lambda_m)\big]$$

$$= \frac{1}{q^{d-2}}\#[qA_F\cap(\Lambda_m)\big] = \frac{1}{q^{d-2}}i(A_F,q,q\lambda_n),\ \text{where } A_F \text{ is as in Lemma 3.4.} \tag{4.3}$$

If $\lambda_n\notin S_F$, by Equation (4.3) and Lemma 3.7(2), we have

$$\psi_n(\lambda) = i(\partial_{D,F},q,q\lambda_n)/q^{d-2} - c^{(1)}_{\lambda_n}(n)/q$$

such that $|c^{(1)}_{\lambda_n}(n)|<C_1$ for some constant C_1, independent of λ and n. Hence

$$\begin{aligned}
\psi_n(\lambda) &= \frac{1}{q^{d-2}}\#[q(\partial_{D,F})\cap(\Lambda_{q\lambda_n})\big] - \frac{c^{(1)}_{\lambda_n}(n)}{q}\\
&= \frac{1}{q^{d-2}}\#\big[\cup_{\substack{\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j\\ \mu\in\mu_{D,F}}} q(E\cap H_{F,\mu})\cap(\Lambda_{q\lambda_n})\big] - \frac{c^{(1)}_{\lambda_n}(n)}{q}\\
&= \frac{1}{q^{d-2}}\sum_{\mu\in\mu_{D,F}}\ \sum_{\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j}\#[q(E\cap H_{F,\mu})\cap(\Lambda_{q\lambda_n})]\\
&+\frac{1}{q^{d-2}}\sum_{\alpha\in J}\epsilon_\alpha\#[qK_\alpha\cap(\Lambda_{q\lambda_n})] - \frac{c^{(1)}_{\lambda_n}(n)}{q}
\end{aligned}$$

where J is the index set indexing the (finite) intersections of elements of the set $\cup_j\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j$, further intersecting with $H_{F,\mu}$ for some $\mu\in\mu_{D,F}$, i.e., $K_\alpha = E_{\alpha_1}\cap\cdots\cap E_{\alpha_l}\cap H_{F,\mu}$ for $E_{\alpha_i}\in\cup_j\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}$ with $l\geq 2$ and for some $\mu\in\mu_{D,F}$; $\epsilon_\alpha\in\{1,-1\}$, depending on the $K_\alpha\in J$. Hence $\psi_n(\lambda)$

$$= \frac{1}{q^{d-2}}\sum_{\substack{\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j\\ \mu\in\mu_{D,F}}} i(E\cap H_{F,\mu},q,q\lambda_n) + \frac{1}{q^{d-2}}\sum_{\alpha\in J}\epsilon_\alpha i(K_\alpha,q,q\lambda_n) - \frac{c^{(1)}_{\lambda_n}(n)}{q}$$

$$= \frac{1}{q^{d-2}}\sum_{E,\mu} i((E\cap H_{F,\mu})_{\lambda_n},q) + \frac{1}{q^{d-2}}\sum_{\alpha\in J}\epsilon_\alpha i((K_\alpha)_{\lambda_n},q) - \frac{c^{(1)}_{\lambda_n}(n)}{q}. \tag{4.4}$$

From Equation (4.4) and using Lemma 3.8, we have for all $\lambda_n\notin T_F$,

$$\begin{aligned}
\psi_n(\lambda) \;=\; &\sum_{\mu\in\mu_{D,F}}\ \sum_{\{E\in\mathcal{F}(P_j)\mid E\neq P_i\cap P_j\}_j}\mathrm{rVol}_{d-2}(E\cap H_{F,\mu}\cap\{z=\lambda_n\})\\
&+\frac{c^{(2)}_{\lambda}(n)}{q} + \frac{1}{q^{d-2}}\sum_{\alpha\in J}\epsilon_\alpha i((K_\alpha)_{\lambda_n},q) - \frac{c^{(1)}_{\lambda_n}(n)}{q};
\end{aligned}\tag{4.5}$$

for real number $c_\lambda^{(2)}(n)$ such that $|c_\lambda^{(2)}(n)| \leq C_2$ for some constant C_2 independent of λ and n. Note that $\dim(K_\alpha) \leq d-2$ for all $\alpha \in J$. Suppose $K_\alpha = E_{\alpha_1} \cap \cdots \cap E_{\alpha_r} \cap H_{F,\mu}$ for some $\mu \in \mu_{D,F}$ and $E_{\alpha_i} \in \cup_j \{E \in \mathcal{F}(P_j) \mid E \neq P_i \cap P_j\}$ with $r \geq 2$. If $\dim((K_\alpha)_\lambda) = d-2$ for some $\lambda \in [0,\infty)$, then $\dim(E_{\alpha_1} \cap E_{\alpha_2}) = \dim((E_{\alpha_1} \cap E_{\alpha_2})_\lambda) = d-2$. Hence $E_{\alpha_1} \cap E_{\alpha_2} \subseteq \{z = \lambda\}$, i.e., $\lambda \in v(\mathcal{P}_D)$. Therefore, for all $\lambda \in [0,\infty)$ and $q = p^n \in \mathbb{N}$ with $\lambda_n \in S \setminus (v(\mathcal{P}_D) \cup S_F \cup T_F) = S \setminus (v(\mathcal{P}_D) \cup T_F)$ (by Remark 3.10), we have $\psi_n(\lambda) =$

$$\sum_{\mu \in \mu_{D,F}} \sum_{\{E \in \mathcal{F}(P_j) | E \neq P_i \cap P_j\}_j} \mathrm{rVol}_{d-2}(E \cap H_{F,\mu} \cap \{z = \lambda_n\}) + \frac{c_\lambda^{(3)}(n)}{q} + \frac{c_\lambda^{(2)}(n)}{q} - \frac{c_{\lambda_n}^{(1)}(n)}{q};$$

such that $|c_\lambda^{(3)}(n)| < C_3$, for some constant C_3, independent of λ and $n \in \mathbb{N}$ [MT20, Lemma 49]. By Remark 4.3, for all $\lambda \in [0,\infty)$, $q = p^n \in \mathbb{N}$ with $\lambda_n \in S \setminus (v(\mathcal{P}_D) \cup T_F)$, we have

$$\psi_n(\lambda) \;=\; \sum_{\mu \in \mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D) \cap H_{F,\mu} \cap \{z = \lambda_n\}) + \frac{c_\lambda^{(3)}(n)}{q} + \frac{c_\lambda^{(2)}(n)}{q} - \frac{c_{\lambda_n}^{(1)}(n)}{q}.$$

Hence the lemma. □

Lemma 4.6. *For $\lambda \in [0,\infty)$,*

$$g_n(I,\mathbf{m})(\lambda) = g_n(R,\mathbf{m})(\lambda) - f_n(R/I,\mathbf{m}/I)(\lambda) + \psi_n(\lambda).$$

Proof. Let $m = \lfloor q\lambda \rfloor$. We have

$$\begin{aligned} g_n(I,\mathbf{m})(\lambda) &= \frac{1}{q^{d-2}}[\ell(I/\mathbf{m}^{[q]}I)_m - f_{I,\mathbf{m}}(m/q)q^{d-1}] \\ &= \frac{1}{q^{d-2}}[\ell(I/\mathbf{m}^{[q]} \cap I)_m + \ell\big((\mathbf{m}^{[q]} \cap I)/(\mathbf{m}^{[q]}I)\big)_m - f_{I,\mathbf{m}}(m/q)q^{d-1}]. \end{aligned}$$

Using the additive property of HK density function [Tri18, Proposition 2.14], we have $f_{I,\mathbf{m}}(\lambda) = f_{R,\mathbf{m}}(\lambda)$ for all $\lambda \in [0,\infty)$. Hence we have

$$g_n(I,\mathbf{m})(\lambda) = \frac{1}{q^{d-2}}[\ell(I/\mathbf{m}^{[q]} \cap I)_m - f_{R,\mathbf{m}}(m/q)q^{d-1}] + \psi_n(\lambda). \tag{4.6}$$

Note that

$$\ell(I/\mathbf{m}^{[q]} \cap I)_m = \#\left[\big([C_D \setminus C_F] \setminus \cup_{u \in P_D} q(u,1) + C_D\big) \cap (\Lambda_m)\right]$$

$$\begin{aligned} &= \#\left[\big(C_D \setminus \cup_{u \in L(P_D)} q(u,1) + C_D\big) \cap (\Lambda_m)\right] - \#\left[\big(C_F \setminus \cup_{u \in L(P_D)} q(u,1) + C_D\big) \cap (\Lambda_m)\right] \\ &= \#(q\mathcal{P}_D \cap \Lambda_m) - \#\left[\big(C_F \setminus \cup_{u \in L(P_D)} q(u,1) + C_D\big) \cap (\Lambda_m)\right]. \end{aligned} \tag{4.7}$$

By Equation (4.6), Equation (4.7) and Remark 2.4, it follows that

$$\begin{aligned} g_n(I,\mathbf{m})(\lambda) &= \frac{1}{q^{d-2}}\left[\#(q\mathcal{P}_D \cap \Lambda_m) - f_{R,\mathbf{m}}(m/q)q^{d-1}\right] - f_n(R/I,\mathbf{m}/I)(\lambda) + \psi_n(\lambda) \\ &= g_n(R,\mathbf{m})(\lambda) - f_n(R/I,\mathbf{m}/I)(\lambda) + \psi_n(\lambda). \end{aligned}$$

□

Proof of Theorem 1.1. By Remark 4.4, the function Ψ_F is compactly supported and is continuous on $[0,\infty)\setminus v(\mathcal{P}_D)\cup T_F$. Therefore, by [Tri18, Theorem 1] and Theorem 2.1, the function $g_{I,\mathbf{m}} = g_{R,\mathbf{m}} + f_{R/I,\mathbf{m}/I} + \Psi_F$ is also continuous on $[0,\infty)\setminus v(\mathcal{P}_D)\cup T_F$. We set $v_{\mathcal{P}_D,F} = v(\mathcal{P}_D)\cup T_F$.

Following a similar argument as in [MT20, Lemma 39(1)] and using Lemma 4.5, we see that for any compact set $V\subseteq[0,\infty)\setminus v_{\mathcal{P}_D,F}$, the sequence of functions $\{\psi_n|_V\}$ converges uniformly to the function Ψ_F. Hence the sequence of functions $\{g_n^{(I,\mathbf{m})}|_V\}$ converges uniformly to the function $g_{I,\mathbf{m}}|_V$ follows from Lemma 4.6, [Tri18, Theorem 1.1] and Theorem 2.1. Following a similar argument given in the proof of [MT20, Corollary 3], we get $\int_0^\infty g_{I,\mathbf{m}}(\lambda)d\lambda = \beta(I,\mathbf{m})$.

Corollary 4.7. *With the notations as in Theorem 1.1, for a projectively normal toric pair (X,D), we have*

$$\sum_{\{F \mid F \text{ is facet of } P_D\}} g_{p_F,\mathbf{m}}(\lambda) = (r-2)g_{R,\mathbf{m}}(\lambda)$$

for all $\lambda\in[0,\infty)\setminus v_{\mathcal{P}_D,F}$, where $r\in\mathbb{N}$ is the number of facets of the polytope P_D.

Proof. From Theorem 1.1, for $\lambda\in[0,\infty)\setminus v_{\mathcal{P}_D,F}$, we have

$$\sum_{F\in\mathcal{F}(P_D)} g_{p_F,\mathbf{m}}(\lambda) = (r)g_{R,\mathbf{m}}(\lambda) - \sum_{F\in\mathcal{F}(P_D)} f_{R/p_F,\mathbf{m}/p_F}(\lambda) + \sum_{F\in\mathcal{F}(P_D)} \Psi_F(\lambda)$$

where r is the number of facets of P_D. Hence

$$\sum_{F\in\mathcal{F}(P_D)} g_{p_F,\mathbf{m}}(\lambda) = (r)g_{R,\mathbf{m}}(\lambda) - \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D)\cap\partial(C_D)\cap\{z=\lambda\})$$
$$+\mathrm{rVol}_{d-2}([\partial(\mathcal{P}_D)\setminus\partial(C_D)]\cap\{z=\lambda\}) = (r-2)g_{R,\mathbf{m}}(\lambda).$$

The last equation follows from the description of $g_{R,\mathbf{m}}$ in Theorem 2.1. □

Definition 4.8. *For the ideal $I=p_F$, define another function $\alpha_{I,\mathbf{m}}:[0,\infty)\longrightarrow\mathbb{R}$ by setting*

$$\alpha_{I,\mathbf{m}}(\lambda) = g_{I,\mathbf{m}}(\lambda) - g_{R,\mathbf{m}}(\lambda), \text{ for } \lambda\in[0,\infty).$$

Clearly $\int_0^\infty \alpha_{I,\mathbf{m}}(\lambda) = \tau_{\mathbf{m}}(I)$. Extend this to define a map

$$\alpha_{\mathbf{m}}:\mathrm{Cl}(R)\longrightarrow\mathcal{L}^1([0,\infty)) \text{ (the space of integrable functions } f:[0,\infty)\longrightarrow\mathbb{R})$$

such that it is a group homomorphism. Thus our result gives explicit description of the map $\tau_{\mathbf{m}}:\mathrm{Cl}(R)\longrightarrow\mathbb{R}$ defined in [HMM04, Theorem 1.9]*. In the next section we compute these functions for some toric pairs.*

5. SOME EXAMPLES AND PROPERTIES

Example 5.1. *In this example, we compute the β-density functions for the toric pair $(X,D)=(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(l))$ for all $l\in\mathbb{N}$. The polytope P_D can be taken to be the line segment $[0,l]$ (up to translation by integral points). The cone*

$$C_D = \mathrm{Cone}\langle(0,1),(l,1)\rangle = \{(x,y)\mid 0\le x\le ly\}\subset\mathbb{R}^2.$$

Let $(R, \mathbf{m})$ be the associated homogeneous coordinate ring and let I_1 and I_2 be the monomial prime ideals associated to the facets $C_{F_1} = C_D \cap \{x = 0\}$ and $C_{F_2} = C_D \cap \{x = ly\}$ of C_D, respectively. One has

$$g_{R,\mathbf{m}}(\lambda) = \begin{cases} 1 & \text{if } 0 \leq \lambda < 1, \\ -l & \text{if } 1 \leq \lambda < 1 + \frac{1}{l}, \\ 0 & \text{if } \lambda > 1 + \frac{1}{l}, \end{cases}$$

and

$$g_{I_1,\mathbf{m}}(\lambda) = g_{I_2,\mathbf{m}}(\lambda) = 0 \text{ for all } \lambda \geq 0.$$

Example 5.2. *In this example we compute the β-density functions for the toric pair $(X, D) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. The polytope P_D is the convex hull of the points $(0,0), (0,1)$ and $(1,0)$ in $\mathbb{R}^2$ (up to translation by integral points). The cone*

$$C_D = \text{Cone}\langle(0,0,1), (0,1,1), (1,0,1)\rangle = \{(x,y,z) \mid x, y \geq 0, x + y \leq z\} \subset \mathbb{R}^2.$$

Let $(R, \mathbf{m})$ be the associated homogeneous coordinate ring and let J_1, J_2 and J_3 be the monomial prime ideals associated to the facets $C_{F_1} = C_D \cap \{x = 0\}$, $C_{F_2} = C_D \cap \{y = 0\}$ of C_D and $C_{F_3} = C_D \cap \{x + y = z\}$, respectively. One has

$$g_{J_i,\mathbf{m}}(\lambda) = \begin{cases} \lambda/2 & \text{if } 0 \leq \lambda < 1, \\ -\lambda + 3/2 & \text{if } 1 \leq \lambda < 2, \\ \lambda/2 - 3/2 & \text{if } 2 \leq \lambda < 3, \\ 0 & \text{if } \lambda \geq 3, \end{cases}$$

and

$$g_{R,\mathbf{m}}(\lambda) = 3 g_{J_i,\mathbf{m}}(\lambda) \text{ for all } \lambda \geq 0.$$

Remark 5.3. *By Theorem 2.1, for $\lambda \in [0, \infty)$,*

$$g_{R,\mathbf{m}}(\lambda) = \text{rVol}_{d-2} \frac{(\partial(\mathcal{P}_D) \cap \partial(C_D) \cap \{z = \lambda\})}{2} - \frac{\text{rVol}_{d-2}\left([\partial(\mathcal{P}_D) \setminus \partial(C_D)] \cap \{z = \lambda\}\right)}{2}$$
$$= \frac{1}{2}\sum_{i=1}^{r} f_{R/p_{F_i},\mathbf{m}/p_{F_i}}(\lambda) - \frac{1}{2}\sum_{i=1}^{r} \Psi_{F_i}(\lambda),$$

where $\{F_i\}_{i=1}^r$ are the facets of the polytope P_D.

In the next example we compute the functions $f_{R/p_{F_i},\mathbf{m}/p_{F_i}}$ and Ψ_{F_i}, which enables us to describe the β-density functions and τ-density functions of the ring and of the monomial prime ideals of height one.

Example 5.4. *We compute the β-density function and the τ-density function of the monomial prime ideals of height one for the Hirzebruch surface $X = \mathbb{F}_a$, which is a ruled surface over $\mathbb{P}^1_K$, where K is a field of characteristic $p > 0$. See* [Ful93] *for a detailed description of the surface as a toric variety. The T-Cartier divisors are given by*

$$D_i = V(v_i), \ i = 1,2,3,4, \quad \text{where} \ \ v_1 = e_1, v_2 = e_2, v_3 = -e_1 + ae_2, v_4 = -e_2$$

and $V(v_i)$ denotes the T-orbit closure corresponding to the cone generated by v_i. We know the Picard group is generated by $\{D_i \mid i = 1,2,3,4\}$ over $\mathbb{Z}$. One can check $Pic(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_4$ and $D = cD_1 + dD_4$ is ample if and only if $c, d > 0$. Then

$$P_D = \{(x,y) \in M_{\mathbb{R}} \mid x \geq -c, 0 \leq y \leq d, x \leq ay\}.$$

The description of the HK*-density function and the β-density function of the aasociated homogeneous coordinate ring $(R, \mathbf{m})$ can be found in* [Tri16], [MT19] *and* [MT20]. *The facets of the polytope P_D are given by the hyperplanes $x = 0$, $y = 0$, $x = ay + c$ and $y = d$. We denote them by F_1, F_2, F_3 and F_4 respectively. By Remark 5.3, to compute the β-density function and the τ-density function, it is enough to compute the functions $f_{R/p_{F_i},\mathbf{m}/p_{F_i}}$ and the functions Ψ_{F_i} for $i = 1,2,3,4$. We draw the cross section of the set $\partial(\mathcal{P}_D)$ at $z = \lambda$ level for $\lambda \in [0,\infty)$ and use the interpretation of these functions in Remark 2.4 and Definition 4.2, respectively for the computation. We have*

$$f_{R/p_{F_1},\mathbf{m}/p_{F_1}}(\lambda) = \begin{cases} d\lambda & \text{if } 0 \leq \lambda < 1, \\ d(d+1-d\lambda) & \text{if } 1 \leq \lambda < 1+\frac{1}{d}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{d}, \end{cases}$$

$$f_{R/p_{F_2},\mathbf{m}/p_{F_2}}(\lambda) = \begin{cases} c\lambda & \text{if } 0 \leq \lambda < 1, \\ c(c+1-c\lambda) & \text{if } 1 \leq \lambda < 1+\frac{1}{c}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{c}, \end{cases}$$

$$f_{R/p_{F_3},\mathbf{m}/p_{F_3}}(\lambda) = \begin{cases} ad\lambda & \text{if } 0 \leq \lambda < 1, \\ ad(d+1-d\lambda) & \text{if } 1 \leq \lambda < 1+\frac{1}{d}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{d} \end{cases}$$

and

$$f_{R/p_{F_4},\mathbf{m}/p_{F_4}}(\lambda) = \begin{cases} (ad+c)\lambda & \text{if } 0 \leq \lambda < 1, \\ (ad+c)(1-(ad+c)(\lambda-1)) & \text{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{ad+c}. \end{cases}$$

To compute the functions Ψ_{F_i} for $i = 1,2,3,4$, we consider two different cases.

(1) $c \geq d:$ *We have*

$$\Psi_{F_1}(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda < 1, \\ (c+\frac{ad}{2})(d+1)d(\lambda-1) & \text{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ (c+\frac{ad}{2})(d+1)\frac{1}{a}(c+1-c\lambda) & \text{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{c}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{c}, \end{cases}$$

$$\Psi_{F_2}(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda < 1, \\ (cd+d+\frac{ad^2}{2}+\frac{ad}{2})c(\lambda-1) & \text{if } 1 \leq \lambda < 1+\frac{1}{c}, \\ cd\lambda+\frac{ad^2}{2}+\frac{ad}{2} & \text{if } 1+\frac{1}{c} \leq \lambda < 1+\frac{1}{d}, \\ 0 & \text{if } \lambda \geq 1+\frac{1}{d}, \end{cases}$$

$$\Psi_{F_3}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (c+\frac{ad}{2})(d+1)ad(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ (c+\frac{ad}{2})(d+1)(c+1-c\lambda) & \textit{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{c}, \\ 0 & \textit{if } \lambda \geq 1+\frac{1}{c} \end{cases}$$

and

$$\Psi_{F_4}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (cd+d+\frac{ad^2}{2}-\frac{ad}{2})(ad+c)(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ d(ad+c)(\lambda-1)+c+\frac{ad^2}{2}-\frac{ad}{2} & \textit{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{d}, \\ 0 & \textit{if } \lambda \geq 1+\frac{1}{d}. \end{cases}$$

(2) $c \leq d$: *We have*

$$\Psi_{F_1}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (c+\frac{ad}{2})(d+1)d(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ (c+\frac{ad}{2})(d+1)\frac{1}{a}(c+1-c\lambda) & \textit{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{d}, \\ (cd+\frac{ad^2}{2}-\frac{ad}{2})\frac{1}{a}\big(a+1-(ad+c)(\lambda-1)\big) & \textit{if } 1+\frac{1}{d} \leq \lambda < 1+\frac{a+1}{ad+c}, \\ \frac{c}{a}(c+1-c\lambda) & \textit{if } 1+\frac{a+1}{ad+c} \leq \lambda \leq 1+\frac{1}{c}, \\ 0 & \textit{if } \lambda \geq 1+\frac{1}{c}, \end{cases}$$

$$\Psi_{F_2}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (cd+d+\frac{ad^2}{2}+\frac{ad}{2})c(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{d}, \\ 0 & \textit{if } \lambda \geq 1+\frac{1}{d}, \end{cases}$$

$$\Psi_{F_3}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (c+\frac{ad}{2})(d+1)ad(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ (c+\frac{ad}{2})(d+1)(c+1-c\lambda) & \textit{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{d}, \\ (c+\frac{ad}{2})(d-1)\big(a+1-(ad+c)(\lambda-1)\big) & \\ +c(c+1-c\lambda) & \textit{if } 1+\frac{1}{d} \leq \lambda < 1+\frac{a+1}{ad+c}, \\ c(c+1-c\lambda) & \textit{if } 1+\frac{a+1}{ad+c} \leq \lambda < 1+\frac{1}{c}, \\ 0 & \textit{if } \lambda \geq 1+\frac{1}{c} \end{cases}$$

and

$$\Psi_{F_4}(\lambda) = \begin{cases} 0 & \textit{if } 0 \leq \lambda < 1, \\ (cd+d+\frac{ad^2}{2}-\frac{ad}{2})(ad+c)(\lambda-1) & \textit{if } 1 \leq \lambda < 1+\frac{1}{ad+c}, \\ d(ad+c)(\lambda-1)+c+\frac{ad^2}{2}-\frac{ad}{2} & \textit{if } 1+\frac{1}{ad+c} \leq \lambda < 1+\frac{1}{d}, \\ (c+\frac{ad}{2})(d-1)\big(a+1-(ad+c)(\lambda-1)\big) & \textit{if } 1+\frac{1}{d} \leq \lambda < 1+\frac{a+1}{ad+c}, \\ 0 & \textit{if } \lambda \geq 1+\frac{a+1}{ad+c}. \end{cases}$$

Definition 5.5. *Let R be a Noetherian standard graded ring of dimension $d \geq 2$ with homogeneous maximal ideal $\mathbf{m}$ and let M be a finitely generated non negatively graded R-module. Let $\ell(M_n) = \frac{e_0(M,\mathbf{m})}{(d-1)!}n^{d-1} + \tilde{e}_1(M,\mathbf{m})n^{d-2} + \cdots + \tilde{e}_{d-1}(M,\mathbf{m})$ be the Hilbert polynomial of $(M,\mathbf{m})$. Recall the Hilbert density function $F_M : [0,\infty) \longrightarrow [0,\infty)$, of M as*

$$F_M(\lambda) = \frac{e_0(M,\mathbf{m})}{(d-1)!}\lambda^{d-1} = \lim_{n\to\infty} F_n(\lambda) := \frac{1}{q^{d-1}}\ell(M_{\lfloor q\lambda\rfloor}).$$

Similarly we define the second Hilbert density function $G_M : [0,\infty) \longrightarrow \mathbb{R}$ as

$$G_M(\lambda) = \tilde{e}_1(M,\mathbf{m})\lambda^{d-2} = \lim_{n\to\infty} G_n(\lambda) := \frac{1}{q^{d-2}}\left(\ell(M_{\lfloor q\lambda\rfloor}) - F_M(\frac{\lfloor q\lambda\rfloor}{q})\right).$$

Proposition 5.6. *Let $(R,\mathbf{m})$ and $(S,\mathbf{n})$ be two Noetherian standard graded rings over an algebraically closed field K (of characteristic $p>0$) of dimension $d \geq 2$ and $d' \geq 2$, associated to the toric pairs (X,D) and (Y,D'), resply. For the monomial prime ideal $p_F\#S$ of $R\#S$, we have,*

$$G_{p_F\#S} - g_{p_F\#S,\mathbf{m}\#\mathbf{n}} = (G_{p_F} - g_{p_F,\mathbf{m}})(F_S - f_{S,\mathbf{n}}) + (G_S - g_{S,\mathbf{n}})(F_{p_F} - f_{p_F,\mathbf{m}}).$$

Proof. The proof follows by a similar argument used to prove [MT20, Proposition 44]. □

Remark 5.7. *With notations as above, using Proposition 5.6,* [MT20, Proposition 44] *and* [MT20, Remark 43], *one gets*

$$\alpha_{p_F\#S} = \alpha_{p_F,\mathbf{m}}(F_S - f_{S,\mathbf{n}}) + (G_R - G_{p_F})f_{p_F,\mathbf{m}}.$$

This gives a complete description of the β-density function and the τ-density function for Segre product of toric pairs.

Example 5.8. *Let $\mathbb{M}$ be a 2×3 matrix whose entries are the independent variables $x_1,\cdots,x_6$ and let T be the quotient of the ring $k[x_1,\cdots,x_6]$ by the ideal $I_2(\mathbb{M})$, generated by 2×2 minors of $\mathbb{M}$. In their paper, Huneke, McDermott and Monsky have referred to this example by K. Watanabe where $\beta(T,\mathbf{m}_T) = -1/4$ to show that the map $\tau : \mathrm{Cl}(T) \longrightarrow \mathbb{R}$ is not necessarily a zero map. In this example we compute the map $\tau := \tau_{\mathbf{m}_T}$ for all height one monomial prime ideals. Let $(R,\mathbf{m}_R)$ and $(S,\mathbf{m}_S)$ be the homogeneous coordinate ring for the toric pairs $(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ and $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1))$ respectively. Let $I_1, I_2 \subseteq R$ and $J_1,J_2,J_3 \subseteq S$ be the monomial prime ideals of height one of R and S respectively. For the monomial prime ideals of height one $I_i\#S$ and $R\#J_j$ of $R\#S$, we compute the β-density function with respect to the homogeneous maximal ideal $\mathbf{m}_T = \mathbf{m}_R\#\mathbf{m}_S$. We have*

$$\beta_{R\#S,\mathbf{m}_T}(\lambda) = \begin{cases} 2\lambda^2 & \text{if } 0 \leq \lambda < 1, \\ 2\lambda^2 - 12(\lambda-1)^2 & \text{if } 1 \leq \lambda < 2, \\ 2\lambda^2 - \frac{15}{2}\lambda + \frac{9}{2} & \text{if } 2 \leq \lambda < 3, \\ 0 & \text{if } \lambda \geq 3, \end{cases}$$

$$\beta_{I_i\#S,\mathbf{m}_T}(\lambda) = \begin{cases} \frac{3}{2}\lambda^2 & \text{if } 0 \leq \lambda < 1, \\ \frac{3}{2}\lambda^2 - 9(\lambda-1)^2 & \text{if } 1 \leq \lambda < 2, \\ \frac{3}{2}\lambda^2 - \frac{9}{2}\lambda & \text{if } 2 \leq \lambda < 3, \\ 0 & \text{if } \lambda \geq 3, \end{cases}$$

and

$$\beta_{R\#J_j,\mathbf{m}_T}(\lambda) = \begin{cases} \lambda^2 & \text{if } 0 \leq \lambda < 1, \\ \lambda^2 - 6(\lambda-1)^2 & \text{if } 1 \leq \lambda < 2, \\ \lambda^2 - \frac{9}{2}(\lambda-1) & \text{if } 2 \leq \lambda < 3, \\ 0 & \text{if } \lambda \geq 3. \end{cases}$$

Hence $\tau(I_i\#S, \mathbf{m}_T) = -\frac{1}{2}$ *for* $i = 1, 2$ *and* $\tau(R\#J_j, \mathbf{m}_T) = \frac{1}{2}$ *for* $j = 1, 2, 3$.

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